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Existence and Uniqueness of Periodic Solutions for a Quasilinear Parabolic Problem

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Summary. - *We are concerned with the existence and uniqueness of the nonnegative periodic weak solution to a quasilinear parabolic problem of degenerate type, which describes a mathematical model in petroleum engineering. The existence of periodic solutions is established by means of the Schauder fixed point Theorem applied to the Poincaré map. Instead, the uniqueness of the periodic solution is proved under the assumption that* $b(\varphi^{-1})$ *is Hölder continuous of order* $1/2$*, adapting a technique utilized in the study of nonlinear hyperbolic equations.*

1. Introduction

We are interested to study the existence and uniqueness of the non-negative periodic weak solution for the parabolic degenerate problem

$$u_t = \varphi(u)_{xx} - b(u)_x \quad , \qquad \text{in } Q_T := (0,1)\times(0,T) \tag{1}$$
$$\varphi(u(0,t)) = 0 \quad , \qquad \forall\, t \in (0,T) \tag{2}$$
$$\varphi(u(1,t))_x - b(u(1,t)) = -q(t) \quad , \quad \forall\, t\in(0,T) \tag{3}$$
$$u(x,t+\omega) = u(x,t) \quad , \quad u \ge 0 \quad , \quad \omega > 0 \text{ in } Q_T \ , \ T \ge \omega. \tag{4}$$

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This problem, under the assumptions (H_φ) and (H_b) see Sect. 2, is utilized in petroleum engineering to describe the simplest mathematical model in the secondary recovery of oil by means of an amount on injected water in the oil reservoir.

In this type of model, $u(x,t)$ denotes for any $x \in (0,1)$, the saturation of oil at the time t, while $q(t)$ measure the quantity of injected water into the reservoir. The assumptions that shall be done on $\varphi'(u)$, $\varphi'(u)$ vanishes when the saturation u is minimal $u=0$ and maximal $u=1$, reflect the immiscible character of water–oil fluids and make the problem degenerate (see [3]).

The problem of secondary recovery of oil, has been extensively studied in [5, 6, 7] where the existence of strong global solutions and some its properties as regularity and dependence on initial data for a nonlinear degenerate diffusion–convection variational inequality, has been shown. In [6], the uniqueness of the solution is proven in the BV spaces, under the assumption that $b(\varphi^{-1})$ is Hölder continuous of order $1/2$. In [7], besides to derive on physical grounds the equations of the oil recovery, are studied various models utilized in petroleum engineering. The uniqueness of the weak solution for a problem similar to our, is established adapting a technique used to study the nonlinear hyperbolic equations. This approach shall be followed in our paper to show the uniqueness of the periodic solution.

To deal with the periodic solutions, we begin to consider a quasilinear parabolic problem of nondegenerated type, approximating problem (1)–(4). This nondegenerate problem, is obtained adding a so called artificial viscosity term, substituting φ with $\phi_\varepsilon(s) := \varphi(s)+\varepsilon s$, for any $\varepsilon > 0$. The variational formulation of the approximating problem is:

To find u_ε, periodic with respect to t, such that for a.e. $t \in (0,T)$

$$\begin{aligned} \langle u_{\varepsilon t}, v\rangle_{V'V} &+ \varepsilon((u_\varepsilon, v)) + \int_0^1 \varphi(u_\varepsilon)_x v_x dx - \int_0^1 b(u_\varepsilon) v_x dx + \\ &+ q(t) v(1,t) = 0 \quad , \quad \forall\, v(.,t) \in V \end{aligned} \tag{5}$$

where V is a suitable Hilbert space (see Sect. 2).

Existence of periodic solutions to (5) shall be obtained utilizing the Schauder fixed point theorem for the Poincaré map of the associated initial–boundary value problem for all $T \geq \omega$.

To find u_ε such that for a.e. $t \in (0,T)$

$$\langle u_{\varepsilon t}, v\rangle_{V'V} + \varepsilon((u_\varepsilon, v)) + \int_0^1 \varphi(u_\varepsilon)_x v_x dx - \int_0^1 b(u_\varepsilon) v_x dx + q(t)v(1,t) = 0 \quad , \quad \forall\, v(.,t) \in V \tag{5}$$

$$u_\varepsilon(x,0) = u_{0\varepsilon} \quad , \quad \text{a.e. in } (0,1) \ , \tag{6}$$

with $u_{0\varepsilon} \in L^\infty(0,1)$, $0 \le u_{0\varepsilon} \le 1$ a.e. in (0,1).

For (5)–(6) we show the existence and uniqueness of the solution u_ε. To prove the existence of u_ε, we use a classical variational approach, while the uniqueness shall be obtained choosing a suitable test function in (5).

After proving that the Poincaré map for (5)–(6) has a fixed point u_ε, the existence of periodic solutions for (1)–(4) is obtained passing to the limit as $\varepsilon \to 0$ on u_ε. This is the reason why we look for estimates independent of ε.

Finally, the uniqueness of the periodic solution, is showed by means of the assumption

$$b(\varphi^{-1}) \tag{7}$$

is Hölder continuous of order 1/2.

2. Variational Formulation

a) Assumptions

The following assumptions will be made throughout

$$\begin{cases} \varphi \in C^2([0,1]) \ , \ \varphi(0) = \varphi'(0) = \varphi'(1) = 0 \ , \ \varphi'(s) > 0 \text{ for } 0 < s < 1 \\ \text{and } \varphi^{-1} \text{ is Hölder continuous of order } \alpha \in (0,1) \end{cases} \tag{H_φ}$$

$$b \in C^{0,1}([0,1]) \ , \quad \text{such that } b(0) = -1 \le b(s) \le b(1) = 1 \ , \quad \text{for all } s \in [0,1] \tag{H_b}$$

$$q \in C^1([0,T]) \ , \ q(t) \ge 0 \ , \ q(t+\omega) = q(t) \ , \quad \text{for all } t \in [0,T] \ . \tag{H_q}$$

b) Functional Framework of Modelling

Let V be the Hilbert space defined by

$$V := \{v \in H^1(0,1) : v(0) = 0\}$$

with inner product

$$((u,v)) := \int_0^1 u_x v_x dx \ .$$

Let $H = L^2(0,1)$, since $H = H'$, its dual, we can identify V', the dual space of V, to a super space containing H i.e. $V \subset H \subset V'$. The embedding of V in H is dense with continuous injection. Denote with $|\cdot|$ the norm in V and with $\langle .,. \rangle_{V'V}$ the pairing of duality V', V. The norm and the inner product in H are denoted respectively with $\|\cdot\|$ and $(.,.)$. Consider the Hilbert space

$$W(0,T) := \{v \in L^2(0,T;V) \ , \ v_t \in L^2(0,T;V')\}$$

endowed with the usual norm (see [9]).

The variational formulation for problem (1)–(4) is the following:

To find $u \in L^\infty(Q_T) \cap C([0,T]\,; L^2(0,1))$, $0 \le u \le 1$, $u(x,t+\omega) = u(x,t)$, $\forall\, t \in [0,T]$ and a.e. $x \in (0,1)$, $\varphi(u) \in L^2(0,T;V)$, $u_t \in L^2(0,T;V')$ such that for a.e. $t \in (0,T)$

$$\langle u_t, v\rangle_{V'V} + \int_0^1 \varphi(u)_x v_x dx - \int_0^1 b(u) v_x dx + q(t)v(1,t) = 0 \quad , \qquad \forall\, v(.,t) \in V \ . \tag{8}$$

To solve (8), we consider the approximating problem, obtained adding the artificial viscosity term:

To find $u_\varepsilon \in W(0,T)$, $0 \le u_\varepsilon \le 1$, $u_\varepsilon(x,t+\omega) = u_\varepsilon(x,t)$, $\forall\, t \in [0,T]$ and a.e. $x \in (0,1)$, $\phi_\varepsilon(u_\varepsilon) \in L^2(0,T;V)$ such that for a.e. $t \in (0,T)$

$$\begin{aligned} \langle u_{\varepsilon t}, v\rangle_{V'V} &+ \varepsilon((u_\varepsilon, v)) + \int_0^1 \varphi(u_\varepsilon)_x v_x dx - \int_0^1 b(u_\varepsilon) v_x dx + \\ &+ q(t)v(1,t) = 0 \quad , \quad \forall\, v(.,t) \in V \end{aligned} \tag{5}$$

The existence of periodic solutions for (5) shall be proved by means of the Schauder fixed point theorem for the Poincaré map of the associated initial–boundary value problem for all $T \ge \omega$.

To find $u_\varepsilon \in W(0,T)$, $0 \le u_\varepsilon \le 1$ $\forall t \in [0,T]$ and a.e. in (0,1), $\phi_\varepsilon(u_\varepsilon) \in L^2(0,T;V)$ such that for a.e. $t \in (0,T)$

$$\begin{aligned}\langle u_{\varepsilon t}, v\rangle_{V'V} \quad &+ \quad \varepsilon((u_\varepsilon, v)) + \int_0^1 \varphi(u_\varepsilon)_x v_x dx - \int_0^1 b(u_\varepsilon) v_x dx + \\ &+ \quad q(t)v(1,t) = 0 \quad , \quad \forall\, v(.,t) \in V \qquad (5)\end{aligned}$$

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x) \quad , \quad \text{a.e. in } (0,1) \qquad (9)$$

with $u_{0\varepsilon} \in L^\infty(0,1)$, $0 \le u_{0\varepsilon}(x) \le 1$, a.e. in (0,1).

We begin our study, resolving the problem (5)–(9).

3. Existence of periodic solutions for the approximating problem

PROPOSITION 3.1. *If* $(H_\varphi)-(H_q)$ *hold,* $u_{0\varepsilon} \in L^\infty(0,1)$*,* $0 \le u_{0\varepsilon}(x) \le 1$ *a.e. in* $(0,1)$*, there exists at least a solution* u_ε *for* (5)–(9)*.*

Proof. By classical results (see [7]), one knows that for all $\varepsilon > 0$ and for all $g \in W(0,T)$, $0 \le g \le 1$ a.e. in Q_T, there exists a unique $U_\varepsilon(g)$ such that: $U_\varepsilon(g) \in W(0,T)$, $0 \le U_\varepsilon(g) \le 1$ a.e. in Q_T verifying for a.e. $t \in (0,T)$

$$\begin{aligned}\langle U_{\varepsilon t}, v\rangle_{V'V} \quad &+ \quad \varepsilon((U_\varepsilon, v)) + \int_0^1 \varphi'(g) U_{\varepsilon x} v_x dx - \int_0^1 b(g) v_x dx + \\ &+ \quad q(t)v(1,t) = 0 \quad , \quad \forall\, v(.,t) \in V \qquad (10)\end{aligned}$$

$$U_\varepsilon(x,0) = u_{0,\varepsilon}(x) \quad , \quad \text{a.e. in } (0,1)\ . \qquad (11)$$

The existence of solutions u_ε for (5)–(9) follows by the Schauder fixed point theorem, after proving that the map from $W(0,T)$ into itself, which transforms g to $U_\varepsilon(g)$ is such that

a) it leaves invariant the nonempty, convex, weakly compact set $K := \{v \in W(0,T),\ 0 \le v \le 1 \text{ a.e. in } Q_T,\ v(0) = u_{0\varepsilon} \text{ a.e. in (0,1)},\ \|v\|_{L^2(0,T;V)} \le c_1,\ \|v_t\|_{L^2(0,T;V')} \le c_2\}$, with c_1, c_2, suitable constants.

b) it is weakly–sequentially continuous from K to K for the $\sigma(W(0,T); W'(0,T))$ topology.

Taking $v = \phi_\varepsilon(u_\varepsilon)$ as test function in (5) (see [12]), by a result of [1] one has

$$\langle u_{\varepsilon t}, \phi_\varepsilon(u_\varepsilon)\rangle_{V'V} = \partial/\partial t \left(\int_0^1 \left(\int_0^{u_\varepsilon(x,t)} \phi_\varepsilon(\tau)d\tau\right) dx\right) .$$

If we integrate (5) on $(0,T)$ we will have

$$\int_0^T \partial/\partial \left(\int_0^1 \left(\int_0^{u_\varepsilon(x,t)} \phi_\varepsilon(\tau)d\tau\right) dx\right) dt + \int_0^T \int_0^1 |\phi_\varepsilon(u_\varepsilon)_x|^2 dxdt - \\ - \int_0^T \int_0^1 b(u_\varepsilon)\phi_\varepsilon(u_\varepsilon)_x dxdt + \int_0^T q(t)\phi_\varepsilon(u_\varepsilon(1,t))dt = 0 \ , \quad (12)$$

by which

$$\int_0^1 \int_0^{u_\varepsilon(x,T)} \phi_\varepsilon(\tau)d\tau dx + \|\phi_\varepsilon(u_\varepsilon)\|^2_{L^2(0,T;V)} = \\ = \int_0^1 \int_0^{u_{0\varepsilon}(x)} \phi_\varepsilon(\tau)d\tau dx + \int_0^1 \int_0^T b(u_\varepsilon)\phi_\varepsilon(u_\varepsilon)_x dtdx - \\ - \int_0^T q(t)\phi_\varepsilon(u_\varepsilon(1,t))dt \ . \quad (13)$$

Since $\int_0^1 \int_0^{u_\varepsilon(x,T)} \phi_\varepsilon(\tau)d\tau dx \geq 0$, and $\|\phi_\varepsilon\|L^\infty(0,|u_{0\varepsilon}|)$ is bounded independently of ε, the Young inequality gives us

$$\|\phi_\varepsilon(u_\varepsilon)\|^2_{L^2(0,T;V)} \leq \int_0^1 \phi_\varepsilon(u_{0\varepsilon}(x))u_{0\varepsilon}(x)dx + \\ +(1/2)\int_0^T \int_0^1 b(u_\varepsilon)^2 dxdt + (1/2)\int_0^T \int_0^1 |\phi_\varepsilon(u_\varepsilon)_x|^2 dxdt + \\ +QT\phi_\varepsilon(1) \ , \quad (14)$$

($Q := \max\{q(t)$, in $[0,T]\}$)
thus

$$(1/2)\|\phi_\varepsilon(u_\varepsilon)\|^2_{L^2(0,T;V)} \leq \phi_\varepsilon(1) + (1/2)T + QT\phi_\varepsilon(1) \quad (15)$$

that is

$$\|\phi_\varepsilon(u_\varepsilon)\|^2_{L^2(0,T;V)} \leq C \ , \quad (16)$$

with C independent of ε. (Later on, we shall denote with C various constants independent of ε).

Moreover, from (16) we get

$$\|\varphi(u_\varepsilon)\|^2_{L^2(0,T,V)} + \varepsilon^2\|u_\varepsilon\|^2_{L^2(0,T;V)} \le C \tag{17}$$

and from (5) one has

$$\|u_{\varepsilon t}\|_{L^2(0,T;V')} \le C\ . \tag{18}$$

By (17) it follows that $\varphi(u_\varepsilon)$ is bounded in $L^2(0,T;V)$, therefore it is bounded in $L^2(0,T;W^{s,2}(0,1))$, $\forall\, s \in (0,1)$. Since φ^{-1} is Hölder continuous of order $\alpha \in (0,1)$ and $\varphi^{-1}(0)=0$, then $u_\varepsilon \in W^{\alpha s,2/\alpha}(0,1)$ for a.e. $t \in (0,T)$. By classical results (see [10], [2]), one has that u_ε is bounded in $L^{2/\alpha}(0,T;W^{\alpha s,2/\alpha}(0,1))$ and because of the compactness theorem (see [10]), the injection J of $\{v \in L^{2/\alpha}(0,T;W^{\alpha s,2/\alpha}(0,1)),\ v_t \in L^{2/\alpha}(0,T;L^{2/\alpha}(0,1))\}$ in $L^{2/\alpha}(0,T;L^{2/\alpha}(0,1))$ is compact.

Therefore

$$u_\varepsilon \longrightarrow u \quad , \quad \text{in } L^2(Q_T) \text{ and a.e. in } Q_T \text{ with } 0 \le u \le 1 \text{ a.e. in } Q_T \tag{19}$$

$$u_{\varepsilon t} \rightharpoonup u_t \quad , \quad \text{in } L^2(0,T;V') \tag{20}$$

$$\varphi(u_\varepsilon) \rightharpoonup \varphi(u) \quad , \quad \text{in } L^2(0,T;V) \tag{21}$$

$$\phi_\varepsilon(u_\varepsilon) \rightharpoonup \varphi(u) \quad , \quad \text{in } L^2(0,T;V) \tag{22}$$

$$\phi_\varepsilon(u_\varepsilon)_x \longrightarrow \varphi(u)_x \quad , \quad \text{in } L^2(Q_T) \tag{23}$$

□

PROPOSITION 3.2. *If the assumptions* $(H_\varphi)-(H_q)$ *hold and* $u_{0\varepsilon} \in L^\infty(0,1)$, $0 \le u_{0\varepsilon} \le 1$ *a.e. in* $(0,1)$, *then there exists a unique solution* u_ε *to* (5)–(9).

Proof. If u_ε and w_ε solve (5)–(9), then

$$\langle u_{\varepsilon t} - w_{\varepsilon t}, v\rangle_{V'V} + \varepsilon((u_\varepsilon - w_\varepsilon, v)) + \int_0^1 (\varphi(u_\varepsilon)_x - \varphi(w_\varepsilon)_x)v_x dx - q(t)\int_0^1 (b(u_\varepsilon)-b(w_\varepsilon))v_x dx = 0\ . \tag{24}$$

For $\eta > 0$ define

$$\operatorname{sgn}_\eta(x) \begin{array}{ll} = 1 & , \quad x > \eta \\ = \dfrac{x}{\eta} & , \quad |x| \le \eta \\ = -1 & , \quad x < -\eta\ . \end{array} \tag{25}$$

Choosing $v = \mathrm{sgn}_\eta(\phi_\varepsilon(u_\varepsilon) - \phi_\varepsilon(w_\varepsilon))$ in (24), this yields

$$\int_0^1 (\phi_\varepsilon(u_\varepsilon) - \phi_\varepsilon(w_\varepsilon))_x v_x dx \geq 0 \tag{26}$$

$$\begin{aligned} \left| \int_0^1 (b(u_\varepsilon) - b(w_\varepsilon))(\mathrm{sgn}_\eta(\phi_\varepsilon(u_\varepsilon) - \phi_\varepsilon(w_\varepsilon)))_x dx \right| &\leq \\ \leq \frac{\hat{c}}{\eta} \int_{\{|z_\varepsilon - s_\varepsilon| \leq \eta\}} |z_\varepsilon - s_\varepsilon| \, |(z_\varepsilon - s_\varepsilon)_x| dx &\leq \\ \leq \hat{c} \int_{\{|z_\varepsilon - s_\varepsilon| \leq \eta\}} |(z_\varepsilon - s_\varepsilon)_x| dx & \end{aligned} \tag{27}$$

($\hat{c}$ is the Lipschitz constant of $b(\phi_\varepsilon^{-1})$ and $z_\varepsilon = \phi_\varepsilon(u_\varepsilon), s_\varepsilon = \phi_\varepsilon(w_\varepsilon)$).

By a result of [11],

$$\lim_{\eta \to 0^+} \int_{\{|z_\varepsilon - s_\varepsilon| \leq \eta\}} |(z_\varepsilon - s_\varepsilon)_x| dx = 0 \ ,$$

hence, integrating (24) on $(0, t)$ and passing to the limit as $\eta \to 0^+$, because of (26) and (27), we obtain

$$\int_0^1 (u_\varepsilon(x,t) - w_\varepsilon(x,t)) \mathrm{sgn}(\phi_\varepsilon(u_\varepsilon(x,t)) - \phi_\varepsilon(w_\varepsilon(x,t))) dx \leq 0 \ , \tag{28}$$

since $\mathrm{sgn}(z) \to \mathrm{sgn}(z)$ in $L^p(0,1)$, $\forall\, 1 \leq p \leq \infty$ when $\eta \to 0^+$ (see [11]). Since $\phi_\varepsilon(\cdot)$ is an increasing function, $\mathrm{sgn}(\phi_\varepsilon(u_\varepsilon) - \phi_\varepsilon(w_\varepsilon)) = \mathrm{sgn}(u_\varepsilon - w_\varepsilon)$. Thus (28) gives us

$$\int_0^1 |u_\varepsilon(x,t) - w_\varepsilon(x,t)| dx \leq 0 \ , \tag{29}$$

this conclude the prove. □

We write the nonvariational formulation of (5)–(9) as follows

$$c_\varepsilon(v_\varepsilon)_t = v_{\varepsilon xx} - b(c_\varepsilon(v_\varepsilon))_x \quad , \qquad \text{in } Q_T \tag{30}$$

$$v_\varepsilon(0,t) = 0 \quad , \qquad \forall\, t \in (0,T) \tag{31}$$

$$v_{\varepsilon x}(1,t) - b(c_\varepsilon(v_\varepsilon(1,t)) = -q(t) \quad , \qquad \forall\, t \in (0,T) \tag{32}$$

$$v_\varepsilon(x,0) = v_{0\varepsilon}(x) \quad , \qquad \forall\, x \in (0,1) \tag{33}$$

where $\phi_\varepsilon(u_\varepsilon) = v_\varepsilon$ and $u_\varepsilon = \phi_\varepsilon^{-1}(v_\varepsilon) = c_\varepsilon(v_\varepsilon)$.

Define the closed and convex set

$$A := \{w \in C([0,1]) : 0 \le w(x) \le M \quad , \quad \forall\, x \in [0,1]\}$$

and the Poincaré map defined by

$$F(v_{0\varepsilon}(\cdot)) = v_\varepsilon(\cdot, \omega)$$

where $v_\varepsilon(x,t)$ is the unique solution of (30)–(33).

Then, the map F verifies

i) $F(A) \subset A$;

ii) $F|_A$ is continuous;

iii) F is relatively compact in $C([0,1])$.

Condition i) it follows by Proposition 3.1. To show ii) we need of the following result.

PROPOSITION 3.3. *Assume the assumptions of Proposition 3.2 and let $v_{0\varepsilon}^n, v_{0\varepsilon} \in A$ with $v_{0\varepsilon}^n \to v_{0\varepsilon}$ uniformly on $[0,1]$ as $n \to \infty$. Then, if $v_\varepsilon^n, v_\varepsilon$ are the solutions of (30)–(33) with initial data $v_{0\varepsilon}^n$, respectively, $v_{0\varepsilon}$, we get that $v_\varepsilon^n(x,t) \to v_\varepsilon(x,t)$ uniformly on $[0,1]$ as $n \to \infty$, $\forall\, t \in [0,T]$.*

Proof. By

$$\begin{aligned}(c_\varepsilon(v_\varepsilon^n)_t \quad &- \quad c_\varepsilon(v_\varepsilon)_t, \zeta) + \int_0^1 (v_\varepsilon^n - v_\varepsilon)_x \zeta_x dx - \\ &- \int_0^1 (b(c_\varepsilon(v_\varepsilon^n)) - b(c_\varepsilon(v_\varepsilon)))\zeta_x dx = 0 \qquad (34)\end{aligned}$$

with $\zeta = \mathrm{sgn}_\eta(v_\varepsilon^n - v_\varepsilon)$, where $\mathrm{sgn}_\eta(\cdot)$ has been given in (25), one has

$$\int_0^1 |c_\varepsilon(v_\varepsilon^n(x,t)) - c_\varepsilon(v_\varepsilon(x,t))|dx \le \int_0^1 |v_{0\varepsilon}^n(x) - v_{0\varepsilon}(x)|dx \ . \qquad (35)$$

Therefore, $c_\varepsilon(v_\varepsilon^n(.,t))$ strongly converges to $c_\varepsilon(v_\varepsilon(.,t))$ in $L^1(0,1)$ and a.e. in (0,1) when $n \to \infty$. Since $0 \le v_\varepsilon^n(.,t) \le M$, by the Lebesgue theorem, we conclude that $v_\varepsilon^n(.,t) \to v_\varepsilon(.,t)$ in $L^p(0,1)$,

$\forall\, 1 \le p \le \infty$. Moreover, $v_\varepsilon^n(.,t), v_\varepsilon(.,t) \in C([0,1])$ which implies the uniform convergence.

Finally, we prove that F is relatively compact in $C([0,1])$.

Using regularization arguments i.e. convolutions with mollifiers functions, it is possible to approximate $v_{0\varepsilon}$ and b as follows

$$\begin{cases} v_{0\varepsilon}^s \in C^2([0,1]) \ , \quad 0 \le v_{0\varepsilon}^s(x) \le 1 \ , \quad \forall\, x \in [0,1] \ , \quad v_{0\varepsilon}^s(0) = 0 \\ v_{0\varepsilon}^s \longrightarrow v_{0\varepsilon} \ \text{ uniformly on } [0,1] \\ |(v_{0\varepsilon}^s(x))'| \le M \ , \quad \forall\, x \in [0,1] \\ b_s \in C^1([0,1]) \ , \quad b_s \longrightarrow b \ \text{ uniformly on the compact sets} \\ (v_{0\varepsilon}^s(1))' - b_s(c_\varepsilon(v_{0\varepsilon}^s(1))) = -q(0) \ . \end{cases} \tag{36}$$

Under this assumptions, by classical results (see [8]), there exists a unique solution $v_\varepsilon^s \in C^{2+\alpha,1+\alpha/2}(Q_T)$, $\alpha \in (0,1)$ for the problem

$$c_\varepsilon(v_\varepsilon^s)_t = v_{\varepsilon xx}^s - b_s(c_\varepsilon(v_\varepsilon^s))_x \quad , \qquad \text{in } Q_T \tag{37}$$

$$v_\varepsilon^s(0,t) = 0 \quad , \qquad \forall\, t \in (0,T) \tag{38}$$

$$v_{\varepsilon x}^s(1,t) - b_s(v_\varepsilon^s(1,t)) = -q(t) \quad , \qquad \forall\, t \in (0,T) \tag{39}$$

$$v_\varepsilon^s(x,0) = v_{0\varepsilon}(x) \quad , \qquad \text{in } [0,1] \tag{40}$$

we can prove this result. □

Lemma 3.4. *There exists a constant $M_1 > 0$ such that*

$$|v_{\varepsilon x}^s(x,t)| \le M_1 \quad , \quad \text{in } Q_T \tag{41}$$

Proof. Define $V(x,t) = v_{\varepsilon x}^s - b_s(c_\varepsilon(v_\varepsilon^s))$, thus $V(x,t)$ satisfies $V_x = (v_{\varepsilon x}^s - b_s(c_\varepsilon(v_\varepsilon^s)))_x$ and

$$V_t = v_{\varepsilon xt}^s - b_s'(c_\varepsilon(v_\varepsilon^s))(c_\varepsilon(v_\varepsilon^s))_t \ . \tag{42}$$

Deriving (37) with respect to x, one has

$$c_\varepsilon''(v_\varepsilon^s)v_{\varepsilon x}^s v_{\varepsilon t}^s + c_\varepsilon'(v_\varepsilon^s)v_{\varepsilon xt}^s = V_{xx} \ , \tag{43}$$

by (42) we get

$$c_\varepsilon'(v_\varepsilon^s)V_t + b_s'(c_\varepsilon(v_\varepsilon^s))c_\varepsilon'(v_\varepsilon^s)(c_\varepsilon(v_\varepsilon^s))_t + c_\varepsilon''(v_\varepsilon^s)v_{\varepsilon x}^s v_{\varepsilon t}^s = V_{xx}$$

which implies

$$c_\varepsilon'(v_\varepsilon^s)V_t = V_{xx} - c_\varepsilon'(v_\varepsilon^s)b_s'(c_\varepsilon(v_\varepsilon^s))(c_\varepsilon(v_\varepsilon^s))_t - c_\varepsilon''(v_\varepsilon^s)v_{\varepsilon x}^s v_{\varepsilon t}^s$$

i.e.

$$V_t = (V_{xx}/c'_\varepsilon(v^s_\varepsilon)) - V_x(b'_s(c_\varepsilon(v^s_\varepsilon)) + (c''_\varepsilon(v^s_\varepsilon)v^s_{\varepsilon x})/{c'_\varepsilon}^2(v^s_\varepsilon)) \ .$$

By our assumptions, $V \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ (see [4]), since $V(x,t)$ is also a weak solution, by Proposition 3.1 we have that $V(0,t) \geq 0$, $V(1,t) \leq 0$ and $V(x,0) = v'_{0\varepsilon}(x) - q(0)b_s(c_\varepsilon(v_{0\varepsilon}(x))$ is bounded on $(0,1)$ by assumption (36). $V(x,t)$ is uniformly bounded with respect to s and ε.

Hence,

$$\max\{|V(x,t)| \text{ in } \bar{Q}_T\} \leq L$$

which implies

$$|v^s_{\varepsilon x}(x,t)| \leq M_1 \ .$$

Because of (41) and the boundedness of b_s, we can apply a result of [13] (see Proposition 3.1) where it was showed that $c_\varepsilon(v^s_\varepsilon(x,t))$ is Lipschitz continuous in x uniformly w.r.t. s, and Hölder continuous in t uniformly w.r.t. s. Therefore, for a subsequence, we have that $c_\varepsilon(v^s_\varepsilon) \to c_\varepsilon(\hat{v}_\varepsilon)$ in $C^\beta(\bar{Q}_T)$, $\beta \in (0,1)$, as $s \to \infty$. Moreover, $v^s_\varepsilon \to \hat{v}_\varepsilon$ pointwise and in $L^p(Q_T)$, $1 \leq p \leq \infty$ by the Lebesgue theorem. It is very easy to prove that $\hat{v}_\varepsilon$ solves (37)-(40).

By the uniqueness of the solution to (37)–(40), it follows that $v_\varepsilon = \hat{v}_\varepsilon$. Since (41) is stable with respect to the weak convergence in $L^2(Q_T)$, one has

$$|v_{\varepsilon x}(x,t)| \leq M_1 \ . \tag{44}$$

The estimate (44) implies that $v_\varepsilon(.,\omega)$ is Lipschitz continuous, hence the Poincaré map F is relatively compact in $C([0,1])$. To this point, we can conclude with the following □

THEOREM 3.5. *If* $H_\varphi) - (H_q)$ *hold, then there exists at least one* ω*–periodic solution for* (5).

Proof. We have proved that F is continuous and relatively compact hence, by the Schauder fixed point theorem, there exists a fixed point v_ε for the Poincaré map F. This fixed point is an ω–periodic solution to the approximated problem (5). □

4. Existence and uniqueness of the periodic solution

Because of (19)–(23) and by the Lebesgue theorem, which implies that $b(u_\varepsilon) \to b(u)$ a.e. and in $L^2(Q_T)$, it follows that u is an ω–periodic solution to (8) such that $u \in L^\infty(Q_T) \cap C([0,T]; L^2(\Omega))$.

To show the uniqueness of the periodic solution to (1)–(4), we use the condition

$$|b(\varphi^{-1}(t)) - b(\varphi^{-1}(s))| \leq c|t-s|^{1/2} \tag{7}$$

then,

THEOREM 4.1. *Let $u, \hat{u}$ be two periodic solutions for* (1)–(4) *and* (7) *holds, then*

$$\partial/\partial t \int_0^1 (u(x,t) - \hat{u}(x,t))^+ \, dx \leq 0 \qquad \textit{in the sense of } D'(0,T) \; . \tag{45}$$

Proof. As it is usual for nonlinear first order hyperbolic problem (see [7]), we introduce two arbitrary instants t and τ in $(0,T) \times (0,T)$ and consider u and $\hat{u}$ defined on $(0,1) \times (0,T) \times (0,T)$ as follows $u(x,t,\tau) = u(x,t)$, $\hat{u}(x,t,\tau) = \hat{u}(x,\tau)$.

Let $\xi \in D(0,T)$ such that $\xi \geq 0$ and for any $\delta > 0$ let ρ_δ be a regularizing sequence, such that $\rho_\delta \in D(\mathbf{R})$, $\rho_\delta \geq 0$, $\int_{\mathbf{R}} \rho_\delta(x)dx = 1$, ρ_δ, ρ'_δ uniformly bounded w.r.t. δ and supp ρ_δ, supp $\rho'_\delta \subset [-\delta, \delta]$.

For δ sufficiently small, define

$$\xi_\delta(t,\tau) = \xi((t+\tau)/2)\rho_\delta((t-\tau)/2) \in D((0,T) \times (0,T)) \; .$$

Let $r := \varphi(u) - \varphi(\hat{u})$ and $H_\varepsilon(r) := (r^+)^2/(r^2+\varepsilon)$, $\varepsilon > 0$, it is easy verify that for all $r \in \mathbf{R}$

$$\lim_{\varepsilon \to 0^+} rH'_\varepsilon(r) = 0 \; , \; 0 \leq rH'_\varepsilon(r) \leq 1/2 \; , \; H'_\varepsilon \geq 0 \text{ and } 0 \leq H_\varepsilon(r) \leq 1 \; . \tag{46}$$

Since $u, \hat{u}$ are solutions of (1) we get

$$\begin{aligned} \langle u_t - \hat{u}_\tau, v \rangle_{V'V} &+ \int_0^1 (\varphi(u) - \varphi(\hat{u}))_x v_x dx - \int_0^1 (b(u) - b(\hat{u})) v_x dx + \\ &+ q(t)v(1,t,\tau) - q(\tau)v(1,t,\tau) = 0 \; , \end{aligned} \tag{47}$$

for any $v(\cdot, t, \tau) \in V$ a.e. $(t, \tau) \in (0, T) \times (0, T)$.

Multiplying (47) by $\xi_\delta \geq 0$ and choosing $v = H_\varepsilon(r)$, after an integration on $(0, T) \times (0, T)$, one has

$$\begin{aligned}
&\int_0^T \int_0^T \langle u_t - \hat{u}_\tau, H_\varepsilon(\varphi(u) - \varphi(\hat{u}))\rangle_{V'V} \xi_\delta dt d\tau + \\
&\quad + \int_0^T \int_0^T \int_0^1 |(\varphi(u) - \varphi(\hat{u}))_x|^2 H'_\varepsilon(\varphi(u) - \varphi(\hat{u}))\xi_\delta dx dt d\tau - \\
&\quad - \int_0^T \int_0^T \int_0^1 (b(u) - b(\hat{u}))(\varphi(u) - \varphi(\hat{u}))_x H'_\varepsilon(\varphi(u) - \varphi(\hat{u})) \cdot \\
&\qquad\qquad \cdot \xi_\delta dx dt d\tau + \\
&\quad + \int_0^T \int_0^T (q(t) - q(\tau)) H_\varepsilon(\varphi(u(1,t)) - \varphi(\hat{u}(1,\tau))) \cdot \\
&\qquad\qquad \cdot \xi_\delta dt d\tau = 0 \ . \qquad (48)
\end{aligned}$$

Now, estimate

$$\begin{aligned}
&\int_0^T \int_0^T \int_0^1 (b(u) - b(\hat{u}))(\varphi(u) - \varphi(\hat{u}))_x H'_\varepsilon(\varphi(u) - \varphi(\hat{u}))\xi_\delta dx dt d\tau \leq \\
&\quad \leq \int_0^T \int_0^T \int_0^1 c|s - \hat{s}|^{1/2} |(s - \hat{s})_x| H'_\varepsilon(s - \hat{s})\xi_\delta dx dt d\tau \leq \\
&\quad \leq (1/2) \int_0^T \int_0^T \int_0^1 |(s - \hat{s})_x|^2 H'_\varepsilon(s - \hat{s})\xi_\delta dx dt d\tau + \\
&\quad + c' \int_0^T \int_0^T \int_0^1 |s - \hat{s}| H'_\varepsilon(s - \hat{s})\xi_\delta dx dt d\tau
\end{aligned}$$

by (7) with $s = \varphi(u)$, $\hat{s} = \varphi(\hat{u})$ and the Young inequality.

Thus, (48) yields

$$\begin{aligned}
&\int_0^T \int_0^T \langle u_t - \hat{u}_\tau, H_\varepsilon(\varphi(u) - \varphi(\hat{u}))\rangle_{V'V} \xi_\delta dt d\tau + \\
&\quad + (1/2) \int_0^T \int_0^T \int_0^1 |(s - \hat{s})_x|^2 H'_\varepsilon(s - \hat{s})\xi_\delta dx dt d\tau + \\
&\quad + \int_0^T \int_0^T (q(t) - q(\tau)) H_\varepsilon(\varphi(u(1,t)) - \varphi(\hat{u}(1,\tau)))\xi_\delta dt d\tau \leq \\
&\quad \leq c' \int_0^T \int_0^T \int_0^1 |s - \hat{s}| H'_\varepsilon(s - \hat{s})\xi_\delta dx dt d\tau \ . \qquad (49)
\end{aligned}$$

Setting for any $\varepsilon > 0$

$$\sigma_{\varepsilon 1}(x,y) = \int_y^x H_\varepsilon(\varphi(s)-\varphi(y))ds \ , \ \sigma_{\varepsilon 2}(x,y) = \int_y^x H_\varepsilon(\varphi(x)-\varphi(s))ds$$

$$\lim_{\varepsilon\to 0^+} \sigma_{\varepsilon i}(u,\hat{u}) = (u-\hat{u})^+ \ \text{ a.e. in } \ (0,1)\times(0,T)\times(0,T) \ , \quad i=1,2 \ .$$

By a result of [1],

$$\int_0^T \langle u_t, H_\varepsilon(\varphi(u)-\varphi(\hat{u}))\rangle_{V'V}\xi_\delta dt = -\int_0^T\int_0^1 \sigma_{\varepsilon 1}(u,\hat{u})\partial/\partial t\xi_\delta dxdt \tag{50}$$

and

$$\int_0^T \langle u_\tau, H_\varepsilon(\varphi(u)-\varphi(\hat{u}))\rangle_{V'V}\xi_\delta dt = \int_0^T\int_0^1 \sigma_{\varepsilon 2}(u,\hat{u})\partial/\partial \tau\xi_\delta \ dxd\tau \ . \tag{51}$$

Substituting (50), (51) in (49) we have

$$\begin{aligned} -\int_0^T\int_0^T\int_0^1 (\sigma_{\varepsilon 2}(u,\hat{u})\partial/\partial\tau\xi_\delta + \sigma_{\varepsilon 1}(u,\hat{u})\partial/\partial t\xi_\delta)dxdtd\tau + \\ +\int_0^T\int_0^T (q(t)-q(\tau))H_\varepsilon(\varphi(u(1,t))-\varphi(\hat{u}(1,\tau)))\xi_\delta\, dtd\tau \le \\ \le c'\int_0^T\int_0^T\int_0^1 |s-\hat{s}|H'_\varepsilon(s-\hat{s})\xi_\delta\, dxdtd\tau \ . \end{aligned} \tag{52}$$

Going to the limit as $\delta\to 0^+$ in (52) one has

$$\begin{aligned} -T\int_0^T\int_0^1 ((\xi'(t)\rho(0))/2 - (\xi(t)\rho'(0))/2)\sigma_{\varepsilon 2} + \\ +((\xi'(t)\rho(0))/2 + (\xi(t)\rho'(0))/2)\sigma_{\varepsilon 1})dxdt \le \\ \le c'T\int_0^T\int_0^1 |s-\hat{s}|H'_\varepsilon(s-\hat{s})\xi(t)\rho(0)dxdt \ . \end{aligned} \tag{53}$$

Since $H_\varepsilon(r)$ is an C^1–approximation of the Heaviside function, by the Lebesgue theorem and (46) we conclude that

$$-\int_0^T\int_0^1 \xi'(t)(u(x,t)-\hat{u}(x,t))^+ dxdt \le 0 \tag{54}$$

because $H_\varepsilon(\varphi(u(1,t))-\varphi(\hat{u}(1,t))) \longrightarrow \mathrm{sgn}^+(\varphi(u)-\varphi(\hat{u})) = \mathrm{sgn}^+(u-\hat{u})$ as ε goes to zero and

$$\lim_{\varepsilon\to 0^+} \int_0^T \int_0^1 |s-\hat{s}| H'_\varepsilon(s-\hat{s})\xi(t)\rho(0)dxdt = 0 \ .$$

Therefore,

$$\int_0^T \int_0^1 \xi'(t)(u(x,t)-\hat{u}(x,t))^+ dxdt \geq 0 \ . \tag{55}$$

For any $t \in [0,T]$, $u(.,t) \in L^1(0,1)$ consequently, the bounded function $t \to \int_0^1 (u(.,t)-\hat{u}(.,t))^+dx$ define a distribution. Thus, there exists $\partial/\partial t \int_0^1 (u(x,t)-\hat{u}(x,t))^+dx$ and $\partial/\partial t \int_0^1 (u(x,t)-\hat{u}(x,t))^+dx \leq 0$.

Finally, we obtain the uniqueness of the periodic solution, because (45) implies that $(u(x,.)-\hat{u}(x,.))^+$ as function of t is nonincreasing.

Moreover $t \to (u(x,t)-\hat{u}(x,t))^+$ is periodical, so that it must be constant i.e. $u \leq \hat{u}$. Changing u with $\hat{u}$, the same argument proves that $u \geq \hat{u}$, i.e. $u = \hat{u}$. Concluding, our main result is □

THEOREM 4.2. *If* $(H_\varphi)-(H_q)$ *and (7) hold, there exists a unique* ω*–periodic solution for* (1)–(4).

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