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Representation of Homomorphisms on Function Lattices

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Summary. - *Let $\mathcal{L}$ be a unital vector lattice of continuous functions on a topological space X. We study when every real lattice homomorphism on $\mathcal{L}$ is given by evaluation at some point of X. Some applications are given in order to obtain Banach-Stone type theorems in this context.*

For a topological space X we consider the sets $C(X)$ of all continuous real functions on X, and $C^*(X)$ of all bounded functions in $C(X)$. Let $\mathcal{L} \subset C(X)$ be a unital vector lattice, we say that $\varphi : \mathcal{L} \to \mathbb{R}$ is a *lattice homomorphism* whenever it satisfies:

(i) $\varphi(\lambda f + \mu g) = \lambda\varphi(f) + \mu\varphi(g)$, for all f, $g \in \mathcal{L}$ and all λ, $\mu \in \mathbb{R}$.
(ii) $\varphi(|f|) = |\varphi(f)|$, for all $f \in \mathcal{L}$.
(iii) $\varphi(1) = 1$.

We denote by $Hom(\mathcal{L})$ the set of all lattice homomorphisms on $\mathcal{L}$. Note that every $\varphi \in Hom(\mathcal{L})$ is positive, i.e., $\varphi(f) \geq 0$ when $f \geq 0$, and also φ preserves the supremum and the infimum of functions, i.e., $\varphi(f \vee g) = \varphi(f) \vee \varphi(g)$ and $\varphi(f \wedge g) = \varphi(f) \wedge \varphi(g)$.

In this paper we are mainly interested in the problem as to whether every $\varphi \in Hom(\mathcal{L})$ is given by evaluation at some point of X, that is, if there exists some $x \in X$ such that

$$\varphi(f) = f(x), \text{ for all } f \in \mathcal{L}.$$

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In this case we write $Hom(\mathcal{L}) = X$. The analogous problem for subalgebras of $C(X)$ has been considered for instance in [3] and [10], where several applications are also given. Here we shall obtain some results along the lines of [3]. To avoid repetition, we agree that all topological spaces will always be completely regular and Hausdorff. Recall that $Z(f) = f^{-1}(0)$ denotes the zero-set of a function $f \in C(X)$. For further notation, we refer to Gillman and Jerison [7].

The following Lemma is well known when $\mathcal{L} = C(X)$. For completeness we include a short proof for a general vector lattice $\mathcal{L}$.

LEMMA 1. *Let X be a compact space and let $\mathcal{L} \subset C(X)$ be a unital vector lattice. Then, $Hom(\mathcal{L})$=X.*

Proof. First, we are going to see that every lattice homomorphism $\varphi : \mathcal{L} \to \mathbb{R}$ is point-evaluating, that is, for each $f \in \mathcal{L}$ there exists some $x \in X$ with $\varphi(f) = f(x)$. Otherwise, the function $g = |f - \varphi(f)|$ does not vanish on X, and then $g \geq \varepsilon$, for some $\varepsilon > 0$. But this is a contradiction, since φ is monotone and $\varphi(g) = 0$. Hence, for each $f \in \mathcal{L}$ the zero-set $Z(|f - \varphi(f)|)$ is non-empty.

On the other hand, given $f_1, ..., f_n \in \mathcal{L}$, we have that

$$Z(|f_1 - \varphi(f_1)|) \cap ... \cap Z(|f_n - \varphi(f_n)|) = Z\left(\sum_{i=1}^{n} |f_i - \varphi(f_i)| \right)$$

and therefore the family $\{Z(|f - \varphi(f)|) : f \in \mathcal{L}\}$ has the finite intersection property. By compactness, there exists $x \in \bigcap_{f \in \mathcal{L}} Z(|f - \varphi(f)|)$, and then the proof is complete since $\varphi(f) = f(x)$, for all $f \in \mathcal{L}$. □

It is well known that the Stone-Čech compactification βX of the space X is characterized by the property that every continuous map from X into any compact space can be continuously extended to βX (see e.g. Gillman and Jerison [7]). In particular, every continuous function $f : X \to \mathbb{R}$ admits a (unique) continuous extension $f^\beta : \beta X \to \mathbb{R} \cup \{\infty\}$ (the one-point compactification of $\mathbb{R}$). Note that when f is bounded then f^β is real-valued, and so $f^\beta \in C(\beta X)$. Our main result about the representation of lattice homomorphisms is the following.

THEOREM 2. *Let $\mathcal{L} \subset C(X)$ be a unital vector lattice. For each $\varphi \in Hom(\mathcal{L})$ there exists $\xi \in \beta X$ such that $\varphi(f) = f^\beta(\xi)$, for every $f \in \mathcal{L}$.*

Proof. Define $\mathcal{L}^* = \mathcal{L} \cap C^*(X)$, and let $\mathcal{L}^\beta = \{f^\beta : f \in \mathcal{L}^*\}$. Note that $\mathcal{L}^\beta$ is a unital vector sublattice of $C(\beta X)$. Next, consider the lattice homomorphism $\varphi^\beta : \mathcal{L}^\beta \to \mathbb{R}$ given by $\varphi^\beta(f^\beta) = \varphi(f)$. From Lemma 1, there exists $\xi \in \beta X$ such that $\varphi(f) = f^\beta(\xi)$, for every $f \in \mathcal{L}^*$.

Since $f = f^+ - f^-$, where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$, it is enough to show that $f^\beta(\xi) = \varphi(f)$, for every nonnegative function f in $\mathcal{L}$. So, let $f \in \mathcal{L}$ with $f \geq 0$. For every natural number $n > \varphi(f)$, we have

$$\varphi(f) = \varphi(f \wedge n) = (f \wedge n)^\beta(\xi).$$

Now, taking into account that $f^\beta(\xi) = \lim_{n\to\infty}(f \wedge n)^\beta(\xi)$, it follows at once that $f^\beta(\xi) = \varphi(f)$, as required. □

Our next result, which can be derived directly from Theorem 2, gives a sufficient condition in order to get that every lattice homomorphism on $\mathcal{L}$ is given by evaluation at some point of X.

COROLLARY 3. *Let $\mathcal{L} \subset C(X)$ be a unital vector lattice. Suppose that for each $\xi \in \beta X \setminus X$ there exists $f \in \mathcal{L}$ such that $f^\beta(\xi) = \infty$. Then $Hom(\mathcal{L}) = X$.*

The condition appearing in Corollary 3 seems to be very abstract but it can be easily applied in many cases. For example, let X be a closed subset of $\mathbb{R}^n$ and let $\mathcal{L}$ be any unital vector lattice containing the projection maps $\pi_1, ..., \pi_n$. Then, $Hom(\mathcal{L}) = X$. Indeed, if X is bounded in $\mathbb{R}^n$ then X is compact, and we can apply Lemma 1. On the other hand, if X is unbounded in $\mathbb{R}^n$, the function $g = |\pi_1| + ... + |\pi_n| \in \mathcal{L}$ satisfies $g^\beta(\xi) = \infty$ for every $\xi \in \beta X \setminus X$, and Corollary 3 applies. In particular $\mathcal{L}$ could be the vector lattice $U(X)$ of all uniformly continuous real functions on X.

More generally, if X is a locally compact and σ-compact space, there exists some $g \in C(X)$ with $g^\beta(\beta X \setminus X) = \{\infty\}$. Then, any unital vector lattice $\mathcal{L} \subset C(X)$ which contains such a function g satisfies $Hom(\mathcal{L}) = X$. In particular, suppose that (X, d) is a metric space such that every closed ball is compact. Given $x_0 \in X$, it is easy to see that the uniformly continuous function $g = d(\cdot, x_0)$ satisfies $g^\beta(\beta X \setminus X) = \{\infty\}$. As a consequence, we have in this case that $Hom(U(X)) = X$.

On the other hand, we are going to see that the condition of Corollary 3 is also necessary when $\mathcal{L}$ separates points and closed sets of X (that is, if F is a closed subset of X and $x \in X \setminus F$, there exists some $f \in \mathcal{L}$ so that $f(x) \notin \overline{f(F)}$).

Proposition 4. *Let $\mathcal{L} \subset C(X)$ be a unital vector lattice that separates points and closed sets of X. The following are equivalent:*

(a) *$Hom(\mathcal{L}) = X$.*

(b) *For each $\xi \in \beta X \setminus X$ there exists $f \in \mathcal{L}$ such that $f^\beta(\xi) = \infty$.*

Proof. Suppose that there exists $\xi \in \beta X \setminus X$ such that $f^\beta(\xi) \neq \infty$, for all $f \in \mathcal{L}$. Then, we can define the lattice homomorphism $\varphi_\xi : \mathcal{L} \to \mathbb{R}$ by $\varphi_\xi(f) = f^\beta(\xi)$, for all $f \in \mathcal{L}$. Since $\mathcal{L}$ separates points and closed sets of X, there exists for each $x \in X$ some $f \in \mathcal{L}$ with $f^\beta(\xi) \neq f(x)$. Indeed, if $x \in X$ then $x \neq \xi$ and there exists an open set $U \subset X$ such that $x \in U$ and $\xi \in cl_{\beta X}(X \setminus U)$. So, if f is a function in $\mathcal{L}$ that separates x and the closed subset $F = X \setminus U$, then clearly $f(x) \neq f^\beta(\xi)$. Hence the lattice homomorphism φ_ξ is not given by evaluation at any point of X. □

Recall that a completely regular space is realcompact if, and only if, for every $\xi \in \beta X \setminus X$ there exists some $f \in C(X)$ such that $f^\beta(\xi) = \infty$ (see e.g. Engelking [2], pg. 274). Using this, we can derive the following result.

Corollary 5. *Let X be a realcompact space and let $\mathcal{L} \subset C(X)$ be a unital vector lattice which is uniformly dense in $C(X)$. Then $Hom(\mathcal{L}) = X$.*

The remainder of this paper will be devoted to give some applications. More precisely, we shall see how the fact that the lattice homomorphisms are evaluations can be used to obtain Banach-Stone type theorems for lattices of continuous functions. Recall that the classical Banach-Stone theorem asserts that, for a compact space X, the linear metric structure of $C(X)$ (endowed with the *sup*-norm) determines the topology of X. Further results along this line were obtained by Gelfand and Kolmogoroff [6] and Kaplansky [9]. They proved, respectively, that the topology of a compact space X is determined by the ring structure and by the lattice structure of $C(X)$. These results were extended to the case of a realcompact space X by Hewitt [8] and Shirota [11]. Moreover, Shirota proved in [11] that the lattices $U(X)$ and $U^*(X)$ determine the topology of a complete metric space X, where $U(X)$ denotes the family of all uniformly continuous real functions on X, and $U^*(X)$ denotes the subfamily of all bounded functions in $U(X)$.

THEOREM 6. *For $i = 1, 2$, let $\mathcal{L}_i \subset C(X_i)$ be a unital vector lattice which separates points and closed sets of X_i, and such that $Hom(\mathcal{L}_i) = X_i$. If $\mathcal{L}_1$ is isomorphic to $\mathcal{L}_2$ as unital vector lattices, then X_1 is homeomorphic to X_2.*

Proof. Suppose that $T : \mathcal{L}_2 \to \mathcal{L}_1$ is an isomorphism of unital vector lattices. We define the function $h : X_1 \to X_2$ in the following way: for every $x_1 \in X_1$ consider $\varphi_{x_1} \in Hom(\mathcal{L}_1)$ given by $\varphi_{x_1}(f) = f(x_1)$. Hence $\varphi_{x_1} \circ T \in Hom(\mathcal{L}_2)$, and from the hypothesis the lattice homomorphism $\varphi_{x_1} \circ T$ is given by evaluation at some (unique) $x_2 \in X_2$. We define $h(x_1) = x_2$. Analogously, since T is an isomomorphism, the function h^{-1} exists.

Now, for every function $g \in \mathcal{L}_2$, we have that $g \circ h = T(g)$. Indeed, for every $x_1 \in X_1$,

$$T(g)(x_1) = \varphi_{x_1}(T(g)) = (\varphi_{x_1} \circ T)(g) = \varphi_{h(x_1)}(g) = g(h(x_1)).$$

Now, X_2 is endowed with the weak topology given by $\mathcal{L}_2$, since $\mathcal{L}_2$ separates points and closed sets of X_2, and then h is continuous. The same is true for h^{-1}. □

As an easy consequence of Theorem 6 we obtain the following result essentially due to Kaplansky [9] and Shirota [11].

COROLLARY 7. *Let X and Y be realcompact spaces. Then, X and Y are homeomorphic if, and only if, $C(X)$ and $C(Y)$ are isomorphic as unital vector lattices.*

Next we give two analogous results for lattices of uniformly continuous functions on metric spaces. Let (X, d) be a metric space and let $\mathcal{L} \subset U(X)$ a sublattice of uniformly continuous real functions on X. We say that the family $\mathcal{L}$ is *uniformly separating* if for every pair of subsets $A, B \subset X$ with $d(A,B) > 0$, there exists some $f \in \mathcal{L}$ such that $\overline{f(A)} \cap \overline{f(B)} = \emptyset$.

THEOREM 8. *For $i = 1, 2$, let (X_i, d_i) be a metric space, let $\mathcal{L}_i \subset U(X_i)$ be a unital vector lattice which is uniformly separating, and such that $Hom(\mathcal{L}_i) = X_i$. If $\mathcal{L}_1$ is isomorphic to $\mathcal{L}_2$ as unital vector lattices, then X_1 is uniformly homeomorphic to X_2.*

Proof. As in the proof of Theorem 6, we construct the homeomorphism $h : X_1 \to X_2$. We need to prove that h and h^{-1} are uniformly continuous. Note that the function h satisfies

$$d_1(A,B) = 0 \Rightarrow d_2(h(A), h(B)) = 0.$$

Otherwise, there exist $A, B \subset X_1$ such that $d_1(A,B) = 0$ and $d_2(h(A), h(B)) > 0$. Since $\mathcal{L}_2$ is uniformly separating, there is $g \in \mathcal{L}_2$ with $\overline{g(h(A))} \cap \overline{g(h(B))} = \emptyset$. But this is impossible because $g \circ h \in \mathcal{L}_1$ is a uniformly continuous function, and $d_1(A,B) = 0$.

Now applying a classical result due to Efremovich [1] (see also Engelking [2], pg. 573), it follows that h is uniformly continuous. The same is true for h^{-1}. □

From last Theorem and taking into account that $Hom(U(X)) = X$, whenever X is a metric space whose closed balls are compact (see Examples after Corollary 3), we can deduce that, in this class of metric spaces, the unital vector lattice structure of $U(X)$ determines the uniformity of the metric space X.

COROLLARY 9. *Let X and Y be metric spaces in which every closed ball is compact. Then, X is uniformly homeomorphic to Y if, and only if , $U(X)$ and $U(Y)$ are isomorphic as unital vector lattices.*

The question arises if Corollary 9 can be extended to a more general class of metric spaces. Using different techniques, we prove in [4] that Corollary 9 holds in fact for the class of complete metric spaces. Finally, for a further study of homomorphisms on function lattices, we refer to [5].

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