INVARIANTS IN AFFINE DIFFERENTIAL GEOMETRY (*)

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SOMMARIO. - *In questo lavoro si ottengono soluzioni dell' equazione di Monge-Ampère* $\det(\text{Hess } z) = 1$. *Si da inoltre un risultato del tipo Theorema egregium.*

SUMMARY. - *In this paper one obtains solutions of the Monge-Ampère equation* $\det(\text{Hess } z) = 1$. *Furthermore one formulates a result of type Theorema egregium.*

Suppose A is the $(n+1)$-dimensional real affine space with associated vector space V and denote by M an n-dimensional differentiable manifold without boundary.

A pair (M, x) where $x : M \to A$ is an immersion is called a *hypersurface of A*. Choosing a fixed origin in A we identify A with its vector space V.

A *relative normalization* of a hypersurface (M, x) is a mapping $y : M \to V$ such that in every point of M

(i) the vectors $y, x_1, \ldots, x_n$ are linearly independent;

(ii) the vector y_i is contained in the span of $x_1, \ldots, x_n$ for every $i = 1, 2, \ldots, n$.

As usual an index denotes partial differentiation with respect to a local coordinate system on M.

The triple (M, x, y) is called a *relative normalized hypersurface*. Denote by V^* the dual vector space and by $\langle \, , \, \rangle : V^* \times V \to R$ the non-degenerate scalar product.

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Then the differentiable mapping $X : M \to V^*$ such that in every point of M

(i) $\langle X, x_i \rangle = 0$ for every $i = 1, \ldots, n$ and

(ii) $\langle X, y \rangle = 1$

is called the induced *conormal* of the tangential plane. It is uniquely determined by the hypersurface (M, x, y).

The immersion $x : M \to A$ is said to be *regular* when $\{X, X_1, \ldots, X_n\}$ forms a basis of V^*.

A *metric* on M is induced from (M, x, y) by the formula

$$G_{ij} = -\langle X_i, x_j \rangle, \quad i, j = 1, 2, \ldots, n.$$

G_{ij} is definite whenever $x : M \to A$ is non-degenerate.

The hypersurface (M, x, y) induces the *relative Weingarten tensor* B_{ij} by

$$B_{ij} = \langle X_i, y_j \rangle, \quad i, j = 1, 2, \ldots, n;$$

the *affine mean curvature* H is the first curvature function, i.e.

$$H = \frac{1}{n} B_i^i .$$

A hypersurface (M, x, y) is called *affine maximal* if the affine mean curvature vanishes identically.

An affine maximal hypersurface is an *improper affine sphere* if the relative normalization is constant.

Suppose $\| \quad \| : V^n \to R$ is a fixed determinant form. Define θ_{ij}, and g_{ij}, $i, j = 1, \ldots, n$, by

$$\theta_{ij} = \| \, x_{ij}, x_1, \ldots, x_n \, \| ,$$
$$g_{ij} = |\mathrm{Det}(\theta_{ij})|^{-1/(n+2)} \, \theta_{ij} .$$

Then g_{ij} defines a definite tensor field on M which is by proper choice of orientation positive definite.

The Laplacian of (M, g_{ij}) induces a relative normalization y by

$$y = \frac{1}{n} \Delta x .$$

This normalization is called the *equiaffine normalization* of (M, x).

The above introduction is given by Schneider in [SCHN]. In [SCHN] the Weingarten tensor differs by sign from the above definition.

A wide class of affine hypersurfaces is formed by affine maximal surfaces (cf. [CAL], [SCHI], [SCHN], [SI]).

Suppose Ω is a region in the plane and that an affine surface $\Sigma : \Omega \to A_3$ is given by a differentiable function $z : \Omega \to R$ as a graph over Ω

$$\Sigma(x, y) = \begin{bmatrix} x \\ y \\ z(x, y) \end{bmatrix} .$$

The determinant of the Hessian of z over Ω shall be denoted by d

$$d = \det(\operatorname{Hess} z) = z_{xx} z_{yy} - z_{xy}^2 . \tag{1}$$

The surface is an improper affine sphere if $z : \Omega \to R$ solves the Monge-Ampère equation

$$z_{xx} z_{yy} - z_{xy}^2 = 1 . \tag{2}$$

The affine surface $\Sigma : \Omega \to A_3$ is an affine maximal surface if $z : \Omega \to A_3$ solves the Euler-Lagrange equation

$$d\{z_{xx} d_{yy} + z_{yy} d_{xx} - 2\, z_{xy} d_{xy}\} =$$

$$= \frac{7}{4}\{z_{xx} d_y^2 + z_{yy} d_x^2 - 2\, z_{xy} d_x d_y\} . \tag{3}$$

A non-trivial solution of (2) is induced by the differential

$$dz = tg\, x \sqrt{\cos^2 x + y^2}\, dx + \operatorname{arsh} \left[\frac{y}{\cos x}\right] dy . \tag{4}$$

This exact differential is defined in a region with $\cos x \neq 0$. Otherwise (4) solves the Monge-Ampère equation (2), i.e. the affine surface is an improper affine sphere.

It is convenient to introduce the positive function $\mu : \Omega \longrightarrow R$ and the dual field $W : \Omega \longrightarrow V^*$

$$\mu := \frac{1}{4\sqrt{d}}\,, \quad W = \begin{bmatrix} -z_x \\ -z_y \\ 1 \end{bmatrix} . \tag{5}$$

Then we obtain

$$K = \mu \begin{bmatrix} -z_x \\ -z_y \\ 1 \end{bmatrix} = \mu W, \quad G = \mathrm{Hess}(z) = \mu \begin{bmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{bmatrix} , \tag{6}$$

where $K : \Omega \longrightarrow V^*$ denotes the conormal of the tangential plane.

The support function $\sigma : \Omega \longrightarrow R$ with respect to the origin is given by

$$\sigma(x, y) = -\langle K(x, y),\ \Sigma(x, y)\rangle . \tag{7}$$

A well-known solution of (2) is given by

$$z = \frac{x^2 + y^2}{2}$$

and describes the elliptic paraboloid.

By (7) one has for its support function

$$\sigma = \frac{x^2 + y^2}{2} = z . \tag{8}$$

Another example of a solution of the Monge-Ampère equation (2) is given by

$$z = \frac{2}{3}(x^2 + y)^{3/2} . \tag{9}$$

Using (7) one gets for the induced support function

$$\sigma = \frac{1}{3}(x^2 + y)^{3/2} + (x^2 + y)^{1/2} . \tag{10}$$

Suppose $f : R^+ \to R$ is defined by the integral

$$f(r) = \int_0^r \sqrt{1+\tau^2}\, d\tau \,, \tag{11}$$

and $z : R^2 \backslash \{0\} \to R$ is given by

$$z = f(r) \quad \text{with} \quad r^2 = x^2 + y^2 \,. \tag{12}$$

Then z is a solution of the Monge-Ampère equation (2). By (7) one has for the induced support function

$$\sigma = -f(r) + r\sqrt{1+r^2} \,. \tag{13}$$

In regions with sufficiently large r one has $f(r) \approx \frac{1}{2}r^2$. Thus by (13) $\sigma \approx f(r)$.

Like $z : \Omega \to R$ the determinant of the Hessian $d : \Omega \to R$ also induces an affine surface $\theta : \Omega \to A_3$

$$\theta(x,y) = \begin{bmatrix} x \\ y \\ d(x,y) \end{bmatrix} .$$

We consider here the elliptic case, i.e. $d : \Omega \to R$ is strictly positive and $\theta : \Omega \to A_3$ lies in the "upper" half plane.

For example consider the elliptic paraboloid. Then $d \equiv 1$, i.e. $\theta : \Omega \to A_3$ is a plane.

A solution of the Euler Lagrange equation (3) is given by

$$z(x,y) = -\frac{9}{2}x^{2/3} + \frac{y^2}{2} \,, \quad x > 0 \,, \quad y \in R \,.$$

For this solution one gets

$$d = z_{xx}z_{yy} - z_{xy}^2 = x^{-4/3} \,.$$

Finally we give the following

THEOREM. *Suppose $x : M \to A_3$ is an affine surface with induced Berwald-Blaschke metric G such that the affine mean curvature vanishes and that (M, G) is a complete Riemannian space. Then the Pick invariant is not bounded from below by a positive constant.*

Proof. Because of the affine Theorema egregium $K = 2(J + H)$, where K denotes the Gauss curvature, J the Pick-invariant and H the affine mean curvature. As H vanishes the assertion follows from a theorem of Myers [MYE].

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