UNCONDITIONALLY CONVERGENT SERIES AND SUBSPACES OF $D^m(0,1)$ (*)

by BOLIS BASIT (in Cairo) (**)

SOMMARIO. - *In questo lavoro si dimostra che i sottospazi complementati di $D^m(0,1)$ hanno dimensione finita. Si dimostra altresì che le sottoalgebre di $D^m(0,1)$ che non contengono sottospazi isomorfi a c_0 sono di dimensione* 1 *o* 0.

SUMMARY. - *In this paper we prove that complemented subspaces of $D^m(0,1)$ are finite dimensional. We also show that subalgebras of $D^m(0,1)$ which do not contain isomorphic to c_0 subspaces are of dimension* 1 *or* 0.

I. Introduction

Let B be a Banach space and B^* its dual Banach space. We write $B \supset c_0 (B \not\supset c_0)$ if B contains (does not contain) a subspace isomorphic to c_0. The series $\sum_{n=1}^{\infty} x_n$ of elements of B is weakly unconditionally convergent (w.u.c.) iff $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty, x^* \in B^*$ and it is unconditionally convergent (u.c.) iff each of its permutation is convergent. It is obvious that u.c. implies w.u.c. The sum of w.u.c. series is not necessarily an element of B and it is proved in [6] p. 61, that if the sum of each subseries of a w.u.c. series is an element of B, then the series is u.c. Also, it is shown in [3] that w.u.c. series are u.c. iff $B \not\supset c_0$.

In this paper we study subspaces and subalgebras of the space

(*) Pervenuto in Redazione il 29 luglio 1981.
(**) Indirizzo dell'Autore: Faculty of Science - University of Cairo - Cairo - Egypt.

$D^m(0,1)$ of m times differentiable functions on $[0,1]$. We show that if $B \not\supset c_0$ and B is complemented in $D^m(0,1)$, or B is a closed subalgebra of $D^m(0,1)$ then B is finite dimensional.

We shall use the following notations. ∂ will denote the operator defined by $\partial f = f', f \in D^m(0,1)$. If $B \subset D^m(0,1)$, then $\partial^n B = \{g : g = h^{(n)}, h \in B\}, 1 \leqslant n \leqslant m$. If $f \in D^m(0,1)$, then $A^m(f), A^{m-1}(f)$ will denote the closed subalgebras of $D^m(0,1)$ and $D^{m-1}(0,1)$ generated by f. $A^{m-1}(f, 1)$ will denote the closed subalgebra of $D^{m-1}(0,1)$ generated by f and 1.

II. Subspaces of $D^m(0,1)$ containing c_0

In this section we prove the following four needed lemmas:

LEMMA 2.1. *Let B be a closed subspace of $D^m(0,1)$. Then the set ∂B is a closed subspace of $D^{m-1}(0,1)$. Moreover, $B \supset c_0$ iff $\partial B \supset c_0$.*

PROOF. Without loss of generality we can assume that B contains the constant functions. Hence $D^{m-1}(0,1) - B = \partial(D^m(0,1) - B)$. Since ∂ is a linear continuous operator from $D^m(0,1)$ to $D^{m-1}(0,1)$, it is open [8] p. 75.

Hence $D^{m-1}(0,1) - B$ is open and therefore ∂B is closed and the linearity of ∂ implies that ∂B is a subspace of $D^{m-1}(0,1)$. Now, let $B \supset c_0$. By [3] there exists a w.u.c. series $\sum_{n=1}^{\infty} g_n$ of elements of B which is not u.c. Using the boundedness of ∂, the series $\sum_{n=1}^{\infty} \partial g_n$ of elements of ∂B is w.u.c. Assuming that $\sum_{n=1}^{\infty} \partial g_n$ is u.c., we get that the sum of each of its subseries is an element of ∂B. This implies that the sum of each subseries of $\sum_{n=1}^{\infty} g_n$ is an element of B. By theorem 3.2.3 [6] p. 62, the series $\sum_{n=1}^{\infty} g_n$ is u.c. This is a contradiction which proves that the series $\sum_{n=1}^{\infty} \partial g_n$ is w.u.c. which is not u.c. Again by [3], we get that $\partial B \supset c_0$. Conversely, let $\partial B \supset c_0$. Let $\sum_{n=1}^{\infty} h_n$ be w.u.c. series of elements of ∂B, which is not u.c. Consider the functions $g_n(t) = \int_0^t h_n(x)\,dx, 0 \leqslant t \leqslant 1, n \in N$. Then the series $\sum_{n=1}^{\infty} g_n$ of elements of B is w.u.c. which is not u.c. Hence $B \supset c_0$.*

REMARK 2.1. It follows easily that $B \not\supset c_0$ iff $\partial B \not\supset c_0$.

REMARK 2.2. The set $\partial^n B$ is a closed subspace of $D^{m-n}(0,1)$, $1 \leqslant n \leqslant m$. Moreover, $B \supset c_0$ iff $\partial^n B \supset c_0$. The proof follows easily using Lemma 2.1 n times.

The following result seems to be useful in itself and is needed for Sec. III.

LEMMA 2.2. Let $f \in D^m(0,1)$. Then $f' \in A^{m-1}(f)$.

PROOF. Consider the sequence of Banach spaces $D^m(-\frac{1}{n}, 1+\frac{1}{n})$, $n \in N$. Choose $f_n \in D^m(-\frac{1}{n}, 1+\frac{1}{n})$ such that $f_n(t) = f(t), t \in [0,1]$, $n \in N$. Consider the sequence $h_n(t) = f_n(t+\frac{1}{n}), 0 \leqslant t \leqslant 1, n \in N$. Denote by C_n the Banach subspace of $D^{m-1}(0,1)$ generated by the elements of $A^{m-1}(f)$ and the functions $h_k, k \geqslant n, n \in N$. We have $C_1 \supset C_2 \supset C_3 \supset \ldots$ and $\bigcap_{n=1}^{\infty} C_n = A^{m-1}(f)$.

Since $f^{(k)}(t) = \lim_{n\to\infty} n\,[h_n^{(k-1)}(t) - f^{(k-1)}(t)], t \in [0,1]$, we get that $f^{(m)}$ is a weak limit of a sequence of elements of $\partial^{m-1} C_n, n \in N$. Hence $f^{(m)} \in \partial^{m-1} C_n, n \in N$ and this implies that $f^{(m)} \in \partial^{m-1} A^{m-1}(f)$. Hence $f' \in A^{m-1}(f)$.*

Let A be a subalgebra of $D^m(0,1)$. Then ∂A is not necessarily a subalgebra of $D^{m-1}(0,1)$. We prove

LEMMA 2.3. *Let* $f \in D^m(0,1)$. *Then* $A^{m-1}(f') \subset f'A^{m-1}(f,1) \subset \partial A^{m-1}(f)$.

PROOF. Let $h \in A^{m-1}(f,1)$. There is a sequence $P_n(f) \sum_{k=0}^{\infty} c_{k,n} f^k$ of linear combinations of powers of f which converges to h in the sense of the norm of $D^{m-1}(0,1)$. This implies that $\{f' P_n(f)\}$ converges to $f'h$ in $D^{m-1}(0,1)$. Each of the functions $f' P_n(f)$ is a derivative of another linear combination of powers of f. Therefore $f' P_n(f) \in \partial A^{m-1}(f), n \in N$. Since $\partial A^{m-1}(f)$ is closed, we get $f'h \in \partial A^{m-1}(f)$. This proves that $f' A^{m-1}(f,1) \subset \partial A^{m-1}(f)$. Now, by Lemma 2.2 we have $f' \in A^{m-1}(f) \subset A^{m-1}(f,1)$. Therefore $f' A^{m-1}(f,1)$ is a closed ideal of $A^{m-1}(f,1)$. Since $1 \in A^{m-1}(f,1)$, we conclude that $f' \in f' A^{m-1}(f,1)$, and therefore $A^{m-1}(f') \subset f' A^{m-1}(f,1)$.*

The following lemma is a criterion for a subalgebra of $C(0,1)$ to have isomorphic to c_0 subspaces.

LEMMA 2.4. *Let* $f \in C(0,1)$ *have an infinite range. Then the closed subalgebra* $A^0(f)$ *generated by* f *has isomorphic to* c_0 *subspace (i.e.* $A^0(f) \supset c_0$*).*