Rend. Istit. Mat. Univ. Trieste
Vol. XXXII, 25–41 (2000)

Between Semi–closed Sets and Semi–pre–closed Sets

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Summary. - *In this paper a new class of sets, namely ψ-**closed sets** is introduced for topological spaces. This class falls strtictly in between the class of semi-closed sets and the class of semi-preclosed sets. This class also sits strictly in between the class of semi-closed sets and the class of semi-generalized closed sets. We also introduce and study a new class of spaces, namely **semi-$T_{1/3}$ spaces**. Further we introduce and study ψ-**continuous** maps and ψ-**irresolute** maps.*

1. Introduction

N. Levine [21] and M.E.Abd El-Monsef et al. [1] introduced semi-open sets and β-sets respectively. β-sets are also called as semi-preopen sets by Andrijević [2]. Levine [22] generalized the concept of closed sets to generalized closed sets. Bhattacharya and Lahiri [7] generalized the concept of closed sets to semi-generalized closed sets via semi-open sets. The complement of a semi-open (resp. semi-generalized closed) set is called a semi-closed [8] (resp. semi-generalized open [7]) set. A lot of work was done in the field of generalized closed sets. In this paper we employ a new technique to obtain a new class of sets, called ψ-**closed** sets. This class is obtained by generalizing semi-closed sets via semi-generalized open sets. It is shown that the class of ψ-closed sets properly contains

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1991 AMS Classification: 54 A 05, 54 D 10
Keywords and Phrases: semi-closure, sg-open sets, semi-$T_{1/2}$ spaces, semi-$T_{1/3}$ spaces

the class of semi-closed sets and is properly contained in the class of semi-preclosed sets. Further it is observed that the class of ψ-closed sets is indipendent from the class of preclosed sets, the class of g-closed sets, the class of gα-closed sets and the class of αg-closed sets. Moreover this class sits properly in between the class of semi-closed sets and the class of semi-generalized closed sets.

Bhattacharya and Lahiri [7], Jancović and Reilly [19] and Maki et al. [25] introduced semi-$T_{1/2}$ spaces, semi-T_D and ${}_\alpha T_{1/2}$ spaces respectively. Later Dontchev [13] [14] proved that ${}_\alpha T_{1/2}$, semi-T_D and semi-$T_{1/2}$ separation axioms are equivalent. R. Devi, K. Balachandran and H. Maki [5] and R. Devi, H. Maki and K. Balachandran [4] introduced ${}_\alpha T_b$ spaces and T_b spaces respectively. As an application of ψ-closed sets, we introduced a new class of spaces, namely **semi-$T_{1/3}$** spaces. We also characterize semi-$T_{1/3}$ spaces and show that the class of semi-$T_{1/3}$ spaces properly contains the class of semi-$T_{1/2}$ spaces, the class of ${}_\alpha T_b$ spaces and the class of semi-$T_{1/3}$ spaces.

We also introduce and study two classes of maps, namely **ψ-continuity** and **ψ-irresoluteness**. ψ-continuity falls strictly in between semi-continuity [21] and β-continuity [1]. ψ-continuity also falls strictly in between semi-continuity [21] and sg-continuity [30].

2. Preliminaries

Throughout this paper (X,τ), (Y,τ) and (Z,τ) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), $cl(A)$, $int(A)$ and $C(A)$ denote the closure of A, the interior of A and the complement of A in X respectively.

Let us recall the following definitions, which are useful in the sequel.

DEFINITION 2.1. *A subset A of a space (X,τ) is called*

1. *a* **semi-open** *set [21] if $A \subseteq cl(int(A))$ and a* **semi-closed** *set if $int(cl(A)) \subseteq A$,*

2. *a* **preopen** *set [27] if $A \subseteq int(cl(A))$ and a* **preclosed** *set if $cl(int(A)) \subseteq A$,*

3. *an* α-**open** *set [29] if* $A \subseteq int(cl(int(A)))$ *and an* α-**closed** *set if* $cl(int(cl(A))) \subseteq A$,

4. *a* **semi-preopen** *set [2] (=*β-**open** *[1]) if* $A \subseteq cl(int(cl(A)))$ *and a* **semi-preclosed** *set [2] (=*β-**closed** *[1]) if* $int(cl(int(A))) \subseteq A$,

5. *a* **regular-open** *set if* $A = int(cl(A))$ *and a* **regular-closed** *set if* $cl(int(A)) = A$,

6. *a* **semi-regular** *set [11] if it is both semi-open and semi-closed in* (X, τ),

7. *a* δ-**closed** *set [31] if* $A = cl_\delta(A)$, *where* $cl_\delta(A) = \{x \in X / int(cl(U)) \cap A \neq \emptyset, x \in U \text{ and } U \in \tau\}$.

The semi-closure (resp. α-closure, semi-preclosure) of a subset A of (X, τ) is the intersection of all semi-closed (resp α-closed, semi-preclosed) sets that contain A and is denoted by $scl(A)$ (resp. $\alpha cl(A)$, $spcl(A)$). The union of all semi-open subsets of X is called the semi-interior of A and is denoted by $sint(A)$.

The following definitions are useful in the sequel.

DEFINITION 2.2. *A subset* A *of a space* (X, τ) *is called*

1. *a* **generalized closed** *(briefly* **g-closed***) set [22] if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* (X, τ),

2. *a* **semi-generalized closed** *(briefly* **sg-closed***) set [7] if* $scl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semi-open in* (X, τ), *- the complement of a sg-closed set is called a sg-open set -*

3. *a* **generalized semi-closed** *(briefly* **gs-closed***) set [3] if* $scl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* (X, τ),

4. *an* α-**generalized closed** *(briefly* α**g-closed***) set [26] if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* (X, τ),

5. *a* **generalized** α-**closed** *(briefly* **g**α-**closed***) set [25] if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is* α*-open in* (X, τ),

6. *a* **g**α^{**}-**closed** *set [25] if* $cl(A) \subseteq int(cl(U))$ *whenever* $A \subseteq U$ *and* U *is* α*-open in* (X, τ),

7. *a* **generalized semi-preclosed** *(briefly* **gsp-closed***) set [12] if* $spcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* (X, τ),

8. *a* δ**-generalized closed** *(briefly* δ**g-closed***) set [15] if* $cl_\delta(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* (X, τ),

9. *a* **Q** *set [20] if* $int(cl(A)) = cl(int(A))$.

Definition 2.3. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is said to be*

1. **semi-continuous** *[21] if* $f^{-1}(V)$ *is semi-open in* (X, τ) *for every open set* V *of* (Y, σ),

2. **pre-continuous** *[27] if* $f^{-1}(V)$ *is pre-closed in* (X, τ) *for every closed set* V *of* (Y, σ),

3. α**-continuous** *[28] if* $f^{-1}(V)$ *is* α*-closed in* (X, τ) *for every closed set* V *of* (Y, σ),

4. β**-continuous** *[1] if* $f^{-1}(V)$ *is semi-preopen in* (X, τ) *for every open set* V *of* (Y, σ),

5. **g-continuous** *[6] if* $f^{-1}(V)$ *is g-closed in* (X, τ) *for every closed set* V *of* (Y, σ),

6. **sg-continuous** *[30] if* $f^{-1}(V)$ *is sg-closed in* (X, τ) *for every closed set* V *of* (Y, σ),

7. **gs-continuous** *[10] if* $f^{-1}(V)$ *is gs-closed in* (X, τ) *for every closed set* V *of* (Y, σ),

8. **g**α**-continuous** *[25] if* $f^{-1}(V)$ *is g*α*-closed in* (X, τ) *for every closed set* V *of* (Y, σ),

9. α**g-continuous** *[18] if* $f^{-1}(V)$ *is* α*g-closed in* (X, τ) *for every closed set* V *of* (Y, σ),

10. **gsp-continuous** *[12] if* $f^{-1}(V)$ *is gsp-closed in* (X, τ) *for every closed set* V *of* (Y, σ),

11. **irresolute** *[9] if* $f^{-1}(V)$ *is semi-open in* (X, τ) *for every semi-open set* V *of* (Y, σ),

12. **sg-irresolute** *[30] if* $f^{-1}(V)$ *is sg-closed in* (X,τ) *for every sg-closed set* V *of* (Y,σ)*,*

13. **pre-semi-open** *[9] if* $f(U)$ *is semi-open in* (Y,σ) *for every semi-open set* U *of* (X,τ)*,*

14. **pre-semi-closed** *[9] if* $f(U)$ *is semi-closed in* (Y,σ) *for every semi-closed set* U *of* (X,τ)*.*

Definition 2.4. *A space* (X,τ) *is called a*

1. $T_{1/2}$ *space [22] if every g-closed set is closed,*
2. **semi-**$T_{1/2}$ *space [7] if every sg-closed set is semi-closed,*
3. **semi-**T_D *space [19] if every singleton is either open or nowhere dense,*
4. ${}_{\alpha}T_i$ *space [25] if a space* (X,τ^{α}) *is* T_i*, where* $i = 1/2, 1$*,*
5. ${}_{\alpha}T^{*}_{1/2}$ *space [25] if every* $g\alpha^{**}$*-closed set is* α*-closed,*
6. ${}_{\alpha}T_m$ *space [25] if every* $g\alpha^{**}$*-closed set is closed,*
7. T_b *space [4] if every gs-closed set is closed,*
8. ${}_{\alpha}T_b$ *space [5] if every* αg*-closed set is closed,*
9. **semi-**T_1 *space [23] if, for any* $x,y \in X$ *such that* $x \neq y$*, there exist two semi-open sets* G *and* H *such that* $x \in G$*,* $y \in H$ *but* $x \notin H$ *and* $y \notin G$*,*
10. **feebly-**T_1 *space [19], [24] if every singleton is either nowhere dense or clopen,*
11. $T_{3/4}$ *space [15] if every* δ*-g-closed set is* δ*-closed.*

3. Basic properties of ψ-closed sets

We introduce the following definition:

Definition 3.1. *A suset* A *of* (X,τ) *is called a* ψ**-closed set** *if* $scl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is a sg-open set of* (X,τ)*.*

REMARK 3.2. *If A is ψ-closed and U is sg-open with $A \subseteq U$, then $scl(A) \subseteq sint(U)$. This follows from the Theorem 6 of [7].*

THEOREM 3.3.

1. *Every semi-closed set, and thus every closed set and every α-closed set is ψ-closed.*
2. *Every ψ-closed set is sg-closed, and thus semi-preclosed (by Theorem 2.4(i) in [14]) and also gs-closed.*

Proof. Follows immediately from the definitions. □

The following examples show that these implications are not reversible.

EXAMPLE 3.4. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a,b\}\}$. Then $A = \{a,c\}$. A is ψ-closed. B is not a semi-closed set.*

EXAMPLE 3.5. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b,c\}\}$. Then $B = \{b\}$ is sg-open and sg-closed. Since $scl(B) = \{b,c\}$, B is not ψ-closed.*

Thus the class of ψ-closed sets properly contains the class of semi-closed sets, and thus properly contains the class of α-closed sets and also properly contains the class of closed sets. Also the class of ψ-closed sets is properly contained in the class of sg-closed sets, and hence it is properly contained in the class of semi-preclosed sets and contained in the class of gs-closed sets.

THEOREM 3.6.

1. *ψ-closedness and g-closedness are independent notions.*
2. *ψ-closedness is independent from gα-closedness, αg-closedness and preclosedness.*

Proof. It can be seen by the following examples. □

EXAMPLE 3.7. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{a,c\}\}$ and $C = \{c\}$ and $D = \{a,b\}$. C is a ψ-closed set but not even a g-closed set of (X,τ). D is a g-closed set but not a ψ-closed set of (X,τ).*

The following two examples show that ψ-closedness is indipendent from gα-closedness, αg-closedness and preclosedness.

EXAMPLE 3.8. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ *and* $E = \{a\}$. E *is* ψ*-closed but it is neither a* $g\alpha$*-closed nor an* αg*-closed set. Also* E *is not a preclosed set.*

EXAMPLE 3.9. *Let* X, τ *and* B *ba as in the example 3.5.* B *is not a* ψ*-closed set of* (X, τ)*. However* B *is a* $g\alpha$*-closed set hence it is an* αg*-closed set. Moreover* B *is also a preclosed set of* (X, τ)*.*

The following Theorem characterize the ψ-closed sets.

THEOREM 3.10. *Let* A *be a subset of* (X, τ)*. Then*

1. A *is* ψ*-closed if and only if* $scl(A) - A$ *does not contain any non-empty sg-closed set,*

2. *If* A *is* ψ*-closed and* $A \subseteq B \subseteq scl(A)$*, then* B *is* ψ*-closed.*

Proof. 1. Necessity: Suppose that A is ψ-closed and let F be a nonempty sg-closed set with $F \subseteq scl(A) - A$. Then $A \subseteq X - F$ and so $scl(A) \subseteq X - F$. Hence $F \subseteq X - scl(A)$, a contradiction. Sufficiency: Suppose that for $A \subseteq X$, $scl(A) - A$ does not contain a non-empty sg-closed set. Let U be a sg-open set such that $A \subseteq U$. If $scl(A) \not\subseteq U$, then $scl(A) \cap C(U) \neq \emptyset$. It follows from theorem 2.3 in [16] that $scl(A) \cap C(U)$ is sg-closed, a contradiction.

2. Follows from tha fact that $scl(A) = scl(B)$.

□

THEOREM 3.11. *For a subset* A *of* (X, τ)*, the following conditions are equivalent:*

1. A *is sg-open and* ψ*-closed,*

2. A *is semi-regular.*

COROLLARY 3.12. *For a subset* A *of a space* (X, τ)*, the following conditions are equivalent:*

1. A *is pre-open, sg-open and* ψ*-closed,*

2. A *is regular open,*

3. *A is pre-open, sg-open and semi-closed.*

The following example shows that a subset G of a space (X, τ) need not be a closed set even though G is pre-open, sg-open and a Q-set.

EXAMPLE 3.13. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{a\}\}$ *and* $G = \{a\}$. *Clearly* G *is pre-open, sg-open and a* Q*-set but not a closed set.*

THEOREM 3.14. *For a subset* A *of a space* (X, τ), *the following conditions are equivalent:*

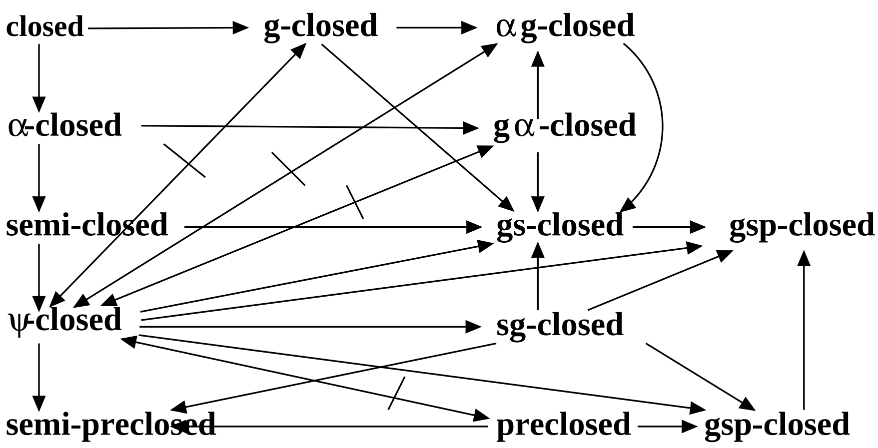
1. *A is clopen,*

2. *A is preopen, sg-open, Q-set and* ψ*-closed.*

Proof. $1 \Rightarrow 2$ is obvious. $2 \Rightarrow 1$: Since A is preopen, sg-open and a ψ-closed set of (X, τ), then by the Theorem 3.12 A is a regular open set. This implies A is open. On the other side, $A = int(cl(A)) = cl(int(A)) \subseteq cl(A)$ since A is a Q-set. So A is closed. Therefore A is a clopen set of (X, τ). □

REMARK 3.15. *Union of two* ψ*-closed sets need not to be* ψ*-closed. Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $A = \{a\}$ *and* $B = \{b\}$. *Both* A *and* B *are* ψ*-closed but* $A \cup B$, *their union, is not a* ψ*-closed set of* (X, τ).

REMARK 3.16. *The following diagram shows tha relationships established between* ψ*-closed sets and some other sets.* $A \longrightarrow B$ *(resp.* $A \longleftrightarrow\!\!\!\!\!\!\!\mid\;\;\; B$*) represents* A *implies* B *but not conversely (resp.* A *and* B *are independent of each other).*



4. Semi-$T_{1/3}$ spaces

We introduce the following definitions:

Definition 4.1. *A space* (X, τ) *is said to be a* **semi-$T_{1/3}$ space** *if every* ψ*-closed set in it is semi-closed.*

Theorem 4.2. *Every semi-*$T_{1/2}$ *space is a semi-*$T_{1/3}$ *space.*

The converse of the above theorem is not true as it can be seen from the following example.

Example 4.3. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. (X, τ) *is not a semi-*$T_{1/2}$ *space since* $\{b\}$ *is a sg-closed set but not a semi-closed set of* (X, τ). *However* (X, τ) *is a semi-*$T_{1/3}$ *space.*

We characterize semi-$T_{1/3}$ spaces in the following Theorem.

Theorem 4.4. *For a space* (X, τ)*, the following conditions are equivalent:*

1. (X, τ) *is a semi-*$T_{1/3}$ *space,*
2. *Every singleton of* X *is either sg-closed or semi-open,*
3. *Every singleton of* X *is either sg-closed or open.*

Proof. 1 ⇒ 2: Let $x \in X$ and suppose that $\{x\}$ is not a sg-closed of (X, τ). Then $X - \{x\}$ is a sg-open set of (X, τ). So X is the only sg-open set of (X, τ). So X is the only sg-open set containing $X - \{x\}$. Hence $X - \{x\}$ is a ψ-closed set of (X, τ). Since (X, τ) is a semi-$T_{1/3}$ space, then $X - \{x\}$ is a semi-closed set of (X, τ) or equivalentely $\{x\}$ is semi-open set of (X, τ).
2 ⇒ 1: Let A be a ψ-closed set of (X, τ). Clearly $A \subseteq scl(A)$. Let $x \in X$. By assumption, $\{x\}$ is either sg-closed or semi-open. Case (i): Suppose $\{x\}$ is sg-closed. By the Theorem 3.10 $scl(A) - A$ does not contain any non-empty sg-closed set. Since $x \in scl(A)$, then $x \in A$. Case (ii): Suppose $\{x\}$ is a semi-open set. Since $x \in scl(A)$, then $\{x\} \cap A \neq \emptyset$. So $x \in A$. Thus in any case, $scl(A) \subseteq A$. Therefore $A = scl(A)$ or equivalentely A is a semi-closed set of (X, τ). Hence (X, τ) is a semi-$T_{1/3}$ space.
2 ⇔ 3: Follows from the fact that a singleton is semi-open if and only if it is open. □

THEOREM 4.5. *Every T_1 space (resp. $T_{3/4}$ space, $T_{1/2}$ space, ${}_\alpha T^*_{1/2}$ space, ${}_\alpha T_m$ space, ${}_\alpha T_1$ space, ${}_\alpha T_{1/2}$ space) is a semi-$T_{1/3}$ space but not conversely.*

Proof. Since every T_1 space (resp. $T_{3/4}$ space, $T_{1/2}$ space, ${}_\alpha T_1$ space, ${}_\alpha T_m$ space, ${}_\alpha T^*_{1/2}$ space, ${}_\alpha T_{1/2}$ space) is a $T_{3/4}$ space [15] (resp. $T_{1/2}$ space [15], semi-$T_{1/2}$ space [7], ${}_\alpha T_{1/2}$ space [25], ${}_\alpha T^*_{1/2}$ space [25], ${}_\alpha T_{1/2}$ space [25], semi-$T_{1/2}$ space [14]), the first assetion is true. The space (X, τ) in the example 4.3 is a semi-$T_{1/3}$ space but not even a semi-$T_{1/2}$ space. □

REMARK 4.6. *Dontchev [13], [14] showed that ${}_\alpha T_{1/2}$, semi-T_D, semi-$T_{1/2}$ separation axioms are equivalent and also that ${}_\alpha T_1$ness and feebly-T_1ness are equivalent. Dontchev and Ganster [15] proved that every space $T_{3/4}$ space is a semi-T_1 space but not conversely.*

THEOREM 4.7. *Every T_b space is a semi-$T_{1/3}$ space and an ${}_\alpha T_b$ space but the respective converses are not true.*

Proof. First we observe that every T_b space is an ${}_\alpha T_b$ space since every αg-closed set is a gs-closed set. Tha fact that every T_b space is a semi-$T_{1/3}$ space follows from the Remark 6.10 of [10] since every T_b

space is a $T_{1/2}$ space. The space in the example 3.8 is an ${}_{\alpha}T_b$ space but not a T_b space. The space in the example 3.5 is a semi-$T_{1/3}$ space but not a T_b space. □

THEOREM 4.8. *Every ${}_{\alpha}T_b$ space is a semi-$T_{1/3}$ but not conversely.*

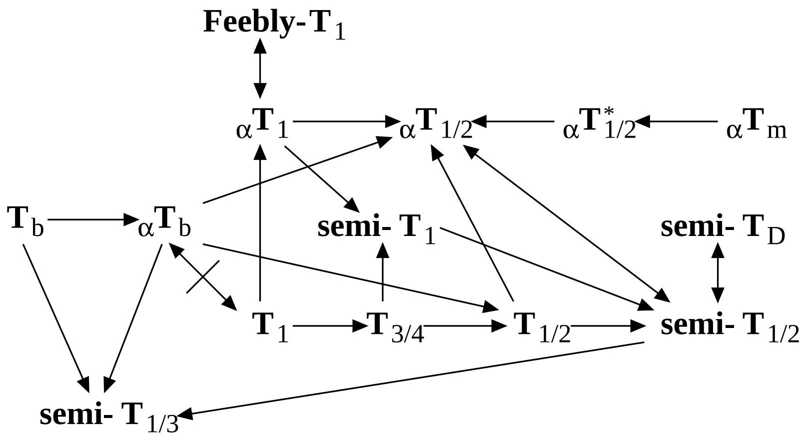
Proof. The first assertion follows from the Theorem 5.3 [5] and the Theorem 4.2 since every $T_{1/2}$ space is a semi-$T_{1/2}$ space. The space in the example 3.5 is a semi-$T_{1/3}$ space but not an ${}_{\alpha}T_b$ space. □

DEFINITION 4.9. *A function $f : (X, \tau) \to (Y, \sigma)$ is called a* **pre-sg-closed** *if $f(U)$ is sg-closed in (Y, σ) for every sg-closed set of (X, τ).*

THEOREM 4.10. *If the domain of a bijective, pre-sg-closed and pre-semi-open map is a semi-$T_{1/3}$ space, then so is the codomain (=range).*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective, pre-sg-closed and pre-semi-open map. Suppose (X, τ) is a semi-$T_{1/3}$ space. Let $y \in Y$. Since f is a bijection, then $y = f(x)$ for some $x \in X$. Since (X, τ) is a semi-$T_{1/3}$ space, then by the Theorem 4.4, $\{x\}$ is either sg-closed or semi-open. If $\{x\}$ is sg-closed, then $\{y\} = f(\{x\})$ is sg-closed since f is a pre-sg-closed map. If $\{x\}$ is semi-open, then $\{y\} = f(\{x\})$ is semi-open since f is a pre-semi-open map. Thus every singleton of Y is either sg-closed or semi-open in (Y, σ). By the Theorem 4.4 again, (Y, σ) is also a semi-$T_{1/3}$ space. □

REMARK 4.11. *The following diagram shows the relationships among the separation axioms considered in this paper. $A \longrightarrow B$ (resp. $A \longleftrightarrow B$, $A \not\longleftrightarrow B$) represents A implies B but B need not imply A always (resp. A and B are equivalent, A and B are indipendent).*



5. Continuous and ψ-irresolute maps

We introduce the following definitions:

DEFINITION 5.1. *A function: $f:(X,\tau)\to(Y,\sigma)$ is called ψ-**continuous** if $f^{-1}(V)$ is a ψ-closed set of (X,τ) for every closed set V of (Y,σ).*

THEOREM 5.2. *1. Every semi-continuous map and thus every continuous map and every α-continuous map is ψ-continuous.*

2. Every ψ-continuous map is sg-continuous and thus β-continuous, gs-continuous and gsp-continuous.

Proof. 1. Let $f:(X,\tau)\to(Y,\sigma)$ be a semi-continuous map. Let V be a closed set of (Y,σ). Since f is semi-continuous, then $f^{-1}(V)$ is a semi-closed set of (X,τ). By the Theorem 3.3, $f^{-1}(V)$ is also a ψ-closed set of (X,τ). Therefore f is a ψ-continuous map.

2. Let $f:(X,\tau)\to(Y,\sigma)$ be a ψ-continuous map. Let V be a closed set of (Y,σ). Since f is ψ-continuous, then $f^{-1}(V)$ is a ψ-closed set of (Y,σ). By the theorem 3.3, $f^{-1}(V)$ is sg-closed and thus β-closed, gs-closed and gsp-closed set of (Y,σ). Therefore f is a sg-continuous map and thus β-continuous, gs-continuous and gsp-continuous.

□

The converse in the above Theorem are not true as it can be seen from the following examples.

EXAMPLE 5.3. *Let* $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, Y, \{a\}, \{b\}\}$ *and* $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. *Let* f *be the identity map from* (X, τ) *into* (Y, σ). f *is not even semi-continuous since* $\{a, c\}$ *is a closed set of* (Y, σ) *but* $f^{-1}(\{a, c\}) = \{a, c\}$ *is not a semi-closed set of* (X, τ). *However* f *is a* ψ*-continuous map.*

EXAMPLE 5.4. *Let* $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\} = \sigma$. *Define* $g : (X, \tau) \to (Y, \sigma)$ *by* $g(a) = c$, $g(b) = a$ *and* $g(c) = c$. g *is not a* ψ*-continuous map since* $\{a\}$ *is a closed set of* (Y, σ) *but* $g^{-1}(\{a\}) = \{b\}$ *is not a* ψ*-closed set of* (X, τ). *However* g *is a* ψ*-continuous map.*

Thus the class of ψ-continuous maps properly contains the class of semi-continuous maps and thus it contains the class of continuous maps the class of α-continuous maps. Also the class of ψ-continuous maps is properly contained in tha class of sg-continuous maps and hence it is contained in the classes of β-continuous maps, gs-continuous maps and gsp-continuous maps.

THEOREM 5.5.
1. *ψ-continuity and g-continuity are independent of each other.*
2. *ψ-continuity is independent from αg-continuity, gα-continuity and precontinuity.*

Proof.
1. Let $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\} = \sigma$. Define $h : (X, \tau) \to (Y, \sigma)$by $h(a) = a$, $h(b) = c$ and $h(c) = b$. h is not g-continuous since $\{b\}$ is a closed set of (Y, σ) but $h^{-1}(\{b\}) = \{c\}$ is not a g-closed set of (X, τ). However h is a ψ-continuous map. Define $\theta : (X, \tau) \to (Y, \sigma)$ by $\theta(a) = c$, $\theta(b) = b$ and $\theta(c) = a$. θ is not ψ-continuous since $\{b, c\}$ is a closed set of (Y, σ) but $\theta^{-1}(\{b, c\}) = \{a, b\}$ is not a ψ-closed set of (X, τ). However θ is a g-continuous map.
2. Let $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{a, c\}\}$. Define $\phi : (X, \tau) \to (Y, \sigma)$ by $\phi(a) = a$, $\phi(b) = b$ and $\phi(c) = c$. ϕ is a ψ-continuous map. ϕ is neither a

pre-continuous nor an αg-continuous map. Moreover ϕ is not a gα-continuous map. The function g in the example 5.4 is not ψ-continuous. However g is pre-continuous, αg-continuous and gα-continuous.

□

The composition of two ψ-continuous maps need not be ψ-continuous as it can be seen from the following Example.

EXAMPLE 5.6. *Let X,Y,τ,σ and ϕ be as in the above result. Let $Z = X$ and $\eta = \{\emptyset, Z, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$. Define $f : (Z,\eta) \to (X,\tau)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Clearly both f and ϕ are ψ-continuous maps. But $\phi \circ f : (Z,\eta) \to (Y,\sigma)$ is not ψ-continuous since $\{b\}$ is a closed set of (Y,σ) but $(\phi \circ f)^{-1}(\{b\}) = f^{-1}(\phi^{-1}(\{b\})) = f^{-1}(\{b\}) = \{a\}$ is not a ψ-closed set of (Z,η).*

We introduce the following definitions:

DEFINITION 5.7. *A function $f(X,\tau) \to (Y,\sigma)$ is called ψ-**irresolute** if $f^{-1}(V)$ is a ψ-closed set of (X,τ) for every ψ-closed set V of (Y,σ).*

Clearly every ψ-irresolute map is ψ-continuous. The converse, however is not true as it can be seen from the following example.

EXAMPLE 5.8. *Let X,Y,τ,σ and f be as in the example 5.3. f is not a ψ-irresolute since $\{a\}$ is a ψ-closed set of (Y,σ) but $f^{-1}(\{a\}) = \{a\}$ is not a ψ-closed set of (X,τ). However f is a ψ-continuous map.*

THEOREM 5.9. *Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\eta)$ be any two functions. Then:*

(i) $g \circ f : (X,\tau) \to (Z,\eta)$ is ψ-continuous if g is continuous and f is ψ-continuous.

(ii) $g \circ f : (X,\tau) \to (Z,\eta)$ is irresolute if g is ψ-irresolute and f is ψ-irresolute.

(iii) $g \circ f : (X,\tau) \to (Z,\eta)$ is ψ-continuous if g is ψ-continuous and f is ψ-irresolute.

Proof. Omitted. □

THEOREM 5.10. *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective ψ-irresolute map. If (X, τ) is a semi-$T_{1/3}$ space, then f is an irresolute map.*

Proof. Let V be a semi-open set of (Y, σ). Then $C(V)$ is a semi-closed set of (Y, σ). By the Theorem 3.3, $C(V)$ is a ψ-closed set of (Y, σ). Since f is a ψ-irresolut map, then $f^{-1}(C(V))$ is a ψ-closed set of (X, τ). Since (X, τ) is a semi-$T_{1/3}$ space, then $f^{-1}(C(V))$ is a semi-closed set of (X, τ). Since f is a bijection, $f^{-1}(V) = C(f^{-1}(C(V)))$. Thus $f^{-1}(V)$ is a semi-open set of (X, τ). Therefore f is an irresolute map. □

THEOREM 5.11. *Let $f : (X, \tau) \to (Y, \sigma)$ be a surjective sg-irresolute and a pre-semi-closed map. Then for every ψ-closed set A of (X, τ), $f(A)$ is a ψ-closed set of (Y, σ).*

Proof. Let A be a ψ-closed set of (X, τ). Let U be a sg-open set of (Y, σ) such that $f(A) \subseteq U$. Since f is a surjective, sg-irresolute map, then $f^{-1}(U)$ is a sg-open set of (X, τ). Then $scl(A) \subseteq f^{-1}(A)$ since A is a ψ-closed set and $A \subseteq f^{-1}(U)$.This implies $f(scl(A)) \subseteq U$. Since f is a pre-semi-closed, then $f(scl(A)) \subseteq scl(f(scl(A)))$. Now $scl(f(A)) \subseteq scl(f(scl(A))) = f(scl(A)) \subseteq U$. Therefore $f(A)$ is a ψ-closed set of (Y, σ). □

THEOREM 5.12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a surjective, ψ-irresolute and a pre-semi-closed map. If (X, τ) is a semi-$T_{1/3}$ space, then (Y, σ) is also a semi-$T_{1/3}$ space.*

Proof. Let A be a ψ-closed set of (Y, σ). Since f is a ψ-irresolute map, then $f^{-1}(A)$ is a ψ-closed set of (X, τ). Since (X, τ) is a semi-$T_{1/3}$ space, then $f^{-1}(A)$ is semi-closed in (X, τ). Then $f(f^{-1}(A))$ is semi-closed in (Y, σ) since f is a pre-semi-closed map. Since f is a surjection, then $A = f(f^{-1}(A))$. Thus A is a semi-closed set of (Y, σ). Therefore (Y, σ) is a semi-$T_{1/3}$ space. □

REFERENCES

[1] M.E. ABD EL-MONSEF, S.N. EL-DEEB, AND R.A. MAHMOUD, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ. **12** (1983), 77–90.

[2] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik **38** (1986), no. 1, 24–32.
[3] S.P. Arya and T. Nour, *Characterizations of s-normal spaces*, Indian J. Pure. Appl. Math. **21** (1990), no. 8, 717–719.
[4] K. Balachandran, R. Devi, and H. Maki, *Semi-generalized closed maps and generalized semi-closed maps*, Mem. Fac. Sci. Kochi Univ. Ser. A. Math. **14** (1993), 41–54.
[5] K. Balachandran, R. Devi, and H. Maki, *Generalized α-closed maps and α-generalized closed maps*, Indian J. Pure. Appl. Math. **29** (1998), no. 1, 37–49.
[6] K. Balachandran, H. Maki, and P. Sundaram, *On generalized continuous maps in topological spaces*, Mem. Fac. Sci. Kochi Univ. Ser. A. Math. **12** (1991), 5–13.
[7] P. Bhattacharya and B.K. Lahiri, *Semi-generalized closed sets in topology*, Indian J. Math. **29** (1987), no. 3, 375–382.
[8] N. Biswas, *On characterizations of semi-continuous functions*, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **48** (1970), no. 8, 399–402.
[9] S.G. Crossley and S.K. Hildebrand, *Semi-topological properties*, Fund. Math. **74** (1972), 233–254.
[10] R. Devi, K. Balachandran, and H. Maki, *Semi-generalized homeomorphism and generalized semi-homeomorphism in topological spaces*, Indian J. Pure. Appl. Math. **26** (1995), no. 3, 271–284.
[11] G. Di Maio and T. Noiri, *On s-closed spaces*, Indian J. Pure. Appl. Math. **18** (1987), no. 3, 226–233.
[12] J. Dontchev, *On generalizing semi-preopen sets*, Mem. Fac. Sci. Kochi Univ. Ser. A. Math. **16** (1995), 35–48.
[13] J. Dontchev, *On point generated spaces*, Question Answers Gen. Topology **13** (1995), 63–69.
[14] J. Dontchev, *On some separation axioms associated with the α-topology*, Mem. Fac. SCi. Kochi Univ. Ser. A. Math. **18** (1997), 31–35.
[15] J. Dontchev and M. Ganster, *On δ-generalized closed sets and $T_{3/4}$ spaces*, Mem. Fac. Sci. Kochi Univ. Ser. A. Math. **17** (1996), 15–31.
[16] J. Dontchev and H. Maki, *On sg-closed sets and semi-λ-closed sets*, Questions and Answers Gen. Topology **15** (1997), 259–266.
[17] W. Dunham, *$T_{1/2}$ spaces*, Kyungpook Math. J. **17** (1977), 161–169.
[18] Y. Gnanambal, *On generalized preregular closed sets in topological spaces*, Indian J. Pure. Appl. Math. **28** (1997), no. 3, 351–360.
[19] D.S. Jancović and I.L. Reilly, *On semi-separation properties*, Indian J. Pure. Appl. Math. **16** (1985), no. 9, 957–964.
[20] N. Levine, *On the commutativity of the closure and the interior op-*

erator in topological spaces, Amer. Math. Montly **68** (1961), 474–477.
[21] N. Levine, *Semi-open sets and semi-continuity in toplogical spaces*, Amer. Math. Montly **70** (1963), 36–41.
[22] N. Levine, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo **19** (1970), no. 2, 89–96.
[23] S.N. Maheswari and R. Prasad, *Some new separation axioms*, Ann. Soc. Sco. Bruxelles Ser. I. **89** (1975), 395–402.
[24] S.N. Maheswari and U. Tapi, *Feebly T_1 spaces*, An. Univ. Timisoara Ser. Stiint. Mat. **16** (1978), no. 2, 173–177.
[25] H. Maki, R. Devi, and K. Balachandran, *Generalized α-closed sets in topology*, Bull. Fukuoka Univ. Ed. Part III **42** (1993), 13–21.
[26] H. Maki, R. Devi, and K. Balachandran, *Associeted topologies of generalized α-closed sets and α-generalized closed sets*, Mem. Fac. Sci. Kochi Univ. Ser. A. Math. **15** (1994), 51–63.
[27] A.S. Mashhour, M.E. Abd El-Monsef, and S.N. El-Deeb, *On pre-continuous and weak pre-continuous mappings*, Proc. Math. and Phys. Soc. Egypt **53** (1982), 47–53.
[28] A.S. Mashhour, I.A. Hasanein, and S.N. El-Deeb, *α-continuous and α-open mappings*, Acta Math. Hung. **41** (1983), no. 3-4, 213–218.
[29] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math. **15** (1965), 961–970.
[30] P. Sundaram, H. Maki, and K. Balachandaran, *Semi-generalized continuous maps and semi-$T_{1/2}$ spaces*, Bull. Fukuoka Univ. Ed. Part. III **40** (1991), 33–40.
[31] N.V. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl. **78** (1968), 103–118.

Received April 26, 1999.