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Indefinite Pettis integral of multifunctions in locally convex spaces

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Abstract. *In this paper, we first prove that indefinite Pettis integral of multifunctions in locally convex spaces is a μ-continuous strong multimeasure. Then, we present a full characterization of strong multimeasures in terms of weak multimeasures.*

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1. Introduction and Preliminaries

The notion of Pettis integrable multifunction was first considered in [11, Chapter 4] and has been pretty recently studied in [1, 7–10, 16, 18, 23, 24, 29, 30]. In the last decades was made a great deal of work about measurable and integrable multifunctions. Some pioneering and highly influential ideas and notions around the matter were inspired by problems arising in Control Theory and Mathematical Economics. We can cite the papers by Aumann [2] and Debreu [12], the monographs by Castaing and Valadier [11], Klein and Thompson [20], and the survey by Hess [17]. There are beautiful research results for gauge integrals of multifunctions in the papers [3–7, 14, 15]. The definitions and further properties of the integrals of multifunctions are depended on the existence of measurable selectors. The best results for the existence of measurable selectors are achieved in the papers [9, 10] by Cascales, Kadets, and Rodríguez.

In this paper we first prove that indefinite Pettis integral of multifunctions in locally convex spaces is a μ-continuous strong multimeasure, Theorem 2.8. Then, we present a full characterization of strong multimeasures in terms of weak multimeasures, Theorem 2.10.

Throughout this paper X is a complete Hausdorff locally convex space with the topology τ and $\mathscr{P}$ the family of all τ-continuous seminorms. For any $p \in \mathscr{P}$, we denote by $\widetilde{X}_p$ the quotient vector space $X/p^{-1}(0)$, by $\varphi_p : X \to \widetilde{X}_p$ the canonical quotient map, by $(\widetilde{X}_p, \widetilde{p})$ the quotient normed space and by $(\overline{X}_p, \overline{p})$

the completion of $(\widetilde{X}_p, \widetilde{p})$. It is easy to see that

$$X' = \bigcup_{p \in \mathscr{P}} X'_p, \tag{1}$$

where X' is the topological dual of X and X'_p is the topological dual of (X, p) $(p \in \mathscr{P})$, and since $X'_p = \{\widetilde{x}'_p \circ \varphi_p : \widetilde{x}'_p \in \widetilde{X}'_p\}$ it follows that

$$X' = \left\{\widetilde{x}'_p \circ \varphi_p : \widetilde{x}'_p \in \widetilde{X}'_p, p \in \mathscr{P}\right\}, \tag{2}$$

where $\widetilde{X}'_p$ is the topological dual of $(\widetilde{X}_p, \widetilde{p})$ $(p \in \mathscr{P})$. Define the continuous linear maps $\widetilde{g}_{pq}$ and $\overline{g}_{pq}$ $(p, q \in \mathscr{P}; p \leq q)$ as follows: for each $p, q \in \mathscr{P}$ such that $p \leq q$, the map $\widetilde{g}_{pq} : \widetilde{X}_q \to \widetilde{X}_p$ is defined by

$$\widetilde{g}_{pq}(\varphi_q(x)) = \varphi_p(x), \text{ for each } x \in X;$$

$\overline{g}_{pq}$ is the continuous linear extension of $\widetilde{g}_{pq}$ to $\overline{X}_q$. We denote by

$$\underleftarrow{\lim}\, \widetilde{g}_{pq}\, \widetilde{X}_q \quad \text{and} \quad \underleftarrow{\lim}\, \overline{g}_{pq}\, \overline{X}_q$$

the *projective limits* of $\{(\widetilde{X}_p, \widetilde{p}) : p \in \mathscr{P}\}$ and $\left\{\left(\overline{X}_p, \overline{p}\right) : p \in \mathscr{P}\right\}$ with respect to the mappings $(\widetilde{g}_{pq})$ and $(\overline{g}_{pq})$ respectively, cf. [26, p.52]. The following lemma is obtained by [26, II.5.4, p.53].

Lemma 1.1. *Let X be a complete Hausdorff locally convex space and let $\mathscr{P}$ be the family of all continuous seminorms. Then,*

$$L = \underleftarrow{\lim}\, \widetilde{g}_{pq}\, \widetilde{X}_q = \underleftarrow{\lim}\, \overline{g}_{pq}\, \overline{X}_q \subset \widetilde{X}_{\mathscr{P}} = \prod_{p \in \mathscr{P}} \widetilde{X}_p \subset \overline{X}_{\mathscr{P}} = \prod_{p \in \mathscr{P}} \overline{X}_p, \tag{3}$$

and the function

$$\varphi : X \to L, \quad \varphi(x) = (\varphi_p(x))_{p \in \mathscr{P}} \tag{4}$$

is an isomorphism of (X, τ) onto (L, τ_L), where τ_L is the induced topology in L by the product topology in $\overline{X}_{\mathscr{P}}$ (or by the product topology in $\widetilde{X}_{\mathscr{P}}$).

By $cwk(X)$ the family of all nonempty convex weakly compact (or $\sigma(X, X')$-compact) subsets of X is denoted; $cwk(X)$ is considered with Minkowski addition: $A \oplus B = \overline{A + B}$; so, $A \oplus B = A + B$ whenever $A, B \in cwk(X)$. Since the function $\varphi_p : (X, \tau) \to (\widetilde{X}_p, \widetilde{p})$ is linear and continuous, we obtain by [25, Proposition II.6.13, p.39] that the function $\varphi_p : (X, \sigma(X, X')) \to (\widetilde{X}_p, \sigma(\widetilde{X}_p, \widetilde{X}'_p))$ is also continuous for every $p \in \mathscr{P}$. Therefore,

$$\varphi_p(cwk(X)) \subset cwk\left(\widetilde{X}_p\right), \quad p \in \mathscr{P},$$

and since $cwk\left(\widetilde{X}_p\right) \subset cwk\left(\overline{X}_p\right)$ it follows that

$$\varphi_p(cwk(X)) \subset cwk\left(\overline{X}_p\right), \quad p \in \mathscr{P}, \tag{5}$$

where

$$\varphi_p(cwk(X)) = \{\varphi_p(C) : C \in cwk(X)\} \text{ and } \varphi_p(C) = \left\{\varphi_p(c) \in \widetilde{X}_p : c \in C\right\}.$$

For any $C \in cwk(X)$ and $x' \in X'$, we write

$$\delta^*(x', C) = \sup\{x'(c) : c \in C\}.$$

Let (Ω, Σ, μ) be a complete probability space.

Definition 1.2. *A function $f : \Omega \to X$ is said to be scalarly measurable if, for each $x' \in X'$ the composition $x' \circ f : \Omega \to \mathbb{R}$ is measurable, i.e. $(x' \circ f)^{-1}(B) \in \Sigma$ for all Borel subsets $B \subset \mathbb{R}$. f is said to be Pettis integrable if*

(i) $x' \circ f \in L_1(\mu)$ for every $x' \in X'$,

(ii) for each $A \in \Sigma$ there exists a vector $\int_A f d\mu \in X$ such that

$$x'\left(\int_A f d\mu\right) = \int_A x' \circ f d\mu \quad \text{for every } x' \in X';$$

the vector $\int_A f d\mu$ is said to be Pettis integral of f over A; the map

$$\nu_f : \Sigma \to X, \quad \nu_f(A) = \int_A f d\mu$$

is said to be the indefinite Pettis integral of f; ν_f is a countably additive μ-continuous vector measure, see Corollary 2.9; a vector measure $\nu : \Sigma \to X$ is said to be μ-continuous if for each $A \in \Sigma$, we have $\mu(A) = 0$ implies that $\nu(A) = 0$. We refer to [13, 21–24, 27] for the detailed information about Pettis integral.

A map $F : \Omega \to cwk(X)$ is called a *multifunction*; a function $f : \Omega \to X$ is called a *selector* of F if $f(\omega) \in F(\omega)$ for every $\omega \in \Omega$.

Definition 1.3. *We say that a multifunction $F : \Omega \to cwk(X)$ is p-scalarly measurable if for every $x'_p \in X'_p$, the map $\delta^*(x'_p, F(\cdot))$ is measurable; F is said to be scalarly measurable if for every $x' \in X'$, the map $\delta^*(x', F(\cdot))$ is measurable; by* (1)*, F is scalarly measurable if and only if F is p-scalarly measurable for every $p \in \mathscr{P}$.*

We say that F is p-scalarly integrable if $\delta^*(x'_p, F(\cdot)) \in L_1(\mu)$ for every $x'_p \in X'_p$; F is said to be scalarly integrable if $\delta^*(x', F(\cdot)) \in L_1(\mu)$ for every $x' \in X'$; (so, F is scalarly integrable if and only if F is p-scalarly integrable for every $p \in \mathscr{P}$).

Given a multifunction $F:\Omega\to cwk(X)$, by virtue of (5) we can define

$$\widetilde{F}_p:\Omega\to cwk\left(\overline{X}_p\right),\quad \widetilde{F}_p(\omega)=\varphi_p(F(\omega)),\quad \omega\in\Omega,\ p\in\mathscr{P},$$

where $\varphi_p(F(\omega))=\left\{\varphi_p(x)\in\widetilde{X}_p : x\in F(\omega)\right\}$.

Definition 1.4. *We say that a multifunction $F:\Omega\to cwk(X)$ is the Pettis integrable if*

(i) F is scalarly integrable;

(ii) for every $A\in\Sigma$ there exists $I_F(A)\in cwk(X)$ such that

$$\delta^*\left(x',I_F(A)\right)=\int_A \delta^*(x',F)d\mu,\ \textit{for every } x'\in X'.$$

We set $I_F(A)=\int_A Fd\mu$ and call $I_F(A)$ Pettis integral of F over A; the map

$$I_F:\Sigma\to cwk(X),\quad I_F(A)=\int_A Fd\mu$$

is said to be the indefinite Pettis integral of F. We will prove that $I_F(\cdot)$ is a μ-continuous strong multimeasure, see Theorem 2.8.

Given a sequence (B_n) in $cwk(X)$, the symbol $\sum_{n=1}^{+\infty}B_n$ denotes a formal series. The series $\sum_{n=1}^{+\infty}B_n$ is said to be *unconditionally convergent* in X if for every choice $b_n\in B_n, n\in\mathbb{N}$, the series $\sum_{n=1}^{+\infty}b_n$ is unconditionally convergent in X. In this case, we set

$$\sum_{n=1}^{+\infty}B_n=\left\{\sum_{n=1}^{+\infty}b_n : b_n\in B_n \text{ for all } n\in\mathbb{N}\right\};$$

it is easy to see that $\sum_{n=1}^{+\infty}B_n$ is a convex subset of X.

A map $M:\Sigma\to cwk(X)$ is said to be a *finitely additive multimeasure* if $M(A\cup B)=M(A)+M(B)$ whenever $A,B\in\Sigma$ are disjoint; M is said to be a *strong multimeasure* (or a *countably additive multimeasure*), if for every pairwise disjoint sequence (A_n) in Σ the series $\sum_{n=1}^{+\infty}M(A_n)$ is unconditionally convergent and

$$M\left(\bigcup_{n=1}^{+\infty}A_n\right)=\sum_{n=1}^{+\infty}M(A_n).$$

The map M is said to be a *weak multimeasure* if $\delta^*(x',M)$ is countably additive for every $x'\in X'$. We are going to prove that M is a strong multimeasure if and only if M is a weak multimeasure which is a well-known result in Banach spaces, see [9, Theorem 3.4]. The last result can be seen as the set-valued version of well-known fact that weakly countably additive vector measures are norm countably additive (Orliz-Pettis theorem, cf. [13, Corollary I.4.4, p.22]). M is said to be *μ-continuous* if for each $A \in \Sigma$, we have that $\mu(A) = 0$ implies that $M(A) = \{0\}$.

2. The Main Results

A series $\sum_{n=1}^{+\infty} c_n$ of elements $c_n \in X$, $n \in \mathbb{N}$, is said to be *unconditionally convergent* if it converges for every rearrangement of its terms, i.e. if the series $\sum_{n=1}^{+\infty} c_{\pi(n)}$ converges whenever π is a one-to-one mapping of $\mathbb{N}$ onto $\mathbb{N}$.

Lemma 2.1. *If series $\sum_{n=1}^{+\infty} c_n$ of elements $c_n \in X$, $n \in \mathbb{N}$ is unconditionally convergent, then all rearrangements have the same sum.*

Proof. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation. We write $\sum_n c_n = a$ and $\sum_n c_{\pi(n)} = a_\pi$. Given $x' \in X'$ the series $\sum_n x'(c_n)$ is unconditionally convergent and its sum is the same for every rearrangement. It follows that $x'(a) = x'(a_\pi)$ for every $x' \in X'$. Hence, $a = a_\pi$ and that finished the proof. □

Corollary 2.2. *If the series $\sum_{n=1}^{+\infty} \varphi_p(c_n)$ is unconditionally convergent for every $p \in \mathscr{P}$, then the series $\sum_{n=1}^{+\infty} c_n$ is also unconditionally convergent.*

Proof. Suppose that $\sum_n \varphi_p(c_n)$ is unconditionally convergent for every $p \in \mathscr{P}$ but there exists a rearrangement $(m_n)_n$ of $\mathbb{N}$ such that $\sum_n c_{m_n}$ is divergent. Since X is complete, it means that $\sum_n c_{m_n}$ does not satisfy the Cauchy condition, i.e. there exists a sequence $(N_k)_k$ of pairwise disjoint sets $N_k \subset \mathbb{N}$ such that $\sup N_k < \inf N_{k+1}$ and the sequence $\left(\sum_{n\in N_k} c_{m_n}\right)_k$ does not converge to zero. Hence, there exists $p \in \mathscr{P}$ such that

$$\lim_{k\to\infty} \widetilde{p}\left(\sum_{n\in N_k} \varphi_p(c_{m_n})\right) = \lim_{k\to\infty} p\left(\sum_{n\in N_k} c_{m_n}\right) \neq 0.$$

The last result contradicts the unconditionally convergence of $\sum_n \varphi_p(c_n)$.

Therefore, $\sum_n c_n$ is unconditionally convergent and this ends the proof. □

Corollary 2.3. *The series $\sum_{n=1}^{+\infty} c_n$ is unconditionally convergent if and only if the series $\sum_{n=1}^{+\infty} \varphi_p(c_n)$ is unconditionally convergent for every $p \in \mathscr{P}$. In this case, we have*

$$\varphi_p\left(\sum_{n=1}^{+\infty} c_n\right) = \sum_{n=1}^{+\infty} \varphi_p(c_n) \quad \textit{for every} \quad p \in \mathscr{P}.$$

The next lemma follows immediately from Corollary 2.3.

Lemma 2.4. *The series* $\sum_{n=1}^{+\infty} B_n$ *is unconditionally convergent in* X *if and only if the series* $\sum_{n=1}^{+\infty} \varphi_p(B_n)$ *is unconditionally convergent in the Banach space* $\overline{X}_p$ *for every* $p \in \mathscr{P}$.

Given $p \in \mathscr{P}$ and $C \in cwk(X)$ we write

$$p(C) = \sup_{c\in C} p(c) \quad \text{and} \quad \widetilde{p}(\varphi_p(C)) = \sup_{c\in C} \widetilde{p}(\varphi_p(c)).$$

A series $\sum_{n=1}^{+\infty} \varphi_p(B_n)$ is unconditionally convergent in the Banach space $\overline{X}_p$ if and only if given $\varepsilon > 0$ there exists $n_{p\varepsilon} \in \mathbb{N}$ such that

$$\widetilde{p}\left(\sum_{i\in S} \varphi_p(B_i)\right) \leq \varepsilon$$

whenever S is a finite subset of $\mathbb{N}\setminus\{1, \ldots, n_{p\varepsilon}\}$, see [7, p.4]. Hence, by equalities

$$\sum_{i\in S} \varphi_p(B_i) = \varphi_p\left(\sum_{i\in S} B_i\right) \quad \text{and} \quad \widetilde{p}(\varphi_p(x)) = p(x) \quad (x \in X)$$

we obtain immediately the following corollary.

Corollary 2.5. *The series* $\sum_{n=1}^{+\infty} B_n$ *is unconditionally convergent if and only if for each* $p \in \mathscr{P}$ *and for each* $\varepsilon > 0$ *there exists* $n_{p\varepsilon} \in \mathbb{N}$ *such that*

$$p\left(\sum_{i\in S} B_i\right) \leq \varepsilon$$

whenever S *is a finite subset of* $\mathbb{N} \setminus \{1, \ldots, n_{p\varepsilon}\}$.

The next lemma is proved in the same manner as [7, Lemma 2.2].

Lemma 2.6. *Let* $\sum_{n=1}^{+\infty} B_n$ *be an unconditionally convergent series and let* $B = \sum_{n=1}^{+\infty} B_n$. *Then* $B \in cwk(X)$.

Proof. Let us consider the mapping

$$T : \prod_{n=1}^{+\infty} (B_n, \sigma_n) \to (X, \sigma(X, X')), \quad T((b_n)_n) = \sum_{n=1}^{+\infty} b_n,$$

where σ_n is the induced topology in B_n by the weak topology $\sigma(X, X')$. It is enough to prove that T is a continuous function, since $\prod_{n=1}^{+\infty}(B_n, \sigma_n)$ is a compact topological space with respect to the product topology by [19, Tychonoff's Theorem] and, therefore, B is $\sigma(X, X')$-compact subset of X.

By $\beta_{\sigma(X,X')}(0)$ a 0-neighborhood base with respect to the weak topology $\sigma(X,X')$ is denoted. Assume that an arbitrary element $b=(b_n)\in\prod_{n=1}^{+\infty}B_n$ and an arbitrary neighborhood $U\in\beta_{\sigma(X,X')}(0)$ are given. Since the function

$$f:(X,\tau)\times(X,\sigma(X,X'))\to(X,\sigma(X,X')),\quad f(x,y)=x+y$$

is continuous, given U there exists a $(0,0)$-neighborhood $U_p(\varepsilon)\times V$ in $(X,\tau)\times(X,\sigma(X,X'))$ such that $U_p(\varepsilon)+V\subset U$, where $U_p(\varepsilon)=\{x\in X:p(x)\le\varepsilon\}$. Hence, by Corollary 2.5 there exists $n_{p\varepsilon}\in\mathbb{N}$ such that

$$p\left(\sum_{i\in S}B_i\right)\le\frac{\varepsilon}{2},$$

whenever S is a finite subset of $\mathbb{N}\setminus\{1,\dots,n_{p\varepsilon}\}$. There exist 0-neighborhoods $W_1,\dots,W_{p\varepsilon}$ in $\beta_{\sigma(X,X')}(0)$ such that $W_1+\ldots+W_{p\varepsilon}\subset V$. Define $C_n=B_n\cap(b_n+W_n)$ for every $1\le n\le n_{p\varepsilon}$, $C_n=B_n$ for every $n>n_{p\varepsilon}$ and $C=\prod_n C_n$. Then C is a neighborhood of $b=(b_n)$ such that for each $b'=(b'_n)\in C$, we have

$$\begin{aligned}T(b')&=\sum_{n=1}^{+\infty}b'_n=\sum_{n=1}^{n_{p\varepsilon}}b'_n+\sum_{n_{p\varepsilon}+1}^{+\infty}b'_n\in\sum_{n=1}^{n_{p\varepsilon}}(b_n+W_n)+\sum_{n_{p\varepsilon}+1}^{+\infty}b'_n\\&=\sum_{n=1}^{n_{p\varepsilon}}b_n+\sum_{n_{p\varepsilon}+1}^{+\infty}b'_n+\sum_{n=1}^{n_{p\varepsilon}}W_n\subset\sum_{n=1}^{+\infty}b_n+\sum_{n_{p\varepsilon}+1}^{+\infty}(b'_n-b_n)+V\\&\subset T(b)+U_p(\varepsilon)+V\subset T(b)+U=T(b)+U.\end{aligned}$$

Since b and U were arbitrary, the last result yields that T is continuous and the proof is finished. □

LEMMA 2.7. *Let $F:\Omega\to cwk(X)$ be a multi-function. Then,*

(i) F is scalarly measurable if and only if $\widetilde{F}_p$ is scalarly measurable for every $p\in\mathscr{P}$,

(ii) F is scalarly integrable if and only if $\widetilde{F}_p$ is scalarly integrable for every $p\in\mathscr{P}$,

(iii) if F is Pettis integrable, then each $\widetilde{F}_p$ is Pettis integrable and

$$\varphi_p\left(\int_A F\,d\mu\right)=\int_A\widetilde{F}_p\,d\mu,\quad A\in\Sigma,p\in\mathscr{P}.\tag{6}$$

Proof. Assume that F is scalarly measurable (scalarly integrable) and let $p\in\mathscr{P}$. Given $\widetilde{x}'_p\in\widetilde{X}'_p$ we have $\delta^*\left(\widetilde{x}'_p,\widetilde{F}_p(\cdot)\right)=\delta^*\left(x'_p,F(\cdot)\right)$ where $x'_p=\widetilde{x}'_p\circ\varphi_p$. So, $\widetilde{F}_p$ is scalarly measurable (scalarly integrable).

Conversely, assume that $\widetilde{F}_p$ is scalarly measurable (scalarly integrable) for every $p \in \mathscr{P}$, and let x' be an arbitrary element of X'. Then, there exists $p \in \mathscr{P}$ such that $x' = x'_p \in X'_p$; further there exists $\widetilde{x}'_p \in \widetilde{X}'_p$ such that $x'_p = \widetilde{x}'_p \circ \varphi_p$. Since

$$\delta^*\left(x', F(\cdot)\right) = \delta^*\left(x'_p, F(\cdot)\right) = \delta^*\left(\widetilde{x}'_p, \widetilde{F}_p(\cdot)\right)$$

and $\delta^*\left(\widetilde{x}'_p, \widetilde{F}_p(\cdot)\right)$ is measurable (integrable) it follows that $\delta^*\left(x', F(\cdot)\right)$ is measurable (integrable). Since x' was arbitrary we infer that F is scalarly measurable (scalarly integrable).

(iii) Assume that F is Pettis integrable and let $p \in \mathscr{P}$, $A \in \Sigma$ and $\widetilde{x}'_p \in \widetilde{X}'_p$. Then, by Definition 1.4 we have

$$\begin{aligned}
\delta^*\left(\widetilde{x}'_p, \varphi_p\left(\int_A F d\mu\right)\right) &= \delta^*\left(\widetilde{x}'_p \circ \varphi_p, \int_A F d\mu\right) = \int_A \delta^*\left(\widetilde{x}'_p \circ \varphi_p, F(\omega)\right) d\mu \\
&= \int_A \delta^*\left(\widetilde{x}'_p \circ \varphi_p, F(\omega)\right) d\mu \\
&= \int_A \delta^*\left(\widetilde{x}'_p, \varphi_p[F(\omega)]\right) d\mu = \int_A \delta^*\left(\widetilde{x}'_p, \widetilde{F}_p(\omega)\right) d\mu.
\end{aligned}$$

The last result together with (ii) yields that $\widetilde{F}_p$ is Pettis integrable and (6) holds, and this ends the proof. □

We are now ready to present the first main result.

THEOREM 2.8. *If $F : \Omega \to cwk(X)$ is a Pettis integrable multifunction, then the indefinite Pettis integral $I_F(\cdot)$ is a μ-continuous strong multimeasure.*

Proof. Let (A_n) be a pairwise disjoint sequence in Σ and let $A = \bigcup_{n=1}^{+\infty} A_n$. By Lemma 2.7 we have that each $\widetilde{F}_p$ is Pettis integrable and $\varphi_p\left(\int_B F d\mu\right) = \int_B \widetilde{F}_p d\mu$ for every $B \in \Sigma$ and $p \in \mathscr{P}$. Therefore, by [9, Theorem 4.1] we obtain that the series $\sum_{n=1}^{+\infty} \int_{A_n} \widetilde{F}_p d\mu$ is unconditionally convergent and

$$\sum_{n=1}^{+\infty} \int_{A_n} \widetilde{F}_p d\mu = \int_A \widetilde{F}_p d\mu, \text{ for every } p \in \mathscr{P}.$$

So,

$$\sum_{n=1}^{+\infty} \varphi_p\left(\int_{A_n} F d\mu\right) = \varphi_p\left(\int_A F d\mu\right), \text{ for every } p \in \mathscr{P}.$$

Hence,

$$\begin{aligned}0 &= \lim_{k\to\infty} \widetilde{p}\left(\sum_{n=1}^{k} \varphi_p\left(\int_{A_n} F d\mu\right) - \varphi_p\left(\int_A F d\mu\right)\right)\\ &= \lim_{k\to\infty} \widetilde{p}\left[\varphi_p\left(\sum_{n=1}^{k}\int_{A_n} F d\mu - \int_A F d\mu\right)\right]\\ &= \lim_{k\to\infty} p\left(\sum_{n=1}^{k}\int_{A_n} F d\mu - \int_A F d\mu\right)\end{aligned}$$

whenever $p \in \mathscr{P}$. Therefore,

$$\sum_{n=1}^{+\infty}\int_{A_n} F d\mu = \int_A F d\mu.$$

This means that I_F is a strong multimeasure.

It remains to prove that I_F is μ-continuous. Let $A \in \Sigma$ be such that $\mu(A) = 0$ and let $x_A \in I_F(A)$. Then for every $x' \in X'$, we have

$$\delta^*(x', I_F(A)) = \int_A \delta^*(x', F(\omega))d\mu = 0$$

and $\delta^*(-x', I_F(A)) = \int_A \delta^*(-x', F(\omega))d\mu = 0$. It follows that $x'(x_A) = 0$ for every $x' \in X'$. Hence, $x_A = 0$ and, therefore, $I_F(A) = \{0\}$. This means that I_F is μ-continuous and this ends the proof. □

Corollary 2.9. *If $f : \Omega \to X$ is a Pettis integrable function, then the indefinite Pettis integral ν_f is a countably additive μ-continuous vector measure.*

Given a map $M : \Sigma \to cwk(X)$, by virtue of (5) we can define

$$\widetilde{M_p} : \Sigma \to cwk\left(\overline{X}_p\right), \quad \widetilde{M_p}(A) = \varphi_p(M(A)), \quad A \in \Sigma,\ p \in \mathscr{P},$$

where $\varphi_p(M(A)) = \left\{\varphi_p(x) \in \widetilde{X}_p : x \in M(A)\right\}$.

Theorem 2.10. *Let $M : \Sigma \to cwk(X)$ be a function. Then the following statements are equivalent:*

(i) M is a strong multimeasure,

(ii) M is a weak multimeasure.

Proof. $(i) \Rightarrow (ii)$ Assume that M is a strong multimeasure, let (A_n) be a pairwise disjoint sequence in Σ with $A = \bigcup_{n=1}^{+\infty} A_n$ and let $x' \in X'$. Then, there exists $p \in \mathscr{P}$ such that $x' = x'_p \in X'_p$. Hence, there exists $\widetilde{x}'_p \in \widetilde{X}'_p$ such that $x'_p = \widetilde{x}'_p \circ \varphi_p$. Thus,

$$\delta^*(x', M(A)) = \delta^*(x'_p, M(A)) = \delta^*(\widetilde{x}'_p \circ \varphi_p, M(A)) = \delta^*(\widetilde{x}'_p, \widetilde{M}_p(A)). \tag{7}$$

By Lemma 2.4 and Lemma 2.6 we obtain that $\widetilde{M}_p$ is a countably additive multimeasure in $\overline{X}_p$ and

$$\widetilde{M}_p(A) = \sum_{n=1}^{+\infty} \widetilde{M}_p(A_n).$$

The last result together with [9, Theorem 3.4] yields

$$\delta^*(\widetilde{x}'_p, \widetilde{M}_p(A)) = \sum_{n=1}^{+\infty} \delta^*(\widetilde{x}'_p, \widetilde{M}_p(A_n)) \quad \text{for every } \widetilde{x}'_p \in \widetilde{X}'_p. \tag{8}$$

Since for each $n \in \mathbb{N}$, $\delta^*(\widetilde{x}'_p, \widetilde{M}_p(A_n)) = \delta^*(x'_p, M(A_n))$, we obtain by (7) and (8) that

$$\delta^*(x', M(A)) = \delta^*(x'_p, M(A)) = \sum_{n=1}^{+\infty} \delta^*(x'_p, M(A_n)) = \sum_{n=1}^{+\infty} \delta^*(x', M(A_n)).$$

This means that $\delta^*(x', M)$ is countably additive.

$(ii) \Rightarrow (i)$ Assume that M is a weak multimeasure. Then, given $p \in \mathscr{P}$ we have that $\delta^*(\widetilde{x}'_p, \widetilde{M}_p)$ is countably additive for every $\widetilde{x}'_p \in \widetilde{X}'_p$. Therefore, by [9, Theorem 3.4], $\widetilde{M}_p$ is a countably additive multimeasure. Hence, by Lemma 2.4 and Lemma 2.6 it follows that M is a countably additive multimeasure and this ends the proof. □

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