A PROJECTIVE HOMOTOPY THEORY FOR NON-ADDITIVE CATEGORIES (*)

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SOMMARIO. - *La teoria d'omotopia di Eckmann e Hilton associata agli oggetti proiettivi nella categoria dei moduli su un anello R, è stato uno dei primi esempi nei quali la teoria dell'omotopia veniva considerata in un contesto non topologico. Una mappa è* nullomotopica *se e solo se può essere fattorizzata attraverso un modulo proiettivo. I moduli proiettivi sono perciò gli oggetti contraibili della teoria. Grazie alla struttura additiva la relazione d'omotopia è determinata dagli oggetti contraibili: le mappe f e g sono omotopiche se e solo se $f-g$ è nullomotopica. Ci si può chiedere se la classe degli oggetti proiettivi di una categoria non additiva determini essa pure una teoria d'omotopia. Interpretando gli "oggetti proiettivi" di una* struttura proiettiva *nel senso di Maranda, diamo una risposta positiva alla domanda; per una categoria puntata* $\mathbf{F}$ *che ha limiti e colimiti finiti e dotata di una struttura proiettiva, costruiamo una classe* fib *di fibrazioni e un'appropriata classe* we *di equivalenze deboli tali che (*$\mathbf{F}$, fib, we*) soddisfino gli assiomi di una* categoria di oggetti fibranti *nel senso di K.S. Brown.*

SUMMARY. - *The homotopy theory due to Eckmann and Hilton associated with the projective objects in the category of modules over a ring R was one of the first examples in which homotopy theory was considered in a non-topological context. A map is* nullhomotopic *if and only if it can be factored through a projective module. The projective modules are the contractible objects in the theory. Because of the additive structure, the homotopy relation is determined by the contractible objects: maps f and g are homotopic if and only if $f-g$ is nullhomotopic. One may ask whether the class of projective objects of a non-additive category also determines a homotopy theory. Interpreting 'projective objects' in terms of a* projective structure *in the sense of Maranda, we give a positive answer to the question: for a pointed category* $\mathbf{F}$ *with finite limits and colimits and equipped with a projective structure, we construct a class* fib *of fibrations and an appropriate class* we *of weak equivalences such that (*$\mathbf{F}$, fib, we*) satisfies the axioms of a* category of fibrant objects *in the sense of K.S. Brown.*

0. Introduction.

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The homotopy theory due to Eckmann and Hilton [5] associated with the projective objects in the category of modules over a ring R was one of the first examples in the literature in which homotopy theory was considered in a non-topological context. In this theory a map is *nullhomotopic* if and only if it can be factored through a projective module; the projective modules are the contractible objects in the theory. Because of the additive structure, the homotopy relation is determined by the contractible objects: maps f and g are homotopic if and only if $f - g$ is nullhomotopic.

One may ask whether the class of projective objects of a non-additive category still determines a homotopy theory. To make the question precise we interpret 'projective object' in terms of a *projective structure* in the sense of Maranda [6] and give a positive answer in the sense that, for a category **F** admitting finite limits and certain colimits and equipped with a projective structure, we construct a class *fib* of fibrations and an appropriate class *we* of weak equivalences such that (**F**, *fib, we*) satisfies the first three axioms of a *fibration category* in the sense of Baues [1]. We go further and show that a category of fractions of **F**, together with associated classes of fibrations and weak equivalences, satisfies all four axioms of a Baues fibration category.

Maranda projective structures are abundant. For example, to every comonad in a category **F**, is associated a projective structure whose projective objects are those objects P at which the counit has a right inverse. The 'fibrations' of the structure are those maps that are converted into retractions by applying the functor part of the comonad. Thus to every comonad in a suitable category **F** is associated a homotopy theory.

For example, if **F** is the category of modules over a ring R, the forgetful functor to the category of sets has a left adjoint giving rise to a comonad in **F**. The projective objects of the associated Maranda structure are exactly the projective modules and the class of fibrations is exactly the class of epimorphisms in **F**. The homotopy theory constructed as above coincides with the classical projective homotopy theory of Eckmann and Hilton.

Dually, if a category **C** (admitting certain finite limits and colimits) is equipped with a Maranda *injective structure* [6], the construction yields a Baues cofibration category structure on **C**, so that each monad in **C** is associated with an 'injective' homotopy theory in **C**. For example, the reduced cone monad in the category of pointed topological spaces yields an injective structure in which the injective objects are the contractible pointed spaces . We prove that the associated homotopy category is equivalent to the pointed homotopy category of topological spaces. In this work there is some conceptual overlap with the paper of Seebach [7] but our technical results are distinct.

1. Projective Structures.

We recall that a projective structure in the sense of Maranda [6] on a category $\mathbf{K}$ is a pair $(\mathcal{P},\mathcal{F})$, where $\mathcal{P}$ is a class of objects of $\mathbf{K}$, the *projective* objects, and a class of maps $\mathcal{F}$, the *fibrations* satisfying the following axioms.

- P1) $\mathcal{P}$ is the class of objects of $\mathbf{K}$ that are projective with respect to all of the fibrations.
- P2) $\mathcal{F}$ is the class of maps of $\mathbf{K}$ with respect to which each object of $\mathcal{P}$ is projective.
- P3) If X is an object of $\mathbf{K}$ there is a map $p : P \to X$ where p is in $\mathcal{F}$ and P is in $\mathcal{P}$.

The following result may be well known.

PROPOSITION 1.1. *If (C, ϵ, ν) is a comonad in $\mathbf{K}$ then there is a projective structure $(\mathcal{P}, \mathcal{F})$, where*

$$\mathcal{P} = \{X | \epsilon X : CX \to X \text{ has a section}\}$$

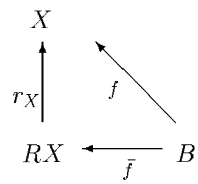
and

$$\mathcal{F} = \{f : X \to Y | Cf : CX \to CY \text{ has a section}\}.$$

Proof. Suppose that $\epsilon X : CX \to X$ has a section h, let $f : A \to B$ belong to $\mathcal{F}$ and suppose that $g : X \to B$ is a map. Then if $k : CB \to CA$ is a section of Cf, the map $t = \epsilon A.k.Cg.h : X \to A$ is such that $f.t = g$, showing that each object of $\mathcal{P}$ is projective with respect to each member of $\mathcal{F}$. Suppose that an object Q of $\mathbf{K}$ is projective with respect to each member of $\mathcal{F}$. Then since νQ is a section of $C\epsilon Q$, ϵQ is a member of $\mathcal{F}$. But then there exists a map $h : Q \to CQ$ such that $\epsilon Q.h = 1_Q$, so that Q is in $\mathcal{P}$, verifying (P1). Next, observe that for any object B in $\mathbf{K}$, νB is a section of ϵCB, so that each CB belongs to $\mathcal{P}$. Suppose that each member of $\mathcal{P}$ is projective with respect to a map $f : A \to B$. Then there exists a map $h : CB \to A$ such that $f.h = \epsilon B$. It follows that $Ch.\nu B$ is a section of Cf, so that f belongs to $\mathcal{F}$, verifying (P2). Moreover, since $\epsilon X : CX \to X$ is a fibration with CX projective, (P3) is also satisfied.

REMARK 1.1.1. The reader may note that the above argument uses neither the associativity nor the naturality property of the comultiplication ν.

Recently Herrlich [4] has drawn attention to the notion of almost reflective subcategory. We recall that a full subcategory **B** of **K** is *almost coreflective* if **B** is closed with respect to retracts and if, for each object X in **K** there exists a map $r_X : RX \to X$, where RX belongs to **B**, and if for each map $f : B \to X$, where B is in **B** there exists



in **B** a (not necessarily unique) map $\bar{f}$ such that $r_X.\bar{f} = f$.

PROPOSITION 1.2. *There is a one to one correspondence between projective structures in* **K** *and almost coreflective subcategories of* **K**.

Proof. Suppose that the full subcategory **B** of **K** is almost coreflective with almost coreflector r. Note that, by the defining property of almost coreflection, every B in **B** is projective with respect to r_X for each X. Applying the dual of [6; Proposition 2] we find that $(\mathcal{P},\mathcal{F})$ is a projective structure in **K**, where $\mathcal{P}$ is the class of all objects in **B** and $\mathcal{F}$ is the class of all maps with respect to which each object of **B** is projective. Conversely suppose that $(\mathcal{P},\mathcal{F})$ is a projective structure in **K**. It follows from axiom (P3) that an almost coreflector exists for the full subcategory **B** whose objects coincide with those in the class $\mathcal{P}$.

2. Homotopy Theories.

We recall that a *fibration category* in the sense of Baues [1] is a category **F** with the structure (**F**, fib, we) subject to axioms (F1), (F2), (F3), (F4), where *fib* and *we* are classes of morphisms in **F** called *fibrations* and *weak equivalences* respectively :

F1) The isomorphisms in **F** are weak equivalences and are also fibrations. For two maps $f : A \to B$, $g : B \to C$ if any two of f, g and gf are weak equivalences, then so is the third. The composition of fibrations is a fibration.

F2a) For a fibration $p : A \to B$ and map $f : Y \to B$ there exists the pullback in **F**

(2.1)
$$\begin{array}{ccc} A \times_B Y & \xrightarrow{\bar{f}} & A \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & B \end{array}$$

and $\bar{p}$ is a fibration. Moreover if f is a weak equivalence, so is $\bar{f}$.

F3) For a map $f : Y \to B$ in **F** there exists a commutative diagram

(2.2)
$$\begin{array}{ccc} Y & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle g}\searrow^{\sim} \quad \nearrow{\scriptstyle p} & \\ & A & \end{array}$$

in which g is a weak equivalence and p is a fibration.

F4) For each object X in **F** there is a trivial fibration $MX \overset{\sim}{\twoheadrightarrow} X$, where MX is *cofibrant* in F, i.e. each trivial fibration $Y \overset{\sim}{\twoheadrightarrow} MX$ admits a section.

We shall also be concerned with the notion due to K S Brown [2] of a *category of fibrant objects* with structure (**K**, *fib, we, e*), where e is a terminal object, which satisfies (F1), (F2a), (F3) and the further axiom :

(A) For all objects X of **K** the unique arrow $X \to e$ is a fibration.

Let **K** be a category which admits finite limits. Then **K** has a terminal object e. Suppose also that **K** has a projective structure ($\mathcal{P}$,$\mathcal{F}$) and an initial object ϕ (which need not coincide with e) satisfying the following condition.

(K) For every P in $\mathcal{P}$ and every X in **K**, the following pushout (i.e. coproduct) exists.

$$\begin{array}{ccc} \phi & \longrightarrow & P \\ \downarrow & & \downarrow{\scriptstyle j_2} \\ X & \xrightarrow[j_1]{} & X \vee P \end{array}$$

DEFINITION 2.3. Set *fib* $= \mathcal{F}$ and let *we* be the closure with respect to axioms (F1) and (F2a) of the class of 'generators' satisfying the following conditions.

G1) The isomorphisms of $\mathbf{K}$ are generators.

G2) For any P in $\mathcal{P}$, the map $P \to e$ is a generator.

G3) For any P in $\mathcal{P}$ and any object X of $\mathbf{K}$, the map $j_1 : X \to X \vee P$ is a generator.

PROPOSITION 2.4. *If the category* $\mathbf{K}$ *admits finite limits and has a projective structure and initial object satisfying condition* (K) *then the structure (*$\mathbf{K}$*,* $\mathcal{F}$*, we, e) satisfies axioms* (F1) , (F2a) , (F3)*. If also there exists a map* $e \to \phi$ *then the structure satisfies axiom* (A)*.*

Proof. (F1) and (F2a) are certainly satisfied. To check (F3), suppose that $f : X \to Y$ is a map in $\mathbf{K}$, let $p : P \to Y$ be a fibration with P projective. By the universal property of coproducts there exists a unique map

$$(f, p) : X \vee P \to Y$$

with the property that $(f,p).j_1 = f$ and $(f,p).j_2 = p$. It is easy to check (from the defining properties of a projective structure) that (f,p) is a fibration. Moreover j_1 is a *we* by condition (G3) so that (F3) is satisfied. Since every retraction is necessarily a fibration, this completes the proof.

REMARK 2.4.1. Baues [1] has shown that, in the presence of axioms (F1) and (F3), his axiom (F2b) is equivalent to (F2a). It follows that the structure referred to in Proposition 2.4 also satisfies (F2b).

REMARK 2.4.2. The structure ($\mathbf{K}$, $\mathcal{F}$, *we*, *e*) obtained as above will in general not also satisfy axiom (F4). The following (trivial but useful) Proposition gives a sufficient condition for the exception to be the case.

PROPOSITION 2.5. *Suppose that the pointed category* $\mathbf{K}$ *satisfies condition* (K) *and admits a structure* ($\mathbf{K}$*, Fib, We, e) which satisfies axioms* (F1), (F2a), (F3*) and* (F4) *in such a way that every object is cofibrant. Suppose also that* $\mathcal{F} \subseteq$ *Fib and that the generators of the class we belong to We. Then the structure* ($\mathbf{K}$*,* $\mathcal{F}$*, we, e) also satisfies axiom* (F4)*.*

Proof. Since $\mathcal{F} \cap we \subseteq Fib \cap We$, every object is cofibrant in $(\mathbf{K}, \mathcal{F}, we, e)$.

EXAMPLE 2.6. Let $\mathbf{K}$ be the category of modules over a ring R, let $\mathcal{P}$ be the class of projective modules and let $\mathcal{F} = Fib$ be the class of surjective homomorphisms. Let We be the class of homotopy equivalences in the sense of Eckmann and Hilton. It is well known that the structure $(\mathbf{K}, Fib, We)$ is a Baues fibration category structure in terms of which every object is cofibrant. By [5; Theorem 13.6 (dual)] the generators of we belong to We. It follows that every object is cofibrant in terms of $(\mathbf{K}, \mathcal{F}, we, e)$. We shall prove in section 4 that the homotopy categories coincide.

3. A Category of Fractions.

In this section we assume that $(\mathbf{K}, fib, we, e)$ is a category with finite limits satisfying axioms (F1), (F2a), and (F3). Let Σ denote the class of *trivial fibrations*, that is to say let

$$\Sigma = fib \cap we. \tag{3.1}$$

Recall that the notation $\mathbf{K}[\Sigma^{-1}]$ refers to the *category of fractions* obtained from $\mathbf{K}$ by freely inverting the arrows of Σ [4]. Modulo set theoretic difficulties $\mathbf{K}[\Sigma^{-1}]$ exists (for objects X and Y of $\mathbf{K}$, $\mathbf{K}[\Sigma^{-1}](X, Y)$ may be a proper class). The objects of $\mathbf{K}[\Sigma^{-1}]$ are the same as those of $\mathbf{K}$ and the morphisms from X to Y are obtained from chains of the form

$$X \underset{s_1}{\longleftarrow} \bullet \xrightarrow{f_1} \bullet \underset{s_2}{\longleftarrow} \bullet \xrightarrow{f_2} \bullet \ldots \bullet \underset{s_n}{\longleftarrow} \bullet \xrightarrow{f_n} Y\ , \tag{3.1.1}$$

where the s_i are all in Σ, subject to the replacement rule:

(3.1.2) if there exist t, w in $\mathbf{K}$ with $f_i.t = s_i.w$ and if $s_{i-1} \in \mathbf{K}$ then, in 3.1.1, the subchain

$$\bullet \underset{s_i-1}{\longleftarrow} \bullet \xrightarrow{f_1} \bullet \underset{s_i}{\longleftarrow} \bullet \xrightarrow{f_{i+1}} \bullet \text{ can be replaced by } \bullet \underset{s_{i-1}.t}{\longleftarrow} \bullet \xrightarrow{f_{i+1}.w} \bullet\ .$$

Specifically, a morphism of $\mathbf{K}[\Sigma^{-1}]$ is an equivalence class of chains of type 3.1.1, with composition of morphisms defined by concatenation of chains subject to the equivalence relation. In $\mathbf{K}[\Sigma^{-1}]$ we define classes *fib* and *we* as follows.

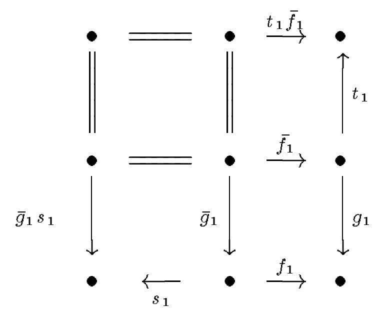
fib (respectively *we*) consists of the equivalence classes of chains of type 3.1.1 in which the forward arrows are all fibrations of (**K**, *fib, we*) (respectively weak equivalences of (**K**, *fib, we*)).

REMARK 3.1.2. Since the pullback of a pair of arrows of form $\bullet \to \bullet \twoheadleftarrow \bullet$ always exists, each chain is equivalent to a chain consisting of exactly one reverse arrow and exactly one forward arrow.

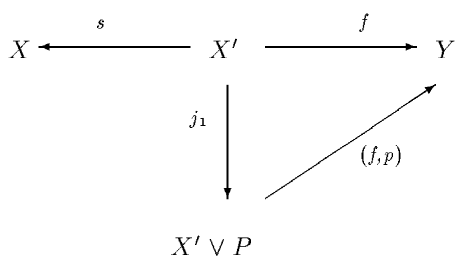
PROPOSITION 3.2. *The structure (*$\mathbf{K}[\Sigma^{-1}]$*, fib, we) satisfies axioms*(F1), (F2a), (F3) and (F4).

Proof. It may first easily be checked that the classes *fib, we* are well defined in the sense that the replacement rule preserves chains of the relevant type. The axiom (F1) follows directly from corresponding properties of (**K**, *fib, we*) via Remark 3.1.2. To check (F2a), note that the following diagram, in which $\bar{f}_1$ (respectively $\bar{g}_1$) refers to the pullback in **K** of f_1 over g_1 (respectively of g_1 over f_1) indicates a pullback of (reduced) chains of the type required.

(3.2.1)



To check (F3), suppose that $X \stackrel{s}{\leftarrow} X' \stackrel{f}{\to} Y$ is a chain from X to Y, where s is in Σ, and let $p : P \to Y$ be a fibration in **K** with P in $\mathcal{P}$. Then, as in the diagram



note that a factorisation of f in **K** induces a factorisation of the desired type in $\mathbf{K}[\Sigma^{-1}]$. The Axiom (F4) is satisfied in $\mathbf{K}[\Sigma^{-1}]$ since the trivial fibrations are equivalence classes of chains of arrows in which both the forward and the reverse arrows belong to Σ. Such classes are invertible.

4. The Homotopy Relations.

We return to the situation of section 2 in which we have a category $\mathbf{K}$, with finite limits and initial object, equipped with a projective structure satisfying condition (K). As indicated in the proof of Proposition 2.4 there is a factorisation

$$(4.1)\qquad \begin{array}{ccc} X & \xrightarrow{\quad\Delta\quad} & X\times X \\ & {\scriptstyle j_1}\searrow \quad \nearrow{\scriptstyle (\Delta,p)} & \\ & X\vee P & \end{array}$$

of the diagonal map Δ, where $p : P \to X \times X$ is a fibration with P projective. Imitating Baues [1; I.1.6 dual] we may define maps $f, g : A \to X$ to be *homotopic*, denoted $f \simeq_{\mathcal{P}} g$ if there is a commutative diagram

$$(4.2)\qquad \begin{array}{ccc} X\vee P & \xrightarrow{\quad(\Delta,p)\quad} & X\times X \\ & {\scriptstyle H}\nwarrow \quad \nearrow{\scriptstyle (f,g)} & \\ & A & \end{array} .$$

When the structure ($\mathbf{K}$, $\mathcal{F}$, *we*, *e*) satisfies axiom (F4) it follows from the (dual of the) Baues theory of homotopy in a cofibration category [1; Chapter II] that the homotopy relation $\simeq_{\mathcal{P}}$ is an equivalence relation and is independent of the choice of P. We can contemplate $\simeq_{\bar{\mathcal{P}}}$, the smallest equivalence relation containing $\simeq_{\mathcal{P}}$. Another way to obtain an equivalence relation is to consider the functor $Q : \mathbf{K} \to \mathbf{K}[\Sigma^{-1}]$ and define $f \simeq_{\Sigma} g$ if and only if $Qf \simeq_{\mathcal{P}} Qg$. It is clear that $f \simeq_{\bar{\mathcal{P}}} g \Rightarrow f \simeq_{\Sigma} g$. However these relations coincide when (F4) is satisfied.

For example, in the situation of example 2.6, we have:

Proposition 4.3. *The relation $\simeq_{\mathcal{P}}$ associated with the class of projective modules and surjective homomorphisms coincides with the Eckmann-Hilton homotopy relation.*

Proof. Recall that maps $f, g : A \to X$ are homotopic in the sense of Eckmann and Hilton if and only if the difference $f - g$ factors through a projective object. Suppose that $f \simeq_{\mathcal{P}} g$. Then there exists a factorisation 4.2. Since the category enjoys a biproduct we have $X \vee_e P \approx X \oplus P$. Hence $f - g = (\pi_1 - \pi_2) \circ (\Delta, p) \circ H = (\pi_1 - \pi_2) \circ \Delta \circ H + (\pi_1 - \pi_2) \circ p \circ H = (\pi_1 - \pi_2) \circ p \circ H$, thus factoring through the projective P. Conversely we show that if $f, g : A \to X$ are homotopic then a factorisation 4.2 exists. Let $p = (p_1, p_2) : P \to X \times_e X$. Then $p_1 - p_2 = \nabla(1_X \oplus -1_X)(p_1, p_2)$, being a composition of surjective homomorphisms, is surjective. It follows that $f - g = (p_1 - p_2)h_2$, for some $h_2 : A \to P$. Let $h_1 = f - p_1 h_2 : A \to X$. Then $H = (h_1, h_2) : A \to X \vee_e P$ is the desired factorisation.

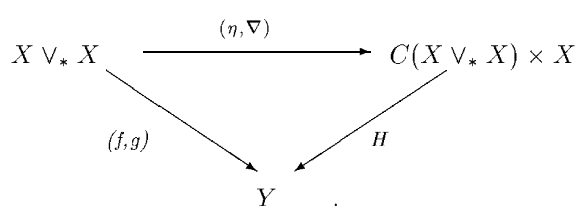
For our second example we compare the homotopy theory of pointed topological spaces arising from the standard reduced cylinder functor I with the injective homotopy theory generated by the reduced cone monad. Recall that the *reduced cylinder* IX on a pointed topological space X (we will use $*$ to denote the base point) is the space obtained from $X \times I$, the product of X with the closed unit interval I, by identifying $(x, 0) \sim (x, 1) \sim (*, t)$ for all x in X and all t in I. Maps f and g from X to Y are *homotopic*, i.e. $f \simeq g$, if there exists $F : IX \to Y$ with $F(x, 0) = fx$ and $F(x, 1) = gx$ and with $F(*, t) = *$, for all x in X and all t in I.

The *reduced cone* functor C results by defining $CX = X \times I/\sim$ where $(x, 1) \sim * \sim (*, t)$, for all x in X and all t in I. For each space X there is an embedding map $\eta X : X \to CX$ and a multiplication map $\mu X : CCX \to CX$, given by $\eta X(x) = (x, 0)$ and by $\mu X((x, s), t) = (x, 1 - (1 - s)(1 - t))$. Then η and μ are respectively the unit and multiplication of a monad (C, η, μ).

With the reduced cone monad is associated an injective structure in which the class $\mathcal{I}$ of injective objects contains exactly the contractible pointed spaces. It seems less easy to characterise the class $cof = \{i | Ci$ has a right inverse$\}$. However one may compare the relation $\simeq_{\mathcal{I}}$ with the relation $\simeq$ associated with the reduced cylinder functor I.

PROPOSITION 4.4. *The relation $\simeq_{\mathcal{I}}$ coincides with $\simeq$ and is an equivalence relation. The associated homotopy categories are naturally equivalent.*

Proof. Dual to 4.2, maps $f, g : X \to Y$ are homotopic via $\simeq_{\mathcal{I}}$ if there is a commutative diagram



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Let $\mu : X \vee_* X \to IX$ be the map given by the equations $\mu(x_1) = (x, 0)$, $\mu(x_2) = (x, 1)$, where we are using x_1 and x_2 to denote the corresponding elements of a typical element x of X in the two components of the pointed sum. We claim that the cylinders represented by μ and $(\eta, \bigtriangledown)$ are equivalent on the grounds that there is a commutative diagram

$$\begin{array}{ccc} X \vee_* X & \xrightarrow{(\eta,\nabla)} & C(X \vee_* X) \times X \\ \| & & \nu \uparrow\downarrow \lambda \\ X \vee_* X & \xrightarrow{\mu} & IX \end{array} , \tag{4.5}$$

where λ and ν are given by equations

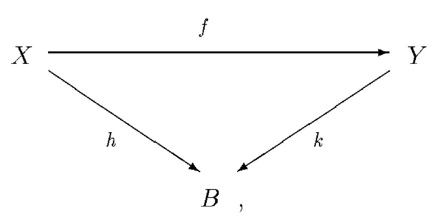
$$\lambda((x_1, t), x') = (x', t/2), \qquad \lambda((x_2, t), x') = (x', 1 - t/2)$$

$$\nu(x, t) = \begin{cases} ((x_1, 2t), x) & (0 \le t \le 1/2) \\ ((x_2, 2 - 2t), x) & (1/2 \le t \le 1). \end{cases}$$

Remark 4.6. In the case of a structure satisfying axiom (F4), Baues [1; Chapter II] proves that the class of weak equivalences *we* coincides with the class of 'homotopy equivalences' determined by the homotopy relation. In situations where the structure $(\mathbf{K}, \mathcal{F},\ we,\ e)$ does not satisfy (F4) there is no reason to expect that *we* will contain all the homotopy equivalences determined by $\simeq_\Sigma$.

5. Projective Homotopy over a Fixed Object.

In this section, given a projective structure $(\mathcal{P}, \mathcal{F})$ in a category $\mathbf{K}$, we study associated homotopy theories in the comma category $\mathbf{K}/B$, where B is a fixed object of $\mathbf{K}$. The objects of $\mathbf{K}/B$ are $\mathbf{K}$-morphisms with codomain B and the morphisms $f : h \to k$ of $\mathbf{K}/B$ are the commutative triangles



where $f : X \to Y$ is a morphism of $\mathbf{K}$. It may be noted that even in $\mathbf{K}$ is additive the category $\mathbf{K}/B$ is generally non-additive. Consider the classes

$$\mathcal{P}_B = \{p : P \to B | P \in \mathcal{P}\} ,$$

$$\mathcal{F}_B = \{f : h \to k | (f : \text{dom} h \to \text{dom} k) \in \mathcal{F}\} . \tag{5.1}$$

The following is easily checked.

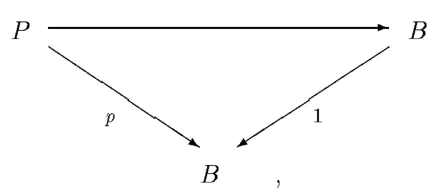
LEMMA 5.2. *The pair* $(\mathcal{P}_B, \mathcal{F}_B)$ *is a projective structure in* $\mathbf{K}_B$.

It is well known that if $\mathbf{K}$ admits finite limits (respectively finite colimits) then so does $\mathbf{K}/B$. It follows that if $\mathbf{K}$ and $(\mathcal{P}, \mathcal{F})$ satisfy the conditions of Proposition 2.4 then the construction may be applied to yield a homotopy theory ($\mathbf{K}/B$, $\mathcal{F}_B$, we_B, e_B) satisfying axioms (F1), (F2a) and (F3). However, if $\mathbf{K}$ is additive this is not the only possible construction of a projective homotopy theory in $\mathbf{K}$/B.

Suppose now that $\mathbf{K}$ is additive and equipped with a projective structure $(\mathcal{P}, \mathcal{F})$. These data induce the following homotopy relation in $\mathbf{K}/B$. Let $f, g : X \to Y$ be maps over B and set $f \simeq_B g$ if the difference $f - g$ can be factored (in $\mathbf{K}$) through a member of $\mathcal{P}$. It is easy to check that $\simeq_B$ is an equivalence relation in $\mathbf{K}/B$. If moreover we define a map $f : X \to Y$ over B to be a *weak equivalence* if there exists a map $g : Y \to X$ over B such that $f_g \simeq_B 1$ and $gf \simeq_B 1$, and set we to be the class of weak equivalences then we may prove the following.

PROPOSITION 5.3. *The structure* $(\mathcal{K}_B, \mathcal{F}_B, we, e_B)$ *satisfies the axioms* (F1), (F2a), (F3).

However the classes we and we_B need not coincide. Indeed, if $p : P \to B$ is a member of $\mathcal{P}_B$ then the unique arrow



,

is, by definition, a member of we_B. (The terminal object of $\mathcal{K}_B$ is the identity $B \to B$.) But it may not be the case that the arrow $p \to 1$ lies in we, since there may be *no* map over B from B to P. For example this would happen in any situation where the image of p, as a subobject of B was not B itself.

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