Rend. Istit. Mat. Univ. Trieste
Volume 42 (2010), 59–64.

On Universal Covers for Four-Dimensional Sets of a Given Diameter

Zsolt Lángi

Abstract. *Makeev proved that among centrally symmetric four-dimensional polytopes, with more than twenty facets and circumscribed about the Euclidean ball of diameter one, there is no universal cover for the family of unit diameter sets. In this paper we examine the converse problem, and prove that each centrally symmetric polytope, with at most fourteen facets and circumscribed about the Euclidean ball of diameter one, is a universal cover for the family of unit diameter sets.*

Keywords: Polytopal Approximation, Universal Cover, Constant-Width Body, Circumscribe, Diameter, Borsuk's Problem.
MS Classification 2010: 52A27, 52B45, 52C17

A convex body C in the Euclidean n-space $\mathbb{R}^n$ is called a *universal cover for sets of diameter* d, if for every set of diameter d there is a congruent copy of C containing it. The problem of finding universal covers for sets of a given diameter, or equivalently, for unit diameter sets, is a long-standing question of discrete geometry. These universal covers are used, in particular, for the solution of Borsuk's problem, that asks the minimal number of subsets of smaller diameters that an n-dimensional set can be partitioned into. For information about Borsuk's problem and its relationship with universal covers, the reader is referred to [10].

As a special case, we may consider universal covers in the family $\mathcal{P}_n$ of centrally symmetric n-dimensional polytopes circumscribed about a Euclidean ball of diameter one. Since every set in the Euclidean space is contained in a constant-width body of the same diameter, we may rephrase this problem as finding polytopes that can be circumscribed about any n-dimensional body of constant width one. This problem is related to Knaster's problem, that asks which finite point sets on the Euclidean sphere $\mathbb{S}^m$ has the property that for any continuous function $f : \mathbb{S}^m \to \mathbb{R}^n$, f is constant on a congruent copy of S (cf., e.g. [6]).

Let D_n denote the dual of the difference body of an n-dimensional regular simplex such that D_n is circumscribed about the Euclidean ball of unit diameter. In 1994, Makeev [7] conjectured that D_n is a universal cover for unit diameter sets. This conjecture is partly motivated by the fact that D_n has $n(n+1)$ facets, and by a result of Makeev [7] that no universal cover in $\mathcal{P}_n$ has more than $n(n+1)$ facets. The $n = 2$ case of Makeev's conjecture is known as Pál's lemma, and has been known since 1920 (cf. [9]). The $n = 3$ case was proven independently by Makeev [8], by G. Kuperberg [6] and by Hausel, Makai, Jr. and Szűcs [5]. In [8] and [6], the main idea of the proofs is that, homologically, for any convex body of constant width one, there are an odd number of congruent copies of D_3 circumscribing it, which yields that geometrically there is at least one. The author of [6] remarks also that in dimension four, homologically, there is zero circumscribed copy of not only D_4, which has twenty facets, but also of the regular cross-polytope, which has sixteen facets.

Our main result is the following. We note that this result, in some sense, is converse to the result in [7], mentioned in the previous paragraph.

THEOREM 1. *Every polytope $P \in \mathcal{P}_4$, with at most fourteen facets, is a universal cover for unit diameter sets.*

To prove this theorem, first we introduce a topological invariant, called Smith index, and recall some estimates regarding it.

Let T be a fixed-point-free involution, with period two, defined on the topological spaces X and Y (for the terminology, cf., for example [3]). Then the pairs (X, T) and (Y, T) are called *T-spaces*, and a continuous mapping $f : X \to Y$ with $Tf = fT$ is called a *T-map*.

Let $S(X)$ be the singular chain complex of X, with Z_2 as the set of coefficients, and let $\rho = I - T = I + T$, where I is the identity operator on $S(X)$. Then, as T, and thus also ρ, commutes with the boundary operator ∂ of $S(X)$, we have that ρ is a chain homomorphism of $S(X)$ into itself. We denote the image of this homomorhism by $S^\rho(X)$. Composing $S^\rho(X)$ with the homology theory functor of Z_2, we obtain the singular Smith homology groups with coefficient group Z_2. We denote the kth group of this theory by $H_k^\rho(X)$, and the corresponding homology group of $S(X)$ by $H_k(X)$. Theorem 1.6 of [4] states that if (X, T) is a Haussdorff T-space and X^* is its orbit space, then $H_k^\rho(X, T) \approx H_k(X^*, Z_2)$, for every value of k, where $H_k(X^*, Z_2)$ is the ordinary kth singular homology group of X^*.

Now, let i denote the inclusion operator, and Δ_k be the boundary operator of $H_k^\rho(X)$. Then

$$\ldots \xleftarrow{i} H_{k-1}^\rho(X) \xleftarrow{\Delta_k} H_k^\rho(X) \xleftarrow{\rho} H_k(X) \xleftarrow{i} H_k^\rho(X) \xleftarrow{\Delta_{k+1}} \ldots \tag{1}$$

is an exact sequence. This follows from the fact that Z_2 is a field, and thus, the couple (i, ρ) is a direct couple, and hence the observation follows from Theorem

2.7, p. 128 of [3].

Furthermore, we may define a homomorphism

$$\tilde{\Delta}_0 : H_0^\rho(X) \to Z_2 \text{ by}$$

$$\tilde{\Delta}_0 = \mathrm{In} \circ \rho^{-1},$$

where In is the Kronecker index homomorphism of the ordinary singular homology theory. This is well defined, as the kernel of ρ is contained in the kernel of In. Then, we define

$$\mathrm{Ind} : H_n^\rho(X) \to Z_2 \text{ by}$$

$$\mathrm{Ind} = \tilde{\Delta}_0 \circ \Delta_1 \circ \Delta_2 \circ \ldots \circ \Delta_n.$$

Finally, the Smith index $\mathrm{Ind}(\emptyset, T)$ is zero. If X is not empty, then $\mathrm{Ind}(X, T)$ is the largest integer such that $\mathrm{Ind}(H_n^\rho(X)) \neq 0$, if it exists, and otherwise it is defined to be infinity.

An important property of Smith index is stated in the following theorem, proved by Geraghty (cf. the remark after Lemma 2.2 of [4]).

THEOREM 2 (Geraghty, 1961). *If there is a T-map $f : (X, T) \to (Y, T)$, then* $\mathrm{Ind}(X, T) \leq \mathrm{Ind}(Y, T)$.

Another important observation that we use later is that if $H_0(X) = Z_2$, and $H_i(X) = 0$ for $i = 2, 3, \ldots, n-1$, then $\mathrm{Ind}(X, T) \geq n$. Indeed, since $H_0(X) = Z_2$ and since the sequence in (1) is exact, we have $H_0^\rho(X) = Z_2$. Thus Δ_1 is onto, and as $H_i(X) = 0$ for $i = 2, 3, \ldots, n-1$, we obtain that Δ_i is onto for $i = 2, 3, \ldots, n$. Since a zero-dimensional ρ-cycle consisting of a point and its T-image has index one, it yields that $\mathrm{Ind}(X, T) \geq n$. In particular, it is well-known that for the Euclidean sphere $\mathbb{S}^n$ with the usual antipodal mapping T, $\mathrm{Ind}(\mathbb{S}^n, T) = n$.

Now recall the notion of Stiefel manifold; that is, the topological space of the orthonormal k-frames in $\mathbb{R}^n$, denoted by $V_{n,k}$. Observe that there is a natural homeomorphism between $V_{n,n-1}$ and $SO(n)$, and between $V_{n,1}$ and $\mathbb{S}^{n-1}$. We use the following estimates, proved by Geraghty, regarding the Smith indices of Stiefel manifolds with respect to the ususal antipodal mapping T (cf. [4]).

THEOREM 3 (Geraghty, 1961). *If s is the largest power of 2 that divides $2n$, then*

$$s - 1 \leq \mathrm{Ind}(SO(2n), T) \leq \mathrm{Ind}(\mathbb{S}^{2n-1}, T) = 2n - 1.$$

In particular, if n is a power of 2, then $\mathrm{Ind}(SO(2n), T) = 2n - 1$.

For completeness, we recall the proof of this estimate from [4].

Proof. Clearly, it is sufficient to prove the general estimate. Consider the Stiefel manifolds $V_{2n,k}$ with the usual antipodal mapping as T. Observe that by deleting the kth member of the frame, we have a sequence of T-maps

$$(SO(2n),T)=(V_{2n,2n-1},T)\to(V_{2n,2n-2},T)\to\ldots\to(V_{2n,1},T)=(\mathbb{S}^{2n-1},T).$$

The orbit space of $SO(2n)$ is the projective special orthogonal group $PSO(2n)$. Thus, the singular Smith homology groups of $SO(2n)$ are the ordinary singular homology groups of $PSO(2n)$.

The Poincaré polynomial of $PSO(2n)$, with the coefficients in Z_2, is

$$P(t)=(1+t)(1+t^2)\ldots(1=t^{s-1})\cdot(1+t+\ldots+t^s+t^s)(1+t^{s+1})\ldots(1+t^{2n-1}),$$

where s is the largest power of 2 dividing $2n$ (cf. [2]).

Furthermore, the Poincaré polynomial of $SO(2n)$ (cf. [1]), with the coefficients in Z_2, is

$$Q(t)=(1+t)(1+t^2)\ldots(1+t^{2n-1}).$$

Now, let the coefficient of t^i in P(t), or in other words the Betti number of $H_i(PSO(2n),Z_2)=H_i^\rho(SO(2n),T)$, be denoted by B_i^ρ, and similarly, let B_i denote the coefficient of t^i in $Q(t)$. Then for $i=0,1,\ldots,s-1$, we have

$$B_i^\rho=\sum_{j=0}^{i}B_j.$$

Thus, in terms of the Betti numbers, the sequence in (1) is

$$0\leftarrow B_0\overset{\rho}{\leftarrow}B_o\overset{\Delta_1}{\leftarrow}B_0+B_1\overset{\rho}{\leftarrow}B_1\overset{i}{\leftarrow}B_0+B_1\overset{\Delta_2}{\leftarrow}B_0+B_1+B_2\overset{\rho}{\leftarrow}\ldots$$

and hence, Δ_i is onto for $i=1,2,\ldots,s-1$. Since $H_0^\rho(SO(2n),T)=Z_2$, we have $\mathrm{Ind}(SO(2n),T)\geq s-1$. Since $\mathrm{Ind}(\mathbb{S}^{2n-1},T)=2n-1$, and since T-maps do not decrease the value of Smith index, we have that

$$s-1\leq\mathrm{Ind}(SO(2n),T)\leq\mathrm{Ind}(\mathbb{S}^{2n-1},T)=2n-1.\qquad\square$$

Now we are ready to prove Theorem 1.

Proof. Consider a convex body $C\subset\mathbb{R}^4$ of constant width one. We may assume that P has exactly fourteen facets. Let $K_1,K_2,\ldots,K_7$ denote the seven infinite strips bounded by pairs of parallel facet-hyperplanes of P, and note that $P=\bigcap_{i=1}^7K_i$. Observe that the width of any of these strips is one. Note also that in any system of vectors spanning $\mathbb{R}^4$, there are four that also span $\mathbb{R}^4$. Applying this observation for the normal vectors of the facet-hyperplanes of P, we obtain that, among $K_1,K_2,\ldots,K_7$, there are four strips such that the hyperplanes bisecting them intersect in a singleton. We may assume that these strips are

K_1, K_2, K_3 and K_4, which yields the (unique) existence of a translation vector x such that $C \subset x + \bigcap_{i=1}^{4} K_i$.

Consider an arbitrary element $\tau \in SO(4)$. We define a function $g : SO(4) \to \mathbb{R}^3$ in the following way. Let x_τ denote the unique vector with the property that $C \subset x_\tau + \bigcap_{i=1}^{4} \tau(K_i)$. Then the three coordinates of $g(\tau)$ are the signed distances, from x_τ, of the three hyperplanes bisecting $\tau(K_5)$, $\tau(K_6)$ and $\tau(K_7)$.

Let T denote the usual antipodal mapping, and note that by Theorem 3, we have $\mathrm{Ind}(SO(4), T) = 3$. As by Theorem 2, T-maps do not decrease the value of Smith index, and as $\mathrm{Ind}(\mathbb{S}^2, T) = 2$, there is no T-map from $(SO(4), T)$ to $(\mathbb{S}^2, T)$. Similarly like in the proof of the classical Borsuk-Ulam theorem, from this it can be shown that for any T-map $g : SO(4) \to \mathbb{R}^3$, there is a point mapped to the origin (this property follows also from Theorem 4.2 of [4]). □

We note that in our consideration, we have shown also the following, more general statement.

THEOREM 4. *Every polytope $P \in \mathcal{P}_{2m}$ with at most $2m + 2\,\mathrm{Ind}(SO(2m), T)$ facets is a universal cover for sets of diameter one.*

Acknowledgements. The author is indebted to M. Naszódi and an anonymous referee for their helpful remarks.

REFERENCES

[1] A. BOREL, *Sur la cohomologie des variétés de Stiefel et de certains groupes de Lie*, C.R. Acad. Sci. Paris **232** (1951), 1628–1630.

[2] ARMAND BOREL, *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes*, Amer. J. Math. **76** (1954), 273–342.

[3] S. EILENBERG AND N. STEENROD, *Foundations of algebraic topology*, Princeton University Press, Princeton (1952).

[4] M.A. GERAGHTY, *Applications of Smith index to some covering and frame theorems*, Nederl. Akad. Wetensch. Proc. Ser. A **64 23** (1961), 219–228.

[5] T. HAUSEL, E. MAKAI, JR. AND A. SZŰCS, *Inscribing cubes and covering by rhombic dodecahedra via equivariant topology*, Mathematika **47** (2000), 371–397; arXiv:math/9906066v2.

[6] G. KUPERBERG, *Circumscribing constant-width bodies with polytopes*, New York J. Math. **5** (1999), 91–100; arXiv:math/9809165v3.

[7] V.V. MAKEEV, *Inscribed and circumscribed polygons of a convex body*, Mat. Zametki **55** (1994), 128–130.

[8] V.V. MAKEEV, *On affine images of a rhombo-dodecahedron circumscribed about a three-dimensional convex body*, Zap. Nauchn. Sem. S. Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **246** (1997), 191–195.

[9] JULIUS PÁL, *Über ein elementares variationsproblem*, Math. Fys. Meddel. Danske Videnskab. Selskab. **3** (1920), 10–35.

[10] A.M. Raĭgorodskiĭ, *Around the Borsuk conjecture*, Sovrem. Mat. Fundam. Napravl. **23** (2007), 147–164.

Author's address:

Zsolt Lángi
Department of Geometry
Budapest University of Technology
Egry József u. 1., 1111 Budapest, Hungary
E-mail: zlangi@math.bme.hu

Received July 15, 2010
Revised October 22, 2010