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An Agmon-Douglis-Nirenberg Type Result for Some Non Linear Equations

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Summary. - *We consider local distributional solutions* $u \in W^{1,r}_{loc}(\Omega)$ *of non linear elliptic equations of the type*

$$-divA(x, Du) = -divf(x) + g(x)$$

and we prove that $u \in W^{2,r}_{loc}(\Omega)$ *when* r *is sufficiently close to 2 which is the exponent related to the growth conditions of the operator (see assumptions (2) and (3))*

1. Introduction

A classical result, due to Agmon,Douglis and Nirenberg (see [1]), shows that the weak solutions $u \in W_0^{1,r}$ of second order linear elliptic equations with regular coefficients and right hand side in $L^r(\Omega)$ belong to $W^{2,r}(\Omega)$ for any $r > 1$. It is well known also (see [6]) that weak solutions to second order non linear elliptic equations in divergence form, with right hand side in $L^2_{\rm loc}(\Omega)$, under suitable regularity and growth assumptions, belong to $W^{2,2}_{\rm loc}(\Omega)$. As far as we know the

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non linear case has not yet been considered for r different from 2. In this paper we consider the non linear problem

$$\begin{cases} u \in W^{1,r}_{\text{loc}}(\Omega) & r > 1 \\ -\text{div}A(x, Du) = -\text{div}f(x) + g(x) \end{cases} \tag{1}$$

where Ω is an open bounded set in $\mathbb{R}^n$, $f \in \left(W^{1,r}_{\text{loc}}(\Omega)\right)^n$, $g \in L^r_{\text{loc}}(\Omega)$ and $A : (x, \xi) \in \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-function satisfying

$$\frac{\partial A_i}{\partial \xi_j}(x, \xi)\lambda_i\lambda_j \geq a|\lambda|^2 \tag{2}$$

$$|D_x A(x, \xi)| \leq b(|k_1(x)| + |\xi|) \qquad k_1(x) \in L^r(\Omega) \tag{3}$$

$$|A(x, \xi)| \leq c(|k_2(x)| + |\xi|) \qquad k_2(x) \in L^r(\Omega) \tag{4}$$

$$|D_\xi A(x, \xi)| \leq d \tag{5}$$

for every $\xi, \lambda \in \mathbb{R}^n$ and for every $x \in \Omega$, with a, b, c, d positive constants.

We deal with solutions u of (1) in the sense of distributions for $|2 - r|$ small enough and we prove that $u \in W^{2,r}_{\text{loc}}(\Omega)$.

We point out that we are concerned with regularity properties of the distributional solutions.

Existence of distributional solutions for problem (1) if $r > 2$ but $|r - 2|$ small enough is a consequence of classical results (see [9], [10]). Some results concerning existence of distributional solutions for $r < 2$ and $|r - 2|$ small enough have been proved in [2], [8] and [11].

More precisely we prove:

THEOREM 1.1. *Assume that A satisfies (2)-(5). There exist* r_1, r_2 *with* $r_1 < 2 < r_2$ *such that, if* $r_1 < r < r_2$ *and* u *is a distributional solution of problem (1) with* $f \in (W^{1,r}_{loc}(\Omega))^n$ *and* $g \in L^r_{loc}(\Omega)$*, then* $u \in W^{2,r}_{loc}(\Omega)$ *and the following estimate holds:*

$$\int_{B_R} |D^2u|^r dx \leq c\left(\frac{1}{R^{2r}}\int_{B_{2R}} |u|^r dx + \frac{1}{R^r}\int_{B_{2R}} |Du|^r dx + \int_{B_{2R}} \left(|g|^r + |Df|^r + |k_1|^r\right) dx\right). \tag{6}$$

The main tool in our proof is the Hodge decomposition (see Lemma 2.3 in Section 2) which allows us to find a good test function to use in the weak form of the equation. Indeed in our case classical test functions, which are in the same Sobolev space of the solution u, cannot be used if $r < 2$ and give only $W^{2,2}_{\rm loc}$-regularity if $r > 2$.

Similar arguments have been used to prove higher integrability of the gradient of distributional solutions of non linear equations of the type (1) (see [4], [3] and [8]). The analogous result for non linear operators whose growth is $p \neq 2$ is still an open problem.

2. Notation and preliminaries

Let Ω be a bounded open set in $\mathbb{R}^n$ with boundary $\partial\Omega$ and $u : \Omega \to \mathbb{R}$ a given function, we denote by

$$Du = \left(\frac{\partial u}{\partial x_1}, \dots , \frac{\partial u}{\partial x_n}\right) \quad \text{and} \quad D^2 u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,n} .$$

Moreover, if we consider functions $A(x, \xi)$ depending on $x \in \Omega$ and $\xi \in \mathbb{R}^n$, we shall denote by $D_x A(x, \xi)$ the vector $\big(\frac{\partial}{\partial x_1} A(x, \xi), \dots , \frac{\partial}{\partial x_n} A(x, \xi)\big)$ and by $D_\xi A(x, \xi)$ the vector $\big(\frac{\partial}{\partial \xi_1} A(x, \xi), \dots , \frac{\partial}{\partial \xi_n} A(x, \xi)\big)$. Finally for $h \in \mathbb{R}$ we set

$$\tau_{h,s} u = \frac{u(x + he_s) - u(x)}{h} \qquad s = 1, 2, \dots , n$$

where $e_1 = (1, 0, 0, \dots , 0), \dots , e_n = (0, 0, \dots , 1)$.

The function $\tau_{h,s}(u)$ is defined in the set

$$\Omega_{h,s} = \big\{x \in \Omega : x + he_s \in \Omega\big\} .$$

When no confusion may arise we write τ_h and Ω_h instead of $\tau_{h,s}$ and $\Omega_{h,s}$.

The operator τ_h verifies the following properties:

1. If $u \in W^{1,p}(\Omega)$ then $\tau_h u \in W^{1,p}(\Omega_h)$ and $\dfrac{\partial}{\partial x_i}(\tau_h u) = \tau_h \dfrac{\partial}{\partial x_i} u$.

2. If at least one of the functions u and v has support in Ω_h, then

$$\int u \tau_h v \, dx = - \int v \tau_{-h} u \, dx .$$

3. $\tau_h(uv) = u(x+he_s)\tau_h v + v\tau_h u\,.$

We shall use the following propositions which are proved in [6].

PROPOSITION 2.1. *For every* $\Omega' \Subset \Omega$, *if* $u \in W^{1,p}(\Omega)$ *and* $|h| < \frac{1}{2}\mathrm{dist}(\Omega', \partial\Omega)$ *then*

$$\|\tau_h u\|_{p,\Omega'} \le \left\|\frac{\partial u}{\partial x_s}\right\|_{p,\Omega}. \tag{7}$$

PROPOSITION 2.2. *Let* $u \in L^p(\Omega)$, $1 < p < +\infty$, *if there exists a constant* k *such that for every* h *with* $|h| < h_0$ *it results*

$$\|\tau_h u\|_{p,\Omega_h} \le k\,,$$

then

$$\frac{\partial u}{\partial x_s} \in L^p(\Omega) \quad \text{and} \quad \left\|\frac{\partial u}{\partial x_s}\right\|_{p,\Omega} \le k\,.$$

We shall denote by $B_\mu(x_0)$, $x_0 \in \mathbb{R}^n$, $\mu > 0$ the open ball $\{x \in \mathbb{R}^n : |x - x_0| < \mu\}$ and we shall omit the index x_0 when no confusion may arise. The following lemmas will be used in the proof of our theorem in Section 3.

LEMMA 2.3. *Let* $B \subset \mathbb{R}^n$ *be a ball and* $u : B \to \mathbb{R}$ *with* $u \in W_0^{1,r}(B)$, $r > 1$ *and let* $-1 < \delta < r - 1$. *Then there exists* $\phi : B \to \mathbb{R}$ *and* $H : B \to \mathbb{R}^n$ *such that* $H \in L^{r/(1+\delta)}(B)$, $\phi \in W_0^{1,r/(1+\delta)}(B)$ *and*

$$|Du|^\delta Du = D\phi + H\,.$$

Moreover

$$\|H\|_{L^{r/(1+\delta)}(B)} \le \tilde{c}(r,n)|\delta|\,\|Du\|_{L^r(B)}^{1+\delta}. \tag{8}$$

We point out that the constant $\tilde{c}$ *above does not depend explicitely on the center and on the radius of the ball.*

For details on this lemma see [8].

LEMMA 2.4. *Let* $f : [R, 2R] \to [0, +\infty)$ *be a bounded function satisfying*

$$f(\rho) \le \theta f(\sigma) + \frac{A}{(\sigma - \rho)^q} + B$$

for some constants $A, B \geq 0$, $q \geq 1$, $0 < \theta < 1$ *and for every* ρ, σ *such that* $0 < R \leq \rho < \sigma \leq 2R$ *then*

$$f(R) \leq c(\theta, q)\Big(\frac{A}{R^q} + B\Big)$$

where

$$c(\theta, q) = \frac{2^{1+q}}{1-\theta}\left[\Big(\frac{2}{1+\theta}\Big)^{1/q} - 1\right]^{-q}$$

is increasing with respect to q.

For details on this lemma see [1] and [5].

3. Proof of the theorem

In this section we shall denote by C a constant which may vary from line to line.

Our aim is to prove the following estimate

$$\int_{B_R} |D\tau_h u|^r dx \leq C\left[\frac{1}{R^{2r}}\int_{B_{2R}} (|u|^r + |Du|^r)dx + \right. \\ \left. + \int_{B_{2R}} (|Df|^r + |g|^r + |k_1|^r)dx\right]$$

which implies, by Proposition 2.2, that $u \in W^{2,r}_{\rm loc}(\Omega)$ and (6) holds true.

The weak form of the equation is

$$\int_\Omega A(x, Du)\frac{\partial\psi}{\partial x_i}dx = \int_\Omega f_i\frac{\partial\psi}{\partial x_i}dx + \int_\Omega g\psi\, dx \tag{9}$$

$\forall\, \psi \in W^{1,\frac{r}{r-1}}(\Omega)$ with supp$\psi \Subset \Omega$.

Let $x_0 \in \Omega$, $R \leq \rho < \sigma \leq 2R$, $R < 1$, with $2R < \text{dist}(x_0, \partial\Omega)$.

We consider a cut-off function $\eta \in C^\infty_0(B_\sigma)$ such that $\eta \equiv 1$ on B_ρ, $|D\eta| \leq \dfrac{C}{\sigma - \rho}$, $|D^2\eta| \leq \dfrac{C}{(\sigma-\rho)^2}$ (see [12]).

We have that $\eta u \in W^{1,r}_0(B_\sigma)$, supp$(\eta u) \Subset B_\sigma$ and $\tau_h(\eta u) \in W^{1,r}_0(B_\sigma)$, supp$\tau_h(\eta u) \Subset B_\sigma$, if $|h|$ is small enough.

Let us now consider $\gamma < \sigma$ such that $\mathrm{supp}\tau_h(\eta u) \Subset B_\gamma \subset B_\sigma$. By Lemma 2.3, there exist $\phi \in W_0^{1,\frac{r}{r-1}}(B_\gamma)$ and a vector H with $\mathrm{div}H = 0$, such that

$$|D\tau_h(\eta u)|^{r-2} D\tau_h(\eta u) = D\phi + H\,. \tag{10}$$

Moreover the following estimates hold

$$\|D\phi\|_{L^{\frac{r}{r-1}}(B_\gamma)} \le C\|D\tau_h(\eta u)\|^{r-1}_{L^r(B_\gamma)} \tag{11}$$

$$\|H\|_{L^{r/(r-1)}(B_\gamma)} \le C|r-2|\|D\tau_h(\eta u)\|^{r-1}_{L^r(B_\gamma)}\,. \tag{12}$$

By extending ϕ in B_σ with zero value, we get $\phi \in W_0^{1,r}(B_\sigma)$, $\mathrm{supp}\phi \Subset B_\sigma$.

We remark that $-\tau_{-h}\phi \in W_0^{1,\frac{r}{r-1}}(B_\sigma)$ and we can use it as test function in (9), therefore we get

$$-\int_{B_\sigma} A_i(x,Du)\tau_{-h}\frac{\partial\phi}{\partial x_i}dx = -\int_{B_\sigma} f_i\tau_{-h}\frac{\partial\phi}{\partial x_i}dx - \int_{B_\sigma} g\tau_{-h}\phi\,dx$$

and also

$$\begin{aligned}
-\int_{B_\sigma} A_i(x,D(\eta u))\tau_{-h}\frac{\partial\phi}{\partial x_i}dx = \\
-\int\big[A_i(x,D(\eta u)) - A_i(x,Du)\big]\tau_{-h}\frac{\partial\phi}{\partial x_i}dx \\
-\int_{B_\sigma} f_i\tau_{-h}\frac{\partial\phi}{\partial x_i}dx - \int_{B_\sigma} g\tau_{-h}\phi\,dx \\
= I + II + III\,.
\end{aligned} \tag{13}$$

By (2), (3), (5), recalling the property 2, the integrals at the left hand side can be estimated as follows:

$$
\begin{aligned}
&\int_{B_\sigma} \tau_h A_i(x, D(\eta u)) \frac{\partial \phi}{\partial x_i}\, dx = \\
&\quad = \int_{B_\sigma} \left[\frac{1}{h} \int_0^1 \frac{d}{dt} A_i(x + the_s, D(\eta u) + thD\tau_h(\eta u)) dt \right] \frac{\partial \phi}{\partial x_i}\, dx = \\
&\quad = \int_{B_\sigma} \left[\int_0^1 \frac{\partial A_i}{\partial x_s} + \frac{\partial A_i}{\partial \xi_j} \tau_h \Big(\frac{\partial(\eta u)}{\partial x_j} \Big) dt \right] \frac{\partial \phi}{\partial x_i}\, dx = \qquad (*) \\
&\quad = \int_{B_\sigma} \left[\int_0^1 \frac{\partial A_i}{\partial x_s} + \frac{\partial A_i}{\partial \xi_j} \tau_h \Big(\frac{\partial(\eta u)}{\partial x_j} \Big) dt \right] \left(|D\tau_h(\eta u)|^{r-2} \tau_h \Big(\frac{\partial(\eta u)}{\partial x_i} \Big) - H_i \right) \geq \\
&\quad \geq a \int_{B_\sigma} |D\tau_h(\eta u)|^r dx - d \int_{B_\sigma} |D\tau_h(\eta u)|\, |H|\, dx + \\
&\qquad - C \int_{B_\sigma} \big(|k_1(x)| + |D(\eta u)| + |hD\tau_h(\eta u)| \big) \big(|D\tau_h(\eta u)|^{r-1} + |H| \big)\, dx\, .
\end{aligned}
$$

Let us now estimate the terms at the right hand side of (13).

If we put

$$
\mathcal{A}_i = A_i\big(x + the_s, D(\eta u)(x) + thD\tau_h(\eta u)(x)\big) - A_i\big(x + the_s, Du(x) + thD\tau_h u(x)\big)\, ,
$$

recalling (3) and (5) we get

$$
\begin{aligned}
|I| &= \Big| \int_{B_\sigma - B_\rho} \frac{1}{h} \left[\int_0^1 \frac{d}{dt} \mathcal{A}_i dt \right] \frac{\partial \phi}{\partial x_i}\, dx \Big| \\
&\leq C \int_{B_\sigma - B_\rho} [|k_1(x + he_s) + |D(\eta u)| + |Du| + |D\tau_h(\eta u)| + |D\tau_h u|] \cdot \\
&\qquad \cdot \big[|D\tau_h(\eta u)|^{r-1} + |H| \big]\, dx \\
|II| &\leq \int_{B_\sigma} |\tau_h f|\, \big[|D\tau_h(\eta u)|^{r-1} + |H| \big]\, dx \\
|III| &\leq \int_{B_\sigma} |g|\, |\tau_{-h} \phi| dx\, .
\end{aligned}
$$

(*) The functions $\frac{\partial A_i}{\partial x_s}$, $\frac{\partial A_i}{\partial \xi_j}$ are evaluated in $(x + the_s, D(\eta u) + thD\tau_h(\eta u))$.

Putting together the previous estimates we have

$$
\begin{aligned}
a\int_{B_\sigma} |D\tau_h(\eta u)|^r dx \le d\int_{B_\sigma} |D\tau_h(\eta u)|\,|H|\,dx\, + \\
&+ C\int_{B_\sigma} (|k_1(x)| + |D(\eta u)| + |hD\tau_h(\eta u)|)\left(|D\tau_h(\eta u)|^{r-1} + |H|\right) dx\, + \\
&+ C\int_{B_\sigma - B_\rho} [|k_1(x + he_s)| + |D(\eta u)| + Du|]\cdot\left[|D\tau_h(\eta u)|^{r-1} + |H|\right] dx\, + \\
&+ \int_{B_\sigma - B_\rho} [|D\tau_h(\eta u)| + |D\tau_h u|]\,|D\tau_h(\eta u)|^{r-1} dx\, + \\
&+ \int_{B_\sigma - B_\rho} |D\tau_h(\eta u) + D\tau_h u|\,|H| dx\, + \\
&+ \int_{B_\sigma} |\tau_h f|\left(|D\tau_h(\eta u)|^{r-1} + |H|\right) dx + \int_{B_\sigma} |g|\,|\tau_{-h}\phi| dx\,.
\end{aligned}
$$

By using Young's inequality several times and Proposition 2.1, denoting by ε and $c(\varepsilon)$ suitable constants to be chosen, we get, for $|h| = \varepsilon$

$$
\begin{aligned}
a\int_{B_\sigma} |D\tau_h(\eta u)|^r dx \le \varepsilon\int_{B_\sigma} |D\tau_h(\eta u)|^r dx + c(\varepsilon)\Bigg[\int_{B_\sigma} |H|^{r/(r-1)} dx\, + \\
&+ \int_{B_R} |k_1|^r dx + \int_{B_\sigma} |Du|^r dx + \frac{c}{\sigma - \rho}\int_{B_\sigma} |u|^r dx + \int_{B_\sigma} |Df|^r dx \\
&+ \int_{B_\sigma} |g|^r dx + \int_{B_\sigma - B_\rho} |D\tau_h u|^r dx + \int_{B_\sigma - B_\rho} |D\tau_h(\eta u)|^r dx\,.
\end{aligned}
\tag{14}
$$

By properties 1 and 3 the last integral in (14) can be estimated as follows,

$$
\begin{aligned}
&\int_{B_\sigma - B_\rho} (|\tau_h\eta|\,|Du| + \eta|D\tau_h u| + |\tau_h u|\,|D\eta| + |u|\,|D\tau_h\eta|)^r\, dx \le \\
&\le \frac{C}{(\sigma - \rho)^r}\int_{B_\sigma - B_\rho} |Du|^r dx + \int_{B_\sigma - B_\rho} |D\tau_h u|^r dx + \frac{C}{(\sigma - \rho)^{2r}}\int_{B_\sigma - B_\rho} |u|^r dx.
\end{aligned}
\tag{15}
$$

We can choose ε and r in such a way that $\varepsilon + c(\varepsilon)\tilde{c}|r-2| < \frac{a}{2}$ (so we find r_1 and r_2 which is in the statement of the theorem). By using (12), (14) and (15), we get

$$\int_{B_\sigma} |D\tau_h(\eta u)|^r dx \leq$$
$$\leq c\left[\int_{B_{2R}} \big(|k_1|^r + |Df|^r + |g|^r\big)dx + \frac{1}{(\sigma-\rho)^r}\int_{B_{2R}} |Du|^r dx \right.$$
$$\left. \frac{1}{(\sigma-\rho)^{2r}}\int_{B_{2R}} |u|^r dx + \int_{B_\sigma - B_\rho} |D\tau_h u|^r dx\right]. \tag{16}$$

Recalling that $\eta \equiv 1$ on B_ρ, adding the term $c\int_{B_\rho} |D\tau_h u|^r dx$ to both sides we have

$$(c+1)\int_{B_\rho} |D\tau_h u|^r dx \leq c\left[\int_{B_{2R}} \big(|k_1|^r + |Df|^r + |g|^r\big)dx\right.$$
$$\left. + \frac{1}{(\sigma-\rho)^r}\int_{B_{2R}} |Du|^r dx + \frac{1}{(\sigma-\rho)^{2r}}\int_{B_{2R}} |u|^r\, dx + \int_{B_\sigma} |D\tau_h u|^r\, dx\right]. \quad ∎$$

By Lemma 2.4, with $\theta = \frac{c}{c+1}$, we get

$$\int_{B_R} |D\tau_h u|^r \leq c\left[\frac{1}{R^r}\int_{B_{2R}} |Du|^r dx\right.$$
$$\left. + \frac{1}{R^{2r}}\int_{B_{2R}} |u|^r dx + \int_{B_{2R}} \big(|k_1|^r + |Df|^r + |g|^r\big)dx\right]$$

and the proof is complete.

References

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, Comm. Pure Appl.Math. **XII** (1959), 623–727.

[2] L. Boccardo, *Quelques problem de Dirichlet avec donnes dans des grands espaces des Sobolev*, to appear on C.R.A.S.

[3] D. Giachetti, F. Leonetti, and R. Schianchi, *On the regularity of very weak minima*, Proc. Royal Soc. Edinburgh Sect. A **126** (1996), 287–296.

[4] D. Giachetti, F. Leonetti, and R. Schianchi, *Boundary regularity and uniqueness for very weak ⊣ armonic functions*, Atti del Seminario Matematico e Fisico dell'Università di Modena **Suppl. Vo. XLVI** (1998).

[5] M. Giaquinta and E. Giusti, *On the regularity of the minima of variational integrals*, Acta Math. **148** (1982), 31–46.

[6] E. Giusti, *Equazioni ellittiche del secondo ordine*, Quaderni dell'U.M.I., Pitagora, 1978.

[7] T. Iwaniec, *p-harmonic tensors and quasi regular mappings*, Ann. Math. **136** (1992), 589–624.

[8] T. Iwaniec and C. Sbordone, *Weak minima of variational integrals*, J. Reine Angew. Math. **454** (1994), 143–161.

[9] J. Leray and J.L. Lions, *Quelques resultas de Visik sur lesproblémes elliptiques non lineaires par la méthode de Minty-Browder*, Bull. Soc. Math. France **93** (1965), 97–107.

[10] N. Meyers and A. Elcrat, *Some results on regularity for solutions of non linear elliptic systems and quasi regular functions*, Duke Math. J. **42** (1975), 121–136.

[11] F. Murat, *Equations elliptiques non lineaires avec second membre L^1 ou mesure*, Acts du 26éme Congrés National d'Analyse Numerique (les Karellis 1994).

[12] Protter and Morrey, *A first course in real analysis*, Springer-Verlag, New York, 1977.

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