On the Effective Calculation of Holonomy Groups

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in memory of Franco Tricerri

SOMMARIO. - *Estendendo risultati noti solo per il fibrato tangente, si dà una definizione equivalente di gruppo di olonomia infinitesimale, definizione che permette di descrivere tale gruppo in modo esplicito. Successivamente si mostrano alcune applicazioni e un esempio.*

SUMMARY. - *Extending results previously known only for the tangent bundle, we provide an equivalent definition of infinitesimal holonomy group, which represents an efficient way to explicitly describe it. Then we offer some applications and an example.*

0. Introduction

Holonomy groups play an increasingly important role in Differential Geometry: in fact they provide the natural reduction of a given connection and thus, they lead to a deeper understanding of its geometry. This makes especially interesting to determine effective computation procedures.

In this paper, extending results previously known only for the tangent bundle, we provide an equivalent definition of infinitesimal holonomy group, which, at the same time, represents an efficient way to explicitly describe it.

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The argument is developed through a linear representation of the structure group, but same ideas work directly on principal bundles.

Given a linear connection ∇ of a vector bundle $E \xrightarrow{\pi} M$ with standard fibre F, and using also a connection D over M, we can define $\tilde{\nabla}$ in this way

$$\left(\tilde{\nabla}_{V_1}\alpha\right)(X,Y) = \nabla_{V_1}(\alpha(X,Y)) - \alpha(D_{V_1}X,Y) - \alpha(X,D_{V_1}Y)$$

where $\alpha \in \Lambda^2(M) \otimes F$ and $\nabla_{V_1}(\alpha(X,Y)) = [\nabla_{V_1}, \alpha(X,Y)]$.

In this way it is possible to demonstrate that infinitesimal holonomy of ∇ is generated by the curvature of $\tilde{\nabla}$ and its successive covariant derivatives.

In the second part, we offer some applications and some explicit computations.

This paper is dedicated to Franco Tricerri, under whose guide I took my degree and I began this work.

1. Covariant derivatives

Let $G = \mathrm{GL}(m,\mathbb{R})$ act on $\mathbb{R}^m$ by linear transformations and let $P = P(M,G)$ be a principal bundle; then $E = P \times_G \mathbb{R}^m$ is called *vector bundle* of rank r *associated* to P. Let σ be a section of E. Any element u of $P_x = \pi^{-1}(x)$ can be interpreted as a map $u : \mathbb{R}^m \to E_x$ simply setting $u(\xi) = [u,\xi]$ (see [dB]). We can define a function $f : P \to \mathbb{R}^m$ in this way

$$f(u) = u^{-1}(\sigma(\pi(u))).$$

Given an element X of T_xM, let X^* be the horizontal lift of X with respect to X as the vector given by

$$\nabla_X\sigma = u(X^*f) \tag{1}$$

(see [KN], vol. I, p. 115).

At the end let us consider a global section σ of E and a vector field X on M. We define the covariant derivative $\nabla_X\sigma$ of σ with respect to a vector tangent field at M as

$$(\nabla_X\sigma)_x = \nabla_{X_x}\sigma\,.$$

It can be proved that ∇_X is a linear connection over E. Viceversa it is possible to prove that given a linear connection ∇ over E it always exists an unique connection form ω over P that induces ∇.

Let E be a vector bundle associated to P and, how we know, $E \otimes E^* = \mathrm{End}\,(E)$ with standard fibre $\mathrm{End}\,(\mathbb{R}^m)$. If we calculate the covariant derivative of a section F of $E \otimes E^*$ as described above, we will obtain

$$(\nabla_X F)(\sigma) = [\nabla_X, F]\,\sigma\,,$$

where the bracket is the commutator of the operator ∇_X and F.

2. Infinitesimal holonomy of a connection of a vector bundle

Let M be a smooth manifold and let us consider a principal bundle $P(M, G)$ with a connection Γ. We denote by $\Phi(u)$ the *holonomy group of* Γ *with reference point* u of P and by $\Phi^0(u)$ the *holonomy restricted group of* Γ *with reference point* u. Moreover g' is the *infinitesimal holonomy* and $\Phi'(u)$ the connected Lie subgroup of the structure group G called the *infinitesimal holonomy group* at u. An important fact is the following:

PROPOSITION 2.1. *If* $\dim g'(u) =$ *constant, then* $g'(u) = g(u)$.

Let us consider a vector bundle E over M with standard fibre F. If ∇ is a connection over E and D a linear connection over M (in the applications it will nearly always be the Levi Civita connection), given $\alpha \in \Lambda^2(M) \otimes F$, we can define a connection $\tilde{\nabla}$ in this way:

$$(\tilde{\nabla}_{V_1}\alpha)(X, Y) = \nabla_{V_1}(\alpha(X, Y)) - \alpha(D_{V_1}X, Y) - \alpha(X, D_{V_1}Y)$$

where $\nabla_{V_1}(\alpha(X, Y))(\sigma) = [\nabla_{V_1}, \alpha(X, Y)](\sigma)$, and, more generally,

$$\begin{aligned}(\tilde{\nabla}^k_{V_1\ldots V_k}\alpha)(X, Y) = {} & \nabla_{V_k}((\tilde{\nabla}^{k-1}_{V_1\ldots V_{k-1}}\alpha)(X, Y)) + \\ & - (\tilde{\nabla}^{k-1}_{V_1\ldots V_{k-1}}\alpha)(D_{V_k}X, Y) + \\ & - (\tilde{\nabla}^{k-1}_{V_1\ldots V_{k-1}}\alpha)(X, D_{V_k}Y) + \\ & - \sum_{i=1}^{k-1}(\tilde{\nabla}^{k-1}_{V_1\ldots\nabla_{V_k}V_i\ldots V_{k-1}}\alpha)(X, Y)\,,\end{aligned}$$

where

$$\nabla_{V_k}((\tilde{\nabla}^{k-1}_{V_1\ldots V_{k-1}}\alpha)(X,Y))(\sigma) = [\nabla_{V_k}, (\tilde{\nabla}^{k-1}_{V_1\ldots V_{k-1}}\alpha)(X,Y)](\sigma) .$$

To calculate the infinitesimal holonomy we have the following

THEOREM 2.2. *The Lie algebra $g'(u)$ of the infinitesimal holonomy group $\Phi'(u)$ is spanned by all the elements of the form*

$$(\tilde{\nabla}^{k}_{V_1\ldots V_k}R)_{XY} ,$$

where R is the curvature, X, Y, $V_1, \ldots, V_k$ are in T_xM and $0 \le k < \infty$.

Proof. The proof is obtained by two lemmas. We state that

$$(\nabla_{V_k}\ldots\nabla_{V_1}(R_{XY}))(\sigma) = [\nabla_{V_k}, \nabla_{V_{k-1}}\ldots\nabla_{V_1}R_{XY}](\sigma) .$$

LEMMA 1. *By tensor field of type A_k (resp. B_k) we mean a tensor field of type (1.1) of the form $\nabla_{V_k}\ldots\nabla_{V_1}(R_{XY})$ (resp. $(\tilde{\nabla}^{k}_{V_1\ldots V_k}R)_{XY}$), where X, Y, $V_1, \ldots, V_k$ are arbitrary vector fields on M. Then any tensor field of type A_k (resp. B_k) is a linear combination (with differential functions as coefficients) of a finite number of tensor fields of type B_j (resp. A_j) with $0 \le j \le k$.*

Proof of Lemma 1. For induction on k.

LEMMA 2. *If X, Y, $V_1, \ldots, V_k$ are vector fields on M and if X^*, Y^*, $V_1^*, \ldots, V_k^*$ are their horizontal lifts in $L(M)$, then*

$$(\nabla_{V_k}\ldots\nabla_{V_1}(R_{XY}))_x\sigma = u\circ(V_k^*\ldots V_1^*(2\Omega(X^*,Y^*)))_u\circ u^{-1}(\sigma) ,$$

with $x = \pi(u)$ and σ section of $E\otimes E^*$.

Proof of Lemma 2. This lemma follows from Section 1 and from the fact that $R_{XY}Z = u(2\Omega(X^*,Y^*))(u^{-1}Z)$ (see [KN], vol. I, p. 133); we take R_{XY} and $2\Omega(X^*,Y^*)$ as φ and f like in Section 1 and we procede for induction on k (Ω is the curvature form).

By definition $g'(u)$ is spanned by the values at u of all the $\mathrm{End}(\mathbb{R}^m)$ -valued functions of the form $f = V_k\ldots V_1(\Omega(X,Y))$ $(k = 0, 1, 2, \ldots)$, where X, Y, $V_1, \ldots, V_k$ are arbitrary vector fields on P. The Theorem 2.2 follows from the two previous lemmas.

3. Normal holonomy

Let $(M, <,>)$ be a Riemannian connected manifold, $i : M \to N$ an immersion with N of constant curvature, and let $N(M)$ be the normal bundle of M induced by i. We denote the metric of N and the usual metric on the fibres of $N(M)$ both $<,>$. Moreover we denote by $\mathcal{X}(M)^\perp$ the C^∞-sections of $N(M)$.

Let $R^\perp$ be the normal curvature, that is to say

$$R^\perp_{XY}\xi = [\nabla^\perp_X, \nabla^\perp_Y]\xi - \nabla^\perp_{[X,Y]}\xi \,,$$

where $\nabla^\perp$ is the normal connection and ξ is in $\mathcal{X}(M)^\perp$, and let A_ξ be the Weingarten operator with reference the vector normal field ξ.

It is necessary to consider a point p of M and a curve $\gamma : [0,1] \to M$ piecewise differentiable such that $\gamma(1) = p$.

Let us denote by P_γ the parallel displacement along γ with reference to the normal connection and let us define the tensor $\gamma^*(\mathcal{R}^\perp)$ of type (1,3) in $N(M)_p$ in this way

$$\gamma^*(\mathcal{R}^\perp)(v,w)z = P_\gamma(\mathcal{R}^\perp_q(P_\gamma^{-1}(v), P_\gamma^{-1}(w))P_\gamma^{-1}(z) \,,$$

where $\gamma(0) = q$ and $\mathcal{R}^\perp_p(\xi_1,\xi_2)\xi_3 = \sum_{j=1}^n R^\perp_p(A_{\xi_1}(e_j), A_{\xi_2}(e_j))\xi_3$, ξ_1, ξ_2, ξ_3 in $\mathcal{X}(M)^\perp$ and $e_1, \ldots e_n$ orthonormal base of T_pM.

If $\mathcal{S}$ is the subspace of the tensors of type (1,3) of $N(M)_p$ spanned by all the $\gamma^*(\mathcal{R}^\perp)$, then

THEOREM 3.1. *The Lie algebra of the restricted normal holonomy group Φ^* at p coincides with the linear space*

$$\{R(u,v) \,:\, R \in \mathcal{S}\,,\ u,v \in N(M)_p\}$$

Proof. See [OL], p. 815 and p. 817.

4. An example

We want to calculate normal holonomy of $SO(n)$, thought as submanifold of $\mathbb{R}^{n,n}$ (any element of $SO(n)$ is a real matrix $n \times n$).

The tangent plane at the identity is $T_eSO(n) = \{A \in \mathbb{R}^{n,n} : A = -A^T\}$.

Let us consider the left product $L_x : a \to ax$, for translation we obtain that the tangent plane of $SO(n)$ at x is

$$T_x SO(n) = \{xA : A = -A^T\}\,.$$

Now, we take two vector fields X and Y that are tangent to $SO(n)$ at x

$$\begin{array}{ll} X_x = xA & A = -A^T \\ Y_x = xB & B = -B^T \end{array}$$

and we consider the Levi Civita connection ∂ in $\mathbb{R}^n$. Then

PROPOSITION 4.1. *$\partial_X Y = xAB$.*

Proof. Let us consider two vector fields X and Y tangent to $SO(n)$ at x:

$$X = xA = x_{im}A_{mj}\frac{\partial}{\partial x_{ij}} = X_{ij}\frac{\partial}{\partial x_{ij}} \quad \text{with} \quad x_{im}A_{mj} = X_{ij}\,,$$
$$Y = xB = x_{im}B_{mj}\frac{\partial}{\partial x_{ij}} = Y_{ij}\frac{\partial}{\partial x_{ij}} \quad \text{with} \quad x_{im}B_{mj} = Y_{ij}\,,$$

then,

$$\begin{aligned} \partial_X Y &= \sum_{i,j,h,k,m} X_{ij}\frac{\partial}{\partial x_{ij}}(x_{hm}B_{mk})\frac{\partial}{\partial x_{hk}} \\ &= \sum_{i,j,k} X_{ij}B_{jk}\frac{\partial}{\partial x_{ik}} \\ &= \sum_{i,j,m} X_{im}B_{mj}\frac{\partial}{\partial x_{ij}}\,. \end{aligned}$$

It follows that

$$\partial_X Y = XB = xAB$$

in fact

$$\sum_{i,j,m} X_{im}B_{mj}\frac{\partial}{\partial x_{ij}} = \sum_{i,j,m} x_{ik}A_{km}B_{mj}\frac{\partial}{\partial x_{ij}} = xAB\,.$$

$\diamondsuit$

Let us define a metric on the manifold $SO(n)$ with those inducted by the Euclidean metric of $\mathbb{R}^{n,n}$. We know that the metric of $\mathbb{R}^{n,n}$ is given by

$$< A, B > = \sum_{i,j} a_{ij} b_{ij} = \mathrm{tr}\,(AB^T) = \mathrm{tr}\,(A^T B)\,,$$

with $A = (a_{ij})\,, B = (b_{ij})$ in $\mathbb{R}^{n,n}$.

So given X, Y vector fields tangent to $SO(n)$ at x, we define

$$< X, Y > = \mathrm{tr}\,(XY^T)\,.$$

We note that, because of $xx^T = \mathrm{id}$

$$\begin{aligned} < X, Y > &= \mathrm{tr}\,(XY^T) = \mathrm{tr}\,(xA(xB)^T) = \mathrm{tr}\,(xAB^T x^T) \\ &= \mathrm{tr}\,(AB^T) = < A, B > = < X_e, Y_e > \ . \end{aligned}$$

Remark. $T_e SO(n)^\perp = \{A \in \mathbb{R}^{n,n} : A = A^T\}$, i.e. the orthogonal complement of $T_e SO(n)$ is spanned by symmetric matrices. In the same way $T_x SO(n)^\perp = \{xA : A \in \mathbb{R}^{n,n} \text{ and } A = A^T\}\,.$

Let X and Y be vector fields tangent to $SO(n)$ at x. In the Proposition 4.1 we have seen that $\partial_X Y = xAB$.
It is possible to split $\partial_X Y$ into the tangent part and normal part.

Proposition 4.2.

$$\begin{aligned} tan\,(\partial_X Y) &= \frac{1}{2}\,x(AB - BA) \\ nor\,(\partial_X Y) &= \frac{1}{2}\,x(AB + BA) \end{aligned}$$

Proof. First of all we observe that

$$\frac{1}{2}\,x\,(AB - BA) + \frac{1}{2}\,x(AB + BA) = xAB = \partial_X Y$$

and so

$$\begin{aligned} \frac{1}{2}\,(AB - BA)^T &= \frac{1}{2}(B^T A^T - A^T B^T) \\ &= \frac{1}{2}\,(BA - AB) = -\frac{1}{2}\,(AB - BA) \end{aligned}$$

i.e. $\frac{1}{2}\,x(AB - BA)$ is in $T_xSO(n)$. Moreover

$$\begin{aligned}\frac{1}{2}\,(AB + BA)^T &= \frac{1}{2}\,(B^T A^T + A^T B^T)\\ &= \frac{1}{2}\,(BA + AB) = \frac{1}{2}\,(AB + BA)\end{aligned}$$

so $\frac{1}{2}\,x(AB + BA)$ is in $T_xSO(n)^\perp$. ♢

Then, if we denote by ∇ the Levi Civita connection in $SO(n)$ and with α the second fundamental form, it is clear that

$$\begin{aligned}\nabla_X Y &= \frac{1}{2}\,x(AB - BA)\,,\\ \alpha\,(X, Y) &= \frac{1}{2}\,x(AB + BA)\,.\end{aligned}$$

Now, let X be a vector field tangent to $SO(n)$ at x and ξ a vector field normal to $SO(n)$ at x, with

$$\begin{aligned}X &= xA & A &= -A^T\\ \xi &= xN & N &= N^T\end{aligned}$$

Proposition 4.3. *$\partial_X \xi = xAN$.*

Proof. It is similar to those of Proposition 4.1.

It is possible to split $\partial_X \xi$ into tangent part and normal part.

Proposition 4.4.

$$\begin{aligned}tan\,(\partial_X \xi) &= \tfrac{1}{2}x(AN + NA)\,,\\ nor\,(\partial_X \xi) &= \tfrac{1}{2}x(AN - NA)\,.\end{aligned}$$

Proof. It is similar to that of Proposition 4.2.

Then, if we denote by A_ξ the Weingarten operator and by $\nabla^\perp$ the normal connection in $SO(n)$, by the Weingarten formula we obtain

$$\begin{aligned}A_\xi(X) &= -\frac{1}{2}\,x(AN + NA)\,,\\ \nabla^\perp_X \xi &= \frac{1}{2}\,x(AN - NA)\,.\end{aligned}$$

If $R^\perp_{XY}$ is the normal curvature in $SO(n)$, we have

PROPOSITION 4.5. $R^{\perp}_{XY}\xi = \frac{1}{4}\,x[N,[A,B]]$.

Proof. As

$$[X,Y] = x[A,B] = xAB - xBA$$

and

$$R^{\perp}_{XY}\xi = \nabla^{\perp}_X(\nabla^{\perp}_Y\xi) - \nabla^{\perp}_Y(\nabla^{\perp}_X\xi) - \nabla^{\perp}_{[X,Y]}\xi$$

the proof is only a calculation. $\diamondsuit$

For the Theorem 2.2 the infinitesimal holonomy $g'(u)$ of $SO(n)$, that coincides with holonomy algebra of $SO(n)$, being $\dim g'(u)$ constant, is spanned by $(\tilde{\nabla}^k_{V_1\ldots V_k}R^{\perp})_{XY}$ where, in this case, E is the normal bundle of $SO(n)$ and $\nabla = \nabla^{\perp}$.

PROPOSITION 4.6. $(\tilde{\nabla}_{V_1}R^{\perp})_{XY}\xi = 0$.

Proof. Let us consider V_1, X, Y vector fields tangent to $SO(n)$ at x and ξ vector field normal to $SO(n)$ at x, with

$$\begin{aligned} V_1 &= xV & V &= -V^T\,, \\ X &= xA & A &= -A^T\,, \\ Y &= xB & B &= -B^T\,, \\ \xi &= xN & N &= N^T\,. \end{aligned}$$

We know that

$$\begin{aligned} \nabla^{\perp}_X\xi &= \frac{1}{2}\,x(AN - NA)\,, \\ R^{\perp}_{XY}\xi &= \frac{1}{4}\,x[N,[A,B]]\,, \\ D_XY &= \frac{1}{2}\,x(AB - BA)\,, \end{aligned}$$

$$(\tilde{\nabla}_{V_1}R^{\perp})_{XY}\xi = [\tilde{\nabla}_{V_1}, R^{\perp}_{XY}]\xi - R^{\perp}_{D_{V_1}X,Y}\xi - R^{\perp}_{X,D_{V_1}Y}\xi\,.$$

Calculating separately these terms we obtain

$$\begin{aligned} [\nabla^{\perp}_{V_1}, R^{\perp}_{XY}]\xi &= \frac{1}{8}\,x(V[N,[A,B]] - [N,[A,B]]V) + \\ &\quad - \frac{1}{8}\,x[VN - NV,[AB]] \\ R^{\perp}_{D_{V_1}X,Y}\xi &= \frac{1}{8}\,x[N,[VA - AV,B]] \\ R^{\perp}_{X,D_{V_1}Y}\xi &= \frac{1}{8}\,x[N,[A,VB - BV]]\,. \end{aligned}$$

Then

$$(\tilde{\nabla}_{V_1} R^{\perp})_{XY}\xi = 0\,.$$

$\diamondsuit$

Let $so(n)$ be the Lie algebra of $SO(n)$. Then the normal holonomy g of $SO(n)$, considerated as submanifold of $\mathbb{R}^{n,n}$, coincides with $so(n)$:

PROPOSITION 4.7. $g(e) = so(n)$.

Proof. For Proposition 4.6 and for Proposition 4.5 it is clear that

$$g(e) = g'(e) = \text{Span}\,\{R^{\perp}_{XY}|_{\text{Id}}\} = \text{Span}\,\{\text{ad}_{[A,B]}\}$$

with A and B in $so(n)$.
But the Lie algebra $g = so(n)$ of $SO(n)$ is simple for $n \neq 4$ and for $n \geq 3$. Then the derived algebra $[g, g]$ coincides with g.
If $n = 4$ then $so(4) = so(3) \otimes so(3)$ and again $[g, g] = g$.
It follows that $g(e) = so(n)$. $\diamondsuit$

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