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On Hyperbolic 3-Orbifolds of Small Volume and Small Heegaard Genus

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Summary. - *In the present note we shall give geometric descriptions of the orientable hyperbolic 3-orbifolds of smallest known volumes. As in the case of hyperbolic 3-manifolds, the hyperbolic 3-orbifolds of smallest volumes are still not known but there is some evidence that our list should be complete (however in some cases the volumes have not yet been computed). We note that the natural candidates for the ten orientable hyperbolic 3-manifolds of smallest volumes have been described in [6] (all of Heegaard genus two). In the following, we shall consider only orientable orbifolds. Computations of volumes are based on the recent papers [11], [9] and [14].*

Introduction

As in the case of hyperbolic 3-manifolds ([15]), every sequence of hyperbolic 3-orbifolds of bounded volumes has a subsequence which converges to, and is obtained by surgery on, some cusped hyperbolic 3-orbifold ([1], [4]). By [1], the unique orientable hyperbolic 3-orbifold of smallest volume with a nonrigid cusp (a cusp on which surgery can be performed) is the Picard orbifold $\mathcal{P}$ which is the quotient of hyperbolic 3-space by the Picard group $PSL(2, \mathbb{Z}[i])$, of approximate volume 0.305322. It has exactly one cusp, and its singular set is shown in Figure 1a (the space is the 3-sphere minus one point, the numbers associated to the edges are the branching orders, and

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an edge without number has branching order two). It follows that among all hyperbolic 3-orbifolds whose volumes are smaller than that of the Picard orbifold, only finitely many are not obtained by surgery on the Picard orbifold. The computation of the volumes of the hyperbolic 3-orbifolds obtained by surgery on the Picard orbifold has recently been carried out by Petrov ([14]) by direct computations, and independently in [11] by reducing it to the case of hyperbolic links and then using SnapPea (see section 2). We note that also the volumes of orbifolds obtained by small surgeries on the second and third smallest cusped orbifolds (as given in [1], [4]) are computed in [11].

The Picard orbifold belongs to the class of orbifolds of *pyramidal type* whose singular sets are pyramids as in Figure 1a, but with other branching orders associated to the edges of the square constituting the base of the pyramid (the cusp being its cone point). The simplest orbifolds obtained by surgery on the cusps of these pyramidal orbifolds are the *tetrahedral orbifolds* associated to the nine Lanner tetrahedra (uniformized by the orientation-preserving subgroups of index two in the Coxeter groups generated by the reflections in the faces of a hyperbolic Coxeter tetrahedron, i.e. whose angles are submultipes of π). The singular set is the 1-skeleton of the tetrahedron, the space the 3-sphere. Also, one obtains the quotients of these tetrahedral orbifolds by (orientation-preserving isometric) involutions wich we shall call *quasitetrahedral orbifolds*. Various of the smallest known 3-orbifolds are of tetrahedral or quasitetrahedral type. The volumes of these orbifolds are readily computed, see e.g. the appendix of [12]. The volumes of some other hyperbolic orbifolds obtained by small surgeries on the pyramidal orbifolds have been computed by Petrov ([14]).

Arithmetic hyperbolic orbifolds of small volume are discussed in [5], and one expects that the smallest hyperbolic 3-orbifolds are of this type. The smallest arithmetic orbifold is the quasitetrahedral orbifold $\mathcal{O}_1$ in LIST I ([3]). The smallest cusped orbifold, with one rigid cusp, is the arithmetic tetrahedral orbifold $\mathcal{O}_{10}$ ([2],[13]).

A natural notion of Heegaard splitting and Heegaard number for closed 3-orbifolds is discussed in [9], and again one would suspect that the smallest volume hyperbolic 3-orbifolds belong to the 3-orbifolds

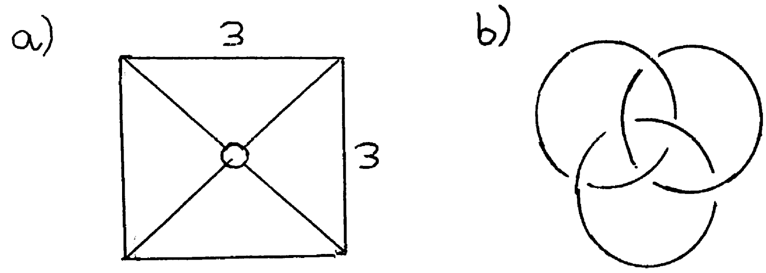


Figure 1

of extremal Heegaard number (resp. are 2-fold quotients of these by involutions which exchange the two handlebody-orbifolds of an extremal Heegaard splitting). In fact the smallest known 3-orbifolds are all extremal with respect to the Heegaard number (or of a closely related type).

The preceding discussion naturally leads to the consideration of the following three classes of hyperbolic 3-orbifolds, in increasing order of complexity:

- *tetrahedral and quasitetrahedral orbifolds;*

- *other orbifolds obtained by small surgeries on the Picard orbifold and other pyramidal orbifolds, and on the smallest orbifolds with a non-rigid cusp;*

- *other orbifolds wich are extremal with respect to the Heegaard number.*

At the end of the paper, we shall give three lists of small orbifolds which will be denoted by $\mathcal{O}_1$, $\mathcal{O}_2$ etc., in order of increasing volume. The 14 smallest volumes of orientable hyperbolic 3-orbifolds wich we find are the following:

List of the volumes of the smallest known hyperbolic 3-orbifolds:

0.039050	0.040890	
0.052654	0.065965	
0.066191	0.071770	
0.078100	0.078559	
0.081780	0.084578	(→ the smallest cusped volume)
0.085770	0.093326	
0.102843	0.105309	

In the following, we shall present two orbifolds $\mathcal{O}_3(5/2; 2, 3)$ and $\mathcal{O}_3(5/2; 3, 2)$ (see section 3 and LIST III) whose volume we do not know at present but suspect to belong to the above list of smallest volumes. It is interesting to compare this list with the list of small volume arithmetic orbifolds given in [5].

1. Tetrahedral and quasitetrahedral orbifolds

The singular sets of the smallest orbifolds of these two types are given in LIST I (the underlying space is always the 3-sphere). The orbifold $\mathcal{O}_{10}$ has one rigid cusp; in fact, by [2] it is the unique cusped hyperbolic 3-orbifold of minimal volume (see also [13] for the smallest cusped hyperbolic 3-orbifolds).

2. Other small orbifolds obtained by surgery on the Picard orbifold and the pyramidal orbifolds

These are given in LIST II. We denote the orbifold obtained by (x, y)-surgery on the cusp of the Picard orbifold by $\mathcal{P}(x, y)$ (see [4] for the notion of surgery on orbifolds).

The orientation-preserving symmetry group of the Borromean rings $\mathcal{B}$ (see Figure 1b) is the symmetric or octahedral group $\mathbb{S}_4$ of order 24. The action of $\mathbb{S}_4$ on the complement of the Borromean rings extends to the manifold $\mathcal{B}(p, q)$ obtained by (p, q)-surgery on all three components of the Borromean rings. It is shown in [11] that the quotient $\mathcal{B}(p, q)/\mathbb{S}_4$ is the 3-orbifold $\mathcal{P}(p-2q, p+2q)$ obtained by $(p - 2q, p + 2q)$-surgery on the cusp of the Picard orbifold. Now the volumes of the orbifolds $\mathcal{P}(x, y)$ can be computed by applying SnapPea (see e.g. [6]) to the Borromean rings. Note that, given (x, y), we do not always get integer solutions for $(p,q) = ((y+x)/2, (y-x)/4)$; in this case $\mathcal{B}(p,q)$ has to be interpreted as a cone manifold (see [7],[8]) for which the computation of SnapPea still works. For example, for $(x,y) = (1,4)$ we get $(p,q) = (5/2, 3/4) = (2.5, 0.75) = (1/4) \times (10,3)$. Performing surgery with these coefficients on all three components of the Borromean rings we obtain the cone manifold $\mathcal{B}(10,3)$, with an angle of $2\pi/(1/4) = 8\pi$ around the central curves of the three surgered solid tori. SnapPea gives the volume 0.981369 for this cone manifold (this is also the volume of the second smallest known hyperbolic 3-manifold $\mathcal{M}_2$, obtained by $(5,1)$-surgery on the figure-8 knot, see [6]). Dividing by 24 we obtain the volume 0.040890 for the second smallest known orbifold $\mathcal{O}_2 = \mathcal{P}(1,4)$; as noted in [5] this is commensurable with the manifold $\mathcal{M}_2$.

The volumes for small surgeries on the other pyramidal orbifolds are computed in [14]. The smallest volume found is 0.066191 for the orbifold $\mathcal{O}_5 = \mathcal{O}_3(3/1; 2, 4)$ (see LIST II).

3. Other small hyperbolic 3-orbifolds of extremal Heegaard number

Their singular sets shown in LIST III are obtained by adjoining to a 2-bridge link $L(a/b)$ of type a/b the standard upper and lower unknotting tunnels, with branching orders 2, 2, 2 and 3 associated to the four strings of the plat resp. braid. If one associates branching orders two also to the two tunnels, their volumes have been computed in [9]. Following [9], we will denote these orbifolds by $\mathcal{O}_3(a/b) = \mathcal{O}_3(a/b; 2, 2)$. Let $\mathcal{L}_3(a/b)$ be the orbifold whose space is the 3-sphere and whose singular set, of branching order three, is the 2-bridge link $L(a/b)$. Each 2-bridge link has a canonical $\mathbb{D}_2$-symmetry by the dihedral group of order four, and it is shown in [9] that the quotient $\mathcal{L}_3(a/b)/\mathbb{D}_2$ is the orbifold $\mathcal{O}_3(a/b)$. Applying SnapPea to the 2-bridge links and dividing by four one obtains the volumes of the orbifolds $\mathcal{O}_3(a/b)$.

The computation of volumes in general, i.e. for other branching orders associated to the two tunnels, is still open. For example, we do not know the volumes of the orbifolds $\mathcal{O}_3(5/2; 2, 3)$ and $\mathcal{O}_3(5/2; 3, 2)$ in LIST III but we suppose that they belong to the above list of smallest volumes. Note that the orbifold $\mathcal{O}_3(5/2) = \mathcal{O}_3(5/2;2,2)$ is euclidean. In fact it has a regular $\mathbb{D}_2$-covering by the euclidean orbifold $\mathcal{L}_3(5/2)$ whose singular set, of branching order three, is the figure-8 knot $L(5/2)$ whose 3-fold cyclic branched covering is the euclidean Hantzsche-Wendt manifold (see [17]). So the Hantzsche-Wendt manifold is a regular $\mathbb{D}_6$-covering of the orbifold $\mathcal{O}_3(5/2)$ (where $\mathbb{D}_6$ denotes the dihedral group of order 12). Similarly, the orbifold $\mathcal{O}_8 = \mathcal{O}_3(7/2)$ has a regular $\mathbb{D}_2$-covering by the orbifold $\mathcal{L}_3(7/2)$ and a regular $\mathbb{D}_6$-covering by the hyperbolic Weeks-Matveev-Fomenko manifold $\mathcal{M}_1$ of smallest known volume which is the 3-fold cyclic branched covering of the knot $5_2 = L(7/2)$ (see [6],[10]).

4. Concluding remarks

With the exception of $\mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_7$ and $\mathcal{O}_9$, all orbifolds $\mathcal{O}_i$ are extremal with respect to the Hegaard number: they admit a Heegaard splitting along a 2-orbifold of signature $(0;2,2,2,3)$, i.e. a sphere with four branch points of orders 2, 2, 2 and 3, into two handlebody 3-orbifolds (see also [9]). As the non-spherical triangular 2-orbifolds (i.e. of signatures $(0;m,n,l)$ with $1/m+1/n+1/l \leq 1$) do not occur as boundaries of handlebody 3-orbifolds, the signature $(0;2,2,2,3)$ is minimal for Heegaard splittings along hyperbolic 2-orbifolds (the splittings along euclidean and spherical 2-orbifolds are very special and do not occur for hyperbolic 3-orbifolds).

The orbifolds $\mathcal{O}_7$ and $\mathcal{O}_9$ are not of this minimal type themselves but are 2-fold coverings of 3-orbifolds of minimal type (of the two smallest orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$, see LIST I and II). Finally, the orbifolds $\mathcal{O}_3$ and $\mathcal{O}_4$ have 2-fold coverings by 3-orbifolds of minimal type (LIST II and III) such that the covering involutions exchange the two handlebody 3-orbifolds of the minimal Heegaard splittings.

For our lists of orbifolds, we took into consideration all "small" orbifolds of minimal type (i. e. defined by small parameters), and similarly all small orbifolds obtained by surgery on the Picard orbifold and the other pyramidal orbifolds. By [1], the Picard orbifold is the smallest hyperbolic orbifold with a non-rigid cusp; this is the quotient of the complement of the Borromean rings by its isometry group, the octahedral group $\mathbb{S}_4$. However, the second and third smallest hyperbolic orbifolds with a non-rigid cusp are not of pyramidal type (the singular set of the second smallest orbifold, with one cusp, is shown in Figure 2). It follows from [1], considering volumes, that they are obtained as the quotients of the complements of the hyperbolic 3-component 6-crossing link 6_1^3, resp. the 4-component 8-crossing link 8_2^4, by their isometry groups which are the dihedral groups $\mathbb{D}_6$ of order 6, resp. the dihedral group $\mathbb{D}_8$ of order 16. Thus, as in the case of the Picard orbifold and the Borromean rings, the volumes of the orbifolds obtained by surgery on these orbifolds may be obtained from SnapPea by performing equivariant surgery on these two links (see [11]). The only two orbifolds of volume smaller than 0.1 which we found are the orbifolds $\mathcal{O}_4$ and $\mathcal{O}_8$ which correspond to $(-3, 1/2)$ and $(-3, 2)$-surgery on all three components of the link 6_1^3. In fact $(-3, 2)$-surgery on this link gives the Weeks-Matveev-Fomenko manifold $\mathcal{M}_1$, of smallest known volume 0.942707 ([6]); the $\mathbb{D}_6$-action on the complement of 6_1^3 extends to an isometric $\mathbb{D}_6$-action on $\mathcal{M}_1$, and by [6] this gives the full isometry group of $\mathcal{M}_1$. As noted already in section 3, the quotient $\mathcal{M}_1/\mathbb{D}_6$ is the orbifold $\mathcal{O}_8 = \mathcal{O}_3(7/2)$ which is obtained also by $(1, 3)$-surgery on the cusp of the orbifold in Figure 2.

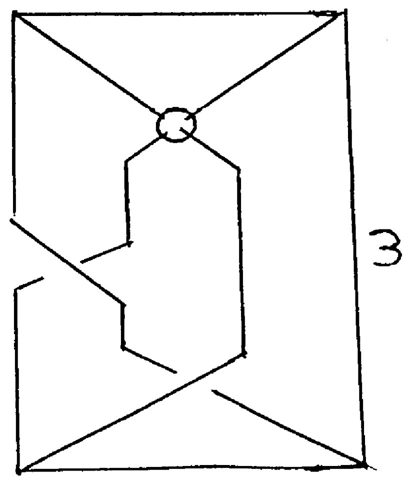


Figure 2

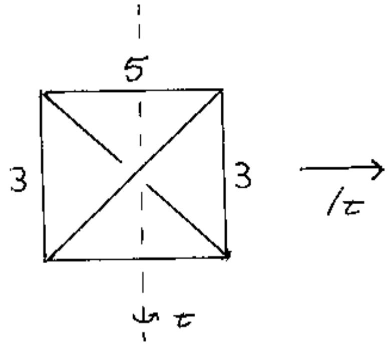
Finally, we comment on the ranks of the fundamental groups of the orbifolds $\mathcal{O}_i$. The universal covering group (or orbifold fundamental group) of the 2-orbifold with signature $(0;2,2,2,3)$ is a Fuchsian group with the same signature, acting on hyperbolic 2-space (see [16]). This Fuchsian group has a presentation of the form

$$< x_1, x_2, x_3, x_4 \mid x_1^2 = x_2^2 = x_3^2 = x_4^3 = x_1x_2x_3x_4 = 1 >$$

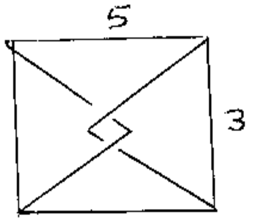
and is one of the few exceptional Fuchsian groups for which the algebraic rank (the minimal number of generators) is smaller than expected, i.e. does not coincide with the "geometric rank", see [16], chapter 4.16. In fact, the Fuchsian group with signature $(0;2,2,2,3)$ has rank two (generated by x_1x_2 and x_2x_3). As the orbifold fundamental group of each 3-orbifold with a minimal Heegaard splitting along a 2-orbifold of signature $(0;2,2,2,3)$ is a surjective image of the Fuchsian group with the same signature, the fundamental groups of all minimal 3-orbifolds have rank two, in particular those of the orbifolds $\mathcal{O}_i$, with the above four exceptions. The orbifold $\mathcal{O}_7$ has a Heegaard splitting along a 2-orbifold of signature $(0;2,2,2,5)$, and the corresponding Fuchsian group belongs again to the exceptional cases and has rank two ($\mathcal{O}_7$ has another Heegaard splitting of type $(0;3,2,2,3)$ but the rank of the Fuchsian group is three here). We do not know the rank in the remaining cases $\mathcal{O}_3, \mathcal{O}_4$ and $\mathcal{O}_9$.

We close with the

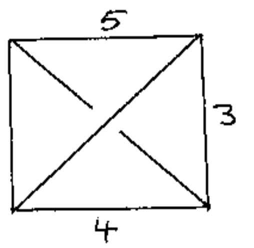
Problem. Compute the volumes of the orbifolds $\mathcal{O}_3(5/2;2,3)$ and $\mathcal{O}_3(5/2;3,2)$, and of other small orbifolds of type $\mathcal{O}_n(a/b;p,q)$, for p or q different from two.



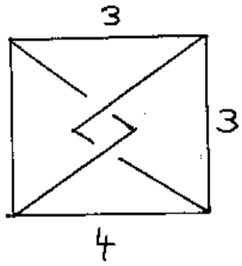
$\mathcal{O}_7$ 0.078100



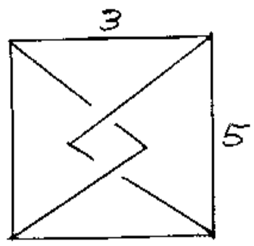
$\mathcal{O}_1$ 0.039050



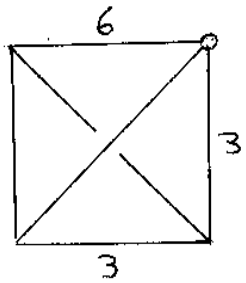
$\mathcal{O}_6$ 0.071770



$\mathcal{O}_{11}$ 0.085770



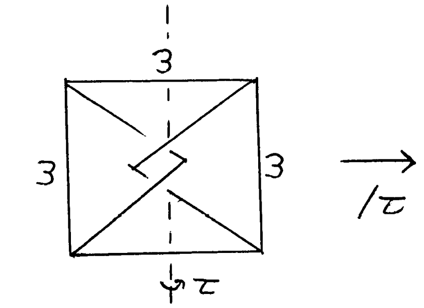
$\mathcal{O}_{12}$ 0.093326



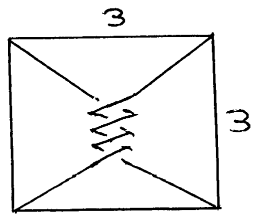
$\mathcal{O}_{10}$ 0.084578

(the smallest cusped orbifold)

LIST I

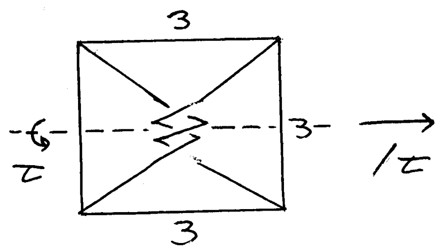


$\mathcal{O}_9$ 0.081780

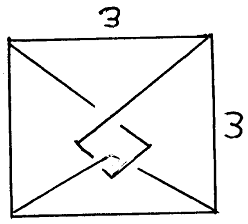


$\mathcal{O}_2$ 0.040890

$\mathcal{P}(1,4)$

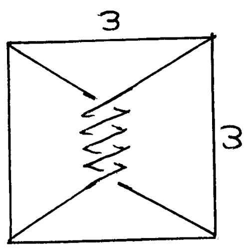


$\mathcal{O}_{14}$ 0.105309



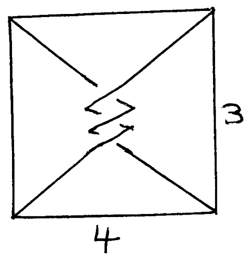
$\mathcal{O}_3$ 0.052654

$\mathcal{P}(2,3)$



$\mathcal{O}_{13}$ 0.102843

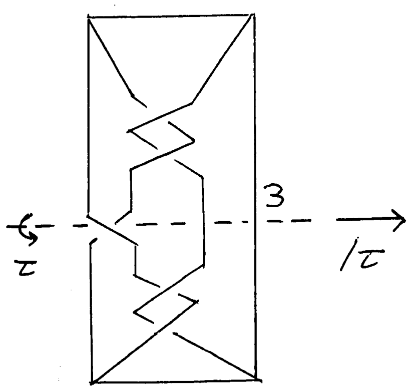
$\mathcal{P}(1,5)$



$\mathcal{O}_5$ 0.066191

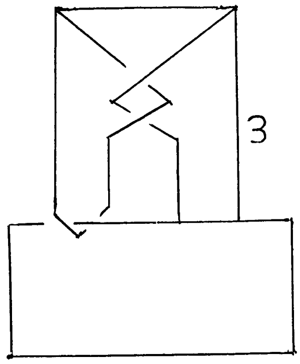
$\mathcal{O}_3(3/1;2,4)$

LIST II

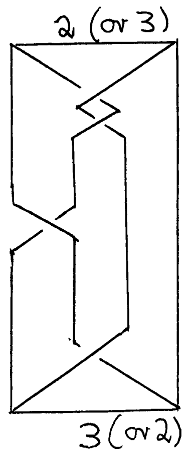


0.131930

$\mathcal{O}_3(8/3) = \mathcal{O}_3(8/3;2,2)$



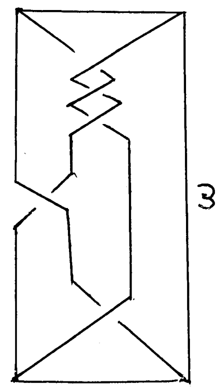
$\mathcal{O}_4$ 0.065965



vol = ?

$\mathcal{O}_3(5/2;2,3)$

$\mathcal{O}_3(5/2;3,2)$



$\mathcal{O}_8$ 0.078559

$\mathcal{O}_3(7/2) = \mathcal{O}_3(7/2;2,2)$

LIST III

References

[1] C.C. Adams, *Limit volumes of hyperbolic 3-orbifolds*, J.Diff.Geom **34** (1991), 115–141.

[2] C.C. Adams, *Noncompact hyperbolic 3-orbifolds of small volume*, Topology '90 (A.W. Reid B. Apanasov, W.D. Neumann and L.Siebenmann, eds.), Ohio State University Mathematical Research Institute Publications, De Gruyter, 1 ed., 1992, pp. 1–15.

[3] T. Chinburg and E. Friedman, *The smallest arithmetic hyperbolic three-orbifold*, Invent. math. **86** (1986), 507–527.

[4] W. Dunbar and R. Meyerhoff, *Volumes of hyperbolic 3-Orbifolds*, Indiana Univ. Math. J. **43** (1994), 611–637.

[5] F.W. Gehring, C. Maclachlan, G.J. Martin, and A.W. Reid, *Arithmeticity, discreteness and volume*, Trans. Amer. Math. Soc. **349** (1997), 3611–3643.

[6] C.D. Hodgson and J.R. Weeks, *Symmetries, isometries and length spectra of closed hyperbolic 3-manifolds*, Experimental Math. **3** (1994), 261–274.

[7] S.P. Kerkhoff, *Deformations of hyperbolic cone manifolds*, Proc. 37th Taniguchi Symposium on Topology and Teichmüller spaces, Finland 1995 (S.Kojima et al., ed.), World Scientific Publishing Co., 1996, pp. 101–114.

[8] S. Kojima, *Hyperbolic 3-manifolds singular along knots*, Chaos, Solitons & Fractals **9** (1998), 765–777.

[9] M. Mecchia and B. Zimmermann, *On a class of hyperbolic 3-orbifolds of small volume and small Heegaard genus associated to 2-bridge links*, Rend. Circ. Mat. Palermo **49** (2000), 41–60.

[10] A. Mednykh and A. Vesnin, *Covering properties of small volume hyperbolic 3-manifolds*, J. Knot Theory Ramif. **7** (1998), 381–392.

[11] A. Mednykh, A. Vesnin, and B. Zimmermann, *Surgery on small volume hyperbolic 3-orbifolds*, Siberian J. Math **42** (2001), 318–331.

[12] G.R. Meyerhoff, *A lower bound for the volume of hyperbolic 3-orbifolds*, Duke Math. J. **57** (1988), 185–203.

[13] W.D. Neumann and A.W. Reid, *Notes on Adams' small volume orbifolds*, Topology '90 (B. Apanasov, W.D. Neumann, A.W. Reid, and L. Siebenmann, eds.), vol. 1, Ohio State University Mathematical Research Institute Publications, De Gruyter, 1992, pp. 311–314.

[14] V.S. Petrov, *Dehn surgery on hyperbolic π-orbifolds*, Preprint Novosibirsk.

[15] W. Thurston, *The Geometry and Topology of 3-Manifolds*, Lecture Notes Princeton University (1978).

[16] H. Zieschang, E. Vogt, and H.D. Coldewey, *Surfaces and Pla-*

nar Discontinuous Groups, Lecture Notes in Math. (1980), 835.
[17] B. Zimmermann, *On the Hantzsche-Wendt manifold*, Monatsh. Math. **110** (1990), 321–327.

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