ON DELTA SEQUENCES (*)

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SOMMARIO. - *Si discute la metrizzabilità di convergenze definite da varie famiglie di delta sequenze. Si dimostra poi un teorema sull'esistenza di convoluzione infinita.*

SUMMARY. - *Metrizability of convergences defined by various families of delta sequences is discussed. A theorem on existence of infinite convolution is proved.*

Introduction.

Delta sequences (called also «approximate identities» or «summability kernels») appear in many branches of mathematics, but probably the most important applications are those in the theory of generalized functions. The basic use of delta sequences is the regularization of generalized function. Futhermore, delta sequences can be used to define convolution and product of generalized functions (see e.g. [1], [7], [8]). T. K. Boehme in [3] has used delta sequences to define «regular operators» (as a subalgebra of Mikusiński Operators). His idea was utilized in the construction of Boehmians (see [9], [10]), where delta sequences play the crucial role.

The first two sections of this note are devoted to the metrizability of a convergence (called Δ-convergence) defined by various

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families of delta sequences. Δ-convergence appears naturally when studying properties of generalized functions called Boehmians (see [10], [11], [12] and [13]).

In the last section we discuss the infinite convolution. The concept was first used in [2] and then widely exploited in [4], [5], [10], [11] and [13]. A theorem on existence of infinite convolution for a large class of sequences is proved.

1. - Abstract approach to delta sequences.

Let (E, p) be a linear (over R or C) quasi-normed space (i. e. $p(f)=0$ iff $f=0$, $p(f_n-f)\to 0$ and $|\alpha_n-\alpha|\to 0$ implies $p(\alpha_n f_n-\alpha f)\to 0$, $p(f+g)\leqq p(f)+p(g)$) and let $F_\circ$ be a subset of E. Suppose that to each pair of elements $f\in E$ and $\varphi\in F_\circ$ there is assigned an element $f*\varphi\in E$ such that the following conditions are satisfied.

(1) If $\varphi, \psi\in F_\circ$, then $\varphi*\psi\in F_\circ$ and $\varphi*\psi=\psi*\varphi$.

(2) If $f\in E$ and $\varphi,\psi\in F_\circ$, then $f*(\varphi*\psi)=(f*\varphi)*\psi$.

(3) If $f,g\in E$ and $\varphi\in F_\circ$, then $(f+g)*\varphi=f*\varphi+g*\varphi$.

(4) If $\lambda\in R$ (or C), $f\in E$ and $\varphi\in F_\circ$, then $\lambda(f*\varphi)=(\lambda f)*\varphi$.

(5) If $p(f_n)\to 0$, then $p(f_n*\varphi)\to 0$ for every $\varphi\in F_\circ$.

Let Λ be a positive functional on $F_\circ$ such that

(6) $\Lambda(\varphi*\psi)\leqq\Lambda(\varphi)+\Lambda(\psi)$ for every $\varphi,\psi\in F_\circ$.

(7) If $p(f_n-f)\to 0$ $(f, f_n\in E)$ and $\Lambda(\delta_n)\to 0$ $(\delta_n\in F_\circ)$, then $p(f_n*\delta_n-f)\to 0$.

A sequence $\delta_n\in F_\circ (n=1,2,\dots)$ for which $\Lambda(\delta_n)\to 0$ will be called a *delta sequence*. The family of all delta sequences will be denoted by Δ.

A sequence $f_n\in E(n=1,2,\dots)$ is said to be *Δ-convergent* to zero if $p(f_n*\delta_n)\to 0$ for some $(\delta_1,\delta_2,\dots)\in\Delta$. In this case we write $\Delta\text{-}\lim f_n=0$. *If* $\Delta\text{-}\lim(f_n-f)=0$, then we write $\Delta\text{-}\lim f_n=f$.

It can be easily checked that Δ-convergence is linear (i.e. Δ-lim $f_n=f$, $\Delta\text{-}\lim g_n=g$ and $|\alpha_n-\alpha|\to 0$ implies $\Delta\text{-}\lim(f_n+g_n)=f+g$ and $\Delta\text{-}\lim\alpha_n f_n=\alpha f$) and that Δ-convergence is weaker than the convergence defined by p (i.e. $p(f_n-f)\to 0$ implies $\Delta\text{-}\lim f_n=f$). Moreover we have the following.

THEOREM 1.1 - *Δ-convergence is of quasi-norm type.*

Proof can be found in [10].

2. - Examples.

Throughout the note we use the following notation

R	the real line
C	the complex plane
$C(R^q)$	the space of continuous functions on R^q
$C_\circ(R^q)$	the space of continuous functions with compact support in R^q
$L_1(R^q)$	the space of Lebesgue integrable functions on R^q
$\|x\| =$	$(x_1^2 + \ldots + x_q^2)^{1/2}$ for $x = (x_1, \ldots, x_q) \in R^q$
$B_\varepsilon =$	$\{x \in R^q : \|x\| < \varepsilon\}$ for $\varepsilon > 0$
$\operatorname{supp}\varphi$	the support of φ for $\varphi \in C_\circ(R^q)$
$s(\varphi) =$	$\inf\{\varepsilon > 0 : \operatorname{supp}\varphi \subset B_\varepsilon\}$ for $\varphi \in C_\circ(R^q)$.

In the following examples the space E is a function space and the product $f * \varphi$ denotes the convolution of functions

$$(f * \varphi)(x) = \int f(u)\,\varphi(x-u)\,du.$$

Now we recall some known properties of the convolution product that will be used later.

LEMMA 2.1 - *If E and $F_\circ$ are such that for every $f \in E$ and $\varphi \in F_\circ$ the convolution $f * \varphi$ exists in $E (f * \varphi \in E)$, and $\varphi * \psi \in F_\circ$, whenever $\varphi, \psi \in F_\circ$, then all the conditions* (1)-(4) *are satisfied.*

LEMMA 2.2 - *If $\varphi, \psi \in C_\circ(R^q)$, then* $\operatorname{supp}\varphi * \psi \subset \operatorname{supp}\varphi + \operatorname{supp}\psi$.

LEMMA 2.3 - *If $\varphi, \psi \in L_1(R^q)$, then $\varphi * \psi \in L_1(R^q)$ and*

$$\int |\varphi * \psi| \leqq \int |\varphi| \cdot \int |\psi|.$$

Example 1 - Let $E = C(R^q)$ and let p be a quasi-norm which defines the uniform convergence on compact subsets of R^q (e.g. $p(f) = \sum_{n=1}^{\infty} 2^{-n} \|f\|_n / (1 + \|f\|_n)$ where $\|f\|_n = \sup\{|f(x)| : x \in B_n\}$).

Let Δ_1 be the family of all sequences $\delta_n \in C_\circ(R^q)$ $(n = 1, 2, \ldots)$ that satisfy the following conditions

A₁ $\int \delta_n = 1$ for all $n \in N$

B₁ $\delta_n \geqq 0$ for all $n \in N$

C₁ The support of δ_n shrinks to zero as $n \to \infty$.

To show that the family Δ_1 fits the abstract approach presented in Section 1 take

$$F_\circ = \{\varphi \in C_\circ(R^q) : \int \varphi = 1 \text{ and } \varphi \geqq 0\}$$

and define

$$\Lambda_1(\varphi) = s(\varphi).$$

It can be verified that all conditions (1)-(7) are satisfied and delta sequences defined by Λ_1 are exactly those in Δ_1.

Remarks. Condition (5) follows immediately from

LEMMA 2.4 - *If $f \in C(R^q)$ and $\varphi \in C_0(R^q)$, then for every $n \in N$*

$$\| f * \varphi \|_n \leqq \| f \|_m \cdot \int | \varphi |$$

where $m \geqq n + s(\varphi)$.

Condition (6) is a consequence of Lemma 2.2, and condition (7) follows from the following lemma (see e.g. [1], Part II 3.1.2).

LEMMA 2.5 - *Let $(\delta_n) \in \Delta_1$, and let the sequence $f_n \in C(R^q)$ converge to f uniformly on compact subsets of R^q. Then*

$$\| f_n * \delta_n - f \|_k \rightarrow 0 \text{ as } n \rightarrow \infty$$

for every $k \in N$.

Example 2 - Let E, F_0 and p be as Example 1. The family Δ_2 is defined by the following conditions

A_2 $\int \delta_n = 1$ for all $n \in N$

B_2 $\int | \delta_n | \rightarrow 1$ as $n \rightarrow \infty$

C_2 The support of δ_n shrinks to zero as $n \rightarrow \infty$.

To describe the family Δ_2 by a functional take

$$F_0 = \{\varphi \in C_0(R^q) : \int \varphi = 1\}$$

and define

$$\Lambda_2(\varphi) = s(\varphi) + \ln \int | \varphi | .$$

All conditions (1)-(7) are satisfied and $(\delta_n) \in \Delta_2$ iff $\Lambda_2(\delta_n) \rightarrow 0$.

It can be shown that Δ_1-convergence is essentially stronger than Δ_2-convergence; (a proof can be found in [11]).

Remarks - Condition (5) follows from Lemma 2.4. To prove (6) use Lemma 2.3 and note that $\ln \int | \varphi * \psi | \leqq \ln \int | \varphi | + \ln \int | \psi |$ and that $\ln \int | \varphi | \geqq 0$ for any $\varphi \in C_0(R^q)$ such that $\int \varphi = 1$. Since Lemma 2.5 holds also for $(\delta_n) \in \Delta_2$, it implies (7).

Example 3 - Let $E = L_1(R^q)$ and $p(f) = \int | f |$. Define a family of delta sequences Δ_3 by the following conditions

A_3 $\int \delta_n = 1$ for all $n \in N$,

B_3 $\int |\delta_n| \to 1$ as $n \to \infty$,

C_3 for every $\varepsilon > 0$, $\int\limits_{|x|>\varepsilon} |\delta_n| \to 0$ as $n \to \infty$.

Put $F_\circ = \{\varphi \in L_1(R^q) : \int \varphi = 1\}$. To define a functional Λ_3 that generates the family Δ_3 we first introduce an auxiliary function

$$\Phi(x) = \min\{|x|, 1\} \quad (\text{for } x \in R^q)$$

and a functional

$$\psi(\varphi) = \int \Phi(x) |\varphi(x)| \, dx \quad (\text{for } \varphi \in F_\circ).$$

LEMMA 2.6 - *Let* $\delta_1, \delta_2, \ldots (\delta_n \in L_1(R^q))$ *satisfy* B_3. *Then* $\delta_1, \delta_2, \ldots$ *satisfies* C_3 *iff* $\psi(\delta_n) \to 0$ *as* $n \to \infty$.

Proof. Assume C_3. Take $\varepsilon > 0$. Then

$$\int \Phi(x) |\delta_n(x)| \, dx = \int\limits_{|x|>\varepsilon} \Phi(x) |\delta_n(x)| \, dx + \int\limits_{|x|<\varepsilon} \Phi(x) |\delta_n(x)| \, dx$$

$$\leqq \int\limits_{|x|>\varepsilon} |\delta_n(x)| \, dx + \varepsilon \cdot \int\limits_{|x|<\varepsilon} |\delta_n(x)| \, dx.$$

By C_3, there exists $k_\circ \in N$ such that $\int\limits_{|x|>\varepsilon} |\delta_n| \, dx < \varepsilon$ for all $n > k_\circ$.

By B_3, there exists $m_\circ \in N$ such that

$$\int\limits_{|x|<\varepsilon} |\delta_n(x)| \, dx < 2 \text{ for all } n > m_\circ.$$

Hence, for $n > \max\{k_\circ, m_\circ\}$ we have

$$\int \Phi(x) |\delta_n(x)| \, dx < 3\varepsilon.$$

Suppose now that $\Psi(\delta_n) \to 0$ as $n \to \infty$. Take $0 < \varepsilon < 1$. Then

$$\int\limits_{|x|>\varepsilon} |\delta_n(x)| \, dx \leqq \frac{1}{\varepsilon} \int\limits_{|x|>\varepsilon} \Phi(x) |\delta_n(x)| \, dx \leqq \frac{1}{\varepsilon} \Psi(\delta_n) \to 0.$$

The functional Ψ cannot be used directly to define Λ_3 because it does not satisfy the triangle inequality (i.e. there are $f, g \in F_\circ$ such that $\Psi(f * g) > \Psi(f) + \Psi(g)$). However, since $\Psi(\varphi_n) \to 0$ and $\Psi(\psi_n) \to 0$ imply $\Psi(\varphi_n * \psi_n) \to 0$ we can use the following

LEMMA 2.7 (see [6]) - *Let* $(X, *)$ *be a semigroup and let* Ψ *be a functional on* X *such that*

$$\Psi(x_n) \to 0, \ \Psi(y_n) \to 0 \text{ implies } \Psi(x_n * y_n) \to 0.$$

Then there exists a functional Ω *on* X *such that*

$$\Omega(x) = 0 \text{ iff } \Psi(x) = 0,$$

$$\Omega(x_n) \to 0 \text{ iff } \Psi(x_n) \to 0 \text{ and}$$

$$\Omega(x * y) \leqq \Omega(x) + \Omega(y).$$

Consequently, the functional

$$\Lambda_3(\varphi) = \ln \int |\varphi(x)|\, dx + \Omega(\varphi)$$

defines equivalently the family Δ_3 and all conditions (1)-(7) are satisfied.

Remarks - Condition (5) follows directly from Lemma 2.3. Condition (6) is guaranteed by Lemma 2.3 and the construction of Ω. Condition (7) can be proved as follows:

$$\begin{aligned} p(f*\delta_n - f) &= \int \left| \int f(x-u)\,\delta_n(u)\,du - f(x)\int \delta_n(u)\,du \right| dx \\ &\leqq \int\int |f(x-u)-f(u)|\;|\delta_n(u)|\,du\,dx \\ &= \int |\delta_n(u)| \left(\int |f(x-u)-f(x)|\,dx\right) du \\ &\leqq \int_{|u|\leqq\varepsilon} |\delta_n(u)| \left(\int |f(x-u)-f(x)|\,dx\right) du + \int_{|u|>\varepsilon} |\delta_n(u)|\,du \cdot 2p(f) \to 0 \end{aligned}$$

as $n\to\infty$ (which follows by the Lebesgue theorem and conditions B_3, C_3).

In the next two examples the families of delta sequences cannot be described by a functional Λ, but what concerns Δ-convergence they do not give anything essentially new. Note that Δ-convergence can be defined for any family of sequences (of elements of $F_\circ$), not necessarily defined by a functional Λ.

Example 4 - Let E, $F_\circ$ and p be as in Example 1. Define a family Δ_4 by the following conditions

A_4 $\int \delta_n = 1$ for all $n \in N$

B_4 $\int |\delta_n| < M$ for some $M > 0$ and for all $n \in N$

C_4 the support of δ_n shrinks to zero as $n \to \infty$.

Obviously, there are more sequences in Δ_4 than in Δ_2. On the other hand, we have the following

THEOREM 2.8 - *Δ_2-convergence and Δ_4-convergence are equivalent.*

Proof. Evidently, Δ_2-convergence implies Δ_4-convergence.

Suppose now, $\Delta_4\text{-}\lim f_n = 0$. Since Δ_2-convergence is metrizable (Theorem 1.1), it suffices to find a subsequence f_{p_n} of f_n such that $\Delta_2\text{-}\lim f_{p_n} = 0$. Let $(\varphi_1, \varphi_2, \ldots) \in \Delta_4$ be such that the sequence of convolutions $f_n * \varphi_n$ converges to zero uniformly on compact subsets of R^q. Let $\psi_1, \psi_2, \ldots$ be a delta sequence from Δ_2. Then, for each $k \in N$, the sequence $\psi_k * \varphi_n$ converges to ψ_k (as $n\to\infty$) uniformly on R^q, (the supports of $\psi_k * \varphi_n$ are commonly bounded). Hence $\int |\psi_k * \varphi_n| \to 1$ as $n \to \infty$. Thus, for each $n \in N$, there exists $p_n \in N$ such that

$$1 \leqq \int |\psi_n * \varphi_{p_n}| \leqq 1 + 1/n.$$

Clearly, we can assume that $p_n \to \infty$. Define $\delta_n = \psi_n * \varphi_{p_n} (n = 1, 2 \ldots)$. Then $(\delta_1, \delta_2, \ldots) \in \Delta_2$ and the sequence of convolutions $f_{p_n} * \delta_n$ converges to zero uniformly on compact subsets of R^q. Therefore, $\Delta_2\text{-}\lim f_{p_n} = 0$, which completes the proof.

Example 5 - The simplest method to obtain a delta sequence is to take a function $\varphi \in C_\circ$ such that $\int \varphi = 1$ and define

$$\delta_n(x) = n^q \varphi(nx) \quad (n = 1, 2, \ldots).$$

It is easy to see that the obtained delta sequence belongs to the family Δ_2. (If $\varphi \geqq 0$, then $(\delta_1, \delta_2, \ldots) \in \Delta_1$). Define Δ_5 to be the family of all sequences obtained in the described way:

$$\Delta_5 = \{ (\delta_1, \delta_2, \ldots) : \delta_n(x) = n^q \varphi(nx), \ \varphi \in C_\circ(R^q) \text{ and } \int \varphi = 1 \}.$$

LEMMA 2.9 - *Let $f_n \in C(R^q)$ $(n = 1, 2, \ldots)$. The following conditions are equivalent:*

(a) *The sequence f_n is Δ_2-convergent to zero.*

(b) *Each subsequence of the sequence f_n contains a subsequence Δ_5-convergent to zero.*

In the proof of the above lemma we use the concept of infinite convolution. All necessary definitions and the proof can be found in Section 3.

Lemma 2.9 gives an answer to a problem posed by Professor Andrzej Kamiński.

3. - Infinite convolutions.

Let $E, p, F_\circ, *$ be as in Section 1 and let $\varphi_n (n = 1, 2, \ldots)$ be a sequence of elements of $F_\circ$. Define $\psi_n = \varphi_1 * \ldots * \varphi_n$. If the sequence ψ_n converges to some $\psi \in E$ (i.e. $p(\psi - \psi_n) \to 0$), then we write $\psi = \varphi_1 * \varphi_2 * \ldots$

THEOREM 3.1 - *Let $E, p, F_\circ, *$ satisfy all the conditions* (1)-(7) *(see Section* 1*). Assume additionally*

(a) *If $p(\varphi_n - \varphi) \to 0$, $\varphi_n \in F_\circ$ and $\Lambda(\varphi_n) \leqq M$, then $\varphi \in F_\circ$ and $\Lambda(\varphi) \leqq M$.*

(b) *If $p(f_n) \to 0$, $\varphi_n \in F_\circ$ and $\Lambda(\varphi_n) \leqq M$, then $P(f_n * \varphi_n) \to 0$.*

Let $\delta_1, \delta_2, \ldots$ be a delta sequence such that $\Lambda(\delta_n) < \infty$. Then for each $n \in N$ the infinite product $\delta_n * \delta_{n+1} * \ldots$ exists and the sequence $\varphi_n = \delta_n * \delta_{n+1} * \delta_{n+2} *$ is a delta sequence.

Proof of the above theorem can be found in [10].

COROLLARY - *Under the assumptions of Theorem* 3.1, *every delta* *sequence δ_n contains a subsequence δ_{p_n} such that for each $n \in N$ the infinite product $\delta_{p_n} * \delta_{p_{n+1}} * \ldots$ exists and the sequence*

$$\varphi_n = \delta_{p_n} * \delta_{p_{n+1}} * \ldots$$

is a delta sequence.

It can be shown that in Examples 1 and 2 all the assumptions in Theorem 3.1 are satisfied. Although, the theorem cannot be applied directly to Λ_3, the family Δ_3 has the property described in the corollary. To prove that we will use the following auxiliary lemma.

LEMMA 3.2 - *Let $\varphi_1, \ldots, \varphi_n \in L_1(R^q)$, $\varepsilon_1, \ldots, \varepsilon_n > 0$ and $\varepsilon \geqq \varepsilon_1 + \ldots + \varepsilon_n$. Denote*

$$\lambda_i^1 = \| \varphi_i \| = \int | \varphi_i | \text{ and } \lambda_i^2 = \int_{|x| > \varepsilon_i} | \varphi_i | \text{ for } i = 1, \ldots, n.$$

Then

$$\int_{|x| > \varepsilon} | \varphi_1 * \ldots * \varphi_n | \leqq (\lambda_1^1 + \lambda_1^2) \ldots (\lambda_n^1 + \lambda_n^2) - \lambda_1^1 \ldots \lambda_n^1 .$$

Proof. For $i = 1, \ldots, n$, define

$$\varphi_i^1(x) = \begin{cases} \varphi_i(x) & \text{if } |x| < \varepsilon_i \\ 0 & \text{if } |x| \geqq \varepsilon_i \end{cases}$$

and $\varphi_i^2(x) = \varphi_i(x) - \varphi_i^1(x)$.

Note that

$$\| \varphi_1^{k_1} * \ldots * \varphi_n^{k_n} \| \leqq \| \varphi_1^{k_1} \| \ldots \| \varphi_n^{k_n} \| \leqq \lambda_1^{k_1} \ldots \lambda_n^{k_n}$$

for $k_i = 1, 2$ and $i = 1, 2, \ldots, n$. Moreover, since $s(\varphi_1^1 * \ldots * \varphi_n^1) < \varepsilon$, we have

$$\int_{|x| > \varepsilon} | \varphi_1^1 * \ldots * \varphi_n^1 | = 0 .$$

Hence

$$\begin{aligned} \int_{|x| > \varepsilon} | \varphi_1 * \ldots * \varphi_n | &= \int_{|x| > \varepsilon} | (\varphi_1^1 + \varphi_1^2) * \ldots * (\varphi_n^1 + \varphi_n^2) | \\ &\leqq \int_{|x| > \varepsilon} | \varphi_1^1 * \ldots * \varphi_1^n | + \int_{|x| > \varepsilon} | \Sigma \varphi_1^{k_1} * \ldots * \varphi_n^{k_n} | \\ &\leqq \Sigma \ \| \varphi_1^{k_1} * \ldots * \varphi_n^{k_n} \| \\ &\leqq \Sigma \lambda_1^{k_1} \ldots \lambda_n^{k_n} \\ &= (\lambda_1^1 + \lambda_1^2) \ldots (\lambda_n^1 + \lambda_n^2) - \lambda_n^1 \ldots \lambda_n^1 \end{aligned}$$

where the summation Σ is taken over all systems $(k_1, \ldots, k_n)$ of indices $k_i = 1, 2$ $(i = 1, \ldots, n)$ except $(1, \ldots, 1)$.

THEOREM 3.3 - *Let* $\varphi_n \in L_1(R^q)$ *and* $\int \varphi_n = 1$ *for* $n = 1, 2 \ldots$ *If*

(a) $\sum_{n=1}^{\infty} \ln \int |\varphi_n| = c < \infty$ *and*

(b) $\int_{|x|>2^{-n}} |\varphi_n| < 2^{-n}$ for $n = 1, 2, \ldots,$

then for each $n \in N$ *the infinite convolution* $\delta_n = \varphi_n * \varphi_{n+1} * \ldots$ *exists and the sequence* $\delta_1, \delta_2, \ldots$ *is a delta sequence* $((\delta_1, \delta_2, \ldots) \in \Delta_3)$.

Proof. Fix $n \in N$. Denote $\psi_k = \varphi_n * \ldots * \varphi_{n+k}$ for $k = 1, 2, \ldots$ We are going to prove that $\psi_1, \psi_2, \ldots$ is a Cauchy sequence in $L_1(R^q)$.

Let $\|\varphi\| = \int |\varphi|$. If $p_1, p_2, \ldots$ is an increasing sequence of indices, then

$$\begin{aligned} \|\psi_{p_{k+1}} - \psi_{p_k}\| &= \|\varphi_{n+1} * \ldots * \varphi_{p_k} * (\varphi_{p_k+1} * \ldots * \varphi_{p_{k+1}} * \varphi_n - \varphi_n)\| \\ &\leqq \|\varphi_{n+1}\| \ldots \|\varphi_{p_k}\| \cdot \|(\varphi_{p_k+1} * \ldots * \varphi_{p_{k+1}}) * \varphi_n - \varphi_n\| \\ &\leqq e^c \cdot \|(\varphi_{p_k+1} * \ldots * \varphi_{p_{k+1}}) * \varphi_n - \varphi_n\| . \end{aligned}$$

To prove that the last term converges to zero as $k \to \infty$ it suffices to show that $\varphi_{p_k+1} * \ldots * \varphi_{p_{k+1}}$ $(k = 1, 2, \ldots)$ is a delta sequence from Δ_3.

Since $\int \varphi_j = 1$ for all $j \in N$, we have also $\int \varphi_{p_k+1} * \ldots * \varphi_{p_{k+1}} = 1$. Condition B_3 follows directly from (a). To prove C_3 we will use Lemma 3.2. Let ε be a positive number. Then for all sufficiently large $k \in N$ we have

$$\int_{|x|>\varepsilon} |\varphi_{p_k+1} * \ldots * \varphi_{p_{k+1}}|$$

$$\leqq (\|\varphi_{p_k+1}\| + 2^{-p_k-1}) \ldots (\|\varphi_{p_{k+1}}\| + 2^{-p_{k+1}}) - \|\varphi_{p_k+1}\| \ldots \|\varphi_{p_{k+1}}\|$$

by Lemma 3.2. Note that $\|\varphi_i\| \geqq 1$ and, by (a), the infinite product $\prod_{i=1}^{\infty} \|\varphi_i\|$ is convergent. Hence, denoting $\|\varphi_i\| = 1 + \eta_i$,

$$\sum_{i=1}^{\infty} (\eta_i + 2^{-i}) = \tfrac{1}{2} + \sum_{i=1}^{\infty} \eta_i < \infty$$

which, in turn, means that the product $\prod_{i=1}^{\infty} (\|\varphi_i\| + 2^{-i})$ is convergent. Thus

$$(\|\varphi_{p_k+1}\| + 2^{-p_k-1}) \ldots (\|\varphi_{p_{k+1}}\| + 2^{-p_{k+1}}) \to 1$$

as $k \to \infty$. Since also $\|\varphi_{p_k+1}\| \ldots \|\varphi_{p_{k+1}}\| \to 1$, we have

$$\int_{|x|>\varphi} |\varphi_{p_k+1} * \ldots * \varphi_{p_{k+1}}| \to 0 \text{ as } k \to \infty .$$

It remains to prove that $\delta_1, \delta_2, \ldots$ is a delta sequence. Since $\int \varphi_n * \ldots * \varphi_{n+k} = 1$ for every $n, k \in N$, we have also $\int \delta_n = 1$. Condition B_3 follows from (a). To prove C_3 note that, for any $\varepsilon > 0$, Lemma 3.2 implies

$$\int_{|x|>\varepsilon} | \varphi_n * \ldots * \varphi_{n+k} |$$
$$\leqq (\int |\varphi_n| + 2^{-n}) \ldots (|\varphi_{n+k}| + 2^{-n-k}) - \int |\varphi_n| \ldots \int |\varphi_{n+k}|$$

for all sufficiently large $n \in N$ and for all $k \in N$. As $k \to \infty$ both products converge, by (a) and (b). Moreover, as $n \to \infty$, the difference converges to zero, which completes the proof.

Proof of Lemma 2.9 - Since $\Delta_5 \subset \Delta_2$ and Δ_2-convergence is metrizable (Theorem 1.1), condition (b) implies condition (a).

Assume (a). Let $(\delta_1, \delta_2, \ldots) \in \Delta_2$ be such that the sequence of convolutions $f_n * \delta_n$ converges to zero uniformly on compact subsets of R^q. Let $p_1, p_2, \ldots$ be an increasing sequence of positive integers such that $\Lambda_2(\delta_{p_n}) < n^{-3}$. Define $\varphi_n(x) = \frac{1}{n^q} \delta_{p_n}(\frac{1}{n} x)$ for $n = 1, 2, \ldots$ Then $\Lambda_2(\varphi_n) < n^{-2}$ and hence, by Theorem 4.1, the infinite convolution $\varphi = \varphi_1 * \varphi_2 * \ldots$ exists. We will prove that the delta sequence $\psi_n(x) = n^q \varphi(nx)$ has the desired property, i.e. the sequence of convolutions $f_{p_n} * \psi_n$ converges to zero unifromly on compact subsets of R^q.

For clarity of the proof we will set $m^q \eta(mx) = \eta^m(x)$. First note that

(1) $\quad \eta_1^m * \ldots * \eta_k^m = (\eta_1 * \ldots * \eta_k)^m .$

(For $k = 2$, the above equality can be obtained by a simple substitution in the integral. For $k > 2$ use induction with respect to k).

Since $\varphi = \lim_{n\to\infty} \varphi_1 * \ldots * \varphi_n$, by (1), we have

$$\begin{aligned}
\varphi^m &= \lim_{n\to\infty} (\varphi_1 * \ldots * \varphi_n)^m \\
&= \lim_{n\to\infty} \varphi_1^m * \ldots * \varphi_n^m \\
&= \varphi_m^m * (\lim_{n\to\infty} \varphi_1^m * \ldots * \varphi_{m-1}^m * \varphi_{m+1}^m * \ldots * \varphi_n^m).
\end{aligned}$$

Denote the second term by γ_m. Since

$$\varphi_m^m = \delta_{p_m}, \text{ we have } \varphi^m = \delta_{p_m} * \gamma_m .$$

Therefore,

$$f_{p_n} * \psi_n = (f_{p_n} * \delta_{p_n}) * \gamma_n$$

where $f_{p_n} * \delta_{p_n}$ converges to zero uniformly of compact subsets of R^q; $\int \gamma_n = 1$ and $\Lambda_2(\gamma_n) < M$ for all $n \in N$ and form some $M > 0$. Thus $f_{p_n} * \psi_n$ converges to zero uniformly on compact sets, which completes the proof.

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