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In 2008 the *Dipartimento di Matematica e Informatica*, the owner of the journal, decided to renew it. The name of the journal however remained unchanged, but the subtitle *An International Journal of Mathematics* was added. The journal can be obtained by subscription, or by reciprocity with other similar journals. Currently more than 100 exchange agreements with mathematics departments and institutes around the world have been entered in.

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An effective criterion for the additive decompositions of forms

EDOARDO BALLICO

ABSTRACT. *We give an effective criterion for the identifiability of additive decompositions of homogeneous forms of degree d in a fixed number of variables. Asymptotically for large d it has the same order of the Kruskal's criterion adapted to symmetric tensors given by L. Chiantini, G. Ottaviani and N. Vannieuwenhoven. We give a new case of identifiability for $d = 4$.*

Keywords: symmetric tensor rank, additive decomposition of polynomials, Waring decomposition, identifiability.
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1. Introduction

Let $\mathbb{C}[z_0, \dots, z_n]_d$ denote the complex vector space of all homogeneous degree d polynomials in the variables z_0, \dots, z_n . An *additive decomposition* (or a Waring decomposition) of a form $f \in \mathbb{C}[z_0, \dots, z_n]_d \setminus \{0\}$ is a finite sum

$$f = \sum \ell_i^d \quad (1)$$

with each $\ell_i \in \mathbb{C}[z_0, \dots, z_n]_1$. The minimal number $R(f)$ of summands in an additive decomposition of f is called the *rank* of f . The form f is said to be *identifiable* if it has a unique decomposition (1), up to a permutation of the summands. Often it is called an additive decomposition of f a finite sum

$$f = \sum c_i \mu_i^d \quad (2)$$

with $c_i \in \mathbb{C}$ and $\mu_i \in \mathbb{C}[z_0, \dots, z_n]_1$. Taking $b_i \in \mathbb{C}$ such that $b_i^d = c_i$ and setting $\ell_i := b_i \mu_i$ we see that the two definitions coincide and that (1) and (2) have the same number of non-zero summands. Degree d forms in the variables z_0, \dots, z_n correspond to symmetric tensors of format $(n+1) \times \dots \times (n+1)$ (d times), i.e. to symmetric elements of $(\mathbb{C}^{n+1})^{\otimes d}$. An additive decomposition (2) of f is said to be *non redundant* or *irredundant* if there are no index i such that $c_i \ell_i^d$ is a linear combination of the other $c_j \ell_j^d$'s, $j \neq i$. See [23] for a long list of possible applications and the language needed. Obviously it is interesting

to know when a non redundant decomposition of f has only $R(f)$ summands, because just from knowing the non redundant decomposition we would know that f has no shorter additive decompositions. More important (as stressed in [17, 18]) is to know if f is identifiable.

In [18] L. Chiantini, G. Ottaviani and N. Vannieuwenhoven stressed the importance (even for arbitrary tensors) of effective criteria for the identifiability and gave a long list of practical applications (with explicit examples even in Chemistry). We add to the list of potential applications the tensor networks ([13, 14, 25], at least for tensors without symmetries. For the case of bivariate forms, see [11]; for bivariate forms the identifiability of a form only depends on its rank and, for generic bivariate forms, on the parity of d by a theorem of Sylvester ([21, Theorem 1.5.3 (ii)]).

L. Chiantini, G. Ottaviani and N. Vannieuwenhoven stressed the importance of the true effectivity of the criterion to be tested as it happens in the case of the famous Kruskal's criterion for the tensor decomposition ([22]). They reshaped the Kruskal's criterion to the case of additive decompositions ([18, Theorem 4.6 and Proposition 4.8]) and proved that it is effective (for $d \geq 5$) for ranks at most $\sim n^{\lfloor (d-1)/2 \rfloor}$. The upper bound to which our criterion applies has the same asymptotic order when $d \gg 0$, but we hope that it is easy and efficient. Then in [3] E. Angelini, L. Chiantini and N. Vannieuwenhoven considered the case $d = 4$ and added the analysis of one more rank. Among the huge number of papers considering mostly "generic" identifiability we also mention [1, 2, 4, 15, 16, 17, 19]. An effective criterion should be something machine-testable in a reasonable time and that to be applied to the form f only requires data from the additive decomposition (1). In our case we need the forms ℓ_i 's in the right hand side of (1) (we only need them up to a scalar multiple, but we need them exactly, not approximately) and the computation of the rank of a matrix with ρ rows and $\binom{n+t}{n}$ columns, where ρ is the number of summands in (1) and $t \leq \lfloor d/2 \rfloor$ (but t may be lower for lower ρ); see Remark 2.1 for more details.

To state our results we need the following geometric language for instance fully explained in [18, 23].

Set $\mathbb{P}^n := \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_1$. Thus points of the n -dimensional complex space p correspond to non-zero linear forms, up to a non-zero multiplicative constant. Set $r := \binom{n+d}{n} - 1$. Thus $\mathbb{P}^r := \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_d$ is an r -dimensional projective space. Let $\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}^r$ denote the order d Veronese embedding, i.e. the map defined by the formula $[\ell] \mapsto [\ell^d]$. An additive decomposition (1) or (2) with k non-proportional non-zero terms corresponds to a subset $S \subset \mathbb{P}^n$ such that $|S| = k$ and $[f] \in \langle \nu_d(S) \rangle$, where $\langle \cdot \rangle$ denote the linear span. This decomposition is called *non redundant* and we say that the set $\nu_d(S)$ *irredundantly spans* $[f]$ if $[f] \in \langle \nu_d(S) \rangle$ and $[f] \notin \langle \nu_d(S') \rangle$ for each $S' \subsetneq S$. For any integer $t \geq 0$ each $p \in \mathbb{P}^n$ gives a linear condition to the vector space $\mathbb{C}[z_0, \dots, z_n]_t$ by taking

$p_1 = (a_0, \dots, a_n) \in \mathbb{C}^{n+1}$ with $[p_1] = p$ and evaluating each $f \in \mathbb{C}[z_0, \dots, z_n]_t$ at p_1 . When we perform this evaluation for all points of a finite set $S \subset \mathbb{P}^n$ we get $|S|$ linear equations and the rank of the corresponding matrix does not depend on the choice of the representatives of the points of S .

We prove the following result.

THEOREM 1.1. *Fix $q \in \mathbb{P}^r$ and take a finite set $S \subset \mathbb{P}^n$ such that $\nu_d(S)$ irredundantly spans q .*

(a) *If $|S| \leq \binom{n+\lfloor d/2 \rfloor}{n}$ and S gives $|S|$ independent conditions to the complex vector space $\mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2 \rfloor}$, then q has rank $|S|$.*

(b) *If $|S| \leq \binom{n+\lfloor d/2 \rfloor - 1}{n}$ and S gives $|S|$ independent conditions to the complex vector space $\mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2 \rfloor - 1}$, then S is the unique set evincing the rank of q .*

In Remark 2.1 we explain why Theorem 1.1 effectively determines the rank of q (and in the set-up of (b) the identifiability of f , i.e. the uniqueness statement often called “uniqueness of additive decomposition ” for homogeneous polynomials or for symmetric tensors). Indeed, to check that S satisfies the assumptions of part (a) (resp. part (b)) of Theorem 1.1 it is sufficient to check that a certain matrix with $|S|$ rows and $\binom{n+\lfloor d/2 \rfloor}{n}$ (resp $\binom{n+\lfloor d/2 \rfloor - 1}{n}$) columns has rank $|S|$. This matrix has rank $|S|$ if S is sufficiently general, but the test is effective for a specific set S .

See [7] and [8] for results similar to Theorem 1.1 for tensors; roughly speaking [8, Corollary 3.10, Remark 3.11 and their proof] is equivalent to part (a) of Theorem 1.1. Part (a) of Theorem 1.1 is good, but one could hope to get part (b) when $|S| < \binom{n+\lfloor d/2 \rfloor}{n}$, adding some other easily testable assumptions on S . We prove the following strong result (an essential step for the proof of part (b) of Theorem 1.1). To state it we recall the following notation: for any finite set $E \subset \mathbb{P}^n$ and any $t \in \mathbb{N}$ let $H^0(\mathcal{I}_E(t))$ denote the set of all $f \in \mathbb{C}[z_0, \dots, z_n]_t$ such that $f(p) = 0$ for all $p \in E$. The set $H^0(\mathcal{I}_E(t))$ is a vector space of dimension at least $\binom{n+t}{n} - |E|$. Set $|\mathcal{I}_E(t)| := \mathbb{P}H^0(\mathcal{I}_E(t))$.

THEOREM 1.2. *Fix $q \in \mathbb{P}^r$ and take a finite set $S \subset \mathbb{P}^n$ such that $\nu_d(S)$ irredundantly spans q . Assume $|S| < \binom{n+\lfloor d/2 \rfloor}{n}$ and that S gives $|S|$ independent conditions to $\mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2 \rfloor}$. Take any $A \subset \mathbb{P}^n$ such that $|A| = |S|$ and A induces an additive decomposition of f . Then $H^0(\mathcal{I}_A(\lfloor d/2 \rfloor)) = H^0(\mathcal{I}_S(\lfloor d/2 \rfloor))$.*

Theorem 1.2 does not assure that S is the only set evincing the rank of q , i.e. the uniqueness of the summands in an additive decomposition of f with $R(f)$ terms, but it shows where the other sets A giving potential additive decomposition with $R(f)$ summands may be located: they are contained in the base locus of $|\mathcal{I}_S(\lfloor d/2 \rfloor)|$. The results in [3] (in particular [3, Theorem 6.2 and 6.3, Proposition 6.4]) for $d = 4$ show that non-uniqueness occurs if and only if the base locus of $|\mathcal{I}_S(2)|$ allows the existence of A .

In the last section we take $d = 4$. E. Angelini, L. Chiantini and N. Vanieuwenhoven consider the case $d = 4$ and $|S| = 2n + 1$ with an additional geometric property (linear general position or LGP for short; section 3 for its definition). For $d = 4$ and $|S| = 2n + 1$ they classified the set S in LGP for which identifiability holds (see Theorem 3.1 for a summary of [3, Theorems 6.2 and 6.3]). In section 3 using Theorem 1.2 we classify another family of sets S with $|S| = 2n + 1$ and for which identifiability holds (Theorem 3.2).

REMARK 1.3: The results used to prove Theorem 1.1 (and summarized in Lemma 2.3 and Remark 2.4) work verbatim for a zero-dimensional scheme $A \subset \mathbb{P}^n$. The key is that in Lemma 2.3 and Remark 2.4 or in [6, Lemma 5.1] (or equivalently [9, Lemmas 2.4 and 2.5]) we may allow that one of the two schemes is not reduced. Under the assumption of part (a) of Theorem 1.1 the cactus rank of q (see [10, 12, 26] for its definition and its uses) is $|S|$. Under the assumptions of part (b) of Theorem 1.1 S is the only zero-dimensional subscheme of \mathbb{P}^n evincing the cactus rank of q . However for our proofs it is important that S (i.e. the scheme to be tested) is a finite set, not a zero-dimensional scheme. Now assume that W is a zero-dimensional scheme and take $q \in \langle \nu_d(W) \rangle$ such that $q \notin \langle \nu_d(W') \rangle$ for any $W' \subsetneq W$. Assume that W is not reduced, that $\deg(W) \leq \binom{n+\lfloor d/2-1 \rfloor}{n}$ and that W gives $\deg(W)$ independent conditions to $\mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2-1 \rfloor}$. Quoting either [6, Lemma 5.1] or [9, Lemmas 2.4 and 2.5] we get that q has rank $> \deg(W)$.

REMARK 1.4: The interested reader may check that the proof works with no modification if instead of \mathbb{C} we take any algebraically closed field containing \mathbb{Q} . Since it uses only linear systems, it works over any field $K \supseteq \mathbb{Q}$ if as an additive decomposition of $f \in K[z_0, \dots, z_n]_d$ we take an expression (2) with $c_i \in K$ and $\ell_i \in K[z_0, \dots, z_n]_1$. Thus for the real field \mathbb{R} when d is odd we may take the usual definition (1) of additive decomposition, while if d is even we allow $c_i \in \{-1, 1\}$. Theorem 1.1 applied to \mathbb{C} says that $|S|$ is the complex rank of q , too, and in set-up of part (b) uniqueness holds even if we allow complex decompositions.

REMARK 1.5: In the proofs of our results we use nothing about the form f or the point $q = [f] \in \mathbb{P}^r$. All our assumptions are on the set S and they apply to all $q \in \langle \nu_d(S) \rangle$ irredundantly spanned by $\nu_d(S)$. In all our results the set $\nu_d(S)$ is linearly independent (i.e. its elements are linearly independent) and hence the set of all $q \in \mathbb{P}^r$ irredundantly spanned by $\nu_d(S)$ is the complement in the $(|S| - 1)$ -dimensional linear space $\langle \nu_d(S) \rangle$ of $|S|$ codimension 1 linear subspaces. To test that $\nu_d(S)$ irredundantly spans q it is sufficient to check the rank of a matrix with $|S|$ rows and $\binom{n+d}{n}$ columns. To the best of our knowledge this check (or a very similar one) must be done for all criteria of effectivity for forms ([3]).

2. The proofs of Theorems 1.1 and 1.2

Fix $q \in \mathbb{P}^r = \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_d$. The rank $r_X(q)$ of q is the minimal cardinality of a finite set $S \subset \mathbb{P}^n$ such that $q \in \langle \nu_d(S) \rangle$. By the definition of Veronese embedding we have $r_X(q) = R(f)$ for any $f \in \mathbb{C}[z_0, \dots, z_n]_d$ such that $[f] = q$. Let $\mathcal{S}(X, q)$ denote the set of all $S \subset \mathbb{P}^n$ such that $q \in \langle \nu_d(S) \rangle$ and $|S| = r_X(q)$. We say that q is *identifiable with respect to X* or that q is *X -identifiable* if $|\mathcal{S}(X, q)| = 1$. By the construction of the order d Veronese embedding of X , $|\mathcal{S}(X, q)| = 1$ if and only if any form $f \in \mathbb{C}[z_0, \dots, z_n]_d$ with $[f] = q$ is identifiable. Recall that a finite subset $E \subset \nu_d(\mathbb{P}^n)$ irredundantly spans q if $q \in \langle E \rangle$ and $q \notin \langle E' \rangle$ for any $E' \subsetneq E$. Note that if E irredundantly spans a point of \mathbb{P}^r , then it is linearly independent, i.e. $\dim \langle E \rangle = |E| - 1$. If $E = \nu_d(A)$ for some $A \subset \mathbb{P}^n$, E is linearly independent if and only if A induces $|A|$ independent conditions to $\mathbb{C}[z_0, \dots, z_n]_d$. For each $S \in \mathcal{S}(X, q)$ the set $\nu_d(S)$ irredundantly spans q . For any $Z \subset \mathbb{P}^n$ and any $t \in \mathbb{Z}$ set $h^0(\mathcal{I}_Z(t)) := \dim H^0(\mathcal{I}_Z(t))$.

REMARK 2.1: Fix an integer $t \geq 0$ and a finite subset A of \mathbb{P}^n . We write $h^1(\mathcal{I}_A(t))$ for the difference between $|A|$ and the number of independent conditions that A imposes to the $\binom{n+t}{n}$ -dimensional vector space $\mathbb{C}[z_0, \dots, z_n]_t$. For any multiindex $\alpha = (a_0, \dots, a_n) \in \mathbb{N}^{n+1}$ set $z^\alpha := z_0^{a_0} \dots z_n^{a_n}$ and $\|\alpha\| = a_0 + \dots + a_n$. The integer $\|\alpha\|$ is the degree of the monomial z^α . The vector space $\mathbb{C}[z_0, \dots, z_n]_t$ of all degree t homogeneous polynomials in z_0, \dots, z_n has the monomials z^α with $\|\alpha\| = t$ as a basis. We explain why to compute the non-negative integer $h^1(\mathcal{I}_A(t))$ we only need to compute the rank of the matrix with $|A|$ rows and $\binom{n+t}{n}$ columns. Since $h^1(\mathcal{I}_A(t)) = |A| - \binom{n+t}{n} + h^0(\mathcal{I}_A(t))$, it is sufficient to compute the integer $h^0(\mathcal{I}_A(t))$. Set $a := |A|$ and $b := \binom{n+t}{n}$. We order the points p_1, \dots, p_a of A and the monomials z^α with $\|\alpha\| = t$. We call w_1, \dots, w_b these monomials with the chosen ordering. The integer $a - h^1(\mathcal{I}_A(t))$ is the rank of the $a \times b$ matrix $M = (a_{ij})$ with as entry a_{ij} the value of w_j at p_i .

REMARK 2.2: Fix $q \in \mathbb{P}^r = \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_d \setminus \nu_d(\mathbb{P}^n)$ and take $A \subset \mathbb{P}^n$ such that $\nu_d(A)$ irredundantly spans q . The condition “ $q \notin \nu_d(\mathbb{P}^n)$ ” is equivalent to “ $r_X(q) > 1$ ”. Since $\nu_d(A)$ spans irredundantly at least one point of \mathbb{P}^r , it is linearly independent, i.e. $h^1(\mathbb{P}^r, \mathcal{I}_{\nu_d(A)}(1)) = 0$. Since $q \in \langle \nu_d(A) \rangle$ and $q \notin \nu_d(A)$, we have $h^1(\mathbb{P}^r, \mathcal{I}_{\nu_d(A) \cup \{q\}}(1)) > 0$. Since $h^1(\mathbb{P}^r, \mathcal{I}_{\nu_d(A)}(1)) = 0$ and $|\nu_d(A) \cup \{q\}| = |\nu_d(A)| + 1$, we have $h^1(\mathbb{P}^r, \mathcal{I}_{\nu_d(A) \cup \{q\}}(1)) = 1$.

Fix $f \in \mathbb{C}[z_0, \dots, z_n]_d \setminus \{0\}$ and let $q = [f] \in \mathbb{P}^r = \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_d$, $r = \binom{n+d}{n} - 1$, be the point associated to f . Take $S \subset \mathbb{P}^n$ such that $\nu_d(S)$ irredundantly spans q . Fix any $A \subset \mathbb{P}^n$ evincing the rank of f . We have $|A| \leq |S|$. Set $Z := A \cup B$. Z is a finite subset of \mathbb{P}^n and $|Z| \leq |A| + |S|$. To prove part (a) of Theorem 1.1 we need to prove that $|A| = |S|$. To prove part

(b) we need to prove that $A = S$. In the proof of part (a) we have $A \neq S$, because $|A| < |S|$. To prove part (b) of the theorem it is sufficient to get a contradiction from the assumption $A \neq S$.

We recall (with the same proof) [5, Lemma 1].

LEMMA 2.3. *Fix $q \in \mathbb{P}^r = \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_d$ and assume the existence of $A, B \subset \mathbb{P}^n$ such that $\nu_d(A)$ and $\nu_d(B)$ irredundantly span q and $A \neq B$.*

Then $A \cup B$ does not impose $|A \cup B|$ independent conditions to $\mathbb{C}[z_0, \dots, z_n]_d$, i.e. $h^1(\mathcal{I}_{A \cup B}(d)) \neq 0$.

Proof. For all linear subspaces $U, W \subseteq \mathbb{P}^r$ the Grassmann's formula says that

$$\dim(U \cap W) + \dim(U + W) = \dim U + \dim W$$

with the convention $\dim \emptyset = -1$. Since $\nu_d(A)$ (resp. $\nu_d(B)$) irredundantly spans q , we have $\dim \langle \nu_d(A) \rangle = |A| - 1$ (resp. $\dim \langle \nu_d(B) \rangle = |B| - 1$). Since $A \neq B$, we have $A \cap B \subsetneq A$ and $A \cap B \subsetneq B$. Since $q \in \langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle$ and $q \notin \langle \nu_d(A \cap B) \rangle$, we have $\langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle \supsetneq \langle \nu_d(A \cap B) \rangle$. Since $\nu_d(A)$ and $\nu_d(B)$ are linearly independent and $\langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle \supsetneq \langle \nu_d(A \cap B) \rangle$, the Grassmann's formula gives that $\nu_d(A \cup B)$ is not linearly independent, i.e. $A \cup B$ does not impose $|A \cup B|$ independent conditions to $\mathbb{C}[z_0, \dots, z_n]_d$. \square

REMARK 2.4: We explain the particular case of [6, Lemma 5.1] or [9, Lemmas 2.4 and 2.5] we need. Fix $q \in \mathbb{P}^r = \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_d$ and take finite sets $A, B \subset \mathbb{P}^n$ such that $\nu_d(A)$ and $\nu_d(B)$ irredundantly span q . In particular both A and B are linearly independent. Set $Z := A \cup B$. We fix $G \in \mathbb{C}[z_0, \dots, z_n]_t$, $1 \leq t \leq d$. We assume that $Z \setminus Z \cap G$ gives $|Z \setminus Z \cap G|$ independent conditions to $\mathbb{C}[z_0, \dots, z_n]_{d-t}$, i.e. we assume $h^1(\mathbb{P}^n, \mathcal{I}_{Z \setminus Z \cap G}(d-t)) = 0$. By either [6, Lemma 5.1] or [9, Lemmas 2.4 and 2.5] we have $A \setminus A \cap G = B \setminus B \cap G$. In particular if $A \subset G$, then $B \subset G$.

Proof of part (a) of Theorem 1.1: Recall that we have $\dim \mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2 \rfloor} = \binom{n + \lfloor d/2 \rfloor}{n}$. Since $|A| < |S| \leq \binom{n + \lfloor d/2 \rfloor}{n}$, there is $g \in \mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2 \rfloor}$ such that $g(p) = 0$ for all $p \in A$. Let $G \subset \mathbb{P}^n$ be the degree $\lfloor d/2 \rfloor$ hypersurface $\{g = 0\}$ of \mathbb{P}^n . Since $A \subset G$, we have $Z \setminus Z \cap G = S \setminus S \cap G$. Thus $Z \setminus Z \cap G$ gives independent conditions to forms of degree $\lfloor d/2 \rfloor$. Thus it gives independent conditions to forms of degree $\lceil d/2 \rceil = d - \lfloor d/2 \rfloor$. Since $A \subset G$, Remark 2.4 gives $S \subset G$. Since this is true for all $g \in \mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2 \rfloor}$ such that $g(p) = 0$ for all $p \in A$, we get that if $g|_A = 0$ and g has degree $\lfloor d/2 \rfloor$, then $g|_S = 0$. Since S gives $|S|$ independent linear conditions to $\mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2 \rfloor}$, A gives at least $|S|$ linear independent conditions to $\mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2 \rfloor}$, contradicting the inequality $|A| < |S|$. \square

Proof of Theorem 1.2. To prove Theorem 1.2 we may assume $A \neq S$. Since $|A| = |S| < \binom{n+\lfloor d/2 \rfloor}{n}$, there is $g \in \mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2 \rfloor}$ such that $g|_A \equiv 0$. The proof of part (a) of Theorem 1.1 gives $g|_S \equiv 0$. Thus $H^0(\mathcal{I}_A(\lfloor d/2 \rfloor)) \subseteq H^0(\mathcal{I}_S(\lfloor d/2 \rfloor))$. Since $H^0(\mathcal{I}_S(\lfloor d/2 \rfloor))$ has codimension $|A|$ in $\mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2 \rfloor}$, we get $H^0(\mathcal{I}_A(\lfloor d/2 \rfloor)) = H^0(\mathcal{I}_S(\lfloor d/2 \rfloor))$. \square

Proof of part (b) of Theorem 1.1: We have $H^0(\mathcal{I}_A(\lfloor d/2 \rfloor)) = H^0(\mathcal{I}_S(\lfloor d/2 \rfloor))$ by Theorem 1.2. To get $A = S$ it is sufficient to prove that for each $p \in \mathbb{P}^n \setminus A$ there is $g \in H^0(\mathcal{I}_S(\lfloor d/2 \rfloor))$ such that $g(p) \neq 0$. Thus it is sufficient to prove that the sheaf $\mathcal{I}_S(\lfloor d/2 \rfloor)$ is generated by its global sections. The assumption that S gives $|S|$ independent conditions to $\mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2 \rfloor - 1}$ is translated in cohomological terms as $h^1(\mathbb{P}^n, \mathcal{I}_S(\lfloor d/2 \rfloor - 1)) = 0$. The sheaf $\mathcal{I}_S(\lfloor d/2 \rfloor)$ is generated by its global sections (and in particular for each $p \in \mathbb{P}^n \setminus S$ there is $f \in H^0(\mathcal{I}_S(\lfloor d/2 \rfloor))$ such that $f(p) \neq 0$) by the Castelnuovo-Mumford's lemma ([20, Corollary 4.18], [24, Theorem 1.8.3]). \square

3. The case $d = 4$

Set $X := \nu_d(\mathbb{P}^n) \subset \mathbb{P}^r$.

A finite set $S \subset \mathbb{P}^n$ is said to be in *linearly general position* (or in LGP, for short) if $\dim \langle A \rangle = \min\{n, |A| - 1\}$ for each $A \subseteq S$. If $|S| \geq n + 1$ the set S is in LGP if and only if each $A \subseteq S$ with $|A| = n + 1$ spans \mathbb{P}^n .

In this section we take $d = 4$ and hence $r = \binom{n+4}{n} - 1$.

We recall a summary of [3, Theorems 6.2 and 6.3].

THEOREM 3.1. ([3, Theorems 6.2 and 6.3]). *Fix a finite set $S \subset \mathbb{P}^n$ in LGP such that $|S| = 2n + 1$ and take $q \in \mathbb{P}^r$, $r = \binom{n+4}{n} - 1$, such that $\nu_4(S)$ irredundantly spans q .*

1. q has rank $2n + 1$.
2. Assume the existence of $B \subset \mathbb{P}^n$ such that $|B| = 2n + 1$ and $B \neq S$. Then $B \cup S$ is contained in a rational normal curve of \mathbb{P}^n .

We prove the following result.

THEOREM 3.2. *Fix a finite set $S \subset \mathbb{P}^n$ such that $|S| = 2n + 1$ and take $q \in \mathbb{P}^r$, $r = \binom{n+4}{n} - 1$, such that $\nu_4(S)$ irredundantly spans q . Assume that S is not in LGP, but there is $S' \subset S$ such that $|S'| = 2n$ and S' is in LGP. The point q has rank $2n + 1$. Let e be the dimension of a minimal subspace $N \subset \mathbb{P}^n$ such that $|N \cap S| \geq e + 2$. The point $q \in \mathbb{P}^r$ is identifiable if and only if $e \geq 2$. If $e = 1$, then $\dim \mathcal{S}(X, q) = 1$.*

To prove Theorem 3.2 we need some elementary observations.

REMARK 3.3: Take $A \subset \mathbb{P}^m$, $m \geq 1$, such that $|A| = m+2$ and A is in LGP. It is classically known that any two such sets are projectively equivalent; we provide a linear algebra proof of this fact. We order the points p_0, \dots, p_{m+1} of $|A|$. Since p_0, \dots, p_m are $m+1$ linearly independent points, up to a change of homogeneous coordinates we may assume that p_0, \dots, p_m are the $m+1$ coordinate points $(1 : 0 : \dots : 0), \dots, (0 : \dots : 0 : 1)$. Write $p_{m+1} = (w_0 : \dots : w_m)$ for some $w_i \in \mathbb{C}$. The assumption that A is in LGP is equivalent to $w_i \neq 0$ for all i . We make the invertible projective transformation $z_i \mapsto w_i^{-1}z_i$, which leave fixed each p_i , $0 \leq i \leq m$, and maps p_{m+1} to the point $(1 : 1 : \dots : 1)$. Thus each $B \subset \mathbb{P}^m$ in LGP such that $|B| = m+2$ is projectively equivalent to the set consisting of the coordinate points, plus the point $(a_0 : \dots : a_m)$ with $a_i = 1$ for all i . In particular A is projectively equivalent to a general subset of \mathbb{P}^m with cardinality $m+2$. Thus $h^1(\mathcal{I}_A(2)) = 0$.

Claim 1: The set A is the set-theoretic base locus of $|\mathcal{I}_A(2)|$ if and only if $m \geq 2$.

Proof of Claim 1: First assume $m = 1$. In this case we have $\mathcal{O}_{\mathbb{P}^1}(2)(-A) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ and hence $h^0(\mathcal{O}_{\mathbb{P}^1}(2)(-A)) = 0$. Now assume $m = 2$. In this case Claim 1 is equivalent to say that 4 points of a plane, no 3 of them collinear, are the complete intersection of 2 conics; not only this is easy, but (since we proved that we may assume that A is general in \mathbb{P}^2), it is true because 2 general plane conics intersects transversally. Now assume $m > 2$. Fix $o \in \mathbb{P}^m \setminus A$. Let $A' \subset A$ be a subset of A such that $o \in \langle A' \rangle$ and with $|A'|$ minimal (it exists, because A spans \mathbb{P}^m). Since $o \notin A$, we have $m+1 \geq |A'| > 1$. Take $a \in A'$. Since A is in LGP, there is a hyperplane $H \subset \mathbb{P}^m$ such that $|A \cap H| = m$ (and hence H is spanned by $A \cap H$), $A' \setminus \{a\} \subset H$ and $a \notin H$. Set $\{a, b\} := A \setminus A \cap H$. Since $A' \cap H = A' \setminus \{a\}$ and the set A' is linearly independent, we have $H \cap \langle A' \rangle = \langle A' \setminus \{a\} \rangle$. Thus $o \notin H$. Assume for the moment $o \notin \langle \{a, b\} \rangle$. Thus $o \notin M$ for a general hyperplane $M \supseteq \langle \{a, b\} \rangle$. The hyperquadric $H \cup M$ contains A , but $o \notin H \cap M$. Hence $o \notin \mathcal{B}$. Now assume $o \in \langle \{a, b\} \rangle$. Since $|A'|$ is minimal and $|A'| > 1$, we have $|A'| = 2$. Write $A' = \{a, c\}$. Since $o \in \langle \{a, c\} \rangle \cap \langle \{a, b\} \rangle$ and $o \neq a$, the 3 points a, b, c are collinear, a contradiction.

REMARK 3.4: Take $A \subset \mathbb{P}^m$, $m \geq 1$, such that $|A| = m+1$ and A spans \mathbb{P}^m . Up to a projective transformation we may assume that A is the union of the coordinates points of \mathbb{P}^m . As in Remark 3.3 by induction on m we see that $h^1(\mathcal{I}_A(2)) = 0$ and that A is the base locus of the linear system $|\mathcal{I}_A(2)|$.

REMARK 3.5: Take $A \in \mathcal{S}(X, q)$ and any $A' \subsetneq A$, $A' \neq \emptyset$. Set $A'' := A \setminus A'$. In particular $|A| \geq 2$ and hence $q \notin X$. Since A evinces the X -rank of q , it is linearly independent and $h^1(\mathbb{P}^r, \mathcal{I}_{A \cup \{q\}}(1)) = 1$ (Remark 2.2). Since $A'' \subsetneq A$, we have $q \notin \langle A'' \rangle$. Thus $\langle A' \rangle \cap \langle A'' \cup \{q\} \rangle$ is a single point, q' , and q' is the only element of $\langle A' \rangle$ such that $q \in \langle \{q'\} \cup A'' \rangle$. In the same way we see the existence of a single point $q'' \in \langle A'' \rangle$ such that $q \in \langle A' \cup \{q''\} \rangle$. We have $q \in \langle \{q', q''\} \rangle$. Since $A \in \mathcal{S}(X, q)$, we have $A' \in \mathcal{S}(X, q')$ and $A'' \in \mathcal{S}(X, q'')$. If we only assume

that A irredundantly spans q the same proof gives the existence and uniqueness of q' and q'' such that A' irredundantly spans q' and A'' irredundantly spans q'' .

LEMMA 3.6. *Let $H \subset \mathbb{P}^m$, $m \geq 2$, be a hyperplane. Take a finite set $S \subset \mathbb{P}^m$ such that $|S \setminus S \cap H| = 1$. Take homogeneous coordinates z_0, \dots, z_m of \mathbb{P}^m such that $H = \{z_m = 0\}$.*

(i) *If $S \cap H$ imposes independent conditions to $\mathbb{C}[z_0, \dots, z_{m-1}]_2$, then S imposes independent conditions to $\mathbb{C}[z_0, \dots, z_m]_2$.*

(ii) *If $S \cap H$ is the base locus of $|\mathcal{I}_{S \cap H}(2)|$, then S is the base locus of $|\mathcal{I}_S(2)|$.*

Proof. Set $\{p\} := S \setminus S \cap H$ and call \mathcal{B} the base locus of $|\mathcal{I}_S(2)|$. We have the residual exact sequence of H :

$$0 \rightarrow \mathcal{I}_p(1) \rightarrow \mathcal{I}_S(2) \rightarrow \mathcal{I}_{S \cap H, H}(2) \rightarrow 0 \quad (3)$$

Since $\{p\}$ imposes independent conditions to $\mathbb{C}[z_0, \dots, z_m]_1$, we get part (i) and that the restriction map $\rho : H^0(\mathcal{I}_S(2)) \rightarrow H^0(H, \mathcal{I}_{S \cap H, H}(2))$ is surjective. Assume that $S \cap H$ is the base locus of $|\mathcal{I}_{S \cap H}(2)|$. Since ρ is surjective, we get $\mathcal{B} \cap H = S \cap H$. Fix $o \in \mathbb{P}^n \setminus H$ such that $o \neq p$. Take a hyperplane $M \subset \mathbb{P}^m$ such that $p \in M$ and $o \notin M$. The reducible quadric $H \cup M$ shows that $o \notin \mathcal{B}$. \square

Proof of Theorem 3.2: Let $H \subset \mathbb{P}^n$ be a hyperplane containing N and spanned by points of S' . Since S' is in LGP and $|S| = |S'| + 1$, we have $|S \cap H| = n + 1$, $|S' \cap H| = n$, $S \setminus S \cap H = S' \setminus S' \cap H$, and $|S' \setminus S' \cap H| = n$. Since S' is in LGP, $S' \setminus S' \cap H$ spans a hyperplane, M , and $S' \cap H \cap M = \emptyset$. Set $A := S' \cap H$ and $B := S' \cap M$. Note that $S \subset H \cup M$, $n \leq |M \cap S| \leq n + 1$ and $|S \cap M| = n + 1$ if and only if $S \setminus S' \subset H \cap M$, i.e. if and only if $N \subseteq H \cap M$. Set $\mathcal{B} := \{p \in \mathbb{P}^n \mid h^0(\mathcal{I}_{S \cup \{p\}}(2)) = h^0(\mathcal{I}_S(2))\}$. Since $S \subset H \cup M$ and $H \cup M$ is a quadric hypersurface, we have $S \subseteq \mathcal{B} \subseteq H \cup M$. Consider the residual exact sequences of H and M :

$$0 \rightarrow \mathcal{I}_{S \setminus S \cap H}(1) \rightarrow \mathcal{I}_S(2) \rightarrow \mathcal{I}_{S \cap H, H}(2) \rightarrow 0 \quad (4)$$

$$0 \rightarrow \mathcal{I}_{S \setminus S \cap M}(1) \rightarrow \mathcal{I}_S(2) \rightarrow \mathcal{I}_{S \cap M, M}(2) \rightarrow 0 \quad (5)$$

Note that \mathcal{B} contains the base locus \mathcal{B}_1 of $\mathcal{I}_{S \cap H, H}(2)$ and the base locus \mathcal{B}_2 of $\mathcal{I}_{S \cap M, M}(2)$.

By Remark 3.4 we have $h^1(H, \mathcal{I}_{S \cap H, H}(2)) = h^1(M, \mathcal{I}_{S \cap M, M}(2)) = 0$. By the long cohomology exact sequence of (4) we get $h^1(\mathcal{I}_S(2)) = 0$. Theorem 1.1 gives that q has rank $2n + 1$. By the long cohomology exact sequences of (4) and (5) the restriction maps $\rho : H^0(\mathcal{I}_S(2)) \rightarrow H^0(H, \mathcal{I}_{S \cap H, H}(2))$ and $\rho' : H^0(\mathcal{I}_S(2)) \rightarrow H^0(M, \mathcal{I}_{S \cap M, M}(2))$ are surjective. Thus $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Since $S \cap M$ is linearly independent, we have $\mathcal{B}_2 = S \cap M$. Take $F \in \mathcal{S}(X, q)$ such

that $F \neq S$ (if any). In the case $e \geq 2$ we need to find a contradiction. In the case $e = 1$ we need a description of all F 's sufficiently explicit to prove that $\dim \mathcal{S}(X, q) = 1$. More precisely, in the case $e = 1$ we will prove the existence of a subset $F_2 \subset F$ such that all $E \in \mathcal{S}(X, q)$ are of the form $F_2 \cup E_1$ with E_1 depending on E , F_2 the same for all $E \in \mathcal{S}(X, q)$ (and in particular $F_2 \subset S$) and E_1 coming from a bivariate form q' associated to q . Since q has rank $\leq 2n + 1$, we have $|F| \leq 2n + 1 < \binom{n+2}{2}$. Thus there is $G \in |\mathcal{I}_F(2)|$. Set $Z := S \cup F$. Fix any $G \in |\mathcal{I}_F(2)|$. Since $Z \setminus Z \cap G \subseteq S$ and $h^1(\mathcal{I}_S(2)) = 0$, we have $h^1(\mathcal{I}_{Z \setminus Z \cap G}(2)) = 0$. Since $F \subset G$, Lemma 2.3 and Remark 2.4 give $S \subset G$. Since this is true for all $G \in |\mathcal{I}_F(2)|$, we get $|\mathcal{I}_S(2)| \supseteq |\mathcal{I}_F(2)|$. Since $|S| \geq |F|$ and $h^1(\mathcal{I}_S(2)) = 0$, we get again $|F| = |S|$ and also that $|\mathcal{I}_S(2)| = |\mathcal{I}_F(2)|$. Since F is contained in the base locus of $|\mathcal{I}_F(2)|$, we get $F \subseteq \mathcal{B}$.

(a1) Assume $e \geq 2$. By Remark 3.3 $S \cap N$ is the base locus of $\mathcal{I}_{S \cap N}(2)$. Applying (if $e < n - 1$) $n - 1 - e$ times Lemma 3.6 we get $\mathcal{B}_1 = S \cap H$. Thus $\mathcal{B} = S$. Hence $F \subseteq S$, a contradiction.

(a2) Assume $e = 1$. In this case \mathcal{B} contains the line N . The proof of Lemma 3.6 gives $\mathcal{B}_1 = N \cup (S \cap H)$. By Theorem 1.2 we have $F \subset N \cap (S \setminus S \cap N)$. Thus $F = A_1 \cup A_2$ with $A_1 \subset N$, $A_2 \subseteq S \setminus S \cap N$ and $A_1 \cap A_2 = \emptyset$. We apply Remark 3.5 with $A = F$ and $A' = N \cap S$ and get $q' \in \langle \nu_d(S \cap N) \rangle$ and $q'' \in \langle \nu_4(S \setminus S \cap N) \rangle$ such that $q \in \langle \{q', q''\} \rangle$. Since $|S \cap N| = 3$, Sylvester's theorem q' has rank 3 with respect to degree 4 rational normal curve $\nu_4(N)$. We get $|F \cap N| \leq 3$. Since $|F| = |S|$, we get that each element of $\mathcal{S}(X, q)$ is the union of $S \setminus S \cap N$ and an element of $\mathcal{S}(X, q')$. By Sylvester's theorem ([21, §1.5]) we have $\dim \mathcal{S}(\nu_4(N), q') = 1$. A word about this case. We worked taking $F \neq S$. In principle if $\mathcal{S}(X, q)$ is a singleton we got a contradiction, not the proof that $\dim \mathcal{S}(X, q) = 1$. However, we may add $S \setminus S \cap N$ to any $E_1 \in \mathcal{S}(X, q')$ to get an element of $\mathcal{S}(X, q)$, and so it is never a singleton. \square

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Author's address:

Edoardo Ballico
Dipartimento di Matematica
Università di Trento
38123 Povo (TN), Italy
E-mail: ballico@science.unitn.it

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Third Hankel determinant for a subclass of analytic functions of reciprocal order defined by Srivastava-Attiya integral operator

KHALIED ABDULAMEER CHALLAB, MASLINA DARUS
AND FIRAS GHANIM

ABSTRACT. *The aim of this paper is to investigate coefficient estimates, Fekete-Szegő inequality, and upper bound of third Hankel determinant for a subclass of analytic functions of reciprocal order defined by Srivastava-Attiya integral operator.*

Keywords: Analytic functions, Zeta function, Srivastava-Attiya integral operator, Fekete-Szegő inequality, reciprocal order, Third Hankel.
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1. Introduction

Let A be the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1)$$

in the open unit disc $U = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.
The Lipschitz- Lerch zeta function is a series characterized as follows

$$R(a, x, s) \equiv \sum_{k=0}^{\infty} \frac{e^{2k\pi ix}}{(a+k)^s}, \quad s, x, a \in \mathbb{C},$$

with conditions $1 - a \notin \mathbb{N}$ and $x \geq 0$. The series converges $\forall s \in \mathbb{C}$ if $x > 0$ and represents an entire function of s . The series converges absolutely for $\Re(s) > 1$ if $x = 0$. Lerch [23] and Lipschitz [26] studied this type of function with regard to Dirichlet's well known theorem on primes in arithmetic progression. The Lipschitz-Lerch zeta function reduces to the meromorphic Hurwitz zeta function $\zeta(s, a)$ if $x \in \mathbb{Z}$ with one single pole at $s = 1$ [37, Section 2, 3 Eq(2)].

By using a different notation for the Lipschitz-Lerch zeta function, Bateman gave the following function [3]:

$$\Phi(z, s, a) \equiv \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s},$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1). \quad (2)$$

The equation (2) is connected to Lipschitz-Lerch zeta function by the relation $\Phi(e^{2k\pi ix}, s, a) = R(a, x, s)$ and called later Hurwitz-Lerch zeta function.

Also, the Riemann zeta function $\zeta(s)$, the Hurwitz (or generalized) zeta function $\zeta(s, a)$ and the Lerch zeta function $\ell_s(\xi)$ are defined respectively as follows (see, for details, [3, Chapter 1] and [37, Chapter 2]):

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1, s, 1) = \zeta(s, 1), \quad (\Re(s) > 1),$$

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \Phi(1, s, a), \quad (\Re(s) > 1; a \in \mathbb{C} \setminus \mathbb{Z}_0^-),$$

and

$$\ell_s(\xi) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i \xi}}{(n+1)^s} = \Phi(e^{2\pi i \xi}, s, 1), \quad (\Re(s) > 1; \xi \in \mathbb{R}).$$

In addition, an important function of Analytic Number Theory such as the Polylogarithmic function (or de Jonquière's function) $Li_s(z)$ is given by:

$$Li_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1),$$

$$(s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1).$$

It is known that the Hurwitz-Lerch zeta function $\Phi(z, s, a)$ in (2) can be written as

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} dt$$

$$(\Re(a) > 0; \Re(s) > 0 \text{ when } |z| \leq 1 \text{ (} z \neq 1\text{)}; \Re(s) > 1 \text{ when } z = 1). \quad (3)$$

Besides, since

$$\sum_{n=0}^{\infty} f(n) = \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} f(kn+j), \quad (k \in \mathbb{N}),$$

we have

$$\Phi(z, s, a) = k^{-s} \sum_{j=0}^{k-1} \Phi\left(z^k, s, \frac{a+j}{k}\right) z^j, \quad (k \in \mathbb{N}). \quad (4)$$

By combining (3) and (4), immediately we have:

$$\Phi(z, s, a) = \sum_{j=0}^{k-1} \frac{z^j}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(a+j)t}}{1 - z^k e^{-kt}} dt$$

($k \in \mathbb{N}$; $\Re(a) > 0$; $\Re(s) > 0$ when $|z| \leq 1$ ($z \neq 1$); $\Re(s) > 1$ when $z = 1$). (5)

The above equation is mainly prompted by the sum-integral representation in which the authors introduce an analogous investigation of the following general family of the Hurwitz-Lerch zeta function by using $(\mu)_{\rho n}$ and $(\nu)_{\sigma n}$ (see [24]):

$$\Phi_{\mu, \nu}^{(\rho, \sigma)}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s},$$

($\mu \in \mathbb{C}$; $a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\rho, \sigma \in \mathbb{R}^+$; $\rho < \sigma$ when $s, z \in \mathbb{C}$;
 $\rho = \sigma$ and $s \in \mathbb{C}$ when $|z| < 1$; $\rho = \sigma$ and $\Re(s - \mu + \nu) > 1$ when $|z| = 1$).

Here, and for the remainder of this paper, $(\gamma)_k$ denotes the Pochhammer symbol defined in terms of Gamma function, by

$$(\gamma)_k := \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)} = \begin{cases} \gamma(\gamma + 1) \dots (\gamma + n - 1) & (k = n \in \mathbb{N}; \gamma \in \mathbb{C}) \\ 1 & (k = 0; \gamma \in \mathbb{C} \setminus \{0\}). \end{cases}$$

We then have

$$\Phi_{\nu, \nu}^{(\sigma, \sigma)}(z, s, a) = \Phi_{\mu, \nu}^{(0, 0)}(z, s, a) = \Phi(z, s, a)$$

and

$$\Phi_{\mu, 1}^{(1, 1)}(z, s, a) = \Phi_{\mu}^*(z, s, a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s}. \quad (6)$$

Recently, Goyal and Laddha ([14], p. 100, Eq. (1.5)) studied the generalized Hurwitz-Lerch zeta function $\Phi_{\mu}^*(z, s, a)$ given by (6).

For functions $f \in A$ given by (1) and $g \in A$ ($g(z) = z + \sum_{n=2}^{\infty} b_n z^n$), the Hadamard product (or convolution) of f and g can be defined by

$$(f * g) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U.$$

The Hurwitz-Lerch zeta function $\Phi(z, s, a)$ given in (3) was recently studied by Choi and Srivastava [8], Ferreira and Lopez [12], Garg *et al.* [13], Lin *et al.* [25], Srivastava and Attiya [36], Lin and Srivastava *et al.* [38] and others (see [4, 5, 6, 7]).

Now,

$$J_{s,a} : A \rightarrow A,$$

$$J_{s,a}f(z) = G_{s,a} * f(z), \quad (z \in U; a \in \mathbb{C} \setminus \{Z_0^-\}; s \in \mathbb{C}; f \in A) \quad (7)$$

where, for convenience

$$G_{s,a}(z) := (1+a)^s [\Phi(z, s, a) - a^{-s}] \quad (z \in U). \quad (8)$$

Successfully, by utilizing (1), (7) and (8), we can obtain

$$J_{s,a}f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a} \right)^s a_n z^n.$$

Also let $S^*(\alpha)$ be classes of starlike functions and $K(\alpha)$ classes of convex functions of order α , $0 \leq \alpha < 1$. In 1975, Silverman [33] proved that $f(z) \in S^*(\alpha)$ if the following condition is satisfied:

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha, \quad (z \in U). \quad (9)$$

Geometrical importance of inequality (9) is that $zf'(z)/f(z)$ maps U onto the inside of the circle with radius $1 - \alpha$ and center at 1.

We can define $S_*(\alpha)$ (classes of starlike functions of reciprocal order α) and $K_*(\alpha)$ (classes of convex functions of reciprocal order α), $0 \leq \alpha < 1$, individually by

$$S_*(\alpha) = \left\{ f(z) \in A : \Re \frac{f(z)}{zf'(z)} > \alpha, \quad (z \in U) \right\},$$

$$K_*(\alpha) = \left\{ f(z) \in A : \Re \frac{f'(z)}{zf''(z) + f'(z)} > \alpha, \quad (z \in U) \right\}.$$

In 2008, Nunokawa and his coauthors [29] enhanced inequality (9) for the class $S_*(\alpha)$ and they showed that $f(z) \in S_*(\alpha)$, $0 < \alpha < \frac{1}{2}$, if and only if next inequality holds:

$$\left| \frac{zf'(z)}{f(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad (z \in U).$$

In perspective of these outcomes, we now characterize the accompanying subclass of analytic functions of reciprocal order and study its different properties.

DEFINITION 1.1. A function $f \in A$ is said to be in the class $L(a, s, \gamma)$ with $\gamma \in \mathbb{C} \setminus \{0, \frac{1}{2}\}$ and $a \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}$, if it satisfies the following inequality

$$\Re \left(1 + \frac{1}{\gamma} \left(\frac{J_{s,a}f(z)}{zJ'_{s,a}f(z)} - 1 \right) \right) > 0,$$

where $J_{s,a}f(z) = G_{s,a}(z) * f(z)$.

EXAMPLE 1.2: Let us define the function $J_{s,a}f(z)$ by

$$J_{s,a}f(z) = \frac{z}{(1 + (2\gamma - 1)z)^{2\gamma/(2\gamma-1)}}.$$

This implies that

$$\frac{zJ'_{s,a}f(z)}{J_{s,a}f(z)} = \frac{1 - z}{1 + (2\gamma - 1)z}.$$

Hence

$$1 + \frac{1}{\gamma} \left(\frac{J_{s,a}f(z)}{zJ'_{s,a}f(z)} - 1 \right) = \frac{1 + z}{1 - z},$$

this further implies that

$$\Re \left(1 + \frac{1}{\gamma} \left(\frac{J_{s,a}f(z)}{zJ'_{s,a}f(z)} - 1 \right) \right) = \Re \frac{1 + z}{1 - z} > 0, \quad (z \in U).$$

Noonan and Thomas [28] considered the q th Hankel determinant $H_q(n)$, $q \geq 1, n \geq 1$ for a function $f \in A$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}, \quad a_1 = 1.$$

In the literature, many authors have shed light on the determinant $H_q(n)$, where $H_2(2)$ refer to the second Hankel determinant. After that Janteng *et al.* ([16, 17]), Singh and Singh [35], and many authors have studied sharp upper bounds on $H_2(2)$. Yavuz [39] studied the analytic functions defined by Ruscheweyh derivative and got an upper bound for the second Hankel determinant $|a_2a_4 - a_3^2|$ for it in the unit disc. Mishra and Kund [22] studied a class of analytic functions related to the Carlson-Shaffer operator in the unit disc and estimated the second Hankel determinant for this class. Singh and Mehrok [34] investigated p -valent α -convex functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ in the unit disc and got the sharp upper bound of $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for $f(z)$.

Deniz *et al.* [10] researched bi-starlike and bi-convex functions of order β which are important subclasses of bi-univalent functions and obtained for the second Hankel determinant $H_2(2)$ of these subclasses. Deekonda and Thoutreddy in [9] by using Toeplitz determinants concentrated on the functions belonging to certain subclasses of analytic functions, and obtained an upper bound on the second Hankel determinant $|a_2a_4 - a_3^2|$ for this class. Krishna and Ramreddy [21] by using Toeplitz determinants, considered p -valent starlike and convex functions of order α and obtained an upper bound on the second Hankel determinant $|a_{p+1}a_{p+3} - a_{p+2}^2|$. We refer to $H_3(1)$ as the third Hankel determinant. In 2014 Arif *et al.* [2] studied some families of starlike and convex functions of reciprocal order defined by Al-Oboudi operator and obtained coefficient estimates, Fekete-Szegő inequality, and upper bound on third Hankel determinant for these families. Recently Mishra *et al.* [27] investigated upper bounds on the third Hankel determinants for the starlike and convex functions with respect to symmetric points in the open unit disc. Shanmugam *et al.* [32] investigated the third Hankel determinant, $H_3(1)$, for normalized univalent functions $f(z) = z + a_2z^2 + \dots$ belonging to the class of α starlike functions. In 2015 Prajapat *et al.* [31] focused on the functions belonging to the class of close-to-convex functions and obtained upper bound on third Hankel determinant for this class. Other examples defined on various classes can be read in [1, 18, 19].

In this paper, the authors study the upper bound on $H_3(1)$ for a subclass of analytic functions of reciprocal order by using Toeplitz determinant. Some useful results include coefficient estimates, Fekete-Szegő inequality, and upper bound of third Hankel determinant for the functions belonging to the class $L(a, s, \gamma)$.

To achieve the results, we need the following lemmas:

LEMMA 1.3 ([30]). *If $q(z)$ is a function with $\Re q(z) > 0$ and is of the form*

$$q(z) = 1 + c_1z + c_2z^2 + \dots, \quad (10)$$

then

$$|c_n| \leq 2, \quad \text{for } n \geq 1.$$

LEMMA 1.4 ([20]). *If $q(z)$ is of the form (10) with positive real part, then the following sharp estimate holds:*

$$|c_2 - \nu c_1^2| \leq 2 \max\{1, |2\nu - 1|\}, \text{ for all } \nu \in \mathbb{C}.$$

LEMMA 1.5 ([15]). *If $q(z)$ is of the form (10) with positive real part, then*

$$2c_2 = c_1^2 + (4 - c_1^2)x$$

and

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z,$$

for some x and z satisfy $|x| \leq 1$, $|z| \leq 1$ and $c_1 \in [0, 2]$.

2. Some properties of the class $L(a, s, \gamma)$

THEOREM 2.1. Let $f(z) \in L(a, s, \gamma)$. Then

$$|a_2| \leq \frac{2|\gamma|}{\left(\frac{1+a}{2+a}\right)^s}$$

and for all $n = 3, 4, 5, \dots$

$$|a_n| \leq \frac{2|\gamma|}{(n-1)\left(\frac{1+a}{n+a}\right)^s} \prod_{k=2}^{n-1} \left(1 + \frac{2|\gamma|k}{k-1}\right).$$

Proof. The function $q(z)$ can be characterized as

$$q(z) = 1 + \frac{1}{\gamma} \left(\frac{J_{s,a}f(z)}{zJ'_{s,a}f(z)} - 1 \right),$$

where $J_{s,a}f(z)$ is given by (7) with

$$J_{s,a}f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a}\right)^s a_n z^n,$$

and $q(z)$ is analytic in U with $q(0) = 1$, $\Re q(z) > 0$.

Now, by using (1) and (10), we get

$$z + \sum_{k=2}^{\infty} A_k z^k = \left[1 + \gamma \left(\sum_{k=1}^{\infty} c_k z^k \right) \right] \left(z + \sum_{k=2}^{\infty} k A_k z^k \right),$$

where

$$A_k = \left(\frac{1+a}{k+a}\right)^s a_k. \quad (11)$$

Comparing coefficient of z^n , we get

$$(1-n)A_n = \gamma \{c_{n-1} + 2A_2c_{n-2} + \dots + (n-1)A_{n-1}c_1\}. \quad (12)$$

Using triangle inequality and Lemma 1.3, we obtain

$$|(1-n)A_n| \leq 2|\gamma|\{1 + 2|A_2| + \dots + (n-1)|A_{n-1}|\}. \quad (13)$$

For $n = 2$ and $n = 3$ in (13), we can get the following easily

$$|a_2| \leq \frac{2|\gamma|}{\left(\frac{1+a}{2+a}\right)^s}, \quad |a_3| \leq \frac{|\gamma|(1+4|\gamma|)}{\left(\frac{1+a}{3+a}\right)^s}.$$

Making $n = 4$ in (13), we note that

$$|a_4| \leq \frac{2|\gamma|(1+4|\gamma|)(1+3|\gamma|)}{3\left(\frac{1+a}{4+a}\right)^s}.$$

In general, by using the principle of mathematical induction, we can obtain

$$|A_n| \leq \frac{2|\gamma|}{(n-1)} \prod_{k=2}^{n-1} \left(1 + \frac{2|\gamma|k}{k-1}\right).$$

Presently, using relation (11), we get the required result:

$$|a_n| \leq \frac{2|\gamma|}{(n-1)\left(\frac{1+a}{n+a}\right)^s} \prod_{k=2}^{n-1} \left(1 + \frac{2|\gamma|k}{k-1}\right).$$

□

With $\gamma = 1 - \alpha$ and $s = 0$, we obtain the following result.

COROLLARY 2.2 ([14]). *Let $f(z) \in S_*(\alpha)$. Then, for $n = 3, 4, 5, \dots$, one has*

$$|a_n| \leq \frac{2(1-\alpha)}{(n-1)} \prod_{k=2}^{n-1} \left(1 + \frac{2(1-\alpha)k}{k-1}\right)$$

with $|a_2| \leq 2(1-\alpha)$.

If we make $s = 1$ and $\gamma = 1 - \alpha$, we can get the following easily

COROLLARY 2.3 ([14]). *Let $f(z) \in K_*(\alpha)$. Then, for $n = 3, 4, 5, \dots$, one has*

$$|a_n| \leq \frac{2(1-\alpha)}{(n-1)\left(\frac{1+a}{n+a}\right)} \prod_{k=2}^{n-1} \left(1 + \frac{2(1-\alpha)k}{k-1}\right)$$

with $|a_2| \leq \frac{2(1-\alpha)}{\left(\frac{1+a}{2+a}\right)}$.

THEOREM 2.4. *If $f(z) \in L(a, s, \gamma)$ and is of the form (1). Then*

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma|}{\left(\frac{1+a}{3+a}\right)^s} \max\{1, |2\nu - 1|\},$$

where

$$\nu = 2\gamma \left(\frac{1+a}{3+a}\right)^s \left(\frac{1}{\left(\frac{1+a}{3+a}\right)^s} - \frac{\mu}{\left(\frac{1+a}{2+a}\right)^{2s}} \right). \quad (14)$$

Proof. Let $f(z) \in L(a, s, \gamma)$. Then from (12) we have

$$a_2 = \frac{-\gamma c_1}{\left(\frac{1+a}{2+a}\right)^s}, \quad a_3 = \frac{-\gamma}{2\left(\frac{1+a}{3+a}\right)^s} (c_2 - 2\gamma c_1^2).$$

We now consider

$$|a_3 - \mu a_2^2| = \frac{|\gamma|}{2\left(\frac{1+a}{3+a}\right)^s} \left| c_2 - 2\gamma \left(\frac{1+a}{3+a}\right)^s \left(\frac{1}{\left(\frac{1+a}{3+a}\right)^s} - \frac{\mu}{\left(\frac{1+a}{2+a}\right)^{2s}} \right) c_1^2 \right|.$$

Using Lemma 1.4, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma|}{\left(\frac{1+a}{3+a}\right)^s} \max\{1, |2\nu - 1|\},$$

where ν is given by (14). □

Putting $\mu = 1$, we get

COROLLARY 2.5. *If $f(z) \in L(a, s, \gamma)$. Then*

$$|a_3 - a_2^2| \leq \frac{|\gamma|}{\left(\frac{1+a}{3+a}\right)^s}.$$

THEOREM 2.6. *Let $f(z) \in L(a, s, \gamma)$ and be of the form (1). Then*

$$|a_2 a_4 - a_3^2| \leq \left[\frac{4\left(\frac{1+a}{3+a}\right)^{2s} + |\gamma| \left(28\left(\frac{1+a}{3+a}\right)^{2s} + 24\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \right)}{3\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^{2s} \left(\frac{1+a}{4+a}\right)^s} + \frac{48|\gamma|^2 \left(\left(\frac{1+a}{3+a}\right)^{2s} + \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \right) + 3\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s}{3\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^{2s} \left(\frac{1+a}{4+a}\right)^s} \right] \times |\gamma|^2.$$

Proof. Let $f(z) \in L(a, s, \gamma)$. Then, from (12), we have

$$a_2 = \frac{-\gamma c_1}{\left(\frac{1+a}{2+a}\right)^s}, \quad a_3 = \frac{-\gamma}{2\left(\frac{1+a}{3+a}\right)^s}(c_2 - 2\gamma c_1^2)$$

and

$$a_4 = \frac{-\gamma}{3\left(\frac{1+a}{4+a}\right)^s} \left[c_3 - \frac{7}{2}\gamma c_1 c_2 + 3\gamma^2 c_1^3 \right]. \quad (15)$$

Consider

$$|a_2 a_4 - a_3^2| = \left| \frac{-\gamma c_1}{\left(\frac{1+a}{2+a}\right)^s} \cdot \frac{-\gamma}{3\left(\frac{1+a}{4+a}\right)^s} \left[c_3 - \frac{7}{2}\gamma c_1 c_2 + 3\gamma^2 c_1^3 \right] - \frac{\gamma^2}{4\left(\frac{1+a}{3+a}\right)^{2s}} (c_2 - 2\gamma c_1^2)^2 \right|$$

$$|a_2 a_4 - a_3^2| = \left| \left(\frac{\gamma^2}{12\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^{2s} \left(\frac{1+a}{4+a}\right)^s} \right) \left(4\left(\frac{1+a}{3+a}\right)^{2s} c_1 c_3 - 2\gamma \left(7\left(\frac{1+a}{3+a}\right)^{2s} - 6\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \right) c_2 c_1^2 + 12\gamma^2 \left(\frac{1+a}{3+a}\right)^{2s} - \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s c_1^4 - 3\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s c_2^2 \right) \right|.$$

Now using values of c_2 and c_3 from Lemma 1.5, we obtain

$$|a_2 a_4 - a_3^2| = \frac{\gamma^2}{12\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^{2s} \left(\frac{1+a}{4+a}\right)^s} \times \left| \left\{ \left(\frac{1+a}{3+a}\right)^{2s} - \gamma \left(7\left(\frac{1+a}{3+a}\right)^{2s} - 6\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \right) - \frac{3}{4}\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \right\} c_1^4 + \left\{ 2\left(\frac{1+a}{3+a}\right)^{2s} - \gamma \left(7\left(\frac{1+a}{3+a}\right)^{2s} - 6\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \right) - \frac{3}{2}\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \right\} (4 - c_1^2) c_1^2 x - \left\{ \left(\frac{1+a}{3+a}\right)^{2s} c_1^2 + \frac{3}{4}\left(\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s\right) (4 - c_1^2) \right\} (4 - c_1^2) x^2 + 2c_1 \left(\frac{1+a}{3+a}\right)^{2s} (4 - c_1^2) (1 - |x|^2) z \right|.$$

Applying triangle inequality and replacing c_1 by c , $|x|$ by ρ , and $|z|$ by 1, we get

$$\begin{aligned}
|a_2a_4 - a_3^2| &\leq \frac{|\gamma|^2}{12\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^{2s}\left(\frac{1+a}{4+a}\right)^s} \times \left[\left\{ \left(\frac{1+a}{3+a}\right)^{2s} + |\gamma| \left(7\left(\frac{1+a}{3+a}\right)^{2s}\right. \right. \right. \\
&\quad \left. \left. \left. + 6\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right) + 12|\gamma|^2\left(\left(\frac{1+a}{3+a}\right)^{2s} + \left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right) \right\} \right. \\
&\quad \left. + \frac{3}{4}\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right\}c^4 + \left\{ 2\left(\frac{1+a}{3+a}\right)^{2s} + |\gamma| \left(7\left(\frac{1+a}{3+a}\right)^{2s}\right. \right. \\
&\quad \left. \left. + 6\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right) + \frac{3}{2}\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right\}(4-c^2)c^2\rho \\
&\quad \left. + \left\{ \left(\frac{1+a}{3+a}\right)^{2s}c^2 + \frac{3}{4}\left(\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right)(4-c^2)\right\}(4-c^2)\rho^2 \right. \\
&\quad \left. + 2c\left(\frac{1+a}{3+a}\right)^{2s}(4-c^2)(1-\rho^2) \right] = F(c, \rho).
\end{aligned}$$

Differentiating with respect to ρ , we get

$$\begin{aligned}
\frac{\partial F(c, \rho)}{\partial \rho} &= \frac{|\gamma|^2}{12\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^{2s}\left(\frac{1+a}{4+a}\right)^s} \times \left[\left\{ 2\left(\frac{1+a}{3+a}\right)^{2s} + |\gamma| \left(7\left(\frac{1+a}{3+a}\right)^{2s}\right. \right. \right. \\
&\quad \left. \left. \left. + 6\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right) + \frac{3}{2}\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right\}(4-c^2)c^2 \right. \\
&\quad \left. + \left\{ 2\left(\frac{1+a}{3+a}\right)^{2s}c^2 + \frac{3}{2}\left(\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right)(4-c^2)\right\}(4-c^2)\rho \right. \\
&\quad \left. - 4c\left(\frac{1+a}{3+a}\right)^{2s}(4-c^2)\rho \right].
\end{aligned}$$

Since $\frac{\partial F(c, \rho)}{\partial \rho} > 0$ for $\rho \in [0, 1]$ and $c \in [0, 2]$, the maximize of $F(c, \rho)$ will exist at $\rho = 1$. Let $F(c, 1) = G(c)$, then

$$\begin{aligned}
G(c) &= \frac{|\gamma|^2}{12\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^{2s}\left(\frac{1+a}{4+a}\right)^s} \times \left[\left\{ \left(\frac{1+a}{3+a}\right)^{2s} + |\gamma| \left(7\left(\frac{1+a}{3+a}\right)^{2s}\right. \right. \right. \\
&\quad \left. \left. \left. + 6\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right) + 12|\gamma|^2\left(\left(\frac{1+a}{3+a}\right)^{2s} + \left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right) \right\} \right. \\
&\quad \left. + \frac{3}{4}\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right\}c^4 + \left\{ 2\left(\frac{1+a}{3+a}\right)^{2s}|\gamma| \left(7\left(\frac{1+a}{3+a}\right)^{2s}\right. \right. \\
&\quad \left. \left. + 6\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right) + \frac{3}{2}\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right\}(4-c^2)c^2 \\
&\quad \left. + \left\{ \left(\frac{1+a}{3+a}\right)^{2s}c^2 + \frac{3}{4}\left(\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{4+a}\right)^s\right)(4-c^2)\right\}(4-c^2) \right].
\end{aligned}$$

Now by differentiating with respect to c , we obtain

$$\begin{aligned}
G'(c) = & \frac{|\gamma|^2}{12 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^{2s} \left(\frac{1+a}{4+a}\right)^s} \times \left[4 \left\{ \left(\frac{1+a}{3+a}\right)^{2s} + |\gamma| \left(7 \left(\frac{1+a}{3+a}\right)^{2s} \right. \right. \right. \\
& + 6 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \left. \left. \left. + 12|\gamma|^2 \left(\left(\frac{1+a}{3+a}\right)^{2s} + \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \right) \right. \right. \right. \\
& + \left. \left. \left. \frac{3}{4} \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \right\} c^3 + \left\{ 2 \left(\frac{1+a}{3+a}\right)^{2s} + |\gamma| \left(7 \left(\frac{1+a}{3+a}\right)^{2s} \right. \right. \right. \\
& + 6 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \left. \left. \left. + \frac{3}{2} \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \right\} (8c - 4c^3) \right. \\
& \left. \left. + \left\{ \left(\frac{1+a}{3+a}\right)^{2s} (8c - 4c^3) - 3 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s (4c - c^3) \right\} \right].
\end{aligned}$$

Since $\partial G(c)/\partial c > 0$ for $c \in [0, 2]$, $G(c)$ has a maximum value at $c = 2$ and hence

$$\begin{aligned}
|a_2 a_4 - a_3^2| \leq & \frac{|\gamma|^2}{3 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^{2s} \left(\frac{1+a}{4+a}\right)^s} \\
& \times \left\{ 4 \left(\frac{1+a}{3+a}\right)^{2s} + |\gamma| \left(28 \left(\frac{1+a}{3+a}\right)^{2s} + 24 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \right) \right. \\
& \left. + 48|\gamma|^2 \left(\left(\frac{1+a}{3+a}\right)^{2s} + \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \right) + 3 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s \right\}.
\end{aligned}$$

□

THEOREM 2.7. *Let $f(z) \in L(a, s, \gamma)$ and be of the form (1). Then*

$$\begin{aligned}
|a_2 a_3 - a_4| \leq & \frac{|\gamma|}{3 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{4+a}\right)^s} \\
& \times \left\{ 24|\gamma|^2 \left(\left(\frac{1+a}{4+a}\right)^s + \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) + 2|\gamma| \left(3 \left(\frac{1+a}{4+a}\right)^s \right. \right. \\
& \left. \left. + 7 \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) + 2 \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right\}.
\end{aligned}$$

Proof. From (15), we can write

$$\begin{aligned}
|a_2 a_3 - a_4| = & \frac{|\gamma|}{6 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{4+a}\right)^s} \\
& \times \left| -6\gamma^2 \left(\left(\frac{1+a}{4+a}\right)^s - \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) c_1^3 + \gamma \left(3 \left(\frac{1+a}{4+a}\right)^s \right. \right. \\
& \left. \left. - 7 \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) c_1 c_2 + 2 \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s c_3 \right|.
\end{aligned}$$

Using Lemma 1.5 for the values of c_2 and c_3 , we have

$$\begin{aligned}
|a_2a_3 - a_4| &= \frac{|\gamma|}{6 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{4+a}\right)^s} \\
&\times \left| \left\{ -6\gamma^2 \left(\left(\frac{1+a}{4+a}\right)^s - \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right) + \frac{\gamma}{2} \left(3 \left(\frac{1+a}{4+a}\right)^s \right. \right. \right. \\
&\quad \left. \left. - 7 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right) + \frac{1}{2} \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right\} c_1^3 + \left\{ \frac{\gamma}{2} \left(3 \left(\frac{1+a}{4+a}\right)^s \right. \right. \\
&\quad \left. \left. - 7 \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) + \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right\} (4 - c_1^2) c_1 x \\
&\quad - \left\{ \frac{1}{2} \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right\} c_1 (4 - c_1^2) x^2 \\
&\quad \left. + \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s (4 - c_1^2) (1 - |x|^2 z) \right|.
\end{aligned}$$

Applying triangle inequality and then putting $|z| = 1$, $|x| = \rho$, and $c_1 = c$, we have

$$\begin{aligned}
|a_2a_3 - a_4| &\leq \frac{|\gamma|}{6 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{4+a}\right)^s} \\
&\times \left[\left\{ 6|\gamma|^2 \left(\left(\frac{1+a}{4+a}\right)^s + \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right) \right. \right. \\
&\quad \left. \left. + \frac{|\gamma|}{2} \left(3 \left(\frac{1+a}{4+a}\right)^s + 7 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right) \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right\} c^3 \right. \\
&\quad \left. + \left\{ \frac{|\gamma|}{2} \left(3 \left(\frac{1+a}{4+a}\right)^s + 7 \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) \right. \right. \\
&\quad \left. \left. + \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right\} (4 - c^2) c \rho \right. \\
&\quad \left. + \left\{ \frac{1}{2} \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right\} c (4 - c^2) \rho^2 \right. \\
&\quad \left. + \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s (4 - c^2) (1 - \rho^2) \right] \\
&= F(c, \rho).
\end{aligned}$$

Differentiating with respect to ρ , we get

$$\begin{aligned} \frac{\partial F(c, \rho)}{\partial \rho} &= \frac{|\gamma|}{6 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{4+a}\right)^s} \\ &\times \left[\left\{ \frac{|\gamma|}{2} \left(3 \left(\frac{1+a}{4+a}\right)^s + 7 \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right\} (4-c^2)c \right. \\ &\quad \left. + \left\{ \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right\} c(4-c^2)\rho \right. \\ &\quad \left. - 2 \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s (4-c^2)\rho \right]. \end{aligned}$$

Now since $\frac{\partial F(c, \rho)}{\partial \rho} > 0$ for $c \in [0, 2]$ and $\rho \in [0, 1]$, a maximum of $F(c, \rho)$ will exist at $\rho = 1$ and let $F(c, 1) = G(c)$. Then

$$\begin{aligned} G(c) &= \frac{|\gamma|}{6 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{4+a}\right)^s} \\ &\times \left[\left\{ 6|\gamma|^2 \left(\left(\frac{1+a}{4+a}\right)^s + \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right) \right. \right. \\ &\quad \left. \left. + \frac{|\gamma|}{2} \left(3 \left(\frac{1+a}{4+a}\right)^s + 7 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right\} c^3 \right. \\ &\quad \left. + \left\{ \frac{|\gamma|}{2} \left(3 \left(\frac{1+a}{4+a}\right)^s + 7 \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) \right. \right. \\ &\quad \left. \left. + \frac{3}{2} \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right\} (4-c^2)c \right]. \end{aligned}$$

Now by differentiating with respect to c , we obtain

$$\begin{aligned} G'(c) &= \frac{|\gamma|}{6 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{4+a}\right)^s} \\ &\times \left[3 \left\{ 6|\gamma|^2 \left(\left(\frac{1+a}{4+a}\right)^s + \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right) \right. \right. \\ &\quad \left. \left. + \frac{|\gamma|}{2} \left(3 \left(\frac{1+a}{4+a}\right)^s + 7 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right) + \frac{1}{2} \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right\} c^2 \right. \\ &\quad \left. + \left\{ \frac{|\gamma|}{2} \left(3 \left(\frac{1+a}{4+a}\right)^s + 7 \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) \right. \right. \\ &\quad \left. \left. + \frac{3}{2} \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right\} (4-3c^2) \right]. \end{aligned}$$

Since $\partial G(c)/\partial c > 0$ for $c \in [0, 2]$, $G(c)$ has a maximum value at $c = 2$, hence

$$|a_2a_3 - a_4| \leq \frac{|\gamma|}{3 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{4+a}\right)^s} \\ \times \left\{ 24|\gamma|^2 \left(\left(\frac{1+a}{4+a}\right)^s + \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) \right. \\ \left. + 2|\gamma| \left(3 \left(\frac{1+a}{4+a}\right)^s + 7 \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) \right. \\ \left. + 2 \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right\}.$$

□

THEOREM 2.8. *Let $f(z) \in L(a, s, \gamma)$ and be of the form (1). Then*

$$|H_3(1)| \leq \frac{|\gamma|^3(1+4|\gamma|)}{3 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^{3s} \left(\frac{1+a}{4+a}\right)^s} \\ \times \left[4 \left(\frac{1+a}{3+a}\right)^{2s} + |\gamma| \left(28 \left(\frac{1+a}{3+a}\right)^{2s} + 24 \left(\frac{1+a}{4+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) \right. \\ \left. + 48|\gamma|^2 \left(\left(\frac{1+a}{3+a}\right)^{2s} + \left(\frac{1+a}{4+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) + 3 \left(\frac{1+a}{4+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right] \\ + \frac{4|\gamma|^2(1+4|\gamma|)(1+3|\gamma|)}{9 \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{4+a}\right)^{2s}} \\ \times \left[12|\gamma|^2 \left(\left(\frac{1+a}{4+a}\right)^s + \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) \right. \\ \left. + |\gamma| \left(3 \left(\frac{1+a}{4+a}\right)^s + 7 \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) + \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right] \\ + \frac{|\gamma|^2(29|\gamma| + 92|\gamma|^2 + 96|\gamma|^3 + 3)}{6 \left(\frac{1+a}{5+a}\right)^s \left(\frac{1+a}{3+a}\right)^s}.$$

Proof. Since

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_1a_4| + |a_5||a_3 - a_2^2|,$$

using Corollary 2.5, Theorem 2.6, Theorems 2.7 and a_5 , we have

$$\begin{aligned}
|H_3(1)| &\leq \frac{|\gamma|(1+4|\gamma|)}{\left(\frac{1+a}{3+a}\right)^s} \times \frac{|\gamma|^2}{3\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^{2s} \left(\frac{1+a}{4+a}\right)^s} \\
&\quad \times \left[4\left(\frac{1+a}{3+a}\right)^{2s} + |\gamma| \left(28\left(\frac{1+a}{3+a}\right)^{2s} + 24\left(\frac{1+a}{4+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) \right. \\
&\quad \left. + 48|\gamma|^2 \left(\left(\frac{1+a}{3+a}\right)^{2s} + \left(\frac{1+a}{4+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) + 3\left(\frac{1+a}{4+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right] \\
&+ \frac{2|\gamma|(1+4|\gamma|)(1+3|\gamma|)}{3\left(\frac{1+a}{4+a}\right)^s} \times \frac{|\gamma|}{3\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \left(\frac{1+a}{4+a}\right)^s} \\
&\quad \times \left[24|\gamma|^2 \left(\left(\frac{1+a}{4+a}\right)^s + \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right) \right. \\
&\quad \left. + 2|\gamma| \left(3\left(\frac{1+a}{4+a}\right)^s + 7\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right) + 2\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right] \\
&+ \frac{2|\gamma| \left(\frac{29}{3}|\gamma| + \frac{92}{3}|\gamma|^2 + 32|\gamma|^3 + 1 \right)}{4\left(\frac{1+a}{5+a}\right)^s} \times \frac{|\gamma|}{\left(\frac{1+a}{3+a}\right)^s}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
|H_3(1)| &\leq \frac{(1+4|\gamma|)|\gamma|^3}{3\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^{3s} \left(\frac{1+a}{4+a}\right)^s} \\
&\quad \times \left[4\left(\frac{1+a}{3+a}\right)^{2s} + |\gamma| \left(28\left(\frac{1+a}{3+a}\right)^{2s} + 24\left(\frac{1+a}{4+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) \right. \\
&\quad \left. + 48|\gamma|^2 \left(\left(\frac{1+a}{3+a}\right)^{2s} + \left(\frac{1+a}{4+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right) + 3\left(\frac{1+a}{4+a}\right)^s \left(\frac{1+a}{2+a}\right)^s \right] \\
&+ \frac{4|\gamma|^2(1+4|\gamma|)(1+3|\gamma|)}{9\left(\frac{1+a}{4+a}\right)^{2s} \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s} \\
&\quad \times \left[12|\gamma|^2 \left(\left(\frac{1+a}{4+a}\right)^s + \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right) \right. \\
&\quad \left. + |\gamma| \left(3\left(\frac{1+a}{4+a}\right)^s + 7\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right) + \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^s \right] \\
&+ \frac{|\gamma|^2 \left(29|\gamma| + 92|\gamma|^2 + 96|\gamma|^3 + 3 \right)}{6\left(\frac{1+a}{5+a}\right)^s \left(\frac{1+a}{3+a}\right)^s}.
\end{aligned}$$

This completes the proof. \square

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Authors' addresses:

Khalied Abdulameer Challab
School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor
D. Ehsan, Malaysia
E-mail: khalid_math1363@yahoo.com

Maslina Darus
School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor
D. Ehsan, Malaysia
E-mail: maslina@ukm.edu.my

Firas Ghanim
Department of Mathematics
College of Sciences
University of Sharjah
Sharjah, United Arab Emirates
E-mail: fgahmed@sharjah.ac.ae

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A counterexample to a priori bounds under the Ahmad-Lazer-Paul condition

ALBERTO BOSCAGGIN AND MAURIZIO GARRIONE

ABSTRACT. *In the context of scalar second order ODEs at resonance, we construct a counterexample showing that, in general, the Ahmad-Lazer-Paul condition does not imply a priori bounds for T -periodic solutions.*

Keywords: Ahmad-Lazer-Paul condition, Landesman-Lazer condition, a priori bounds, T -periodic solutions, resonance.
MS Classification 2010: 34C25.

1. Introduction

The forced scalar second order ODE

$$x'' + \lambda x + r(t, x) = 0, \quad (1)$$

where $r : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is T -periodic in the t -variable and bounded, is said to be at resonance if the linear equation $x'' + \lambda x = 0$ associated with (1) has a nontrivial T -periodic solution. As is well known, this occurs if and only if $\lambda = \lambda_k = (2k\pi/T)^2$ for some $k \in \mathbb{N} = \{0, 1, 2, \dots\}$ and prevents, in general, the T -periodic solvability of (1), which can be guaranteed only under additional assumptions on the nonlinear term r (see, e.g., [7, 15]). After the pioneering work [13] dealing with the separate case $r(t, x) = g(x) - e(t)$, two celebrated such assumptions turned out to be the so-called Landesman-Lazer (LL) and Ahmad-Lazer-Paul (ALP) conditions. Denoting by Σ_k the eigenspace associated with the eigenvalue λ_k (namely, the set of T -periodic solutions of the equation $x'' + \lambda_k x = 0$), they read as follows:

- (LL) for every $\varphi \in \Sigma_k \setminus \{0\}$,

$$\int_{\{\varphi > 0\}} \left(\liminf_{x \rightarrow +\infty} r(t, x) \right) \varphi(t) dt + \int_{\{\varphi < 0\}} \left(\limsup_{x \rightarrow -\infty} r(t, x) \right) \varphi(t) dt > 0,$$

where $\{\varphi \geq 0\} = \{t \in [0, T] \mid \varphi(t) \geq 0\}$;

- (ALP) setting $R(t, x) = \int_0^x r(t, s) ds$,

$$\lim_{\substack{\|\varphi\|_\infty \rightarrow +\infty \\ \varphi \in \Sigma_k}} \int_0^T R(t, \varphi(t)) dt = +\infty.$$

Such assumptions are the T -periodic versions of the ones given, respectively, in [12] and in [1] for the Dirichlet problem associated with an elliptic PDE. In the context of the T -periodic problem, they were originally provided, respectively, in [14] and in [16], see also [7] for an exhaustive treatment of the subject. While the Landesman-Lazer condition usually leads to a T -periodic solution of (1) through topological methods relying on a priori bounds and degree theory, the Ahmad-Lazer-Paul one allows to prove T -periodic solvability by variational methods, precisely via the Rabinowitz saddle point theorem.

The aim of this brief note is to highlight that this dichotomy is not only a matter of the chosen techniques but is substantial, in that the a priori bounds on the solutions may actually fail under (ALP). Indeed, we are going to prove the following result.

THEOREM 1.1. *For every $k \in \mathbb{N}$, there exists a bounded, continuous and T -periodic function $r : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying (ALP) and such that equation (1), with $\lambda = \lambda_k$, has a sequence of T -periodic solutions $(u_j)_j$ with*

$$\|u_j\|_\infty \rightarrow +\infty. \quad (2)$$

Of course, the function r appearing in the above statement cannot satisfy condition (LL) (see also Remark 2.2 below).

2. Proof of Theorem 1.1 and related remarks

We start with the easier case $k = 0$ (with corresponding eigenvalue $\lambda_0 = 0$), which will indeed provide the idea for the proof in the more interesting case $k > 0$. Here, Σ_0 is the 1-dimensional space made up by constant functions and (1) reads as

$$x'' + r(t, x) = 0. \quad (3)$$

The corresponding (ALP) condition is given by

$$\lim_{|x| \rightarrow +\infty} \int_0^T R(t, x) dt = +\infty$$

and a natural choice for r is

$$r(t, x) = r(x) = \operatorname{sgn}(x) |\sin(\pi x)|,$$

where we mean $\operatorname{sgn}(0) = 0$. Of course, such a function satisfies (ALP), but (3) has the unbounded sequence of T -periodic solutions $u_j(t) \equiv j$ ($j \in \mathbb{N}$). Notice that all such u_j 's are multiple of the eigenfunction corresponding to λ_0 , that is, $\varphi_0(t) \equiv 1$. This is the key point for the construction of the counterexample when $k \geq 1$.

In this case, take $0 \neq \varphi_k \in \Sigma_k$, with $\|\varphi_k\|_\infty = 1$ (for instance, one may take $\varphi_k(t) = \cos(\sqrt{\lambda_k}t)$) and, after having set $Z_k = \{t \in \mathbb{R} \mid \varphi_k(t) = 0\}$, define

$$r(t, x) = \begin{cases} \operatorname{sgn}(x) |\varphi_k(t)| \left| \sin\left(\frac{\pi x}{\varphi_k(t)}\right) \right| & \text{if } t \notin Z_k, \\ 0 & \text{if } t \in Z_k. \end{cases} \quad (4)$$

Notice that Z_k is discrete and made up by a countable number of points. Of course, r is T -periodic in the first variable and $|r(t, x)| \leq 1$ for every t, x ; moreover, it is immediate to check that for the choice

$$u_j(t) = j\varphi_k(t), \quad j \in \mathbb{N},$$

it holds

$$r(t, u_j(t)) \equiv 0.$$

Thus, u_j satisfies (1) - with $\lambda = \lambda_k$ - and (2). It remains to show that r is continuous and satisfies (ALP).

The continuity at the points (\bar{t}, \bar{x}) with $\bar{t} \notin Z_k$ is obvious; on the other hand, if $\bar{t} \in Z_k$ and $(t_n, x_n) \rightarrow (\bar{t}, \bar{x})$, then $r(t_n, x_n) \rightarrow 0$ since $\operatorname{sgn}(x_n) |\sin(\pi x_n / \varphi_k(t_n))|$ is bounded.

Finally, we show that r satisfies (ALP). To this end, we first notice that $R(t, x) \geq 0$ for every t, x and $R(t, x) = 0$ if and only if $t \in Z_k$ or $x = 0$; moreover, it is easy to check that

$$\lim_{|x| \rightarrow +\infty} R(t, x) = +\infty, \quad \text{for every } t \notin Z_k. \quad (5)$$

Then, we observe that

$$\varphi \in \Sigma_k \iff \varphi(t) = \gamma\varphi_k(t + \theta) \text{ for some } \gamma \geq 0, \theta \in [0, T],$$

so that $\|\varphi\|_\infty = \gamma$ does not depend on θ . Hence, the Ahmad-Lazer-Paul condition (ALP) can be equivalently written as

$$\lim_{\gamma \rightarrow +\infty} \int_0^T R(t, \gamma\varphi_k(t + \theta)) dt = +\infty, \quad \text{uniformly in } \theta \in [0, T]. \quad (6)$$

We then show that r satisfies (6). By contradiction, let us assume that there exist $M > 0$ and two sequences $(\gamma_n)_n, (\theta_n)_n$, with $\gamma_n \rightarrow +\infty$ and $\theta_n \rightarrow \bar{\theta} \in [0, T]$ such that

$$\int_0^T R(t, \gamma_n\varphi_k(t + \theta_n)) dt \leq M \quad (7)$$

for every n . Passing to the inferior limit at both sides and using Fatou's Lemma (recall that $R(t, x) \geq 0$), we obtain

$$\begin{aligned} & \int_{[0, T] \setminus (Z_k \cup \bar{Z})} \liminf_{n \rightarrow +\infty} R(t, \gamma_n \varphi_k(t + \theta_n)) dt \\ & \leq \int_0^T \liminf_{n \rightarrow +\infty} R(t, \gamma_n \varphi_k(t + \theta_n)) dt \leq M, \end{aligned}$$

where $\bar{Z} = \{t \in [0, T] \mid \varphi_k(t + \bar{\theta}) = 0\}$. For every $t \in [0, T] \setminus (Z_k \cup \bar{Z})$, we now have that $|\gamma_n \varphi_k(t + \theta_n)| \rightarrow +\infty$, so that, by (5), we deduce that

$$\liminf_{n \rightarrow +\infty} R(t, \gamma_n \varphi_k(t + \theta_n)) = +\infty \quad \text{for every } t \in [0, T] \setminus (Z_k \cup \bar{Z}).$$

This contradicts (7) and eventually concludes the proof of Theorem 1.1. \square

REMARK 2.1: By defining

$$r(t, x) = \begin{cases} \operatorname{sgn}(x) |\varphi_k(t)|^\alpha \left| \sin\left(\frac{\pi x}{\varphi_k(t)}\right) \right|^\alpha & \text{if } t \notin Z_k, \\ 0 & \text{if } t \in Z_k \end{cases}$$

for α sufficiently large, it is possible to construct similar counterexamples where the nonlinear term enjoys higher regularity.

REMARK 2.2: It is clear that the function r defined in (4) does not satisfy any kind of Landesman-Lazer condition, since

$$\liminf_{x \rightarrow +\infty} r(t, x) = 0 = \limsup_{x \rightarrow -\infty} r(t, x),$$

so that the left-hand side in (LL) is identically equal to 0 (recall that as shown, e.g., in [8], (ALP) is weaker than (LL)). It can also be seen that r satisfies the so-called *potential Landesman-Lazer condition* [18], an intermediate condition between (LL) and (ALP), reading as

$$\int_{\{\varphi > 0\}} \left(\liminf_{x \rightarrow +\infty} \frac{R(t, x)}{x} \right) \varphi(t) dt + \int_{\{\varphi < 0\}} \left(\limsup_{x \rightarrow -\infty} \frac{R(t, x)}{x} \right) \varphi(t) dt > 0$$

for every $\varphi \in \Sigma_k \setminus \{0\}$, so that also in this case the a priori bounds do not work (indeed, the arguments used in [18] are variational).

REMARK 2.3: The previous discussion can be extended with no difficulty to the case of resonance with respect to the Dancer-Fučik spectrum, namely replacing (1) by

$$x'' + \mu x^+ - \nu x^- + r(t, x) = 0,$$

where $\pi/\sqrt{\mu} + \pi/\sqrt{\nu} = T/k$. In this case, the Ahmad-Lazer-Paul condition to be considered has been given in [2, 4, 11], while the Landesman-Lazer one dates back to [6].

REMARK 2.4: We observe that the provided counterexample can be extended to several different situations. For instance, with obvious modifications we can deal with Dirichlet and Neumann boundary conditions; in this case, the proof can be considerably simplified, since the eigenspaces are 1-dimensional and thus there is no dependence on θ in (6). More in general, one could adapt the discussion to the equation

$$(p(t)x')' + \lambda w(t)x + r(t, x) = 0,$$

where $p(t) > 0$ and $w(t) \geq 0$ for every $t \in (0, T)$ and suitable regularity/summability assumptions on p and q are fulfilled, together with boundary conditions of Sturm-Liouville type

$$\begin{cases} \alpha x(0) + \beta x'(0) = 0 \\ \gamma x(T) + \delta x'(T) = 0. \end{cases}$$

It is worth noticing that the search for radial solutions of boundary value problems associated with a second order elliptic PDE enters this setting. Again, the main point is here that there is a sequence of eigenvalues whose associated eigenspace is 1-dimensional and the nodal properties of the corresponding eigenfunctions are known. However, our counterexample becomes much more meaningful if both (LL) and (ALP) provide existence for the considered boundary value problem; from this point of view, existence results of Landesman-Lazer and Ahmad-Lazer-Paul type in this more general setting seem not completely established in literature, especially for what concerns condition (ALP) (cf. [5, 17] and the references therein).

REMARK 2.5: We finally mention that an alternative way to prove the existence of T -periodic solutions under the Landesman-Lazer condition is based on the Poincaré-Bohl theorem, as in [9]. Within this approach, condition (LL) plays a role in showing that large-norm (in the phase-plane) solutions of the Cauchy problems associated with (1) do not have integer rotation number (a property which has been highlighted in [3] and later improved in other contexts, e.g., [10]). Of course, such a property implies a priori bounds for T -periodic solutions¹, so that our example also shows that the Poincaré-Bohl approach fails under the sole Ahmad-Lazer-Paul condition.

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¹However, in general the converse does not hold, as is well seen already in the linear case: the equation $x'' + x = 2 \cos t$ is solved by the unbounded family of functions $\{x_A(t) = (A+t) \sin t\}_{A \in \mathbb{R}}$, having winding number equal to 1 on $[0, 2\pi]$, but does not have 2π -periodic solutions.

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Authors' addresses:

Alberto Boscaggin
Dipartimento di Matematica
Università di Torino
Via Carlo Alberto 10, 10123 Torino
E-mail: alberto.boscaggin@unito.it

Maurizio Garrione
Dipartimento di Matematica
Politecnico di Milano
P.zza Leonardo da Vinci 32, 20133 Milano
E-mail: maurizio.garrione@polimi.it

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Asymptotic behavior for the elasticity system with a nonlinear dissipative term

MOHAMED DILMI, MOURAD DILMI AND HAMID BENSERIDI

ABSTRACT. *We study the asymptotic behavior of an elasticity problem with a nonlinear dissipative term in a bidimensional thin domain Ω^ε . We prove some convergence results when the thickness tends to zero. The specific Reynolds limit equation and the limit of Tresca free boundary conditions are obtained.*

Keywords: A priori estimates, dissipative term, elasticity system, Tresca Law, Reynolds equation, variational formulation.

MS Classification 2010: 35R35, 76F10, 78M35, 35B40, 35J85, 49J40.

1. Introduction and mathematical model

The topic dealing with propagation of elastic waves with dissipative term is a subject of considerable interest due to its industrial applications such as the dynamics of rubbers, silicones, and gels. Furthermore, in quantum mechanics the dissipation term determines the phenomenon according to which a dynamic system (wave, oscillation...) loses energy with time, where this energy turns into heat. Heat production occurs usually when there is friction between two bodies, and is mathematically modeled adding to the equation of motion a term dependent on the velocity. From a theoretical point of view, the mathematics and mechanics of wave phenomena with dissipation is a classical yet still active subject of research, where many studies have been published in this field. We cite among these the article [15], where Lions studied theoretically the problem for the wave equation with Dirichlet boundary conditions and a nonlinear dissipative term $|\frac{\partial u}{\partial t}|^p \frac{\partial u}{\partial t}$, in which the author proved the existence and the uniqueness of the solution. In [11] Georgiev and Todorova studied the nonlinear wave equation involving the nonlinear damping term $|\frac{\partial u}{\partial t}|^{m-1} \frac{\partial u}{\partial t}$ and a source term of type $|u|^{p-1} u$, from large initial data, they proved a global existence theorem for $1 < p \leq m$ and a blow-up result for $1 < m < p$. In [3], Benaïssa and Messaoudi studied the stability of solutions to the nonlinear wave equation with the nonlinear dissipative term $\alpha \left(1 + |\frac{\partial u}{\partial t}|^{m-2}\right) \frac{\partial u}{\partial t}$ and proved for his solution that energy decays exponentially. Lagnese [13], proved some

uniform stability results of elasticity systems with linear dissipative term.

In our paper, we study the asymptotic behavior of the hyperbolic equation governed by a thin, isotropic and homogeneous elastic membrane in the dynamic regime with a dissipative term $(\alpha^\varepsilon + |\frac{\partial u^\varepsilon}{\partial t}|) \frac{\partial u^\varepsilon}{\partial t}$ in a two dimensional thin domain Ω^ε . It is worth noting that the boundary conditions for our problem consist of two conditions: The first is Dirichlet boundary condition on the top and lateral parts, the other condition is Tresca's friction law over lower part of the border. This friction law has a threshold of friction (coefficient of friction) k^ε , when the elastic membrane and the foundation are in contact, the foundation exerts on the elastic membrane a tangential effort which does not exceed the threshold k^ε . As long as the tangential stress has not reached the threshold k^ε , the elastic membrane can not move relative to the foundation and there is blockage. When this threshold is reached, the elastic membrane can move tangentially relative to the foundation and then there is a slip. Some research for initial and boundary value problems involving Tresca friction law can be found in [10, 17].

In the literature, the asymptotic behavior of partial differential equations in a thin domain, particularly those governed by elastic systems has been widely studied. Ciarlet and Destuynder [9] studied equilibrium states of a thin plate $\Omega \times (-\varepsilon, +\varepsilon)$ under external forces where Ω is a smooth domain in \mathbb{R}^2 and ε is a small parameter, to justify the two-dimensional model of the plates. In the paper [16] Paumier studied the asymptotic modeling of a thin elastic plate in unilateral contact with friction against a rigid obstacle (Signorini problem with friction) where he proved that any family of solutions of the three-dimensional problem of Signorini with friction strongly converges towards a unique solution of a two-dimensional problem of plate of the type Signorini without friction. Léger and Miara in [14] justified a mechanical model for an elastic shallow shell in frictionless unilateral contact with an obstacle using the asymptotic analysis. In [5, 6] Benseridi and Dilmi studied the asymptotic analysis of linear elasticity with the nonlinear terms $|u^\varepsilon|^{p-2} u^\varepsilon$ in the stationary case, in [4] they analyzed the asymptotic behavior of a dynamical problem of isothermal elasticity with nonlinear friction of Tresca's type but without including the nonlinear dissipative term. Bayada and Lhalouani [2] investigated the asymptotic and numerical analysis for a unilateral contact problem with Coulomb's friction between an elastic body and a thin elastic soft layer. The reader can also review some articles that are interested in studying the asymptotic analysis of some fluid mechanics problems in a thin domain for the stationary case [1, 7, 8].

Our paper is structured as follows. In Section 1 we present the form of the domain Ω^ε , then we give the basic equations. In Section 2 we derive the variational formulation of the problem and give the theorem of existence and uniqueness of the weak solution. In Section 3 by a scale change we carry out the asymptotic analysis, in which the small parameter (thickness) of the domain tends to

zero. Using Gronwall's lemma and Korn's inequality we establish some parameter independent estimates for the displacement and velocity fields. Finally in Section 4 we go to the limit when the thickness tends to zero, we derive the convergence theorem and find the limiting problem, for which we study the solution.

Let Ω^ε be a bounded domain of \mathbb{R}^2 , where ε is a small parameter which ultimately will tend to zero, the boundary of Ω^ε will be denoted by $\Gamma^\varepsilon = \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon \cup \bar{\omega}$, where Γ_1^ε is the upper boundary of equation $y = \varepsilon h(x)$, $\Gamma_L^\varepsilon = \{x = 0\} \cup \{x = l\}$ is the lateral boundary and $\omega =]0, l[$ is a bounded interval, which constitutes the bottom of the domain Ω^ε . For all $x' = (x, y) \in \mathbb{R}^2$, the domain Ω^ε is given by

$$\Omega^\varepsilon = \{x' \in \mathbb{R}^2 : 0 < x < l, 0 < y < \varepsilon h(x)\},$$

where $h(\cdot)$ is a function of class C^1 defined on $[0, l]$ such that

$$0 < \underline{h} = h_{min} \leq h(x) \leq h_{max} = \bar{h}, \forall x \in [0, l].$$

Let $u^\varepsilon(x', t)$ be the displacement field, then the law of elastic behavior is given by

$$\sigma_{ij}^\varepsilon(u^\varepsilon) = 2\mu d_{ij}(u^\varepsilon) + \lambda \sum_{k=1}^2 d_{kk}(u^\varepsilon) \delta_{ij},$$

$$1 \leq i, j \leq 2; d_{ij}(u^\varepsilon) = \frac{1}{2} \left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i} \right),$$

where δ_{ij} is the Kröner symbol, λ, μ are the Lamé constants and $d_{ij}(\cdot)$ the strain tensor.

The equation which governs the deformations of an isotropic elastic homogeneous body with a nonlinear dissipative term in dynamic regime is the following

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \operatorname{div}(\sigma^\varepsilon(u^\varepsilon)) + \left(\alpha^\varepsilon + \left| \frac{\partial u^\varepsilon}{\partial t} \right| \right) \frac{\partial u^\varepsilon}{\partial t} = f^\varepsilon, \text{ in } \Omega^\varepsilon \times]0, T[, \quad (1)$$

where $|\cdot|$ denotes the Euclidean norm of \mathbb{R}^2 , f^ε represents a force density and $\alpha^\varepsilon \in \mathbb{R}_+$.

To describe the boundary conditions we use the usual notation

$$u_n^\varepsilon = u^\varepsilon \cdot n, \quad u_\tau^\varepsilon = u^\varepsilon - u_n^\varepsilon \cdot n, \quad \sigma_n^\varepsilon = (\sigma^\varepsilon \cdot n) \cdot n, \quad \sigma_\tau^\varepsilon = \sigma^\varepsilon \cdot n - (\sigma_n^\varepsilon) \cdot n,$$

where $n = (n_1, n_2)$ is the unit outward normal to Γ^ε .

- The displacement is known on $\Gamma_1^\varepsilon \times]0, T[$ and on $\Gamma_L^\varepsilon \times]0, T[$

$$u^\varepsilon(x, h(x), t) = 0 \quad \text{on } \Gamma_1^\varepsilon \times]0, T[, \quad (2)$$

$$u^\varepsilon(0, y, t) = u^\varepsilon(l, y, t) = 0 \quad \text{on } \Gamma_L^\varepsilon \times]0, T[.$$

- On ω the velocity is assumed unknown and satisfies the following condition

$$\frac{\partial u^\varepsilon}{\partial t} \cdot n = 0 \quad \text{on }]0, l[\times]0, T[. \quad (3)$$

- There exists friction on ω , this friction is modeled by the nonlinear Tresca's law (see [10])

$$\left. \begin{array}{l} |\sigma_\tau^\varepsilon| < k^\varepsilon \Rightarrow \left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau = 0, \\ |\sigma_\tau^\varepsilon| = k^\varepsilon \Rightarrow \exists \beta > 0 \text{ such that } \left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau = -\beta \sigma_\tau^\varepsilon \end{array} \right\} \text{ on }]0, l[\times]0, T[, \quad (4)$$

where $k^\varepsilon \in C_0^\infty(]0, l[)$, $k^\varepsilon > 0$ does not depend of t .

The problem consists in finding u^ε satisfying (1)-(4) and the following initial conditions

$$u^\varepsilon(x', 0) = \vartheta_0(x'), \quad \frac{\partial u^\varepsilon}{\partial t}(x', 0) = \vartheta_1(x'), \quad \forall x' \in \Omega^\varepsilon. \quad (5)$$

2. Weak formulation

Let $L^p(\Omega)$ be the space of real scalar or real vector functions on Ω whose p^{th} power is absolutely integrable with respect to Lebesgue measure dx' . This is a Banach space with the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_{\Omega} |u|^p dx' \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

The Sobolev space $H^1(\Omega)$ is the space of functions in $L^2(\Omega)$ with first order distributional derivatives also in $L^2(\Omega)$. The norm of this space is

$$\|u\|_{H^1(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

To find the weak formulation, we recall that Tresca's boundary condition (4) is equivalent to

$$\left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau \cdot \sigma_\tau^\varepsilon + k^\varepsilon \left| \left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau \right| = 0. \quad (6)$$

Multiplying (1) by $(\varphi - \frac{\partial u^\varepsilon}{\partial t})$ where φ is test-function, then integrating over Ω^ε and using the Green's formula, we obtain

$$\begin{aligned} & \int_{\Omega^\varepsilon} \frac{\partial^2 u^\varepsilon}{\partial t^2} \left(\varphi - \frac{\partial u^\varepsilon}{\partial t} \right) dx' + \int_{\Omega^\varepsilon} \sigma^\varepsilon \cdot \nabla \left(\varphi - \frac{\partial u^\varepsilon}{\partial t} \right) dx' \\ & - \int_{\Gamma^\varepsilon} \sigma^\varepsilon \cdot n \left(\varphi - \frac{\partial u^\varepsilon}{\partial t} \right) dx' + \int_{\Omega^\varepsilon} \left(\alpha^\varepsilon + \left| \frac{\partial u^\varepsilon}{\partial t} \right| \right) \frac{\partial u^\varepsilon}{\partial t} \left(\varphi - \frac{\partial u^\varepsilon}{\partial t} \right) dx' \\ & = \int_{\Omega^\varepsilon} f^\varepsilon \left(\varphi - \frac{\partial u^\varepsilon}{\partial t} \right) dx', \quad (7) \end{aligned}$$

on the other hand, the boundary condition (2)-(3) implies that

$$\int_{\Gamma^\varepsilon} \sigma^\varepsilon \cdot n \left(\varphi - \frac{\partial u^\varepsilon}{\partial t} \right) dx' = \int_0^l \sigma_\tau^\varepsilon \left(\varphi_\tau - \left(\frac{\partial u^\varepsilon}{\partial t} \right)_\tau \right) dx,$$

going back to (7), we get

$$\begin{aligned} & \int_{\Omega^\varepsilon} \frac{\partial^2 u^\varepsilon}{\partial t^2} \left(\varphi - \frac{\partial u^\varepsilon}{\partial t} \right) dx' + \int_{\Omega^\varepsilon} \sigma^\varepsilon \cdot \nabla \left(\varphi - \frac{\partial u^\varepsilon}{\partial t} \right) dx' \\ & + \int_{\Omega^\varepsilon} \left(\alpha^\varepsilon + \left| \frac{\partial u^\varepsilon}{\partial t} \right| \right) \frac{\partial u^\varepsilon}{\partial t} \left(\varphi - \frac{\partial u^\varepsilon}{\partial t} \right) dx' + \int_0^l k^\varepsilon \left(|\varphi_\tau| - \left| \left(\frac{\partial u^\varepsilon}{\partial t} \right)_\tau \right| \right) dx \\ & = \int_{\Omega^\varepsilon} f^\varepsilon \left(\varphi - \frac{\partial u^\varepsilon}{\partial t} \right) dx' + \int_0^l \sigma_\tau^\varepsilon \left(\varphi_\tau - \left(\frac{\partial u^\varepsilon}{\partial t} \right)_\tau \right) dx \\ & \quad + \int_0^l k^\varepsilon \left(|\varphi_\tau| - \left| \left(\frac{\partial u^\varepsilon}{\partial t} \right)_\tau \right| \right) dx. \end{aligned}$$

Using (6) and the fact that

$$\int_0^l \sigma_\tau^\varepsilon \left(\varphi_\tau - \left(\frac{\partial u^\varepsilon}{\partial t} \right)_\tau \right) dx + \int_0^l k^\varepsilon \left(|\varphi_\tau| - \left| \left(\frac{\partial u^\varepsilon}{\partial t} \right)_\tau \right| \right) dx \geq 0,$$

we get the following variational formulation

$$\left\{ \begin{array}{l} \text{Find } u^\varepsilon, \text{ with } u^\varepsilon(\cdot, t), \frac{\partial u^\varepsilon}{\partial t}(\cdot, t) \in K^\varepsilon \text{ for all } t \in [0, T], \text{ such that} \\ \left(\frac{\partial^2 u^\varepsilon}{\partial t^2}, \varphi - \frac{\partial u^\varepsilon}{\partial t} \right) + a \left(u^\varepsilon, \varphi - \frac{\partial u^\varepsilon}{\partial t} \right) + \alpha^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial t}, \varphi - \frac{\partial u^\varepsilon}{\partial t} \right) \\ \quad + \left(\left| \frac{\partial u^\varepsilon}{\partial t} \right| \frac{\partial u^\varepsilon}{\partial t}, \varphi - \frac{\partial u^\varepsilon}{\partial t} \right) + j^\varepsilon(\varphi) - j^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial t} \right) \\ \quad \geq \left(f^\varepsilon, \varphi - \frac{\partial u^\varepsilon}{\partial t} \right), \quad \forall \varphi \in K^\varepsilon, \\ u^\varepsilon(x', 0) = \vartheta_0(x'), \quad \frac{\partial u^\varepsilon}{\partial t}(x', 0) = \vartheta_1(x'), \end{array} \right. \quad (8)$$

where

$$\begin{aligned} K^\varepsilon &= \{v \in H^1(\Omega^\varepsilon)^2 : v = 0 \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon, v \cdot n = 0 \text{ on } \omega\}, \\ j^\varepsilon(v) &= \int_0^l k^\varepsilon |v| dx, \forall v \in H^1(\Omega^\varepsilon)^2, \\ a(u, v) &= 2\mu \int_{\Omega^\varepsilon} d(u) d(v) dx dy + \lambda \int_{\Omega^\varepsilon} \operatorname{div}(u) \operatorname{div}(v) dx dy, \end{aligned}$$

with

$$d(u) d(v) = \sum_{i,j=1}^2 d_{ij}(u) \cdot d_{ij}(v).$$

THEOREM 2.1. *Under the assumptions*

$$\begin{aligned} f^\varepsilon, \frac{\partial f^\varepsilon}{\partial t}, \frac{\partial^2 f^\varepsilon}{\partial t^2} &\in L^2(0, T, L^2(\Omega^\varepsilon)^2), \\ \vartheta_0 &\in H^1(\Omega^\varepsilon)^2, \quad \vartheta_1 \in H^1(\Omega^\varepsilon)^2, \quad (\vartheta_1)_\tau = 0, \end{aligned} \quad (9)$$

there exists a unique solution u^ε of (8) such that

$$\begin{aligned} u^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} &\in L^\infty(0, T, H^1(\Omega^\varepsilon)^2), \\ \frac{\partial^2 u^\varepsilon}{\partial t^2} &\in L^\infty(0, T, L^2(\Omega^\varepsilon)^2). \end{aligned}$$

The proof of this theorem proceeds in a similar fashion as in Lions [10, 15].

3. Change of the domain and some estimates

In this section, we use the technique of scaling $z = y/\varepsilon$ for studying the asymptotic analysis of the problem (8). This method consists in transposing the initial problem posed in the domain Ω^ε to an equivalent problem posed in a fixed domain Ω independent of ε :

$$\Omega = \{(x, z) \in \mathbb{R}^2 : 0 < x < l, 0 < z < h(x)\},$$

and $\Gamma = \Gamma_1 \cup \Gamma_L \cup \omega$ its boundary. We define on Ω the new unknowns and the data

$$\begin{cases} \hat{u}_1^\varepsilon(x, z, t) = u_1^\varepsilon(x, y, t), \\ \hat{u}_2^\varepsilon(x, z, t) = \varepsilon^{-1} u_2^\varepsilon(x, y, t), \\ \hat{f}_i(x, z, t) = \varepsilon^2 f_i^\varepsilon(x, y, t), \quad i = 1, 2, \\ \hat{k} = \varepsilon k^\varepsilon, \quad \hat{\alpha} = \varepsilon^2 \alpha^\varepsilon, \end{cases}$$

where \hat{f}_i , $i = 1, 2$, \hat{k} and $\hat{\alpha}$ do not depend on ε .

Moreover, we define some function spaces on Ω

$$K = \left\{ \varphi \in H^1(\Omega)^2 : \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_L \text{ and } \varphi \cdot n = 0 \text{ on } \omega \right\},$$

$$\Pi(K) = \left\{ \varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_L \right\},$$

$$V_z = \left\{ v \in L^2(\Omega) : \frac{\partial v}{\partial z} \in L^2(\Omega), v = 0 \text{ on } \Gamma_1 \right\},$$

V_z is a Banach space for the norm

$$\|v\|_{V_z} = \left(\|v\|_{L^2(\Omega)}^2 + \left\| \frac{\partial v}{\partial z} \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

Multiplying (8) by ε then we inject the new variables and the new data, we obtain the following variational formulation on the fixed domain Ω .

$$\left\{ \begin{array}{l} \text{Find } \hat{u}^\varepsilon, \text{ with } \hat{u}^\varepsilon(\cdot, t), \frac{\partial \hat{u}^\varepsilon}{\partial t}(\cdot, t) \in K \text{ for all } t \in [0, T], \text{ such that} \\ \varepsilon^2 \left(\frac{\partial^2 \hat{u}_1^\varepsilon}{\partial t^2}, \hat{\varphi}_1 - \frac{\partial \hat{u}_1^\varepsilon}{\partial t} \right) + \varepsilon^4 \left(\frac{\partial^2 \hat{u}_2^\varepsilon}{\partial t^2}, \hat{\varphi}_2 - \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \right) + \hat{a} \left(\hat{u}^\varepsilon, \hat{\varphi} - \frac{\partial \hat{u}^\varepsilon}{\partial t} \right) \\ + \hat{\alpha} \left(\frac{\partial \hat{u}_1^\varepsilon}{\partial t}, \hat{\varphi}_1 - \frac{\partial \hat{u}_1^\varepsilon}{\partial t} \right) + \hat{\alpha} \varepsilon^2 \left(\frac{\partial \hat{u}_2^\varepsilon}{\partial t}, \hat{\varphi}_2 - \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \right) \\ + \varepsilon^2 \left(\left[\left(\frac{\partial \hat{u}_1^\varepsilon}{\partial t} \right)^2 + \left(\varepsilon \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \right)^2 \right]^{\frac{1}{2}} \frac{\partial \hat{u}_1^\varepsilon}{\partial t}, \hat{\varphi}_1 - \frac{\partial \hat{u}_1^\varepsilon}{\partial t} \right) \\ + \varepsilon^4 \left(\left[\left(\frac{\partial \hat{u}_1^\varepsilon}{\partial t} \right)^2 + \left(\varepsilon \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \right)^2 \right]^{\frac{1}{2}} \frac{\partial \hat{u}_2^\varepsilon}{\partial t}, \hat{\varphi}_2 - \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \right) \\ + \hat{J}(\hat{\varphi}) - \hat{J} \left(\frac{\partial \hat{u}^\varepsilon}{\partial t} \right) \\ \geq \left(\hat{f}_1, \hat{\varphi}_1 - \frac{\partial \hat{u}_1^\varepsilon}{\partial t} \right) + \varepsilon \left(\hat{f}_2, \hat{\varphi}_2 - \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \right), \forall \hat{\varphi} \in K, \\ \hat{u}^\varepsilon(0) = \hat{\vartheta}_0, \quad \frac{\partial \hat{u}^\varepsilon}{\partial t}(0) = \hat{\vartheta}_1, \end{array} \right. \quad (10)$$

where

$$\hat{J}(\hat{\varphi}) = \int_0^l \hat{k} |\hat{\varphi}| dx,$$

and

$$\begin{aligned}
\hat{a} \left(\hat{u}^\varepsilon, \hat{\varphi} - \frac{\partial \hat{u}^\varepsilon}{\partial t} \right) &= 2\mu\varepsilon^2 \int_{\Omega} \left(\frac{\partial \hat{u}_1^\varepsilon}{\partial x} \right) \frac{\partial}{\partial x} \left(\hat{\varphi}_1 - \frac{\partial \hat{u}_1^\varepsilon}{\partial t} \right) dx dz \\
&+ \mu \int_{\Omega} \left(\frac{\partial \hat{u}_1^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_2^\varepsilon}{\partial x} \right) \left[\frac{\partial}{\partial z} \left(\hat{\varphi}_1 - \frac{\partial \hat{u}_1^\varepsilon}{\partial t} \right) + \varepsilon^2 \frac{\partial}{\partial x} \left(\hat{\varphi}_2 - \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \right) \right] dx dz \\
&+ 2\mu\varepsilon^2 \int_{\Omega} \frac{\partial \hat{u}_2^\varepsilon}{\partial z} \cdot \frac{\partial}{\partial z} \left(\hat{\varphi}_2 - \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \right) dx dz \\
&+ \lambda\varepsilon^2 \int_{\Omega} \operatorname{div}(\hat{u}^\varepsilon) \cdot \operatorname{div} \left(\hat{\varphi} - \frac{\partial \hat{u}^\varepsilon}{\partial t} \right) dx dz.
\end{aligned}$$

For the rest of this paper, we will denote by c possibly different positive constants and we establish some estimates for the displacement field \hat{u}^ε in the domain Ω .

THEOREM 3.1. *Under the hypotheses of Theorem 2.1, there exists a constant c independent of ε such that*

$$\begin{aligned}
&\left\| \frac{\partial \hat{u}_1^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon \frac{\partial \hat{u}_1^\varepsilon}{\partial t} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon^2 \frac{\partial \hat{u}_2^\varepsilon}{\partial x} \right\|_{L^2(\Omega)}^2 \\
&+ \left\| \varepsilon \frac{\partial \hat{u}_1^\varepsilon}{\partial x} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon \frac{\partial \hat{u}_2^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon^2 \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \right\|_{L^2(\Omega)}^2 \\
&+ \left\| \varepsilon^{\frac{2}{3}} \frac{\partial \hat{u}_1^\varepsilon}{\partial t} \right\|_{L^3(0,T,L^3(\Omega))}^3 + \left\| \varepsilon^{\frac{5}{3}} \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \right\|_{L^3(0,T,L^3(\Omega))}^3 \leq c, \quad (11)
\end{aligned}$$

$$\begin{aligned}
&\left\| \frac{\partial^2 \hat{u}_1^\varepsilon}{\partial z \partial t} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon \frac{\partial^2 \hat{u}_1^\varepsilon}{\partial t^2} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon^2 \frac{\partial^2 \hat{u}_2^\varepsilon}{\partial x \partial t} \right\|_{L^2(\Omega)}^2 \\
&+ \left\| \varepsilon \frac{\partial^2 \hat{u}_1^\varepsilon}{\partial x \partial t} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon \frac{\partial \hat{u}_2^\varepsilon}{\partial z \partial t} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon^2 \frac{\partial^2 \hat{u}_2^\varepsilon}{\partial t^2} \right\|_{L^2(\Omega)}^2 \leq c. \quad (12)
\end{aligned}$$

Proof. First, we recall some inequalities

- **Poincaré's inequality**

$$\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq \varepsilon \bar{h} \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}.$$

- **Young's inequality**

$$ab \leq \eta^2 \frac{a^2}{2} + \eta^{-2} \frac{b^2}{2}, \quad \forall (a, b) \in \mathbb{R}^2, \forall \eta > 0.$$

- **Korn's inequality** [12]

$$\|d(u^\varepsilon)\|_{L^2(\Omega^\varepsilon)}^2 \geq C_K \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2,$$

where \bar{h} and C_K are constants independent of ε .

Let u^ε be a solution of the problem (8), we take $\varphi = 0$, then

$$\begin{aligned} \left(\frac{\partial^2 u^\varepsilon}{\partial t^2}, \frac{\partial u^\varepsilon}{\partial t} \right) + a \left(u^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} \right) + \left(\left(\alpha^\varepsilon + \left| \frac{\partial u^\varepsilon}{\partial t} \right| \right) \frac{\partial u^\varepsilon}{\partial t}, \frac{\partial u^\varepsilon}{\partial t} \right) + j^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial t} \right) \\ \leq \left(f^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} \right), \end{aligned}$$

whence

$$\frac{1}{2} \frac{d}{dt} \left[\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 + a(u^\varepsilon, u^\varepsilon) \right] + \alpha^\varepsilon \left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 + \left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^3(\Omega^\varepsilon)}^3 \leq \left(f^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} \right).$$

For $s \in]0, t[$ by integration we get

$$\begin{aligned} \left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 + a(u^\varepsilon, u^\varepsilon) + 2 \int_0^t \left\| \frac{\partial u^\varepsilon(s)}{\partial t} \right\|_{L^3(\Omega^\varepsilon)}^3 ds \\ \leq \|\vartheta_1\|_{L^2(\Omega^\varepsilon)}^2 + (2\mu + 3\lambda) \|\nabla \vartheta_0\|_{L^2(\Omega^\varepsilon)}^2 + 2 \int_0^t \left(f^\varepsilon(s), \frac{\partial u^\varepsilon(s)}{\partial t} \right) ds. \quad (13) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} 2 \int_0^t \left(f^\varepsilon(s), \frac{\partial u^\varepsilon(s)}{\partial t} \right) ds \\ = 2(f^\varepsilon, u^\varepsilon) - 2(f^\varepsilon(0), \vartheta_0) - 2 \int_0^t \left(\frac{\partial f^\varepsilon(s)}{\partial t}, u^\varepsilon(s) \right) ds, \end{aligned}$$

using Poincaré's and Young's inequalities, we obtain

$$\begin{aligned} \left| 2 \int_0^t \left(f^\varepsilon(s), \frac{\partial u^\varepsilon(s)}{\partial t} \right) ds \right| \\ \leq \mu C_K \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \frac{4\varepsilon^2 \bar{h}^2}{\mu C_K} \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + 4\varepsilon^2 \bar{h}^2 \|f^\varepsilon(0)\|_{L^2(\Omega^\varepsilon)}^2 \\ + \|\nabla \vartheta_0\|_{L^2(\Omega^\varepsilon)}^2 + \mu C_K \int_0^t \|\nabla u^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)}^2 ds \\ + \frac{4(\varepsilon \bar{h})^2}{\mu C_K} \int_0^t \left\| \frac{\partial f^\varepsilon(s)}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 ds. \quad (14) \end{aligned}$$

By inserting (14) in (13), and using Korn's inequality, we find

$$\begin{aligned}
& \left[\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 + \mu C_K \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \right] + 2 \int_0^t \left\| \frac{\partial u^\varepsilon(s)}{\partial t} \right\|_{L^3(\Omega^\varepsilon)}^3 ds \\
& \leq \|\vartheta_1\|_{L^2(\Omega^\varepsilon)}^2 + (1 + 2\mu + 3\lambda) \|\nabla \vartheta_0\|_{L^2(\Omega^\varepsilon)}^2 + \frac{4\varepsilon^2 \bar{h}^2}{\mu C_K} \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \\
& \quad + 4\varepsilon^2 \bar{h}^2 \|f^\varepsilon(0)\|_{L^2(\Omega^\varepsilon)}^2 + \frac{4(\varepsilon \bar{h})^2}{\mu C_K} \int_0^t \left\| \frac{\partial f^\varepsilon(s)}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 ds \\
& \quad + \int_0^t \left[\left\| \frac{\partial u^\varepsilon(s)}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 + \mu C_K \|\nabla u^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)}^2 \right] ds. \quad (15)
\end{aligned}$$

As

$$\varepsilon^2 \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 = \varepsilon^{-1} \|\hat{f}\|_{L^2(\Omega)}^2,$$

multiplying (15) by ε we deduce that

$$\begin{aligned}
& \varepsilon \left[\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 + \mu C_K \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \right] + 2\varepsilon \int_0^t \left\| \frac{\partial u^\varepsilon(s)}{\partial t} \right\|_{L^3(\Omega^\varepsilon)}^3 ds \\
& \leq \int_0^t \varepsilon \left[\left\| \frac{\partial u^\varepsilon(s)}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 + \mu C_K \|\nabla u^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)}^2 \right] ds + A,
\end{aligned}$$

where A is a constant does not depend of ε with

$$\begin{aligned}
A = & \left\| \hat{\vartheta}_1 \right\|_{L^2(\Omega)}^2 + (1 + 2\mu + 3\lambda) \left\| \nabla \hat{\vartheta}_0 \right\|_{L^2(\Omega)}^2 + 4\bar{h}^2 \left\| \hat{f}(0) \right\|_{L^2(\Omega)}^2 \\
& + \frac{4\bar{h}^2}{\mu C_K} \left\| \hat{f} \right\|_{L^\infty(0,T,L^2(\Omega)^2)}^2 + \frac{4\bar{h}^2}{\mu C_K} \left\| \frac{\partial \hat{f}}{\partial t} \right\|_{L^2(0,T,L^2(\Omega)^2)}^2.
\end{aligned}$$

Now using Gronwall's lemma, we have

$$\varepsilon \left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 + \varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \leq c,$$

from which (11) follows.

The functional $j^\varepsilon(\cdot)$ is convex but nondifferentiable. To overcome this difficulty, we shall use the following approach. Let $j_\zeta^\varepsilon(\cdot)$ be a functional defined by

$$j_\zeta^\varepsilon(v) = \int_0^t k^\varepsilon(x) \phi_\zeta(|v_\tau|^2) dx, \text{ where } \phi_\zeta(\lambda) = \frac{1}{1+\zeta} |\lambda|^{(1+\zeta)}, \zeta > 0.$$

To show (12) we consider the approximate equation as in [15]

$$\begin{aligned} \left(\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}, \varphi \right) + a(u_\zeta^\varepsilon, \varphi) + \left(\left(\alpha^\varepsilon + \left| \frac{\partial u_\zeta^\varepsilon}{\partial t} \right| \right) \frac{\partial u_\zeta^\varepsilon}{\partial t}, \varphi \right) \\ + \left((j_\zeta^\varepsilon)' \left(\frac{\partial u_\zeta^\varepsilon}{\partial t} \right), \varphi \right) = (f^\varepsilon, \varphi), \forall \varphi \in K^\varepsilon, \quad (16) \end{aligned}$$

$$u_\zeta^\varepsilon(x', 0) = \vartheta_0(x'), \quad \frac{\partial u_\zeta^\varepsilon(x', 0)}{\partial t} = \vartheta_1(x').$$

We differentiate (16), in t and we take $\varphi = \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}$, we get

$$\begin{aligned} \left(\frac{\partial^3 u_\zeta^\varepsilon}{\partial t^3}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2} \right) + a \left(\frac{\partial u_\zeta^\varepsilon}{\partial t}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2} \right) + \alpha^\varepsilon \left(\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2} \right) \\ + \left(\left| \frac{\partial u_\zeta^\varepsilon}{\partial t} \right| \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2} \right) + \left(\frac{\partial}{\partial t} (j_\zeta^\varepsilon)' \left(\frac{\partial u_\zeta^\varepsilon}{\partial t} \right), \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2} \right) = \left(\frac{\partial f^\varepsilon}{\partial t}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2} \right), \end{aligned}$$

as $\left(\frac{\partial}{\partial t} (j_\zeta^\varepsilon)' \left(\frac{\partial u_\zeta^\varepsilon}{\partial t} \right), \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2} \right) \geq 0$, we have

$$\frac{1}{2} \frac{d}{dt} \left[\left\| \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2} \right\|_{L^2(\Omega^\varepsilon)}^2 + a \left(\frac{\partial u_\zeta^\varepsilon}{\partial t}, \frac{\partial u_\zeta^\varepsilon}{\partial t} \right) \right] \leq \left(\frac{\partial f^\varepsilon}{\partial t}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2} \right).$$

Integrating this inequality over $(0, t)$ and use Korn's inequality, we get

$$\begin{aligned} \left\| \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2} \right\|_{L^2(\Omega^\varepsilon)}^2 + 2\mu C_K \left\| \nabla \frac{\partial u_\zeta^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 \\ \leq \left\| \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2} (0) \right\|_{L^2(\Omega^\varepsilon)}^2 + (2\mu + 3\lambda) \|\nabla \vartheta_1\|_{L^2(\Omega^\varepsilon)}^2 + 2 \left(\frac{\partial f^\varepsilon}{\partial t}, \frac{\partial u_\zeta^\varepsilon}{\partial t} \right) \\ - 2 \left(\frac{\partial f^\varepsilon}{\partial t} (0), \vartheta_1 \right) - 2 \int_0^t \left(\frac{\partial^2 f^\varepsilon(s)}{\partial t^2}, \frac{\partial u_\zeta^\varepsilon(s)}{\partial t} \right) ds. \end{aligned}$$

On the other hand, using Cauchy-Schwarz's, Poincaré's and Young's inequali-

ties, we obtain

$$\begin{aligned}
2 \left(\frac{\partial f^\varepsilon}{\partial t}, \frac{\partial u_\zeta^\varepsilon}{\partial t} \right) &\leq 2 \left\| \frac{\partial f^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)} \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)} \\
&\leq 2\varepsilon\bar{h} \left\| \frac{\partial f^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)} \left\| \nabla \frac{\partial u_\zeta^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)} \\
&\leq \frac{4(\varepsilon\bar{h})^2}{\mu C_K} \left\| \frac{\partial f^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 + \mu C_K \left\| \nabla \frac{\partial u_\zeta^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2.
\end{aligned}$$

We use the same techniques for the other terms, so we will have the following inequality

$$\begin{aligned}
&\left\| \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2} \right\|_{L^2(\Omega^\varepsilon)}^2 + 2\mu C_K \left\| \nabla \frac{\partial u_\zeta^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 \\
&\leq \left\| \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0) \right\|_{L^2(\Omega^\varepsilon)}^2 + (2\mu + 3\lambda) \|\nabla \vartheta_1\|_{L^2(\Omega^\varepsilon)}^2 \\
&\quad + \mu C_K \left\| \nabla \frac{\partial u_\zeta^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 + \frac{4(\varepsilon\bar{h})^2}{\mu C_K} \left\| \frac{\partial f^\varepsilon}{\partial t}(0) \right\|_{L^2(\Omega^\varepsilon)}^2 \\
&\quad + \frac{4(\varepsilon\bar{h})^2}{\mu C_K} \int_0^t \left\| \frac{\partial^2 f^\varepsilon(s)}{\partial t^2} \right\|_{L^2(\Omega^\varepsilon)}^2 ds + \mu C_K \|\nabla \vartheta_1\|_{L^2(\Omega^\varepsilon)}^2 \\
&\quad + \frac{4(\varepsilon\bar{h})^2}{\mu C_K} \left\| \frac{\partial f^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 + \mu C_K \int_0^t \left\| \nabla \frac{\partial u_\zeta^\varepsilon(s)}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 ds. \quad (17)
\end{aligned}$$

Now let us estimate $\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0)$. From (16) and (9) we deduce

$$\left(\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0), \varphi \right) = (f^\varepsilon(0), \varphi) - a(\vartheta_0, \varphi) - \alpha^\varepsilon(\vartheta_1, \varphi) - (|\vartheta_1| \vartheta_1, \varphi), \forall \varphi \in K^\varepsilon.$$

Therefore

$$\begin{aligned}
& \left| \left(\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0), \varphi \right) \right| \\
& \leq \varepsilon \bar{h} \|f^\varepsilon(0)\|_{L^2(\Omega^\varepsilon)} \|\nabla \varphi\|_{L^2(\Omega^\varepsilon)} + (2\mu + 3\lambda) \|\vartheta_0\|_{H^1(\Omega^\varepsilon)} \|\varphi\|_{H^1(\Omega^\varepsilon)} \\
& \quad + \alpha^\varepsilon \|\vartheta_1\|_{L^2(\Omega^\varepsilon)} \|\varphi\|_{L^2(\Omega^\varepsilon)} + \left(\int_{\Omega^\varepsilon} |\vartheta_1|^4 dx dy \right)^{\frac{1}{2}} \|\varphi\|_{L^2(\Omega^\varepsilon)} \\
& \leq \left(\varepsilon \bar{h} \|f^\varepsilon(0)\|_{L^2(\Omega^\varepsilon)} + (2\mu + 3\lambda) \|\vartheta_0\|_{H^1(\Omega^\varepsilon)} \right) \|\varphi\|_{H^1(\Omega^\varepsilon)} \\
& \quad + \left(\hat{\alpha} \bar{h}^2 \|\nabla \vartheta_1\|_{L^2(\Omega^\varepsilon)} + \varepsilon \bar{h} \left(\int_{\Omega^\varepsilon} |\vartheta_1|^4 dx dy \right)^{\frac{1}{2}} \right) \|\varphi\|_{H^1(\Omega^\varepsilon)}.
\end{aligned}$$

As $\varepsilon^{\frac{3}{2}} \|f^\varepsilon(0)\|_{L^2(\Omega^\varepsilon)} = \|\hat{f}(0)\|_{L^2(\Omega)}$, we multiply this last inequality by $\sqrt{\varepsilon}$. Then using Sobolev embedding $\|v\|_{L^4(\Omega)} \leq c_s \|v\|_{H^1(\Omega)}$, we get

$$\sqrt{\varepsilon} \left\| \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0) \right\|_{L^2(\Omega^\varepsilon)} \leq C',$$

where

$$C' = \bar{h} \|\hat{f}(0)\|_{L^2(\Omega)} + (2\mu + 3\lambda) \|\hat{\vartheta}_0\|_{H^1(\Omega)} + \hat{\alpha} \bar{h}^2 \|\hat{\vartheta}_1\|_{H^1(\Omega)} + \bar{h} c_s \|\hat{\vartheta}_1\|_{H^1(\Omega)}^2$$

is independent of ε . Passing to the limit in (17) when ζ tends to zero, we find

$$\begin{aligned}
& \left[\left\| \frac{\partial^2 u^\varepsilon}{\partial t^2} \right\|_{L^2(\Omega^\varepsilon)}^2 + \mu C_K \left\| \nabla \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 \right] \leq \left\| \frac{\partial^2 u^\varepsilon}{\partial t^2}(0) \right\|_{L^2(\Omega^\varepsilon)}^2 \\
& \quad + (2\mu + 3\lambda + \mu C_K) \|\nabla \vartheta_1\|_{L^2(\Omega^\varepsilon)}^2 + \frac{4(\varepsilon \bar{h})^2}{\mu C_K} \left\| \frac{\partial f^\varepsilon}{\partial t}(0) \right\|_{L^2(\Omega^\varepsilon)}^2 \\
& \quad + \frac{4(\varepsilon \bar{h})^2}{\mu C_K} \int_0^t \left\| \frac{\partial^2 f^\varepsilon(s)}{\partial t^2} \right\|_{L^2(\Omega^\varepsilon)}^2 ds + \frac{4(\varepsilon \bar{h})^2}{\mu C_K} \left\| \frac{\partial f^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 \\
& \quad + \int_0^t \left[\left\| \frac{\partial^2 u^\varepsilon(s)}{\partial t^2} \right\|_{L^2(\Omega^\varepsilon)}^2 + \mu C_K \left\| \nabla \frac{\partial u^\varepsilon(s)}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 \right] ds. \quad (18)
\end{aligned}$$

Multiplying now (18) by ε , we obtain

$$\begin{aligned} & \varepsilon \left[\left\| \frac{\partial^2 u^\varepsilon}{\partial t^2} \right\|_{L^2(\Omega^\varepsilon)}^2 + \mu C_K \left\| \nabla \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 \right] \\ & \leq \int_0^t \varepsilon \left[\left\| \frac{\partial^2 u^\varepsilon(s)}{\partial t^2} \right\|_{L^2(\Omega^\varepsilon)}^2 + \mu C_K \left\| \nabla \frac{\partial u^\varepsilon(s)}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 \right] ds + B, \end{aligned}$$

where B is a constant does not depend of ε

$$\begin{aligned} B = & (2\mu + 3\lambda + \mu C_K) \left\| \nabla \hat{\vartheta}_1 \right\|_{L^2(\Omega)}^2 + (C')^2 + \frac{4\bar{h}^2}{\mu C_K} \left\| \frac{\partial \hat{f}}{\partial t}(0) \right\|_{L^2(\Omega)}^2 \\ & + \frac{4\bar{h}^2}{\mu C_K} \left\| \frac{\partial \hat{f}}{\partial t} \right\|_{L^\infty(0,T,L^2(\Omega)^2)}^2 + \frac{4\bar{h}^2}{\mu C_K} \left\| \frac{\partial^2 \hat{f}}{\partial t^2} \right\|_{L^2(0,T,L^2(\Omega)^2)}^2. \end{aligned}$$

By the Gronwall's lemma, there exists a constant c that does not depend of ε such that

$$\varepsilon \left\| \frac{\partial^2 u^\varepsilon}{\partial t^2} \right\|_{L^2(\Omega^\varepsilon)}^2 + \varepsilon \left\| \nabla \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)}^2 \leq c,$$

we conclude (12). \square

4. Convergence theorem and limiting problem

THEOREM 4.1. *Under the hypotheses of Theorem 3.1, there exists $u_1^* \in L^2(0, T, V_z) \cap L^\infty(0, T, V_z)$, such that*

$$\left. \begin{array}{l} \hat{u}_1^\varepsilon \rightharpoonup u_1^* \\ \frac{\partial \hat{u}_1^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u_1^*}{\partial t} \end{array} \right\} \begin{array}{l} \text{weakly in } L^2(0, T, V_z) \\ \text{and weakly } * \text{ in } L^\infty(0, T, V_z), \end{array} \quad (19)$$

$$\left. \begin{array}{l} \varepsilon \frac{\partial \hat{u}_1^\varepsilon}{\partial x} \rightharpoonup 0 \\ \varepsilon \frac{\partial^2 \hat{u}_1^\varepsilon}{\partial x \partial t} \rightharpoonup 0 \end{array} \right\} \begin{array}{l} \text{weakly in } L^2(0, T, L^2(\Omega)) \\ \text{and weakly } * \text{ in } L^\infty(0, T, L^2(\Omega)), \end{array} \quad (20)$$

$$\left. \begin{array}{l} \varepsilon \frac{\partial \hat{u}_1^\varepsilon}{\partial t} \rightharpoonup 0 \\ \varepsilon \frac{\partial^2 \hat{u}_1^\varepsilon}{\partial t^2} \rightharpoonup 0 \end{array} \right\} \begin{array}{l} \text{weakly in } L^2(0, T, L^2(\Omega)) \\ \text{and weakly } * \text{ in } L^\infty(0, T, L^2(\Omega)), \end{array} \quad (21)$$

$$\left. \begin{array}{l} \varepsilon^2 \frac{\partial \hat{u}_2^\varepsilon}{\partial x} \rightharpoonup 0 \\ \varepsilon^2 \frac{\partial^2 \hat{u}_2^\varepsilon}{\partial x \partial t} \rightharpoonup 0 \end{array} \right\} \begin{array}{l} \text{weakly in } L^2(0, T, L^2(\Omega)) \\ \text{and weakly } * \text{ in } L^\infty(0, T, L^2(\Omega)), \end{array} \quad (22)$$

$$\left. \begin{array}{l} \varepsilon^2 \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \rightharpoonup 0 \\ \varepsilon^2 \frac{\partial^2 \hat{u}_2^\varepsilon}{\partial z \partial t} \rightharpoonup 0 \\ \varepsilon^2 \frac{\partial^2 \hat{u}_2^\varepsilon}{\partial t^2} \rightharpoonup 0 \end{array} \right\} \begin{array}{l} \text{weakly in } L^2(0, T, L^2(\Omega)) \\ \text{and weakly } * \text{ in } L^\infty(0, T, L^2(\Omega)), \end{array} \quad (23)$$

$$\left. \begin{array}{l} \varepsilon^{\frac{2}{3}} \frac{\partial \hat{u}_1^\varepsilon}{\partial t} \rightharpoonup 0 \\ \varepsilon^{\frac{5}{3}} \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \rightharpoonup 0 \end{array} \right\} \text{weakly in } L^3(0, T, L^3(\Omega)). \quad (24)$$

Proof. According to Theorem 3.1, there exists a constant c independent of ε such that

$$\left\| \frac{\partial \hat{u}_1^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \leq c.$$

Using this estimate with the Poincaré inequality in the domain Ω , we get

$$\|\hat{u}_1^\varepsilon\|_{L^2(\Omega)}^2 \leq \bar{h} \left\| \frac{\partial \hat{u}_1^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \leq c.$$

So $(\hat{u}_1^\varepsilon)_\varepsilon$ is bounded in $L^2(0, T, V_z) \cap L^\infty(0, T, V_z)$, which implies the existence of an element u_1^* in $L^2(0, T, V_z) \cap L^\infty(0, T, V_z)$ such that $(\hat{u}_1^\varepsilon)_\varepsilon$ converges weakly to u_1^* in $L^2(0, T, V_z) \cap L^\infty(0, T, V_z)$, the same for $\left(\frac{\partial \hat{u}_1^\varepsilon}{\partial t}\right)_\varepsilon$, thus we obtain (19). For (20)-(24), according to (11), (12) and (19). \square

THEOREM 4.2. *Under the hypotheses of Theorem 4.1, the limit u_1^* satisfies the following variational inequality*

$$\begin{aligned} & \mu \int_{\Omega} \frac{\partial u_1^*}{\partial z} \cdot \frac{\partial}{\partial z} \left(\hat{\varphi}_1 - \frac{\partial u_1^*}{\partial t} \right) dx dz + \hat{\alpha} \int_{\Omega} \frac{\partial u_1^*}{\partial t} \cdot \left(\hat{\varphi}_1 - \frac{\partial u_1^*}{\partial t} \right) dx dz \\ & + \hat{J}(\hat{\varphi}) - \hat{J} \left(\frac{\partial u_1^*}{\partial t} \right) \geq \left(\hat{f}_1, \hat{\varphi}_1 - \frac{\partial u_1^*}{\partial t} \right), \quad \forall \hat{\varphi} \in \Pi(K), \end{aligned} \quad (25)$$

and the parabolic problem

$$\begin{cases} -\mu \frac{\partial^2 u_1^*}{\partial z^2}(t) + \hat{\alpha} \frac{\partial u_1^*}{\partial t}(t) = \hat{f}_1(t), & \text{in } L^2(\Omega), \\ u_1^*(x, z, 0) = \hat{\vartheta}_{0,1}. \end{cases} \quad (26)$$

Proof. As $\hat{J}(\cdot)$ is convex and lower semicontinuous i.e

$$\lim_{\varepsilon \rightarrow 0} \left(\inf \int_0^l \hat{k} \left| \frac{\partial \hat{u}_1^\varepsilon}{\partial t} \right| dx \right) \geq \int_0^l \hat{k} \left| \frac{\partial u_1^*}{\partial t} \right| dx,$$

we pass to the limit when ε tends to zero in (10) and using the convergence results of the Theorem 4.1, we find the following limit inequality

$$\begin{aligned} \mu \int_{\Omega} \frac{\partial u_1^*}{\partial z} \cdot \frac{\partial}{\partial z} \left(\hat{\varphi}_1 - \frac{\partial u_1^*}{\partial t} \right) dx dz + \hat{\alpha} \int_{\Omega} \frac{\partial u_1^*}{\partial t} \cdot \left(\hat{\varphi}_1 - \frac{\partial u_1^*}{\partial t} \right) dx dz \\ + \hat{J}(\hat{\varphi}) - \hat{J} \left(\frac{\partial u_1^*}{\partial t} \right) \geq \int_{\Omega} \hat{f}_1 \left(\hat{\varphi}_1 - \frac{\partial u_1^*}{\partial t} \right) dx dz. \end{aligned} \quad (27)$$

We now choose in the variational inequality (27)

$$\hat{\varphi}_1 = \frac{\partial u_1^*}{\partial t} \pm \psi, \quad \psi \in H_0^1(\Omega),$$

we find

$$\mu \int_{\Omega} \frac{\partial u_1^*}{\partial z} \frac{\partial \psi}{\partial z} dx dz + \hat{\alpha} \int_{\Omega} \frac{\partial u_1^*}{\partial t} \cdot \psi dx dz = \int_{\Omega} \hat{f}_1 \psi dx dz.$$

According to Green's formula, we obtain

$$-\int_{\Omega} \mu \frac{\partial}{\partial z} \left(\frac{\partial u_1^*}{\partial z} \right) \psi dx dz + \int_{\Omega} \hat{\alpha} \frac{\partial u_1^*}{\partial t} \cdot \psi dx dz = \int_{\Omega} \hat{f}_1 \psi dx dz, \quad \forall \psi \in H_0^1(\Omega).$$

Therefore

$$-\mu \frac{\partial^2 u_1^*(t)}{\partial z^2} + \hat{\alpha} \frac{\partial u_1^*(t)}{\partial t} = \hat{f}_1(t), \quad \text{in } H^{-1}(\Omega), \quad \forall t \in]0, T[, \quad (28)$$

and, as $\hat{f}_1 \in L^2(\Omega)$, then (28) is valid in $L^2(\Omega)$. \square

THEOREM 4.3. *Under the same assumptions of Theorem 4.1, the traces*

$$\tau^*(x, t) = \frac{\partial u_1^*}{\partial z}(x, 0, t) \quad \text{and} \quad s^*(x, t) = u_1^*(x, 0, t)$$

satisfy the following inequality

$$\int_0^l \hat{k} \left(\left| \psi + \frac{\partial s^*}{\partial t} \right| - \left| \frac{\partial s^*}{\partial t} \right| \right) dx - \int_0^l \mu \tau^* \psi dx \geq 0, \quad \forall \psi \in L^2(]0, l]), \quad (29)$$

and the following limit form of the Tresca boundary conditions

$$\left. \begin{aligned} \mu |\tau^*| < \hat{k} &\Rightarrow \frac{\partial s^*}{\partial t} = 0, \\ \mu |\tau^*| = \hat{k} &\Rightarrow \exists \beta > 0 \text{ such that } \frac{\partial s^*}{\partial t} = -\beta \tau^*, \end{aligned} \right\} \quad \text{a.e on }]0, l[\times]0, T[. \quad (30)$$

Moreover u_1^ and s^* satisfies the following weak form of the Reynolds equation*

$$\int_0^l \left(\tilde{F} - \frac{h}{2} s^* + \int_0^h u_1^*(x, z, t) dz + \tilde{U}_1 \right) \psi'(x) dx = 0, \quad \forall \psi \in H^1(]0, l]), \quad (31)$$

where

$$\begin{aligned}\tilde{F}(x, h, t) &= \frac{1}{\mu} \int_0^h F(x, z, t) dz - \frac{h}{2\mu} F(x, h, t), \\ \tilde{U}_1(x, h, t) &= -\frac{\hat{\alpha}}{\mu} \int_0^h U_1(x, z, t) dz + \frac{\hat{\alpha}h}{2\mu} U_1(x, h, t), \\ F(x, z, t) &= \int_0^z \int_0^\zeta \hat{f}_1(x, \eta, t) d\eta d\zeta, \\ U_1(x, z, t) &= \int_0^z \int_0^\zeta \frac{\partial u_1^*}{\partial t}(x, \eta, t) d\eta d\zeta.\end{aligned}$$

Proof. For the proof of (29), (30), we follow the same steps as in [1]. To prove (31) by integrating (26) from 0 to z , we see that

$$-\mu \frac{\partial u_1^*}{\partial z}(x, z, t) + \mu \frac{\partial u_1^*}{\partial z}(x, 0, t) + \hat{\alpha} \int_0^z \frac{\partial u_1^*}{\partial t}(x, \eta, t) d\eta = \int_0^z \hat{f}_1(x, \eta, t) d\eta.$$

Integrating again between 0 to z , we obtain

$$\begin{aligned}u_1^*(x, z, t) &= s^* + z\tau^* + \frac{\hat{\alpha}}{\mu} \int_0^z \int_0^\zeta \frac{\partial u_1^*}{\partial t}(x, \eta, t) d\eta d\zeta \\ &\quad - \frac{1}{\mu} \int_0^z \int_0^\zeta \hat{f}_1(x, \eta, t) d\eta d\zeta, \quad (32)\end{aligned}$$

in particular for $z = h(x)$ we get

$$s^* + h\tau^* = -\frac{\hat{\alpha}}{\mu} \int_0^h \int_0^\zeta \frac{\partial u_1^*}{\partial t}(x, \eta, t) d\eta d\zeta + \frac{1}{\mu} \int_0^h \int_0^\zeta \hat{f}_1(x, \eta, t) d\eta d\zeta. \quad (33)$$

Integrating (32) from 0 to h , we obtain

$$\begin{aligned}\int_0^h u_1^*(x, z, t) dz &= hs^* + \frac{1}{2}h^2\tau^* + \frac{\hat{\alpha}}{\mu} \int_0^h \int_0^z \int_0^\zeta \frac{\partial u_1^*}{\partial t}(x, \eta, t) d\eta d\zeta dz \\ &\quad - \frac{1}{\mu} \int_0^h \int_0^z \int_0^\zeta \hat{f}_1(x, \eta, t) d\eta d\zeta dz. \quad (34)\end{aligned}$$

From (33) and (34), we deduce that

$$\int_0^h u_1^*(x, z, t) dz - \frac{h}{2}s^* + \tilde{F} + \tilde{U}_1 = 0,$$

with

$$\begin{aligned}\tilde{F}(x, h, t) &= \frac{1}{\mu} \int_0^h F(x, z, t) dz - \frac{h}{2\mu} F(x, h, t), \\ \tilde{U}_1(x, h, t) &= -\frac{\hat{\alpha}}{\mu} \int_0^h U_1(x, z, t) dz + \frac{\hat{\alpha}h}{2\mu} U_1(x, h, t), \\ F(x, z, t) &= \int_0^z \int_0^\zeta \hat{f}_1(x, \eta, t) d\eta d\zeta, \\ U_1(x, z, t) &= \int_0^z \int_0^\zeta \frac{\partial u_1^*}{\partial t}(x, \eta, t) d\eta d\zeta.\end{aligned}$$

Therefore

$$\int_0^l \left(\int_0^h u_1^*(x, z, t) dz - \frac{h}{2} s^* + \tilde{F} + \tilde{U}_1 \right) \psi'(x) dx = 0.$$

□

THEOREM 4.4. *The solution u_1^* of the limiting problem (25), (26) is unique in $L^2(0, T, V_z) \cap L^\infty(0, T, V_z)$.*

Proof. Suppose that there exist two solutions u_1^* and u_1^{**} of the variational inequality (25), we have

$$\begin{aligned}\mu \int_\Omega \frac{\partial u_1^*}{\partial z} \cdot \frac{\partial}{\partial z} \left(\hat{\varphi} - \frac{\partial u_1^*}{\partial t} \right) dx dz + \hat{\alpha} \int_\Omega \frac{\partial u_1^*}{\partial t} \cdot \left(\hat{\varphi} - \frac{\partial u_1^*}{\partial t} \right) dx dz \\ + \hat{J}(\hat{\varphi}) - \hat{J} \left(\frac{\partial u_1^*}{\partial t} \right) \geq \left(\hat{f}_1, \hat{\varphi} - \frac{\partial u_1^*}{\partial t} \right),\end{aligned}\quad (35)$$

and

$$\begin{aligned}\mu \int_\Omega \frac{\partial u_1^{**}}{\partial z} \cdot \frac{\partial}{\partial z} \left(\hat{\varphi} - \frac{\partial u_1^{**}}{\partial t} \right) dx dz + \hat{\alpha} \int_\Omega \frac{\partial u_1^{**}}{\partial t} \cdot \left(\hat{\varphi} - \frac{\partial u_1^{**}}{\partial t} \right) dx dz \\ + \hat{J}(\hat{\varphi}) - \hat{J} \left(\frac{\partial u_1^{**}}{\partial t} \right) \geq \left(\hat{f}_1, \hat{\varphi} - \frac{\partial u_1^{**}}{\partial t} \right).\end{aligned}\quad (36)$$

We take $\hat{\varphi} = \frac{\partial u_1^{**}}{\partial t}$ in (35), then $\hat{\varphi} = \frac{\partial u_1^*}{\partial t}$ in (36), and by summing the two inequalities, we obtain

$$\begin{aligned}\mu \int_\Omega \frac{\partial}{\partial z} (u_1^* - u_1^{**}) \cdot \frac{\partial}{\partial z} \left(\frac{\partial u_1^*}{\partial t} - \frac{\partial u_1^{**}}{\partial t} \right) dx dz \\ + \hat{\alpha} \int_\Omega \left(\frac{\partial u_1^*}{\partial t} - \frac{\partial u_1^{**}}{\partial t} \right) \cdot \left(\frac{\partial u_1^*}{\partial t} - \frac{\partial u_1^{**}}{\partial t} \right) dx dz \leq 0.\end{aligned}$$

If we put $\bar{W}(t) = u_1^*(t) - u_1^{**}(t)$, this implies

$$\mu \frac{d}{dt} \left\| \frac{\partial \bar{W}}{\partial z} \right\|_{L^2(\Omega)}^2 + \hat{\alpha} \left\| \frac{\partial \bar{W}}{\partial t} \right\|_{L^2(\Omega)}^2 \leq 0.$$

We have $\bar{W}(0) = 0$, then we find

$$\left\| \frac{\partial \bar{W}}{\partial z} \right\|_{L^2(\Omega)}^2 \leq 0.$$

Using Poincaré's inequality, we conclude

$$\|\bar{W}\|_{L^2(0,T,V_z)} = \|\bar{W}\|_{L^\infty(0,T,V_z)} = 0.$$

□

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Authors' addresses:

Mohamed Dilmi
Laboratory of Pure and Applied Mathematics
University of M'sila
Bp 166 M'sila, 28000, Algeria
E-mail: mohamed77dilmi@gmail.com

Mourad Dilmi
Applied Mathematics Laboratory
Department of Mathematics
Setif 1 University, 19000, Algeria
E-mail: mouraddil@yahoo.fr

Hamid Benseridi
Applied Mathematics Laboratory
Department of Mathematics
Setif 1 University, 19000, Algeria
E-mail: m.benseridi@yahoo.fr

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Change of variables' formula for the integration of the measurable real functions over infinite-dimensional Banach spaces

CLAUDIO ASCI

ABSTRACT. *In this paper we study, for any subset I of \mathbf{N}^* and for any strictly positive integer k , the Banach space E_I of the bounded real sequences $\{x_n\}_{n \in I}$, and a measure over $(\mathbf{R}^I, \mathcal{B}^{(I)})$ that generalizes the k -dimensional Lebesgue one. Moreover, we recall the main results about the differentiation theory over E_I . The main result of our paper is a change of variables' formula for the integration of the measurable real functions on $(\mathbf{R}^I, \mathcal{B}^{(I)})$. This change of variables is defined by some functions over an open subset of E_J , with values on E_I , called (m, σ) -general, with properties that generalize the analogous ones of the finite-dimensional diffeomorphisms.*

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1. Introduction

In the mathematical literature, some articles introduced infinite-dimensional measures analogue of the Lebesgue one: see for example the pioneering paper of Jessen [10], that one of Léandre [13], in the context of the noncommutative geometry, that one of Tsilevich et al. [19], which studies a family of σ -finite measures in the space of distributions, that one of Baker [7], which defines a measure on $\mathbf{R}^{\mathbf{N}^*}$ that is not σ -finite, that one of Henstock et al. [9], and that one of Tepper et al. [15]. However, the results obtained do not include an infinite-dimensional change of variables' formula for the integration of the measurable real functions, analogous to that which applies in the finite-dimensional case. For example, in the paper of Accardi et al. [1], the authors describe the transformations of generalized measures on locally convex spaces under smooth

transformations of these spaces, but these measures have no connection with the Lebesgue one. The problem that arises is essentially the following. Consider the integration formula with respect to an image measure, that is

$$\int_E f d(\varphi(\mu)) = \int_S f(\varphi) d\mu,$$

where (S, Σ, μ) and (E, \mathcal{E}) are a measure space and a measurable space, respectively, $\varphi : (S, \Sigma) \rightarrow (E, \mathcal{E})$ and $f : (E, \mathcal{E}) \rightarrow (\mathbf{R}, \mathcal{B})$ are measurable functions. In the particular case in which E and S are open sets, suitably constructed, of two infinite-dimensional measurable spaces Ω_1 and Ω_2 , respectively, on which we can define two families \mathcal{M}_1 and \mathcal{M}_2 of measures analogue of the Lebesgue one, and φ has properties that generalize the analogous ones of the standard finite-dimensional diffeomorphisms, we expect existence of two measure λ_1 in \mathcal{M}_1 and λ_2 in \mathcal{M}_2 such that $\varphi(\mu) = \lambda_1$, while μ has density $|\det J_\varphi|$ (properly defined) with respect to λ_2 .

In order to achieve this result, in the articles [4], [5] and [6], for any subset I of \mathbf{N}^* , we define the Banach space $E_I \subset \mathbf{R}^I$ of the bounded real sequences $\{x_n\}_{n \in I}$, the σ -algebra \mathcal{B}_I given by the restriction to E_I of $\mathcal{B}^{(I)}$ (defined as the product indexed by I of the same Borel σ -algebra \mathcal{B} on \mathbf{R}), and a class of functions over an open subset of E_J , with values on E_I , called (m, σ) -general, with properties similar to those of the finite-dimensional diffeomorphisms. Moreover, for any strictly positive integer k , we introduce over the measurable space $(\mathbf{R}^I, \mathcal{B}^{(I)})$ a family of infinite-dimensional measures $\lambda_{N,a,v}^{(k,I)}$, dependent on appropriate parameters N, a, v , that in the case $I = \{1, \dots, k\}$ coincide with the k -dimensional Lebesgue measure on \mathbf{R}^k . More precisely, in the paper [4], we define some particular linear functions over E_J , with values on E_I , called (m, σ) -standard, while in the article [5] we present some results about the differentiation theory over E_I , and we remove the assumption of linearity for the (m, σ) -standard functions. In the last two papers, we provide a change of variables' formula for the integration of the measurable real functions on $(\mathbf{R}^I, \mathcal{B}^{(I)})$; this change of variables is defined by some particular (m, σ) -standard functions. In the paper [6], we introduce a class of functions, called (m, σ) -general, that generalizes the set of the (m, σ) -standard functions given in [5]. The main result is the definition of the determinant of a linear (m, σ) -general function, as the limit of a sequence of the determinants of some standard matrices.

In this paper, we prove that the change of variables' formula given by the standard finite-dimensional theory and in the papers [4] and [5] can be extended by using the (m, σ) -general functions. In Section 2, we recall the construction of the infinite-dimensional Banach space E_I , with its σ -algebra \mathcal{B}_I and its topologies τ_I and $\tau_{\|\cdot\|_I}$; moreover, we expose the main results about the differentiation theory over this space. In Section 3, we recall some properties of the (m, σ) -general functions defined in [6], and we expose some additional results

about these functions. In Section 4, we present the main theorem of our paper, that is a change of variables' formula for the integration of the measurable real functions on $(\mathbf{R}^I, \mathcal{B}^{(I)})$; this change of variables is defined by the bijective, C^1 and (m, σ) -general functions, with further properties (Theorem 4.5). In Section 5, we expose some ideas for further study in the probability theory.

2. Differentiation theory over infinite-dimensional Banach spaces

Let $I \neq \emptyset$ be a set and let $k \in \mathbf{N}^*$; indicate by τ , by $\tau^{(k)}$, by $\tau^{(I)}$, by \mathcal{B} , by $\mathcal{B}^{(k)}$, by $\mathcal{B}^{(I)}$, by Leb , and by $Leb^{(k)}$, respectively, the euclidean topology on \mathbf{R} , the euclidean topology on \mathbf{R}^k , the topology $\bigotimes_{i \in I} \tau$, the Borel σ -algebra

on \mathbf{R} , the Borel σ -algebra on \mathbf{R}^k , the σ -algebra $\bigotimes_{i \in I} \mathcal{B}$, the Lebesgue measure

on \mathbf{R} , and the Lebesgue measure on \mathbf{R}^k . Moreover, for any set $A \subset \mathbf{R}$, indicate by $\mathcal{B}(A)$ the σ -algebra induced by \mathcal{B} on A , and by $\tau(A)$ the topology induced by τ on A ; analogously, for any set $A \subset \mathbf{R}^I$, define the σ -algebra $\mathcal{B}^{(I)}(A)$ and the topology $\tau^{(I)}(A)$. Finally, if $S = \prod_{i \in I} S_i$ is a Cartesian product, for any

$(x_i : i \in I) \in S$ and for any $\emptyset \neq H \subset I$, define $x_H = (x_i : i \in H) \in \prod_{i \in H} S_i$, and

define the projection $\pi_{I,H}$ on $\prod_{i \in H} S_i$ as the function $\pi_{I,H} : S \longrightarrow \prod_{i \in H} S_i$ given

by $\pi_{I,H}(x_I) = x_H$.

THEOREM 2.1. *Let $I \neq \emptyset$ be a set and, for any $i \in I$, let (S_i, Σ_i, μ_i) be a measure space such that μ_i is finite. Moreover, suppose that, for some countable set $J \subset I$, μ_i is a probability measure for any $i \in I \setminus J$ and $\prod_{j \in J} \mu_j(S_j) \in \mathbf{R}^+$. Then,*

over the measurable space $\left(\prod_{i \in I} S_i, \bigotimes_{i \in I} \Sigma_i \right)$, there is a unique finite measure μ ,

indicated by $\bigotimes_{i \in I} \mu_i$, such that, for any $H \subset I$ such that $|H| < +\infty$ and for any

$A = \prod_{h \in H} A_h \times \prod_{i \in I \setminus H} S_i \in \bigotimes_{i \in I} \Sigma_i$, where $A_h \in \Sigma_h \forall h \in H$, we have $\mu(A) =$

$\prod_{h \in H} \mu_h(A_h) \prod_{j \in J \setminus H} \mu_j(S_j)$. In particular, if I is countable, then $\mu(A) = \prod_{i \in I} \mu_i(A_i)$

for any $A = \prod_{i \in I} A_i \in \bigotimes_{i \in I} \Sigma_i$.

Proof. See the proof of Corollary 4 in Asci [4]. \square

Henceforth, we will suppose that I, J are sets such that $\emptyset \neq I, J \subset \mathbf{N}^*$; moreover, for any $k \in \mathbf{N}^*$, we will indicate by I_k the set of the first k elements of I (with the natural order and with the convention $I_k = I$ if $|I| < k$); furthermore, for any $i \in I$, set $|i|_I = |I \cap (0, i]|$. Analogously, define J_k and $|j|_J$, for any $k \in \mathbf{N}^*$ and for any $j \in J$.

The following theorem generalizes a result proved in Rao [14] (Theorem 3, page 349), and can be considered a generalization of the Tonelli's theorem, in the integration of a function over an infinite-dimensional measure space. The integral of the function is the limit of a sequence of integrals of the same function, with respect to a finite subset of variables.

THEOREM 2.2. *Let (S_i, Σ_i, μ_i) be a measure space such that μ_i is finite, for any $i \in I$, and $\prod_{i \in I} \mu_i(S_i) \in [0, +\infty)$; moreover, let $(S, \Sigma, \mu) = \left(\prod_{i \in I} S_i, \bigotimes_{i \in I} \Sigma_i, \bigotimes_{i \in I} \mu_i \right)$, let $f \in L^1(S, \Sigma, \mu)$ and, for any $H \subset I$ such that $0 < |H| < +\infty$, let the measurable function $f_{H^c} : (S, \Sigma) \rightarrow (\mathbf{R}, \mathcal{B})$ defined by*

$$f_{H^c}(x) = \int_{S_H} f(\cdot, x_{H^c}) d\mu_H,$$

where (S_H, Σ_H, μ_H) is the measure space $\left(\prod_{i \in H} S_i, \bigotimes_{i \in H} \Sigma_i, \bigotimes_{i \in H} \mu_i \right)$. Then, there exists $D \in \Sigma$ such that $\mu(D) = 0$ and such that, for any $x \in D^c$, one has $\lim_{n \rightarrow +\infty} \int_{S_n} f_{I_n^c}(x) = \int_S f d\mu$.

Proof. See the proof of Corollary 3 in Asci [5]. \square

DEFINITION 2.3. *For any set $I \neq \emptyset$, define the function $\|\cdot\|_I : \mathbf{R}^I \rightarrow [0, +\infty]$ by*

$$\|x\|_I = \sup_{i \in I} |x_i|, \forall x = (x_i : i \in I) \in \mathbf{R}^I,$$

and define the vector space

$$E_I = \{x \in \mathbf{R}^I : \|x\|_I < +\infty\}.$$

Moreover, indicate by \mathcal{B}_I the σ -algebra $\mathcal{B}^{(I)}(E_I)$, by τ_I the topology $\tau^{(I)}(E_I)$, and by $\tau_{\|\cdot\|_I}$ the topology induced on E_I by the distance $d_I : E_I \times E_I \rightarrow [0, +\infty)$ defined by $d_I(x, y) = \|x - y\|_I$, for any $x, y \in E_I$; furthermore, for any set

$A \subset E_I$, indicate by $\tau_{\|\cdot\|_I}(A)$ the topology induced on A by $\tau_{\|\cdot\|_I}$. Finally, for any $x_0 \in E_I$ and for any $\delta \in \mathbf{R}^+$, indicate by $B_I(x_0, \delta)$ the set $\{x \in E_I : \|x - x_0\|_I < \delta\}$.

PROPOSITION 2.4. Let H, I be sets such that $\emptyset \neq H \subsetneq I$, and let $A \subset E_H$, $B \subset E_{I \setminus H}$; then:

1. E_I is a Banach space, with the norm $\|\cdot\|_I$.
2. $\tau_{\|\cdot\|_I}(A \times B)$ is the product of the topologies $\tau_{\|\cdot\|_H}(A)$ and $\tau_{\|\cdot\|_{I \setminus H}}(B)$.
3. Let $A = \left(\prod_{i \in I} A_i \right) \cap E_I \neq \emptyset$, where $A_i \in \tau$, for any $i \in I$; then, one has $A \in \tau_{\|\cdot\|_I}$ if and only if there exists $h \in I$ such that $A_i = \mathbf{R}$, for any $i \in I \setminus I_h$.
4. One has $\tau_I \subset \tau_{\|\cdot\|_I}$; moreover, if $|I| = +\infty$, then $\tau_I \subsetneq \tau_{\|\cdot\|_I}$.

Proof. 1. See, for example, the proof of Remark 2 in [4].

2. Indicate by $\tau_{\|\cdot\|_H}(A) \otimes \tau_{\|\cdot\|_{I \setminus H}}(B)$ the product of the topologies $\tau_{\|\cdot\|_H}(A)$ and $\tau_{\|\cdot\|_{I \setminus H}}(B)$; $\forall D \in \tau_{\|\cdot\|_H}(A)$, let $D' \in \tau_{\|\cdot\|_H}$ such that $D = D' \cap A$; then, $\forall x = (x_H, x_{I \setminus H}) \in D' \times E_{I \setminus H}$, there exists $\delta \in \mathbf{R}^+$ such that $x_H \in B_H(x_H, \delta) \subset D'$, $x_{I \setminus H} \in B_{I \setminus H}(x_{I \setminus H}, \delta) \subset E_{I \setminus H}$, and so $x \in B_I(x, \delta) \subset D' \times E_{I \setminus H}$; then, we have $D' \times E_{I \setminus H} \in \tau_{\|\cdot\|_I}$, from which $D \times B = (D' \times E_{I \setminus H}) \cap (A \times B) \in \tau_{\|\cdot\|_I}(A \times B)$; analogously, $\forall E \in \tau_{\|\cdot\|_{I \setminus H}}(B)$, we have $A \times E \in \tau_{\|\cdot\|_I}(A \times B)$, and so $D \times E = (D \times B) \cap (A \times E) \in \tau_{\|\cdot\|_I}(A \times B)$; then, we obtain $\tau_{\|\cdot\|_H}(A) \otimes \tau_{\|\cdot\|_{I \setminus H}}(B) \subset \tau_{\|\cdot\|_I}(A \times B)$.

Conversely, $\forall x = (x_H, x_{I \setminus H}) \in E_I$, $\forall \delta \in \mathbf{R}^+$, we have $B_I(x, \delta) \cap (A \times B) = (B_H(x_H, \delta) \cap A) \times (B_{I \setminus H}(x_{I \setminus H}, \delta) \cap B) \in \tau_{\|\cdot\|_H}(A) \otimes \tau_{\|\cdot\|_{I \setminus H}}(B)$, from which $\tau_{\|\cdot\|_I}(A \times B) \subset \tau_{\|\cdot\|_H}(A) \otimes \tau_{\|\cdot\|_{I \setminus H}}(B)$.

3. We can suppose $|I| = +\infty$. If there exists $h \in I$ such that $A_i = \mathbf{R}$, for any $i \in I \setminus I_h$, then $A = \left(\prod_{i \in I_h} A_i \right) \times E_{I \setminus I_h}$; thus, since $\prod_{i \in I_h} A_i \in \tau_{\|\cdot\|_{I_h}}$, $E_{I \setminus I_h} \in \tau_{\|\cdot\|_{I \setminus I_h}}$, from point 2 we have $A \in \tau_{\|\cdot\|_I}$.

Conversely, suppose that there exists $J \subset I$ such that $|J| = +\infty$ and such that $A_j \neq \mathbf{R}$, $\forall j \in J$; then, since $A \neq \emptyset$, there exists $x \in A$ such that $d_I(x, E_I \setminus A) = 0$, and so $A \notin \tau_{\|\cdot\|_I}$.

4. Let

$$\mathcal{E} = \left\{ A = \left(\prod_{i \in I} A_i \right) \cap E_I : A_i \in \tau, \forall i \in I, \right. \\ \left. A_i = \mathbf{R}, \forall i \in I \setminus I_h, \text{ for some } h \in I \right\};$$

as we observed in the proof of point 3, we have $\mathcal{E} \subset \tau_{\|\cdot\|_I}$; moreover, by definition of τ_I , we have $\tau_I = \tau(\mathcal{E}) \subset \tau_{\|\cdot\|_I}$; furthermore, if $|I| = +\infty$, $\forall x \in E_I$, $\forall \delta \in \mathbf{R}^+$, we have $B_I(x, \delta) \in \tau_{\|\cdot\|_I}$, $B_I(x, \delta) \notin \tau_I$, and so $\tau_I \subsetneq \tau_{\|\cdot\|_I}$.

□

PROPOSITION 2.5. *Let H, I be sets such that $\emptyset \neq H \subset I$, and let $\bar{\pi}_{I,H} : E_I \longrightarrow E_H$ be the function given by $\bar{\pi}_{I,H}(x) = \pi_{I,H}(x)$, for any $x \in E_I$; then:*

1. $\bar{\pi}_{I,H} : (E_I, \tau_{\|\cdot\|_I}) \longrightarrow (E_H, \tau_{\|\cdot\|_H})$ is continuous and open.
2. $\bar{\pi}_{I,H} : (E_I, \tau_I) \longrightarrow (E_H, \tau_H)$ is continuous and open.
3. $\bar{\pi}_{I,H} : (E_I, \mathcal{B}_I) \longrightarrow (E_H, \mathcal{B}_H)$ is measurable.

Proof. Points 1 and 2 are proved, for example, in Proposition 6 in [5]; moreover, the proof of point 3 is analogous to the proof of the continuity of the function $\bar{\pi}_{I,H} : (E_I, \tau_I) \longrightarrow (E_H, \tau_H)$. □

REMARK 2.6: Let H, I, J be sets such that $\emptyset \neq H \subsetneq J$, let $U = U_1 \times U_2 \in \tau_{\|\cdot\|_J}$, where $U_1 \in \tau_{\|\cdot\|_H}$, $U_2 \in \tau_{\|\cdot\|_{J \setminus H}}$, let $\psi : U_1 \subset E_H \longrightarrow E_I$ be a function and let $\varphi : U \subset E_J \longrightarrow E_I$ be the function given by $\varphi(x) = \psi(x_H)$, for any $x = (x_H, x_{J \setminus H}) \in U$; then:

1. ψ is $(\tau_{\|\cdot\|_H}(U_1), \tau_{\|\cdot\|_I})$ -continuous if and only if φ is $(\tau_{\|\cdot\|_J}(U), \tau_{\|\cdot\|_I})$ -continuous.
2. ψ is $(\tau^{(H)}(U_1), \tau_I)$ -continuous if and only if φ is $(\tau^{(J)}(U), \tau_I)$ -continuous.
3. If ψ is $(\mathcal{B}^{(H)}(U_1), \mathcal{B}_I)$ -measurable, then φ is $(\mathcal{B}^{(J)}(U), \mathcal{B}_I)$ -measurable.

Proof. $\forall A \subset E_I$, we have

$$\varphi^{-1}(A) = \left(\bar{\pi}_{J,H}^{-1} \circ \psi^{-1} \right) (A), \psi^{-1}(A) = \left(\bar{\pi}_{J,H} \circ \varphi^{-1} \right) (A);$$

then, from Proposition 2.5, we obtain the statement. □

DEFINITION 2.7. Let $U \in \tau_{\|\cdot\|_J}$, let $x_0 \in U$, let $l \in E_I$ and let $\varphi : U \subset E_J \longrightarrow E_I$ be a function; we say that $\lim_{x \rightarrow x_0} \varphi(x) = l$ if, for any $\varepsilon \in \mathbf{R}^+$, there exists a neighbourhood $N \in \tau_{\|\cdot\|_J}(U)$ of x_0 such that, for any $x \in N \setminus \{x_0\}$, one has $\|\varphi(x) - l\|_I < \varepsilon$.

DEFINITION 2.8. Let $U \in \tau_{\|\cdot\|_J}$ and let $\varphi : U \subset E_J \longrightarrow E_I$ be a function; we say that φ is continuous in $x_0 \in U$ if $\lim_{x \rightarrow x_0} \varphi(x) = \varphi(x_0)$, and we say that φ is continuous in U if, for any $x \in U$, φ is continuous in x .

REMARK 2.9: Let $U \in \tau_{\|\cdot\|_J}$, let $V \in \tau_{\|\cdot\|_I}$ and let $\varphi : U \subset E_J \longrightarrow V \subset E_I$ be a function; then, $\varphi : (U, \tau_{\|\cdot\|_J}(U)) \longrightarrow (V, \tau_{\|\cdot\|_I}(V))$ is continuous if and only if φ is continuous in U .

DEFINITION 2.10. Let $U \in \tau_{\|\cdot\|_J}$, let $V \in \tau_{\|\cdot\|_I}$ and let $\varphi : U \subset E_J \longrightarrow V \subset E_I$ be a function; we say that φ is a homeomorphism if φ is bijective and the functions $\varphi : (U, \tau_{\|\cdot\|_J}(U)) \longrightarrow (V, \tau_{\|\cdot\|_I}(V))$ and $\varphi^{-1} : (V, \tau_{\|\cdot\|_I}(V)) \longrightarrow (U, \tau_{\|\cdot\|_J}(U))$ are continuous.

DEFINITION 2.11. Let $U \in \tau_{\|\cdot\|_J}$, let $A \subset U$, let $\varphi : U \subset E_J \longrightarrow E_I$ be a functions and let $\{\varphi_n\}_{n \in \mathbf{N}}$ be a sequence of functions such that $\varphi_n : U \longrightarrow E_I$, for any $n \in \mathbf{N}$; we say that:

1. The sequence $\{\varphi_n\}_{n \in \mathbf{N}}$ converges to φ over A if, for any $\varepsilon \in \mathbf{R}^+$ and for any $x \in A$, there exists $n_0 \in \mathbf{N}$ such that, for any $n \in \mathbf{N}$, $n \geq n_0$, one has $\|\varphi_n(x) - \varphi(x)\|_I < \varepsilon$.
2. The sequence $\{\varphi_n\}_{n \in \mathbf{N}}$ converges uniformly to φ over A if, for any $\varepsilon \in \mathbf{R}^+$, there exists $n_0 \in \mathbf{N}$ such that, for any $n \in \mathbf{N}$, $n \geq n_0$, and for any $x \in A$, one has $\|\varphi_n(x) - \varphi(x)\|_I < \varepsilon$.

The following concept generalizes Definition 6 in [4] (see also the theory in the Lang's book [12] and that in the Weidmann's book [20]).

DEFINITION 2.12. Let $A = (a_{ij})_{i \in I, j \in J}$ be a real matrix $I \times J$ (eventually infinite); then, define the linear function $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow \mathbf{R}^I$, and write $x \longrightarrow Ax$, in the following manner:

$$(Ax)_i = \sum_{j \in J} a_{ij} x_j, \quad \forall x \in E_J, \quad \forall i \in I, \quad (1)$$

on condition that, for any $i \in I$, the sum in (1) converges to a real number. In

particular, if $|I| = |J|$, indicate by $\mathbf{I}_{I,J} = (\bar{\delta}_{ij})_{i \in I, j \in J}$ the real matrix defined by

$$\bar{\delta}_{ij} = \begin{cases} 1 & \text{if } |i|_I = |j|_J \\ 0 & \text{otherwise} \end{cases},$$

and call $\bar{\delta}_{ij}$ generalized Kronecker symbol. Moreover, indicate by $A^{(L,N)}$ the real matrix $(a_{ij})_{i \in L, j \in N}$, for any $\emptyset \neq L \subset I$, for any $\emptyset \neq N \subset J$, and indicate by ${}^t A = (b_{ji})_{j \in J, i \in I} : E_I \longrightarrow \mathbf{R}^J$ the linear function defined by $b_{ji} = a_{ij}$, for any $j \in J$ and for any $i \in I$. Furthermore, if $I = J$ and $A = {}^t A$, we say that A is a symmetric function. Finally, if $B = (b_{jk})_{j \in J, k \in K}$ is a real matrix $J \times K$, define the $I \times K$ real matrix $AB = ((AB)_{ik})_{i \in I, k \in K}$ by

$$(AB)_{ik} = \sum_{j \in J} a_{ij} b_{jk}, \quad (2)$$

on condition that, for any $i \in I$ and for any $k \in K$, the sum in (2) converges to a real number.

PROPOSITION 2.13. Let $A = (a_{ij})_{i \in I, j \in J}$ be a real matrix $I \times J$; then:

1. The linear function $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow \mathbf{R}^I$ given by (1) is defined if and only if, for any $i \in I$, $\sum_{j \in J} |a_{ij}| < +\infty$.
2. One has $\sup_{i \in I} \sum_{j \in J} |a_{ij}| < +\infty$ if and only if $A(E_J) \subset E_I$ and A is continuous; moreover, if $A(E_J) \subset E_I$, then $\|A\| = \sup_{i \in I} \sum_{j \in J} |a_{ij}|$.
3. If $B = (b_{jk})_{j \in J, k \in K} : E_K \longrightarrow E_J$ is a linear function, then the linear function $A \circ B : E_K \longrightarrow \mathbf{R}^I$ is defined by the real matrix AB .
4. If $A(E_J) \subset E_I$, then, for any $\emptyset \neq L \subset I$, for any $\emptyset \neq N \subset J$, one has $A^{(L,N)}(E_N) \subset E_L$.

Proof. The proofs of points 1 and 2 are analogous to the proof of Proposition 7 in [4]. Moreover, the proof of point 3 is analogous to that one true in the particular case $|I|, |J|, |K| < +\infty$ (see, e.g., the Lang's book [12]). Finally, suppose that $A(E_J) \subset E_I$; let $\emptyset \neq L \subset I$, let $\emptyset \neq N \subset J$, let $x = (x_n : n \in N) \in E_N$ and let $y = (y_j : j \in J) \in E_J$ such that $y_j = x_j, \forall j \in N$, and $y_j = 0$,

$\forall j \in J \setminus N$; we have

$$\begin{aligned} \sup_{i \in L} \left| \left(A^{(L,N)} x \right)_i \right| &= \sup_{i \in L} \left| \sum_{j \in N} a_{ij}(x_j) \right| = \sup_{i \in L} \left| \sum_{j \in J} a_{ij}(y_j) \right| \\ &\leq \sup_{i \in I} \left| \sum_{j \in J} a_{ij}(y_j) \right| = \sup_{i \in I} |(Ay)_i| < +\infty \quad \Rightarrow \quad A^{(L,N)} x \in E_L; \end{aligned}$$

then, point 4 follows. \square

The following definitions (from Definition 2.14 to Definition 2.18) can be found in [5] and generalize the differentiation theory in the finite case (see, e.g., the Lang's book [11]).

DEFINITION 2.14. Let $U \in \tau_{\|\cdot\|_J}$; a function $\varphi : U \subset E_J \longrightarrow E_I$ is called differentiable in $x_0 \in U$ if there exists a linear and continuous function $A : E_J \longrightarrow E_I$ defined by a real matrix $A = (a_{ij})_{i \in I, j \in J}$, and one has

$$\lim_{h \rightarrow 0} \frac{\|\varphi(x_0 + h) - \varphi(x_0) - Ah\|_I}{\|h\|_J} = 0. \quad (3)$$

If φ is differentiable in x_0 for any $x_0 \in U$, φ is called differentiable in U . The function A is called differential of the function φ in x_0 , and it is indicated by the symbol $d\varphi(x_0)$.

DEFINITION 2.15. Let $U \in \tau_{\|\cdot\|_J}$, let $v \in E_J$ such that $\|v\|_J = 1$ and let a function $\varphi : U \subset E_J \longrightarrow \mathbf{R}^I$; for any $i \in I$, the function φ_i is called differentiable in $x_0 \in U$ in the direction v if there exists the limit

$$\lim_{t \rightarrow 0} \frac{\varphi_i(x_0 + tv) - \varphi_i(x_0)}{t}.$$

This limit is indicated by $\frac{\partial \varphi_i}{\partial v}(x_0)$, and it is called derivative of φ_i in x_0 in the direction v . If, for some $j \in J$, one has $v = e_j$, where $(e_j)_k = \delta_{jk}$, for any $k \in J$, indicate $\frac{\partial \varphi_i}{\partial v}(x_0)$ by $\frac{\partial \varphi_i}{\partial x_j}(x_0)$, and call it partial derivative of φ_i in x_0 , with respect to x_j . Moreover, if there exists the linear function defined by the matrix $J_\varphi(x_0) = \left((J_\varphi(x_0))_{ij} \right)_{i \in I, j \in J} : E_J \longrightarrow \mathbf{R}^I$, where $(J_\varphi(x_0))_{ij} = \frac{\partial \varphi_i}{\partial x_j}(x_0)$, for any $i \in I, j \in J$, then $J_\varphi(x_0)$ is called Jacobian matrix of the function φ in x_0 . Finally, if, for any $x \in U$, there exists $J_\varphi(x)$, then the function $x \longrightarrow J_\varphi(x)$ is indicated by J_φ .

DEFINITION 2.16. Let $U \in \tau_{\|\cdot\|_J}$, let $i, j \in J$ and let $\varphi : U \subset E_J \longrightarrow \mathbf{R}$ be a function differentiable in $x_0 \in U$ with respect to x_i , such that the function $\frac{\partial \varphi}{\partial x_i}$ is differentiable in x_0 with respect to x_j . Indicate $\frac{\partial}{\partial x_j} \left(\frac{\partial \varphi}{\partial x_i} \right) (x_0)$ by $\frac{\partial^2 \varphi}{\partial x_j \partial x_i} (x_0)$ and call it second partial derivative of φ in x_0 with respect to x_i and x_j . If $i = j$, it is indicated by $\frac{\partial^2 \varphi}{\partial x_i^2} (x_0)$. Analogously, for any $k \in \mathbf{N}^*$ and for any $j_1, \dots, j_k \in J$, define $\frac{\partial^k \varphi}{\partial x_{j_k} \dots \partial x_{j_1}} (x_0)$ and call it k -th partial derivative of φ in x_0 with respect to x_{j_1}, \dots, x_{j_k} .

DEFINITION 2.17. Let $U \in \tau_{\|\cdot\|_J}$ and let $k \in \mathbf{N}^*$; a function $\varphi : U \subset E_J \longrightarrow \mathbf{R}^I$ is called C^k in $x_0 \in U$ if, in a neighbourhood $V \in \tau_{\|\cdot\|_J}(U)$ of x_0 , for any $i \in I$ and for any $j_1, \dots, j_k \in J$, there exists the function defined by $x \longrightarrow \frac{\partial^k \varphi_i}{\partial x_{j_k} \dots \partial x_{j_1}} (x)$, and this function is continuous in x_0 ; φ is called C^k in U if, for any $x_0 \in U$, φ is C^k in x_0 .

DEFINITION 2.18. Let $U \in \tau_{\|\cdot\|_J}$ and let $V \in \tau_{\|\cdot\|_I}$; a function $\varphi : U \subset E_J \longrightarrow V \subset E_I$ is called diffeomorphism if φ is bijective and C^1 in U , and the function $\varphi^{-1} : V \subset E_I \longrightarrow U \subset E_J$ is C^1 in V .

3. Theory of the (m, σ) -general functions

The following definition introduces a class of functions, called m -general, that generalize the linear functions $(a_{ij})_{i \in I, j \in J} : E_J \longrightarrow E_I$ (see the next Remark 3.15). For example, the equation corresponding to a 1-general function is obtained by formula 1, by substituting the functions $x_j \longrightarrow a_{ij}x_j$ for some functions φ_{ij} .

DEFINITION 3.1. Let $m \in \mathbf{N}^*$ and let $\emptyset \neq U = \left(U^{(m)} \times \prod_{j \in J \setminus J_m} A_j \right) \cap E_J \in \tau_{\|\cdot\|_J}$, where $U^{(m)} \in \tau^{(m)}$, $A_j \in \tau$, for any $j \in J \setminus J_m$. A function $\varphi : U \subset E_J \longrightarrow E_I$ is called m -general if, for any $i \in I$ and for any $j \in J \setminus J_m$, there exist some functions $\varphi_i^{(I,m)} : U^{(m)} \longrightarrow \mathbf{R}$ and $\varphi_{ij} : A_j \longrightarrow \mathbf{R}$ such that

$$\varphi_i(x) = \varphi_i^{(I,m)}(x_{J_m}) + \sum_{j \in J \setminus J_m} \varphi_{ij}(x_j), \forall x \in U.$$

Moreover, for any $\emptyset \neq L \subset I$ and for any $J_m \subset N \subset J$, indicate by $\varphi^{(L,N)}$ the function $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow \mathbf{R}^L$ defined by

$$\varphi_i^{(L,N)}(x_N) = \varphi_i^{(I,m)}(x_{J_m}) + \sum_{j \in N \setminus J_m} \varphi_{ij}(x_j), \forall x_N \in \pi_{J,N}(U), \forall i \in L. \quad (4)$$

Furthermore, for any $\emptyset \neq L \subset I$ and for any $\emptyset \neq N \subset J \setminus J_m$, indicate by $\varphi^{(L,N)}$ the function $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow \mathbf{R}^L$ given by

$$\varphi_i^{(L,N)}(x_N) = \sum_{j \in N} \varphi_{ij}(x_j), \quad \forall x_N \in \pi_{J,N}(U), \quad \forall i \in L. \quad (5)$$

In particular, suppose that $m = 1$; then, let $j \in J$ such that $\{j\} = J_1$ and indicate $U^{(1)}$ by A_j and $\varphi_i^{(I,1)}$ by φ_{ij} , for any $i \in I$; moreover, for any $\emptyset \neq L \subset I$ and for any $\emptyset \neq N \subset J$, indicate by $\varphi^{(L,N)}$ the function $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow \mathbf{R}^L$ defined by formula (5).

Furthermore, for any $l, n \in \mathbf{N}^*$, indicate $\varphi^{(I_l, N)}$ by $\varphi^{(l, N)}$, $\varphi^{(L, J_n)}$ by $\varphi^{(L, n)}$, and $\varphi^{(I_l, J_n)}$ by $\varphi^{(l, n)}$.

The following definition introduces a class of m -general functions $\varphi : U \subset E_J \longrightarrow E_I$, called (m, σ) -general, that will be used to provide a change of variables' formula for the integration of the measurable real functions over $(\mathbf{R}^I, \mathcal{B}^{(I)})$. In fact, the properties of some (m, σ) -general functions generalize the analogous ones of the standard finite-dimensional diffeomorphisms. In particular, if A is a linear (m, σ) -general function, we can define the determinant of A (see the next Theorem 3.18 and Definition 3.19): a concept without sense, if A is an arbitrary matrix $I \times J$.

DEFINITION 3.2. Let $m \in \mathbf{N}^*$, let $\emptyset \neq U = \left(U^{(m)} \times \prod_{j \in J \setminus J_m} A_j \right) \cap E_J \in \tau_{\|\cdot\|_J}$, where $U^{(m)} \in \tau^{(m)}$, $A_j \in \tau$, for any $j \in J \setminus J_m$, and let $\sigma : I \setminus I_m \longrightarrow J \setminus J_m$ be an increasing function; a function $\varphi : U \subset E_J \longrightarrow E_I$ m -general and such that $|J| = |I|$ is called (m, σ) -general if:

1. $\forall i \in I \setminus I_m, \forall j \in J \setminus (J_m \cup \{\sigma(i)\}), \forall t \in A_j$, one has $\varphi_{ij}(t) = 0$; moreover

$$\varphi^{(I \setminus I_m, J \setminus J_m)}(\pi_{J, J \setminus J_m}(U)) \subset E_{I \setminus I_m}.$$

2. $\forall i \in I \setminus I_m, \forall x \in U$, there exists $J_{\varphi_i}(x) : E_J \longrightarrow \mathbf{R}$; moreover, $\forall x_{J_m} \in U^{(m)}$, one has $\sum_{i \in I \setminus I_m} \left\| J_{\varphi_i}^{(I, m)}(x_{J_m}) \right\| < +\infty$.

3. $\forall i \in I \setminus I_m$, the function $\varphi_{i, \sigma(i)} : A_{\sigma(i)} \longrightarrow \mathbf{R}$ is constant or injective; moreover, $\forall x_{\sigma(I \setminus I_m)} \in \prod_{j \in \sigma(I \setminus I_m)} A_j$, one has $\sup_{i \in I \setminus I_m} \left| \varphi'_{i, \sigma(i)}(x_{\sigma(i)}) \right| < +\infty$ and $\inf_{i \in \mathcal{I}_\varphi} \left| \varphi'_{i, \sigma(i)}(x_{\sigma(i)}) \right| > 0$, where $\mathcal{I}_\varphi = \{i \in I \setminus I_m : \varphi_{i, \sigma(i)} \text{ is injective}\}$.

4. If, for some $h \in \mathbf{N}$, $h \geq m$, one has $|\sigma(i)|_{J \setminus J_m} = |i|_{I \setminus I_m}$, $\forall i \in I \setminus I_h$, then, $\forall x_{\sigma(I \setminus I_m)} \in \prod_{j \in \sigma(I \setminus I_m)} A_j$, there exists $\prod_{i \in \mathcal{I}_\varphi} \varphi'_{i, \sigma(i)}(x_{\sigma(i)}) \in \mathbf{R}^*$.

Moreover, set

$$\mathcal{A} = \mathcal{A}(\varphi) = \left\{ h \in \mathbf{N}, h \geq m : |\sigma(i)|_{J \setminus J_m} = |i|_{I \setminus I_m}, \forall i \in I \setminus I_h \right\}.$$

If the sequence $\left\{ J_{\varphi_i^{(I, m)}}(x_{J_m}) \right\}_{i \in I \setminus I_m}$ converges uniformly on $U^{(m)}$ to the matrix $(0 \dots 0)$ and there exists $a \in \mathbf{R}$ such that, for any $\varepsilon > 0$, there exists $i_0 \in \mathbf{N}$, $i_0 \geq m$, such that, for any $i \in \mathcal{I}_\varphi \cap (I \setminus I_{i_0})$ and for any $t \in A_{\sigma(i)}$, one has $\left| \varphi'_{i, \sigma(i)}(t) - a \right| < \varepsilon$, then φ is called strongly (m, σ) -general.

Furthermore, for any $I_m \subset L \subset I$ and for any $J_m \subset N \subset J$, define the function $\overline{\varphi}^{(L, N)} : U \subset E_J \longrightarrow \mathbf{R}^I$ in the following manner:

$$\overline{\varphi}_i^{(L, N)}(x) = \begin{cases} \varphi_i^{(L, N)}(x_N) & \forall i \in I_m, \forall x \in U \\ \varphi_i(x) & \forall i \in L \setminus I_m, \forall x \in U \\ \varphi_{i, \sigma(i)}(x_{\sigma(i)}) & \forall i \in I \setminus L, \forall x \in U \end{cases}.$$

Finally, for any $l, n \in \mathbf{N}$, $l, n \geq m$, indicate $\overline{\varphi}^{(I, N)}$ by $\overline{\varphi}^{(l, N)}$, $\overline{\varphi}^{(L, J_n)}$ by $\overline{\varphi}^{(L, n)}$, $\overline{\varphi}^{(I, J_n)}$ by $\overline{\varphi}^{(l, n)}$, and $\overline{\varphi}^{(m, m)}$ by $\overline{\varphi}$.

DEFINITION 3.3. A function $\varphi : U \subset E_J \longrightarrow E_I$ (m, σ) -general is called (m, σ) -standard (or (m, σ) of the first type) if, for any $i \in I \setminus I_m$ and for any $x_{J_m} \in U^{(m)}$, one has $\varphi_i^{(I, m)}(x_{J_m}) = 0$. Moreover, a function $\varphi : U \subset E_J \longrightarrow E_I$ (m, σ) -standard and strongly (m, σ) -general is called strongly (m, σ) -standard (see also Definition 28 in [5]).

REMARK 3.4: Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m, σ) -general function; then:

1. σ is injective if and only if, for any $i_1, i_2 \in I \setminus I_m$ such that $i_1 < i_2$, one has $\sigma(i_1) < \sigma(i_2)$.
2. σ is bijective if and only if, for any $i \in I \setminus I_m$, one has $|\sigma(i)|_{J \setminus J_m} = |i|_{I \setminus I_m}$.
3. There exists $m_0 \in \mathbf{N}$, $m_0 \geq m$, such that $A_j = \mathbf{R}$, for any $j \in J \setminus J_{m_0}$.

Proof. The statement follows from Definition 3.2 and point 3 of Proposition 2.4. \square

PROPOSITION 3.5. Let $I_m \subset L \subset I$, let $J_m \subset N \subset J$ and let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m, σ) -general function; then, one has $\overline{\varphi}^{(L, N)}(U) \subset E_I$, and the function $\overline{\varphi}^{(L, N)} : U \subset E_J \longrightarrow E_I$ is (m, σ) -general. Moreover, suppose that, for any $j \in J \setminus J_m$, for any $t \in A_j$, one has $\sum_{i \in I \setminus I_m} |\varphi'_{i,j}(t)| < +\infty$; then, for any $n \in \mathbf{N}$, $n \geq m$, $\overline{\varphi}^{(L, N)}$ is (n, τ) -general, where the function $\tau : I \setminus I_n \longrightarrow J \setminus J_n$ is defined by

$$\tau(i) = \begin{cases} \sigma(i) & \text{if } \sigma(i) \in J \setminus J_n \\ \min(J \setminus J_n) & \text{if } \sigma(i) \notin J \setminus J_n \end{cases}, \forall i \in I \setminus I_n. \quad (6)$$

Proof. Since $I_m \subset L \subset I$ and $J_m \subset N \subset J$, $\forall i \in I \setminus I_m$, $\forall x \in U$, we have

$$\left| \overline{\varphi}_i^{(L, N)}(x) \right| \leq \left| \varphi_i^{(I, m)}(x_{J_m}) \right| + \left| \varphi_{i, \sigma(i)}(x_{\sigma(i)}) \right|,$$

and so $\sup_{i \in I \setminus I_m} \left| \overline{\varphi}_i^{(L, N)}(x) \right| < +\infty$; then, $\overline{\varphi}^{(L, N)}(U) \subset E_I$. Moreover, from the definition of $\overline{\varphi}^{(L, N)}$, the function $\overline{\varphi}^{(L, N)} : U \subset E_J \longrightarrow E_I$ is (m, σ) -general. Furthermore, suppose that, for any $j \in J \setminus J_m$, for any $t \in A_j$, one has $\sum_{i \in I \setminus I_m} |\varphi'_{i,j}(t)| < +\infty$; $\forall n \in \mathbf{N}$, $n \geq m$, and $\forall x_{J_n} \in \pi_{J, J_n}(U)$, we have

$$\begin{aligned} \sum_{i \in I \setminus I_n} \left\| J_{(\overline{\varphi}^{(L, N)})_i}^{(I, J_n)}(x_{J_n}) \right\| &\leq \sum_{i \in I \setminus I_n} \left\| J_{\varphi_i}^{(I, J_n)}(x_{J_n}) \right\| \\ &= \sum_{i \in I \setminus I_n} \left\| J_{\varphi_i}^{(I, m)}(x_{J_m}) \right\| + \sum_{j \in J_n \setminus J_m} \left(\sum_{i \in I \setminus I_n} |\varphi'_{i,j}(x_j)| \right) < +\infty; \end{aligned}$$

then, $\overline{\varphi}^{(L, N)}$ is (n, τ) -general, where the function $\tau : I \setminus I_n \longrightarrow J \setminus J_n$ is defined by formula (6). \square

PROPOSITION 3.6. Let $\emptyset \neq L \subset I$, let $\emptyset \neq N \subset J$ such that $J_m \subset N$ or $N \subset J \setminus J_m$, and let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m, σ) -general function; then:

1. For any $x \in U$, there exists the function $J_{\varphi}^{(L, N)}(x) : E_N \longrightarrow \mathbf{R}^L$ if and only if, for any $i \in L \cap I_m$ and for any $j \in N$, there exists the partial derivative $\frac{\partial \varphi_i}{\partial x_j}(x)$, and for any $i \in L \cap I_m$ one has $\sum_{j \in N} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right| < +\infty$; moreover, in this case one has $J_{\varphi}^{(L, N)}(x)(E_N) \subset E_L$, and $J_{\varphi}^{(L, N)}(x)$ is continuous.
2. For any $x \in U$, there exists the function $J_{\varphi}^{(I \setminus I_m, J)}(x) : E_J \longrightarrow E_{I \setminus I_m}$, and it is continuous.

3. Suppose that $I_m \subset L$ and $J_m \subset N$, and let $x \in U$; then, there exists the function $J_{\overline{\varphi}^{(L,N)}}(x) : E_J \longrightarrow \mathbf{R}^I$ if and only if, for any $i \in I_m$ and for any $j \in N$, there exists the partial derivative $\frac{\partial \varphi_i}{\partial x_j}(x)$, and for any $i \in I_m$ one has $\sum_{j \in N} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right| < +\infty$; moreover, in this case one has $J_{\overline{\varphi}^{(L,N)}}(x)(E_J) \subset E_I$, and $J_{\overline{\varphi}^{(L,N)}}(x)$ is continuous and (m, σ) -general.

Proof. 1. From Definition 3.2, $\forall i \in L \cap (I \setminus I_m)$ and $\forall j \in N$, there exists the partial derivative $\frac{\partial \varphi_i^{(L,N)}}{\partial x_j}(x) = \frac{\partial \varphi_i}{\partial x_j}(x)$, and one has

$$\sum_{j \in N} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right| \leq \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| + \left| \varphi'_{i,\sigma(i)}(x_{\sigma(i)}) \right| < +\infty,$$

$$\forall i \in L \cap (I \setminus I_m); \quad (7)$$

then, from Proposition 2.13, there exists the function $J_{\varphi^{(L,N)}}(x) : E_N \longrightarrow \mathbf{R}^L$ if and only if, $\forall i \in L \cap I_m$ and $\forall j \in N$, there exists the partial derivative $\frac{\partial \varphi_i^{(L,N)}}{\partial x_j}(x) = \frac{\partial \varphi_i}{\partial x_j}(x)$, and $\forall i \in L \cap I_m$ one has $\sum_{j \in N} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right| < +\infty$.

Furthermore, since $\sum_{i \in I \setminus I_m} \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| < +\infty$, we have

$$\sup_{i \in L \cap (I \setminus I_m)} \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| < +\infty,$$

and so formula (7) implies

$$\sup_{i \in L \cap (I \setminus I_m)} \sum_{j \in N} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right|$$

$$\leq \sup_{i \in L \cap (I \setminus I_m)} \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| + \sup_{i \in L \cap (I \setminus I_m)} \left| \varphi'_{i,\sigma(i)}(x_{\sigma(i)}) \right| < +\infty;$$

thus, if there exists the function $J_{\varphi^{(L,N)}}(x)$, we obtain $\sup_{i \in L} \sum_{j \in N} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right| < +\infty$; then, from Proposition 2.13, we have $J_{\varphi^{(L,N)}}(x)(E_N) \subset E_L$, and $J_{\varphi^{(L,N)}}(x)$ is continuous.

2. The statement follows from point 1.

3. By Definition 3.2, $\forall i \in I \setminus I_m$ and $\forall j \in J$, there exists the partial derivative $\frac{\partial \bar{\varphi}_i^{(L,N)}}{\partial x_j}(x)$, and one has

$$\begin{aligned} \sum_{j \in J} \left| \frac{\partial \bar{\varphi}_i^{(L,N)}}{\partial x_j}(x) \right| &\leq \sum_{j \in J} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right| \\ &\leq \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| + \left| \varphi'_{i,\sigma(i)}(x_{\sigma(i)}) \right| < +\infty, \quad \forall i \in I \setminus I_m; \end{aligned} \quad (8)$$

then, from Proposition 2.13, there exists the function $J_{\bar{\varphi}^{(L,N)}}(x) : E_J \longrightarrow \mathbf{R}^I$ if and only if, $\forall i \in I_m$ and $\forall j \in J$, there exists the partial derivative $\frac{\partial \bar{\varphi}_i^{(L,N)}}{\partial x_j}(x)$, and $\forall i \in I_m$ one has $\sum_{j \in J} \left| \frac{\partial \bar{\varphi}_i^{(L,N)}}{\partial x_j}(x) \right| < +\infty$; thus, this happens if and only if, $\forall i \in I_m$ and $\forall j \in N$, there exists the partial derivative $\frac{\partial \varphi_i}{\partial x_j}(x)$, and $\forall i \in I_m$ one has $\sum_{j \in N} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right| < +\infty$.

Moreover, from formula (8), we have

$$\begin{aligned} \sup_{i \in I \setminus I_m} \sum_{j \in J} \left| \frac{\partial \bar{\varphi}_i^{(L,N)}}{\partial x_j}(x) \right| \\ \leq \sup_{i \in I \setminus I_m} \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| + \sup_{i \in I \setminus I_m} \left| \varphi'_{i,\sigma(i)}(x_{\sigma(i)}) \right| < +\infty; \end{aligned}$$

then, if there exists the function $J_{\bar{\varphi}^{(L,N)}}(x)$, we obtain

$$\sup_{i \in I} \sum_{j \in J} \left| \frac{\partial \bar{\varphi}_i^{(L,N)}}{\partial x_j}(x) \right| < +\infty;$$

thus, from Proposition 2.13, we have $J_{\bar{\varphi}^{(L,N)}}(x)(E_J) \subset E_I$, and $J_{\bar{\varphi}^{(L,N)}}(x)$ is continuous; furthermore, by Definition 3.2, $J_{\bar{\varphi}^{(L,N)}}(x)$ is (m, σ) -general. \square

PROPOSITION 3.7. *Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m, σ) -standard function; then:*

1. *Suppose that φ is injective, $\pi_{I,H}(\varphi(U)) \in \tau^{(H)}$, for any $H \subset I \setminus I_m$ such that $0 < |H| \leq 2$, the function $\varphi_i : U \longrightarrow \mathbf{R}$ is C^1 , for any $i \in I_m$, and $\det J_{\varphi^{(m,m)}}(\mathbf{x}) \neq 0$, for any $\mathbf{x} \in U^{(m)}$; then the functions $\varphi_{i,\sigma(i)}$, for any $i \in I \setminus I_m$, and $\varphi^{(m,m)}$ are injective, and σ is bijective.*
2. *Suppose that φ is bijective, the function $\varphi_i : U \longrightarrow \mathbf{R}$ is C^1 , for any $i \in I_m$, and $\det J_{\varphi^{(m,m)}}(\mathbf{x}) \neq 0$, for any $\mathbf{x} \in U^{(m)}$; then the functions $\varphi_{i,\sigma(i)}$, for any $i \in I \setminus I_m$, $\varphi^{(m,m)}$ and σ are bijective.*

3. Suppose that $\varphi_{ij}(x_j) = 0$, for any $i \in I_m$, for any $j \in J \setminus J_m$, for any $x_j \in A_j$, φ is injective, and $\pi_{I,H}(\varphi(U)) \in \tau^{(H)}$, for any $H \subset I \setminus I_m$ such that $0 < |H| \leq 2$; then the functions $\varphi_{i,\sigma(i)}$, for any $i \in I \setminus I_m$, and $\varphi^{(m,m)}$ are injective, and σ is bijective.
4. Suppose that $\varphi_{ij}(x_j) = 0$, for any $i \in I_m$, for any $j \in J \setminus J_m$, for any $x_j \in A_j$, and φ is bijective; then the functions $\varphi_{i,\sigma(i)}$, for any $i \in I \setminus I_m$, $\varphi^{(m,m)}$ and σ are bijective.
5. If the functions $\varphi_{i,\sigma(i)}$, for any $i \in I \setminus I_m$, and $\varphi^{(m,m)}$ are injective, and σ is bijective, then φ is injective.
6. If the functions $\varphi_{i,\sigma(i)}$, for any $i \in I \setminus I_m$, $\varphi^{(m,m)}$ and σ are bijective, then φ is bijective.

Proof. The statement follows from Proposition 31, Proposition 32 and Remark 33 in [5]. \square

COROLLARY 3.8. Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m, σ) -general function; then:

1. If $\bar{\varphi}$ is injective and $\pi_{I,H}(\bar{\varphi}(U)) \in \tau^{(H)}$, for any $H \subset I \setminus I_m$ such that $0 < |H| \leq 2$, then the functions $\varphi_{i,\sigma(i)}$, for any $i \in I \setminus I_m$, and $\varphi^{(m,m)}$ are injective, and σ is bijective.
2. If $\bar{\varphi}$ is bijective, then the functions $\varphi_{i,\sigma(i)}$, for any $i \in I \setminus I_m$, $\varphi^{(m,m)}$ and σ are bijective.

Proof. Observe that $\bar{\varphi}$ is (m, σ) -standard, and $\bar{\varphi}_{ij}(x_j) = 0$, for any $i \in I_m$, for any $j \in J \setminus J_m$, for any $x_j \in A_j$; then, from points 3 and 4 of Proposition 3.7, we obtain the statements 1 and 2. \square

PROPOSITION 3.9. Let $m \in \mathbf{N}^*$, let $\emptyset \neq L \subset I$, let $\emptyset \neq N \subset J$ such that $J_m \subset N$ or $N \subset J \setminus J_m$, and let $\varphi : U \subset E_J \longrightarrow E_I$ be a function m -general and such that, for any $i \in L$ and for any $j \in N \setminus J_m$, the functions $\varphi_i^{(I,m)} : (U^{(m)}, \mathcal{B}^{(m)}(U^{(m)})) \longrightarrow (\mathbf{R}, \mathcal{B})$ and $\varphi_{ij} : (A_j, \mathcal{B}(A_j)) \longrightarrow (\mathbf{R}, \mathcal{B})$ are measurable; then:

1. The function

$$\varphi^{(L,N)} : (\pi_{J,N}(U), \mathcal{B}^{(N)}(\pi_{J,N}(U))) \longrightarrow (\mathbf{R}^L, \mathcal{B}^{(L)})$$

is measurable; in particular, suppose that, for any $i \in I$ and for any $j \in J \setminus J_m$, $\varphi_i^{(I,m)}$ and φ_{ij} are measurable functions; then, $\varphi : (U, \mathcal{B}^{(J)}(U)) \longrightarrow (E_I, \mathcal{B}_I)$ is measurable.

2. If φ is (m, σ) -general, $I_m \subset L$ and $J_m \subset N$, then the function $\overline{\varphi}^{(L, N)} : (U, \mathcal{B}^{(J)}(U)) \longrightarrow (E_I, \mathcal{B}_I)$ is measurable.

Proof. 1. $\forall i \in L$ and $\forall M \subset N$ such that $J_m \subset M$ or $M \subset J \setminus J_m$, consider the function $\widehat{\varphi}^{(i, M, N)} : \pi_{J, N}(U) \longrightarrow \mathbf{R}$ defined by

$$\widehat{\varphi}^{(i, M, N)}(x) = \begin{cases} \varphi^{\{\{i\}, M\}}(x_M) & \text{if } M \neq \emptyset \\ 0 & \text{if } M = \emptyset \end{cases}, \forall x \in \pi_{J, N}(U);$$

observe that, $\forall n \in \mathbf{N}$, $n \geq m$, we have

$$\widehat{\varphi}^{(i, N \cap J_n, N)}(x) = \widehat{\varphi}^{(i, N \cap J_m, N)}(x) + \sum_{j \in N \cap (J_n \setminus J_m)} \widehat{\varphi}^{(i, \{j\}, N)}(x), \quad \forall x \in \pi_{J, N}(U); \quad (9)$$

moreover, from Remark 2.6, the functions $\widehat{\varphi}^{(i, N \cap J_m, N)}$ and $\widehat{\varphi}^{(i, \{j\}, N)}$, $\forall j \in N \cap (J_n \setminus J_m)$, are $(\mathcal{B}^{(N)}(\pi_{J, N}(U)), \mathcal{B})$ -measurable; thus, from formula (9), $\widehat{\varphi}^{(i, N \cap J_n, N)}$ is $(\mathcal{B}^{(N)}(\pi_{J, N}(U)), \mathcal{B})$ -measurable; then, since

$$\lim_{n \rightarrow +\infty} \widehat{\varphi}^{(i, N \cap J_n, N)} = \varphi_i^{(L, N)},$$

$\varphi_i^{(L, N)}$ is $(\mathcal{B}^{(N)}(\pi_{J, N}(U)), \mathcal{B})$ -measurable too. Furthermore, let

$$\Sigma(L) = \left\{ B = \prod_{i \in L} B_i : B_i \in \mathcal{B}, \forall i \in L \right\};$$

$\forall B = \prod_{i \in L} B_i \in \Sigma(L)$, we have

$$\left(\varphi^{(L, N)} \right)^{-1}(B) = \bigcap_{i \in L} \left(\varphi_i^{(L, N)} \right)^{-1}(B_i) \in \mathcal{B}^{(N)}(\pi_{J, N}(U)).$$

Finally, since $\sigma(\Sigma(L)) = \mathcal{B}^{(L)}$, $\forall B \in \mathcal{B}^{(L)}$, we obtain $\left(\varphi^{(L, N)} \right)^{-1}(B) \in \mathcal{B}^{(N)}(\pi_{J, N}(U))$, and so $\varphi^{(L, N)} : (\pi_{J, N}(U), \mathcal{B}^{(N)}(\pi_{J, N}(U))) \longrightarrow (\mathbf{R}^L, \mathcal{B}^{(L)})$ is measurable. In particular, suppose that, $\forall i \in I$ and $\forall j \in J \setminus J_m$, the functions $\varphi_i^{(I, m)}$ and φ_{ij} are measurable; then, $\varphi : (U, \mathcal{B}^{(J)}(U)) \longrightarrow (\mathbf{R}^I, \mathcal{B}^{(I)})$ is measurable; thus, since $\varphi(U) \subset E_I$, we obtain that φ is $(\mathcal{B}^{(J)}(U), \mathcal{B}_I)$ -measurable.

2. If φ is (m, σ) -general, $I_m \subset L$ and $J_m \subset N$, from Proposition 3.5, the function $\overline{\varphi}^{(L, N)} : U \subset E_J \longrightarrow E_I$ is (m, σ) -general, and so m -general. Moreover, we have

$$\overline{\varphi}_i^{(L, N)}(x) = \psi_i^{(I, m)}(x_{J_m}) + \sum_{j \in J \setminus J_m} \psi_{ij}(x_j), \quad \forall x \in U, \forall i \in I,$$

where

$$\psi_i^{(I,m)} = \begin{cases} \varphi_i^{(I,m)} & \text{if } i \in L \\ 0 & \text{if } i \in I \setminus L \end{cases},$$

$$\psi_{ij} = \begin{cases} \varphi_{ij} & \text{if } (i,j) \in (I_m \times (N \setminus J_m)) \cup ((I \setminus I_m) \times (J \setminus J_m)) \\ 0 & \text{if } (i,j) \in I_m \times (J \setminus N) \end{cases};$$

furthermore, $\forall i \in I, \forall j \in J \setminus J_m, \psi_i^{(I,m)} : (U^{(m)}, \mathcal{B}^{(m)}(U^{(m)})) \longrightarrow (\mathbf{R}, \mathcal{B})$ and $\psi_{ij} : (A_j, \mathcal{B}(A_j)) \longrightarrow (\mathbf{R}, \mathcal{B})$ are measurable functions, and so, from point 1, $\overline{\varphi}^{(L,N)} : (U, \mathcal{B}^{(J)}(U)) \longrightarrow (\mathbf{R}^I, \mathcal{B}^{(I)})$ is measurable; finally, since $\overline{\varphi}^{(L,N)}(U) \subset E_I$, we obtain that $\overline{\varphi}^{(L,N)}$ is $(\mathcal{B}^{(J)}(U), \mathcal{B}_I)$ -measurable. \square

PROPOSITION 3.10. *Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m, σ) -general function such that σ is bijective and $\pi_{I, I \setminus I_m} \circ \overline{\varphi} : (U, \tau_{\|\cdot\|_J}(U)) \longrightarrow (E_{I \setminus I_m}, \tau_{\|\cdot\|_{I \setminus I_m}})$ is continuous; then, for any $n \in \mathbf{N}, n \geq m, \varphi^{(n,n)} : (\pi_{J, J_n}(U), \tau^{(n)}(\pi_{J, J_n}(U))) \longrightarrow (\mathbf{R}^n, \tau^{(n)})$ is continuous if and only if $\overline{\varphi}^{(n,n)} : (U, \tau_{\|\cdot\|_J}(U)) \longrightarrow (E_I, \tau_{\|\cdot\|_I})$ is continuous.*

Proof. Let $n \in \mathbf{N}, n \geq m$, and suppose that $\varphi^{(n,n)}$ is continuous; moreover, let $B = B_1 \times B_2 \in \tau_{\|\cdot\|_I}$, where $B_1 \in \tau^{(n)}, B_2 \in \tau_{\|\cdot\|_{I \setminus I_n}}$; since σ is bijective, we have

$$\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B) = \left(\varphi^{(n,n)}\right)^{-1}(B_1) \times \pi_{J, J \setminus J_n} \left(\left(\pi_{I, I \setminus I_n} \circ \overline{\varphi}\right)^{-1}(B_2) \right);$$

moreover, since $\varphi^{(n,n)}$ and $\pi_{I, I \setminus I_m} \circ \overline{\varphi}$ are continuous, and $\mathbf{R}^{n-m} \times B_2 \in \tau_{\|\cdot\|_{I \setminus I_m}}$, we have

$$\left(\varphi^{(n,n)}\right)^{-1}(B_1) \in \tau^{(n)}(\pi_{J, J_n}(U)),$$

$$\begin{aligned} \left(\pi_{I, I \setminus I_n} \circ \overline{\varphi}\right)^{-1}(B_2) &= \left(\pi_{I \setminus I_m, I \setminus I_n} \circ (\pi_{I, I \setminus I_m} \circ \overline{\varphi})\right)^{-1}(B_2) \\ &= \left(\pi_{I, I \setminus I_m} \circ \overline{\varphi}\right)^{-1}(\mathbf{R}^{n-m} \times B_2) \in \tau_{\|\cdot\|_J}(U), \end{aligned}$$

and so $\pi_{J, J \setminus J_n} \left(\left(\pi_{I, I \setminus I_n} \circ \overline{\varphi}\right)^{-1}(B_2) \right) \in \tau_{\|\cdot\|_{J \setminus J_n}}(\pi_{J, J \setminus J_n}(U))$, from Proposition 2.5; then, we obtain $\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B) \in \tau_{\|\cdot\|_J}(U)$; finally, from Proposition 2.4, $\forall B \in \tau_{\|\cdot\|_I}$, we have $\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B) \in \tau_{\|\cdot\|_J}(U)$, and so $\overline{\varphi}^{(n,n)}$ is continuous.

Conversely, suppose that $\overline{\varphi}^{(n,n)}$ is continuous; $\forall B \in \tau^{(n)}$, we have $B \times E_{I \setminus I_n} \in \tau_{\|\cdot\|_I}$, and so $\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B \times E_{I \setminus I_n}) \in \tau_{\|\cdot\|_J}(U)$; then, $\left(\varphi^{(n,n)}\right)^{-1}(B) = \pi_{J, J_n} \left(\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B \times E_{I \setminus I_n}) \right) \in \tau^{(n)}(\pi_{J, J_n}(U))$. \square

PROPOSITION 3.11. Let $m \in \mathbf{N}^*$, let $\emptyset \neq L \subset I$, let $\emptyset \neq N \subset J$ such that $J_m \subset N$ or $N \subset J \setminus J_m$, and let $\varphi : U \subset E_J \rightarrow E_I$ be a function m -general and C^1 in $x_0 = (x_{0,j} : j \in J) \in U$; then:

1. The function $\varphi^{(L,N)} : \pi_{J,N}(U) \rightarrow \mathbf{R}^L$ is C^1 in $(x_{0,j} : j \in N)$.
2. If φ is (m, σ) -general, $I_m \subset L$ and $J_m \subset N$, then the function $\bar{\varphi}^{(L,N)} : U \subset E_J \rightarrow E_I$ is C^1 in x_0 .

Proof. See the proof of Proposition 2.28 in [6]. □

PROPOSITION 3.12. Let $\varphi : U \subset E_J \rightarrow E_I$ be a (m, σ) -general function such that $\bar{\varphi} : U \rightarrow \bar{\varphi}(U)$ is a homeomorphism. Then, the functions $\varphi^{(m,m)} : U^{(m)} \rightarrow \varphi^{(m,m)}(U^{(m)})$ and $\varphi_{i,\sigma(i)} : A_i \rightarrow \varphi_{i,\sigma(i)}(A_i)$, for any $i \in I \setminus I_m$, are homeomorphisms, and σ is bijective.

Proof. From Proposition 37 in [5], the statement is true if φ is (m, σ) -standard; moreover, observe that $\bar{\varphi}$ is (m, σ) -standard, $\bar{\varphi} = \overline{(\bar{\varphi})}$, $\varphi^{(m,m)} = (\bar{\varphi})^{(m,m)}$, $\varphi_{i,\sigma(i)} = \bar{\varphi}_{i,\sigma(i)}$, $\forall i \in I \setminus I_m$; then, the statement is true if φ is (m, σ) -general too. □

PROPOSITION 3.13. Let $\varphi : U \subset E_J \rightarrow E_I$ be a (m, σ) -general function. Then, $\bar{\varphi} : U \rightarrow \bar{\varphi}(U)$ is a diffeomorphism if and only if the functions $\varphi^{(m,m)} : U^{(m)} \rightarrow \varphi^{(m,m)}(U^{(m)})$ and $\varphi_{i,\sigma(i)} : A_i \rightarrow \varphi_{i,\sigma(i)}(A_i)$, for any $i \in I \setminus I_m$, are diffeomorphisms, and σ is bijective.

Proof. From Proposition 38 in [5], the statement is true if φ is (m, σ) -standard; then, as we observed in the proof of Proposition 3.12, the statement is true if φ is (m, σ) -general too. □

DEFINITION 3.14. Let $A = (a_{ij})_{i \in I, j \in J} : E_J \rightarrow E_I$ be a linear (m, σ) -general function; $\forall i \in I \setminus I_m$, set $\lambda_i = \lambda_i(A) = a_{i,\sigma(i)}$.

REMARK 3.15: For any $m \in \mathbf{N}^*$, a linear function $A = (a_{ij})_{i \in I, j \in J} : E_J \rightarrow E_I$ is m -general; moreover, if $|J| = |I|$ and $\sigma : I \setminus I_m \rightarrow J \setminus J_m$ is an increasing function, A is (m, σ) -general if and only if:

1. $\forall i \in I \setminus I_m, \forall j \in J \setminus (J_m \cup \{\sigma(i)\})$, one has $a_{ij} = 0$.
2. $\forall j \in J_m, \sum_{i \in I \setminus I_m} |a_{ij}| < +\infty$; moreover, one has $\sup_{i \in I \setminus I_m} |\lambda_i| < +\infty$ and $\inf_{i \in I \setminus I_m : \lambda_i \neq 0} |\lambda_i| > 0$.

3. If $\mathcal{A} \neq \emptyset$, there exists $\prod_{i \in I \setminus I_m: \lambda_i \neq 0} \lambda_i \in \mathbf{R}^*$.

Furthermore, A is strongly (m, σ) -general if and only if A is (m, σ) -general and there exists $a \in \mathbf{R}$ such that the sequence $\{\lambda_i\}_{i \in I \setminus I_m: \lambda_i \neq 0}$ converges to a .

Finally, A is (m, σ) -standard if and only if A is (m, σ) -general and $a_{ij} = 0$, for any $i \in I \setminus I_m$, for any $j \in J_m$.

COROLLARY 3.16. *Let $m \in \mathbf{N}^*$, let $\emptyset \neq L \subset I$, let $\emptyset \neq N \subset J$ and let $A = (a_{ij})_{i \in I, j \in J} : E_J \rightarrow E_I$ be a linear function; then:*

1. The function $A^{(L, N)} : (E_N, \mathcal{B}_N) \rightarrow (E_L, \mathcal{B}_L)$ is measurable; in particular, $A : (E_J, \mathcal{B}_J) \rightarrow (E_I, \mathcal{B}_I)$ is measurable.
2. If A is (m, σ) -general, $I_m \subset L$ and $J_m \subset N$, then the function $\overline{A}^{(L, N)} : (E_J, \mathcal{B}_J) \rightarrow (E_I, \mathcal{B}_I)$ is measurable.

Proof. 1. From Proposition 2.13, we have $A^{(L, N)}(E_N) \subset E_L$; furthermore, from Remark 3.15, A is 1-general; moreover, we have $J_1 \subset N$ or $N \subset J \setminus J_1$; then, from Proposition 3.9, $A^{(L, N)} : (E_N, \mathcal{B}_N) \rightarrow (\mathbf{R}^L, \mathcal{B}^{(L)})$ is measurable, and so $A^{(L, N)} : (E_N, \mathcal{B}_N) \rightarrow (E_L, \mathcal{B}_L)$ is measurable; in particular, $A : (E_J, \mathcal{B}_J) \rightarrow (E_I, \mathcal{B}_I)$ is measurable.

2. The statement follows from Proposition 3.9.

□

Henceforth, we will suppose that $|I| = +\infty$. The following definitions and results (from Proposition 3.17 to Proposition 3.21) can be found in [6] and generalize the standard theory of the $m \times m$ matrices.

PROPOSITION 3.17. *Let $A = (a_{ij})_{i \in I, j \in J} : E_J \rightarrow E_I$ be a linear (m, σ) -general function; then, A is continuous.*

THEOREM 3.18. *Let $A = (a_{ij})_{i \in I, j \in J} : E_J \rightarrow E_I$ be a linear (m, σ) -general function; then, the sequence $\{\det A^{(n, n)}\}_{n \geq m}$ converges to a real number. Moreover, if $\mathcal{A} \neq \emptyset$, by setting $\overline{m} = \min \mathcal{A}$, we have*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \det A^{(n, n)} &= \sum_{p \in I \setminus I_{\overline{m}}} \left(\prod_{q \in I \setminus I_{|p|}} \lambda_q \right) \sum_{j \in J_m} a_{p, j} \left(\text{cof} A^{(|p|, |p|)} \right)_{p, j} \\ &\quad + \det A^{(\overline{m}, \overline{m})} \left(\prod_{q \in I \setminus I_{\overline{m}}} \lambda_q \right). \quad (10) \end{aligned}$$

Conversely, if $\mathcal{A} = \emptyset$, we have $\lim_{n \rightarrow +\infty} \det A^{(n,n)} = 0$.

DEFINITION 3.19. Let $A = (a_{ij})_{i \in I, j \in J} : E_J \rightarrow E_I$ be a linear (m, σ) -general function; define the determinant of A , and call it $\det A$, the real number

$$\det A = \lim_{n \rightarrow +\infty} \det A^{(n,n)}.$$

COROLLARY 3.20. Let $A = (a_{ij})_{i \in I, j \in J} : E_J \rightarrow E_I$ be a linear (m, σ) -general function such that $a_{ij} = 0, \forall i \in I_m, \forall j \in J \setminus J_m$, or A is (m, σ) -standard. Then, if σ is bijective, we have

$$\det A = \det A^{(m,m)} \prod_{i \in I \setminus I_m} \lambda_i.$$

Conversely, if σ is not bijective, we have $\det A = 0$. In particular, if $A = \mathbf{I}_{I,J}$, we have $\det A = 1$.

PROPOSITION 3.21. Let $\varphi : U \subset E_J \rightarrow E_I$ be a (m, σ) -general function and let $x_0 = (x_{0,j} : j \in J) \in U$ such that there exists the function $J_\varphi(x_0) : E_J \rightarrow E_I$; then, $J_\varphi(x_0)$ is (m, σ) -general; moreover, for any $n \in \mathbf{N}, n \geq m$, there exists the linear (m, σ) -general function $J_{\overline{\varphi}^{(n,n)}}(x_0) : E_J \rightarrow E_I$, and one has

$$\det J_\varphi(x_0) = \lim_{n \rightarrow +\infty} \det J_{\overline{\varphi}^{(n,n)}}(x_0).$$

PROPOSITION 3.22. Let $m \in \mathbf{N}^*$, let $n \in \mathbf{N}, n \geq m$, and let $\varphi : U \subset E_J \rightarrow E_I$ be a function m -general such that, for any $i \in I_n$, for any $j_1 \in J_m$ and for any $j_2 \in J_n \setminus J_m$, there exist the functions $\frac{\partial \varphi_i^{(i,m)}}{\partial x_{j_1}} : (U^{(m)}, \mathcal{B}^{(m)}(U^{(m)})) \rightarrow (\mathbf{R}, \mathcal{B})$ and $\frac{\partial \varphi_{ij_2}}{\partial x_{j_2}} : (A_{j_2}, \mathcal{B}(A_{j_2})) \rightarrow (\mathbf{R}, \mathcal{B})$, and they are measurable; then:

1. The function $\det J_{\overline{\varphi}^{(n,n)}} : (\pi_{J,J_n}(U), \mathcal{B}^{(n)}(\pi_{J,J_n}(U))) \rightarrow (\mathbf{R}, \mathcal{B})$ is measurable.
2. Suppose that φ is (m, σ) -general and, for any $i \in I \setminus I_m$, the function

$$\varphi'_{i,\sigma(i)} : (A_{\sigma(i)}, \mathcal{B}(A_{\sigma(i)})) \rightarrow (\mathbf{R}, \mathcal{B})$$

is measurable; then, for any $x \in U$, there exists the function $J_{\overline{\varphi}^{(n,n)}}(x) : E_J \rightarrow E_I$, and it is (m, σ) -general; moreover, the function $\det J_{\overline{\varphi}^{(n,n)}} : (U, \mathcal{B}^{(J)}(U)) \rightarrow (\mathbf{R}, \mathcal{B})$ is measurable.

3. Suppose that φ is (m, σ) -general and, for any $x \in U$, there exists the function $J_\varphi(x) : E_J \rightarrow E_I$; moreover, suppose that, for any $i \in I$, for any $j_1 \in J_m$ and for any $j_2 \in J \setminus J_m$, the functions

$$\frac{\partial \varphi_i^{(I,m)}}{\partial x_{j_1}} : \left(U^{(m)}, \mathcal{B}^{(m)} \left(U^{(m)} \right) \right) \longrightarrow (\mathbf{R}, \mathcal{B})$$

and $\frac{\partial \varphi_{ij_2}}{\partial x_{j_2}} : (A_{j_2}, \mathcal{B}(A_{j_2})) \longrightarrow (\mathbf{R}, \mathcal{B})$ are measurable; then the function $\det J_\varphi : (U, \mathcal{B}^{(J)}(U)) \longrightarrow (\mathbf{R}, \mathcal{B})$ is measurable.

Proof. 1. From Remark 2.6, $\forall i \in I_n, \forall j \in J_n$, the function

$$\frac{\partial \varphi_i^{(I,n)}}{\partial x_j} : \left(\pi_{J, J_n}(U), \mathcal{B}^{(n)}(\pi_{J, J_n}(U)) \right) \longrightarrow (\mathbf{R}, \mathcal{B})$$

is measurable; moreover, we have

$$\left(J_{\varphi^{(n,n)}}(x) \right)_{ij} = \frac{\partial \varphi_i^{(I,n)}}{\partial x_j}(x), \forall x \in \pi_{J, J_n}(U);$$

then, by definition of determinant, the function

$$\det J_{\varphi^{(n,n)}} : \left(\pi_{J, J_n}(U), \mathcal{B}^{(n)}(\pi_{J, J_n}(U)) \right) \longrightarrow (\mathbf{R}, \mathcal{B})$$

is measurable too.

2. If φ is (m, σ) -general, from Proposition 3.5, $\overline{\varphi}^{(n,n)}$ is (m, σ) -general too; then, from Proposition 2.13, $\forall x \in U$, there exists the function $J_{\overline{\varphi}^{(n,n)}}(x) : E_J \rightarrow E_I$, and it is (m, σ) -general, from Remark 3.15.

If $\mathcal{A}(\varphi) = \emptyset, \forall x \in U$, we have $\mathcal{A}(J_{\overline{\varphi}^{(n,n)}}(x)) = \emptyset$, and so $\det J_{\overline{\varphi}^{(n,n)}}(x) = 0$; then, the function $\det J_{\overline{\varphi}^{(n,n)}} : (U, \mathcal{B}^{(J)}(U)) \longrightarrow (\mathbf{R}, \mathcal{B})$ is measurable.

Conversely, if $\mathcal{A}(\varphi) \neq \emptyset$, set $\overline{m} = \min \mathcal{A}(\varphi), \widehat{m} = \max\{n, \overline{m}\}$; observe that $\overline{\varphi}^{(n,n)}$ is (\widehat{m}, ρ) -standard, where the bijective increasing function $\rho : I \setminus I_{\widehat{m}} \rightarrow J \setminus J_{\widehat{m}}$ is defined by $\rho(i) = \sigma(i), \forall i \in I \setminus I_{\widehat{m}}$; thus, $\forall x \in U, J_{\overline{\varphi}^{(n,n)}}(x)$ is (\widehat{m}, ρ) -standard too, and so Corollary 3.20 implies

$$\det J_{\overline{\varphi}^{(n,n)}}(x) = \det \left(J_{\overline{\varphi}^{(n,n)}} \right)^{(\widehat{m}, \widehat{m})} (x_{J_{\widehat{m}}}) \prod_{i \in I \setminus I_{\widehat{m}}} \varphi'_{i, \sigma(i)}(x_{\sigma(i)}), \forall x \in U. \quad (11)$$

If $\widehat{m} > n$, we have $\det \left(J_{\overline{\varphi}^{(n,n)}} \right)^{(\widehat{m}, \widehat{m})} (x_{J_{\widehat{m}}}) = 0$, and so $\det J_{\overline{\varphi}^{(n,n)}}(x) = 0, \forall x \in U$; then, $\det J_{\overline{\varphi}^{(n,n)}} : (U, \mathcal{B}^{(J)}(U)) \longrightarrow (\mathbf{R}, \mathcal{B})$ is measurable. Finally, if $\widehat{m} = n$, from formula (11), we have

$$\det J_{\overline{\varphi}^{(n,n)}}(x) = \det J_{\varphi^{(n,n)}}(x_{J_n}) \prod_{i \in I \setminus I_n} \varphi'_{i, \sigma(i)}(x_{\sigma(i)}), \forall x \in U;$$

moreover, from point 1, the function

$$\det J_{\varphi^{(n,n)}} : \left(\pi_{J,J_n}(U), \mathcal{B}^{(n)}(\pi_{J,J_n}(U)) \right) \longrightarrow (\mathbf{R}, \mathcal{B})$$

is measurable, and so it is $(\mathcal{B}^{(J)}(U), \mathcal{B})$ -measurable, from Remark 2.6; analogously, $\forall i \in I \setminus I_n$, $\varphi'_{i,\sigma(i)} : (U, \mathcal{B}^{(J)}(U)) \longrightarrow (\mathbf{R}, \mathcal{B})$ is measurable; then, $\forall h \in \mathbf{N}$, $h \geq n$, the function $f_h : (U, \mathcal{B}^{(J)}(U)) \longrightarrow (\mathbf{R}, \mathcal{B})$ defined by

$$f_h(x) = \det J_{\varphi^{(n,n)}}(x_{J_n}) \prod_{i \in I_h \setminus I_n} \varphi'_{i,\sigma(i)}(x_{\sigma(i)}), \quad \forall x \in U,$$

is measurable; furthermore, we have $\det J_{\varphi^{(n,n)}}(x) = \lim_{h \rightarrow +\infty} f_h(x)$, $\forall x \in U$, and so $\det J_{\varphi^{(n,n)}} : (U, \mathcal{B}^{(J)}(U)) \longrightarrow (\mathbf{R}, \mathcal{B})$ is measurable too.

3. By assumption and from point 2, $\forall n \in \mathbf{N}$, $n \geq m$, there exists the function $\det J_{\varphi^{(n,n)}} : (U, \mathcal{B}^{(J)}(U)) \longrightarrow (\mathbf{R}, \mathcal{B})$, and it is measurable; moreover, from Proposition 3.21, we have $\det J_{\varphi}(x) = \lim_{n \rightarrow +\infty} \det J_{\varphi^{(n,n)}}(x)$, $\forall x \in U$, and so $\det J_{\varphi} : (U, \mathcal{B}^{(J)}(U)) \longrightarrow (\mathbf{R}, \mathcal{B})$ is measurable. \square

PROPOSITION 3.23. *Let $m \in \mathbf{N}^*$, let $n \in \mathbf{N}$, $n \geq m$, and let $\varphi : U \subset E_J \longrightarrow E_I$ be a function m -general such that $\varphi^{(n,n)}$ is C^1 ; then, the function $\det J_{\varphi^{(n,n)}} : (\pi_{J,J_n}(U), \tau^{(n)}(\pi_{J,J_n}(U))) \longrightarrow (\mathbf{R}, \tau)$ is continuous.*

Proof. Since $\varphi^{(n,n)}$ is C^1 , $\forall i \in I_n$, $\forall j \in J_n$, the function

$$\frac{\partial \varphi_i^{(n,n)}}{\partial x_j} : \left(\pi_{J,J_n}(U), \tau^{(n)}(\pi_{J,J_n}(U)) \right) \longrightarrow (\mathbf{R}, \tau)$$

is continuous; then, by definition of determinant, the function $\det J_{\varphi^{(n,n)}} : (\pi_{J,J_n}(U), \tau^{(n)}(\pi_{J,J_n}(U))) \longrightarrow (\mathbf{R}, \tau)$ is continuous too. \square

4. Change of variables' formula

DEFINITION 4.1. *Let $k \in \mathbf{N}^*$, let $M, N \in \mathbf{R}^+$, let $a = (a_i : i \in I) \in [0, +\infty)^I$ such that $\prod_{i \in I: a_i \neq 0} a_i \in \mathbf{R}^+$, and let $v = (v_i : i \in I) \in E_I$; define the following sets in \mathcal{B}_I :*

$$E_{N,a,v}^{(k,I)} = \mathbf{R}^k \times \prod_{i \in I \setminus I_k} \left[v_i - \frac{N}{2} a_i, v_i + \frac{N}{2} a_i \right];$$

$$E_{M,N,a,v}^{(k,I)} = \prod_{h \in I_k} [v_h - M, v_h + M] \times \prod_{i \in I \setminus I_k} \left[v_i - \frac{N}{2} a_i, v_i + \frac{N}{2} a_i \right].$$

Moreover, define the σ -finite measure $\lambda_{N,a,v}^{(k,I)}$ over $(\mathbf{R}^I, \mathcal{B}^{(I)})$ in the following manner:

$$\lambda_{N,a,v}^{(k,I)} = \text{Leb}^{(k)} \otimes \left(\bigotimes_{i \in I \setminus I_k} \frac{1}{N} \text{Leb} \left(\cdot \cap \left[v_i - \frac{N}{2} a_i, v_i + \frac{N}{2} a_i \right] \right) \right).$$

LEMMA 4.2. Let $k \in \mathbf{N}^*$, let $N \in \mathbf{R}^+$, let $a = (a_i : i \in I) \in [0, +\infty)^I$ such that $\prod_{i \in I : a_i \neq 0} a_i \in \mathbf{R}^+$, and let $v = (v_i : i \in I) \in E_I$; then, for any measurable function $f : (\mathbf{R}^I, \mathcal{B}^{(I)}) \rightarrow (\mathbf{R}, \mathcal{B})$ such that f^+ (or f^-) is $\lambda_{N,a,v}^{(k,I)}$ -integrable, one has

$$\int_{\mathbf{R}^I} f d\lambda_{N,a,v}^{(k,I)} = \int_{E_{N,a,v}^{(k,I)}} f d\lambda_{N,a,v}^{(k,I)}.$$

Proof. See the proof of Lemma 46 in [5]. \square

PROPOSITION 4.3. Let $\varphi : U \subset E_J \rightarrow E_I$ be a (m, σ) -general function such that the function $\bar{\varphi}$ is bijective, and suppose that there exists $\varepsilon = (\varepsilon_i : i \in I \setminus I_m) \in [0, +\infty)^{I \setminus I_m}$ such that $|\varphi_i^{(I,m)}(x_{J_m})| \leq \varepsilon_i$, for any $i \in I \setminus I_m$, for any $x_{J_m} \in U^{(m)}$, and such that $\prod_{i \in I \setminus I_m} (1 + 2\varepsilon_i) \in \mathbf{R}^+$; moreover, let $N \in [1, +\infty)$, let $a = (a_i : i \in I) \in [0, +\infty)^I$ such that $\prod_{i \in I : a_i \neq 0} a_i \in \mathbf{R}^+$, and let $v \in E_I$; then:

1. There exist $b = (b_j : j \in J) \in [0, +\infty)^J$ and $z \in E_J$ such that $\prod_{j \in J : b_j \neq 0} b_j \in \mathbf{R}^+$ and such that, for any $l, n, k \in \mathbf{N}$, $l, n, k \geq m$, one has

$$\begin{aligned} \varphi^{-1} \left(E_{N,a,v}^{(k,I)} \right) &\subset E_{N,b,z}^{(k,J)}, \\ \left(\bar{\varphi}^{(l,n)} \right)^{-1} \left(E_{N,a,v}^{(k,I)} \right) &\subset E_{N,b,z}^{(k,J)}. \end{aligned}$$

In particular, if φ is (m, σ) -standard, the statement is true for any $N \in \mathbf{R}^+$, and one has

$$\begin{aligned} \varphi^{-1} \left(E_{N,a,v}^{(k,I)} \right) &= \left(\bar{\varphi}^{(l,n)} \right)^{-1} \left(E_{N,a,v}^{(k,I)} \right) = E_{N,b,z}^{(k,J)}, \\ \varphi^{-1} \left(E_{N,a,v}^{\circ(k,I)} \right) &= \left(\bar{\varphi}^{(l,n)} \right)^{-1} \left(E_{N,a,v}^{\circ(k,I)} \right) = E_{N,b,z}^{\circ(k,J)}. \end{aligned}$$

2. Suppose that the function φ_{ij} is continuous, for any $i \in I_m$, for any $j \in J \setminus J_m$, and the function $\varphi^{(m,m)} : (U^{(m)}, \tau^{(m)}(U^{(m)})) \longrightarrow (\mathbf{R}^m, \tau^{(m)})$ is open; then, for any $M \in \mathbf{R}^+$, there exists $O \in \mathbf{R}^+$ such that, for any $l, n, k \in \mathbf{N}$, $l, n, k \geq m$, one has

$$\begin{aligned} \varphi^{-1} \left(E_{M,N,a,v}^{(k,I)} \right) &\subset E_{O,N,b,z}^{(k,J)}, \\ \left(\bar{\varphi}^{(l,n)} \right)^{-1} \left(E_{M,N,a,v}^{(k,I)} \right) &\subset E_{O,N,b,z}^{(k,J)}. \end{aligned}$$

In particular, if φ is (m, σ) -standard, the statement is true for any $N \in \mathbf{R}^+$.

Proof. 1. Since $\bar{\varphi}$ is bijective, from Corollary 3.8, the functions $\varphi_{i,\sigma(i)}$, $\forall i \in I \setminus I_m$, and σ are bijective.

Let $N \in [1, +\infty)$, let $a = (a_i : i \in I) \in [0, +\infty)^I$ such that $\prod_{i \in I: a_i \neq 0} a_i \in \mathbf{R}^+$, let $v \in E_I$, and let $\bar{a} = (\bar{a}_i : i \in I \setminus I_m) \in [0, +\infty)^{I \setminus I_m}$, where

$$\bar{a}_i = \begin{cases} \max\{1, a_i\} & \text{if } \varepsilon_i > 0 \\ a_i & \text{if } \varepsilon_i = 0 \end{cases}, \forall i \in I \setminus I_m;$$

define $b = (b_j : j \in J) \in [0, +\infty)^J$ and $z = (z_j : j \in J) \in [0, +\infty)^J$ such that $b_j = z_j = 1$, $\forall j \in J_m$; moreover, $\forall i \in I \setminus I_m$, set

$$\begin{aligned} b_{\sigma(i)} &= \frac{\left| \varphi_{i,\sigma(i)}^{-1} \left(v_i + \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) \right) - \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) \right) \right|}{N}, \\ z_{\sigma(i)} &= \frac{\varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) \right) + \varphi_{i,\sigma(i)}^{-1} \left(v_i + \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) \right)}{2}. \end{aligned} \quad (12)$$

Observe that, $\forall i \in I \setminus I_m$, we have $b_{\sigma(i)} \neq 0$ if and only if $\bar{a}_i \neq 0$; then, since $\sigma(I \setminus I_m) = J \setminus J_m$, we have

$$\begin{aligned} \prod_{j \in J: b_j \neq 0} b_j &= \prod_{j \in J \setminus J_m: b_j \neq 0} b_j = \prod_{i \in I \setminus I_m: \bar{a}_i \neq 0} b_{\sigma(i)} \\ &= \left(\prod_{i \in I \setminus I_m: \bar{a}_i \neq 0} \frac{\left| \varphi_{i,\sigma(i)}^{-1} \left(v_i + \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) \right) - \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) \right) \right|}{N \bar{a}_i (1 + 2\varepsilon_i)} \right) \\ &\quad \cdot \left(\prod_{i \in I \setminus I_m: \bar{a}_i \neq 0} \bar{a}_i \right) \left(\prod_{i \in I \setminus I_m: \bar{a}_i \neq 0} (1 + 2\varepsilon_i) \right). \end{aligned} \quad (13)$$

Moreover, $\forall i \in I \setminus I_m$ the function $\varphi_{i,\sigma(i)}^{-1}$ is derivable on \mathbf{R} ; then, if $\bar{a}_i \neq 0$, the Lagrange theorem implies that, for some $\xi_i \in (v_i - \frac{N}{2}\bar{a}_i(1+2\varepsilon_i), v_i + \frac{N}{2}\bar{a}_i(1+2\varepsilon_i))$, we have

$$\begin{aligned} & \left| \frac{\varphi_{i,\sigma(i)}^{-1}(v_i + \frac{N}{2}\bar{a}_i(1+2\varepsilon_i)) - \varphi_{i,\sigma(i)}^{-1}(v_i - \frac{N}{2}\bar{a}_i(1+2\varepsilon_i))}{N\bar{a}_i(1+2\varepsilon_i)} \right| \\ &= \left| \left(\varphi_{i,\sigma(i)}^{-1} \right)'(\xi_i) \right| = \frac{1}{\left| \varphi'_{i,\sigma(i)}(\varphi_{i,\sigma(i)}^{-1}(\xi_i)) \right|}; \end{aligned} \quad (14)$$

furthermore, $\forall i \in I \setminus I_m$, $\varphi_{i,\sigma(i)}$ is injective, and so $\mathcal{I}_\varphi = I \setminus I_m$; then

$$\prod_{i \in I \setminus I_m: \bar{a}_i \neq 0} \left| \varphi'_{i,\sigma(i)}(\varphi_{i,\sigma(i)}^{-1}(\xi_i)) \right| = \prod_{i \in \mathcal{I}_\varphi: \bar{a}_i \neq 0} \left| \varphi'_{i,\sigma(i)}(\varphi_{i,\sigma(i)}^{-1}(\xi_i)) \right| \in \mathbf{R}^+, \quad (15)$$

from Definition 3.2. Moreover, we have

$$\begin{aligned} \prod_{i \in I \setminus I_m: \bar{a}_i \neq 0} \bar{a}_i &= \left(\prod_{i \in I \setminus I_m: a_i > 1, \varepsilon_i > 0} a_i \right) \left(\prod_{i \in I \setminus I_m: a_i \neq 0, \varepsilon_i = 0} a_i \right) \in \mathbf{R}^+, \\ &\prod_{i \in I \setminus I_m: \bar{a}_i \neq 0} (1+2\varepsilon_i) \in \mathbf{R}^+; \end{aligned}$$

then, from formulas (13), (14) and (15), we obtain $\prod_{j \in J: b_j \neq 0} b_j \in \mathbf{R}^+$.

Moreover, let $x_0 = (x_{0,j} : j \in J) \in U$; $\forall i \in I \setminus I_m$, we have

$$\begin{aligned} & \left| \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{N}{2}\bar{a}_i(1+2\varepsilon_i) \right) \right| \\ &= \left| \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{N}{2}\bar{a}_i(1+2\varepsilon_i) \right) - x_{0,\sigma(i)} + x_{0,\sigma(i)} \right| \\ &\leq \left| \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{N}{2}\bar{a}_i(1+2\varepsilon_i) \right) - \varphi_{i,\sigma(i)}^{-1}(\varphi_{i,\sigma(i)}(x_{0,\sigma(i)})) \right| + |x_{0,\sigma(i)}|; \end{aligned} \quad (16)$$

furthermore, from the Lagrange theorem, there exists $\zeta_i \in (\rho_i, \tau_i)$, where

$$\begin{aligned} \rho_i &= \min \left\{ v_i - \frac{N}{2}\bar{a}_i(1+2\varepsilon_i), \varphi_{i,\sigma(i)}(x_{0,\sigma(i)}) \right\}, \\ \tau_i &= \max \left\{ v_i - \frac{N}{2}\bar{a}_i(1+2\varepsilon_i), \varphi_{i,\sigma(i)}(x_{0,\sigma(i)}) \right\}, \end{aligned}$$

such that

$$\begin{aligned} & \left| \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) \right) - \varphi_{i,\sigma(i)}^{-1} (\varphi_{i,\sigma(i)}(x_{0,\sigma(i)})) \right| \\ &= \left| \left(\varphi_{i,\sigma(i)}^{-1} \right)' (\zeta_i) \right| \left| v_i - \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) - \varphi_{i,\sigma(i)}(x_{0,\sigma(i)}) \right| \\ &= \frac{\left| v_i - \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) - \varphi_{i,\sigma(i)}(x_{0,\sigma(i)}) \right|}{\left| \varphi'_{i,\sigma(i)}(\varphi_{i,\sigma(i)}^{-1}(\zeta_i)) \right|}, \end{aligned}$$

thus, from (16), we obtain

$$\begin{aligned} & \left| \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) \right) \right| \\ & \leq \frac{\left| v_i - \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) - \varphi_{i,\sigma(i)}(x_{0,\sigma(i)}) \right|}{\left| \varphi'_{i,\sigma(i)}(\varphi_{i,\sigma(i)}^{-1}(\zeta_i)) \right|} + |x_{0,\sigma(i)}|. \quad (17) \end{aligned}$$

We have $\sup_{i \in I \setminus I_m} \left| v_i - \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) \right| \leq \|v\|_I + \frac{N}{2} \|\bar{a}\|_I (1 + 2\|\varepsilon\|_I) < +\infty$;
 moreover, from Definition 3.2, we have

$$\begin{aligned} \sup_{i \in I \setminus I_m} |\varphi_{i,\sigma(i)}(x_{0,\sigma(i)})| &= \sup_{i \in I \setminus I_m} \left| \varphi_i^{(I \setminus I_m, J \setminus J_m)} \left((x_0)_{J \setminus J_m} \right) \right| < +\infty, \\ \inf_{i \in I \setminus I_m} \left| \varphi'_{i,\sigma(i)}(\varphi_{i,\sigma(i)}^{-1}(\zeta_i)) \right| &= \inf_{i \in \mathcal{I}_\varphi} \left| \varphi'_{i,\sigma(i)}(\varphi_{i,\sigma(i)}^{-1}(\zeta_i)) \right| > 0; \end{aligned}$$

then, there exists $c \in \mathbf{R}^+$ such that $\sup_{i \in I \setminus I_m} \left| \varphi'_{i,\sigma(i)}(\varphi_{i,\sigma(i)}^{-1}(\zeta_i)) \right|^{-1} \leq c$, and
 so formula (17) implies

$$\begin{aligned} & \sup_{i \in I \setminus I_m} \left| \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) \right) \right| \\ & \leq c \left(\sup_{i \in I \setminus I_m} \left| v_i - \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) \right| + \sup_{i \in I \setminus I_m} |\varphi_{i,\sigma(i)}(x_{0,\sigma(i)})| \right) \\ & \quad + \|x_0\|_J < +\infty. \end{aligned}$$

Analogously, we have

$$\sup_{i \in I \setminus I_m} \left| \varphi_{i,\sigma(i)}^{-1} \left(v_i + \frac{N}{2} \bar{a}_i (1 + 2\varepsilon_i) \right) \right| < +\infty;$$

then, from formula (12), we obtain that $\sup_{i \in I \setminus I_m} |z_{\sigma(i)}| < +\infty$, and so
 $z \in E_J$.

Moreover, let $k \in \mathbf{N}$, $k \geq m$, and let $x = (x_j : j \in J) \in \varphi^{-1} \left(E_{N,a,v}^{(k,I)} \right)$; $\forall i \in I \setminus I_k$, we have

$$\begin{aligned} \varphi_i^{(I,m)}(x_{J_m}) + \varphi_{i,\sigma(i)}(x_{\sigma(i)}) &= \varphi_i(x) \in \left[v_i - \frac{N}{2}a_i, v_i + \frac{N}{2}a_i \right] \\ \Rightarrow \varphi_{i,\sigma(i)}(x_{\sigma(i)}) &\in \left[v_i - \frac{N}{2}a_i - \varphi_i^{(I,m)}(x_{J_m}), v_i + \frac{N}{2}a_i - \varphi_i^{(I,m)}(x_{J_m}) \right] \\ &\subset \left[v_i - \frac{N}{2}\bar{a}_i - \varepsilon_i, v_i + \frac{N}{2}\bar{a}_i + \varepsilon_i \right]; \end{aligned}$$

moreover, since $N \geq 1$, we have $\frac{N}{2}\bar{a}_i + \varepsilon_i \leq \frac{N}{2}\bar{a}_i(1 + 2\varepsilon_i)$, and so $x_{\sigma(i)} \in [\alpha_i, \beta_i]$, where

$$\begin{aligned} \alpha_i &= \min \left\{ \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{N}{2}\bar{a}_i(1 + 2\varepsilon_i) \right), \varphi_{i,\sigma(i)}^{-1} \left(v_i + \frac{N}{2}\bar{a}_i(1 + 2\varepsilon_i) \right) \right\}, \\ \beta_i &= \max \left\{ \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{N}{2}\bar{a}_i(1 + 2\varepsilon_i) \right), \varphi_{i,\sigma(i)}^{-1} \left(v_i + \frac{N}{2}\bar{a}_i(1 + 2\varepsilon_i) \right) \right\}; \end{aligned}$$

thus, formula (12) implies

$$x_{\sigma(i)} \in \left[z_{\sigma(i)} - \frac{N}{2}b_{\sigma(i)}, z_{\sigma(i)} + \frac{N}{2}b_{\sigma(i)} \right]; \quad (18)$$

finally, since $\sigma(I \setminus I_k) = J \setminus J_k$, we obtain $\varphi^{-1} \left(E_{N,a,v}^{(k,I)} \right) \subset E_{N,b,z}^{(k,J)}$.

Furthermore, let $l, n \in \mathbf{N}$, $l, n \geq m$, and let

$$x = (x_j : j \in J) \in \left(\bar{\varphi}^{(l,n)} \right)^{-1} \left(E_{N,a,v}^{(k,I)} \right);$$

$\forall i \in I_l \setminus I_k$, since $\varphi_i(x) = \bar{\varphi}_i^{(l,n)}(x)$, by repeating the previous arguments, we have formula (18); conversely, $\forall i \in I \setminus I_l$, we have

$$\varphi_{i,\sigma(i)}(x_{\sigma(i)}) = \varphi_i(x) \in \left[v_i - \frac{N}{2}a_i, v_i + \frac{N}{2}a_i \right],$$

and so $x_{\sigma(i)} \in [\gamma_i, \delta_i]$, where

$$\begin{aligned} \gamma_i &= \min \left\{ \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{N}{2}a_i \right), \varphi_{i,\sigma(i)}^{-1} \left(v_i + \frac{N}{2}a_i \right) \right\}, \\ \delta_i &= \max \left\{ \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{N}{2}a_i \right), \varphi_{i,\sigma(i)}^{-1} \left(v_i + \frac{N}{2}a_i \right) \right\}; \quad (19) \end{aligned}$$

then, since $[\gamma_i, \delta_i] \subset [\alpha_i, \beta_i]$, we obtain formula (18) again; thus, we have $(\bar{\varphi}^{(l,n)})^{-1} \left(E_{N,a,v}^{(k,I)} \right) \subset E_{N,b,z}^{(k,J)}$.

In particular, if φ is (m, σ) -standard, $\forall i \in I \setminus I_m$, we have $\varepsilon_i = 0$, and so $\bar{a}_i = a_i$; then, $\forall N \in \mathbf{R}^+$, we have

$$\begin{aligned} \varphi_{i,\sigma(i)}^{-1} \left(\left[v_i - \frac{N}{2} a_i, v_i + \frac{N}{2} a_i \right] \right) &= [\gamma_i, \delta_i] \\ &= \left[z_{\sigma(i)} - \frac{N}{2} b_{\sigma(i)}, z_{\sigma(i)} + \frac{N}{2} b_{\sigma(i)} \right]; \quad (20) \end{aligned}$$

thus, $\forall k \in \mathbf{N}$, $k \geq m$, we obtain $\varphi^{-1} \left(E_{N,a,v}^{(k,I)} \right) = E_{N,b,z}^{(k,J)}$, $\varphi^{-1} \left(E_{N,a,v}^{\circ(k,I)} \right) = E_{N,b,z}^{\circ(k,J)}$; analogously, $\forall l, n \in \mathbf{N}$, $l, n \geq m$, from formula (20), we have $(\bar{\varphi}^{(l,n)})^{-1} \left(E_{N,a,v}^{(k,I)} \right) = E_{N,b,z}^{(k,J)}$, $(\bar{\varphi}^{(l,n)})^{-1} \left(E_{N,a,v}^{\circ(k,I)} \right) = E_{N,b,z}^{\circ(k,J)}$.

2. Suppose that the function φ_{ij} is continuous, $\forall i \in I_m$, $\forall j \in J \setminus J_m$, and the function $\varphi^{(m,m)} : (U^{(m)}, \tau^{(m)}(U^{(m)})) \longrightarrow (\mathbf{R}^m, \tau^{(m)})$ is open; since $\bar{\varphi}$ is bijective, from Corollary 3.8, $\varphi^{(m,m)}$ is bijective too; moreover, $\forall M \in \mathbf{R}^+$, consider the set

$$\bar{E}_{M,N,a,v}^{(I)} = \prod_{i \in I} \left[v_i - \frac{\bar{N}}{2} \bar{a}_i, v_i + \frac{\bar{N}}{2} \bar{a}_i \right],$$

where $\bar{N} = \max\{2M, N\} \in [1, +\infty)$, $\bar{a}_i = \max\{1, a_i\}$, $\forall i \in I$. We have $\bar{E}_{M,N,a,v}^{(I)} \subset E_{\bar{N},\bar{a},v}^{(m,I)}$, where $\bar{a} = (\bar{a}_i : i \in I) \in [1, +\infty)^I$; moreover, we have

$$\prod_{i \in I \setminus I_m : \bar{a}_i \neq 0} \bar{a}_i = \prod_{i \in I \setminus I_m : a_i > 1} a_i \in \mathbf{R}^+;$$

then, from point 1, there exist $\bar{b} = (\bar{b}_j : j \in J) \in [0, +\infty)^J$ and $\bar{z} \in E_J$ such that $\prod_{j \in J : \bar{b}_j \neq 0} \bar{b}_j \in \mathbf{R}^+$ and such that

$$\varphi^{-1} \left(\bar{E}_{M,N,a,v}^{(I)} \right) \subset \varphi^{-1} \left(E_{\bar{N},\bar{a},v}^{(m,I)} \right) \subset E_{\bar{N},\bar{b},\bar{z}}^{(m,J)};$$

then, $\forall x = (x_j : j \in J) \in \varphi^{-1} \left(\bar{E}_{M,N,a,v}^{(I)} \right)$, we have $\|x_{J \setminus J_m}\|_{J \setminus J_m} \leq \|\bar{z}\|_{J \setminus J_m} + \frac{\bar{N}}{2} \|\bar{b}\|_{J \setminus J_m} \equiv O_1 \in \mathbf{R}^+$. Moreover, $\forall i \in I_m$, we have

$$\varphi_i(x) = \varphi_i^{(m,m)}(x_{J_m}) + \sum_{j \in J \setminus J_m} \varphi_{ij}(x_j),$$

and so

$$x_{J_m} = \left(\varphi^{(m,m)} \right)^{-1} w_{I_m}, \quad (21)$$

where

$$w_i = \varphi_i(x) - \sum_{j \in J \setminus J_m} \varphi_{ij}(x_j), \quad \forall i \in I_m; \quad (22)$$

furthermore, $\forall i \in I \setminus I_m$, we have

$$\begin{aligned} \varphi_i^{(I,m)}(x_{J_m}) + \varphi_{i,\sigma(i)}(x_{\sigma(i)}) &= \varphi_i(x) \in \left[v_i - \frac{\overline{N}}{2} \overline{a}_i, v_i + \frac{\overline{N}}{2} \overline{a}_i \right] \\ \Rightarrow \varphi_{i,\sigma(i)}(x_{\sigma(i)}) &\in \left[v_i - \frac{\overline{N}}{2} \overline{a}_i - \varphi_i^{(I,m)}(x_{J_m}), v_i + \frac{\overline{N}}{2} \overline{a}_i - \varphi_i^{(I,m)}(x_{J_m}) \right] \\ &\subset \left[v_i - \frac{\overline{N}}{2} \overline{a}_i - \varepsilon_i, v_i + \frac{\overline{N}}{2} \overline{a}_i + \varepsilon_i \right], \end{aligned}$$

and so

$$x_{\sigma(i)} \in [\overline{\alpha}_i, \overline{\beta}_i] \subset A_{\sigma(i)}, \quad (23)$$

where

$$\begin{aligned} \overline{\alpha}_i &= \min \left\{ \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{\overline{N}}{2} \overline{a}_i - \varepsilon_i \right), \varphi_{i,\sigma(i)}^{-1} \left(v_i + \frac{\overline{N}}{2} \overline{a}_i + \varepsilon_i \right) \right\}, \\ \overline{\beta}_i &= \max \left\{ \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{\overline{N}}{2} \overline{a}_i - \varepsilon_i \right), \varphi_{i,\sigma(i)}^{-1} \left(v_i + \frac{\overline{N}}{2} \overline{a}_i + \varepsilon_i \right) \right\}; \end{aligned}$$

then, since $\forall i \in I_m, \forall j \in J \setminus J_m$, the function φ_{ij} is continuous, there exists $O_2 = O_2(\varphi, M, N, a, v) \in \mathbf{R}^+$ such that

$$\sup_{i \in I_m} \sum_{j \in J \setminus J_m} |\varphi_{ij}(x_j)| \leq O_2,$$

and so $\|w_{I_m}\|_{I_m} \leq \|v\|_{I_m} + \frac{\overline{N}}{2} \|\overline{a}\|_{I_m} + O_2 \equiv O_3 \in \mathbf{R}^+$, from (22);

then, since the function $(\varphi^{(m,m)})^{-1}$ is continuous, from (21), we have $\|x_{J_m}\|_{J_m} \leq O_4$, for some $O_4 = O_4(\varphi, M, N, a, v) \in \mathbf{R}^+$ such that

$$\left(\varphi^{(m,m)} \right)^{-1} ([-O_3, O_3]^m) \subset [-O_4, O_4]^m,$$

and so $\|x\|_J \leq \max\{O_1, O_4\}$. Thus, if b, z are the sequences defined by point 1, we have

$$\begin{aligned} \varphi^{-1} \left(\overline{E}_{M,N,a,v}^{(I)} \right) &\subset \prod_{j \in J} [-\max\{O_1, O_4\}, \max\{O_1, O_4\}] \\ &\subset \prod_{j \in J} [z_j - O, z_j + O], \quad (24) \end{aligned}$$

where $O \equiv \max\{O_1, O_4\} + \|z\|_J \in \mathbf{R}^+$; moreover, $\forall k \in \mathbf{N}$, $k \geq m$, we have $E_{M,N,a,v}^{(k,I)} \subset E_{N,a,v}^{(k,I)} \cap \bar{E}_{M,N,a,v}^{(I)}$; then, from formula (24), we obtain

$$\begin{aligned} \varphi^{-1} \left(E_{M,N,a,v}^{(k,I)} \right) &\subset \varphi^{-1} \left(E_{N,a,v}^{(k,I)} \right) \cap \varphi^{-1} \left(\bar{E}_{M,N,a,v}^{(I)} \right) \\ &\subset E_{N,b,z}^{(k,J)} \cap \prod_{j \in J} [z_j - O, z_j + O] \subset E_{O,N,b,z}^{(k,J)}. \end{aligned}$$

Furthermore, let $l, n \in \mathbf{N}$, $l, n \geq m$; from point 1, we have

$$\left(\bar{\varphi}^{(l,n)} \right)^{-1} \left(\bar{E}_{M,N,a,v}^{(I)} \right) \subset \left(\bar{\varphi}^{(l,n)} \right)^{-1} \left(E_{N,\bar{a},v}^{(m,I)} \right) \subset E_{N,\bar{b},\bar{z}}^{(m,J)};$$

then, $\forall x = (x_j : j \in J) \in \left(\bar{\varphi}^{(l,n)} \right)^{-1} \left(\bar{E}_{M,N,a,v}^{(I)} \right)$, we have $\|x_{J \setminus J_m}\|_{J \setminus J_m} \leq O_1$. Moreover, $\forall i \in I_m$, we have

$$\bar{\varphi}_i^{(l,n)}(x) = \varphi_i^{(m,m)}(x_{J_m}) + \sum_{j \in J_n \setminus J_m} \varphi_{ij}(x_j),$$

and so

$$x_{J_m} = \left(\varphi^{(m,m)} \right)^{-1} \bar{w}_{I_m}, \quad (25)$$

where

$$\bar{w}_i = \bar{\varphi}_i^{(l,n)}(x) - \sum_{j \in J_n \setminus J_m} \varphi_{ij}(x_j), \quad \forall i \in I_m; \quad (26)$$

furthermore, $\forall i \in I_l \setminus I_m$, since $\varphi_i(x) = \bar{\varphi}_i^{(l,n)}(x)$, we have formula (23).

Finally, $\forall i \in I \setminus I_l$, we have

$$\begin{aligned} \varphi_{i,\sigma(i)}(x_{\sigma(i)}) &= \bar{\varphi}_i^{(l,n)}(x) \in \left[v_i - \frac{\bar{N}}{2} \bar{a}_i, v_i + \frac{\bar{N}}{2} \bar{a}_i \right] \\ &\Rightarrow x_{\sigma(i)} \in [\bar{\gamma}_i, \bar{\delta}_i], \end{aligned}$$

where

$$\begin{aligned} \bar{\gamma}_i &= \min \left\{ \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{\bar{N}}{2} \bar{a}_i \right), \varphi_{i,\sigma(i)}^{-1} \left(v_i + \frac{\bar{N}}{2} \bar{a}_i \right) \right\}, \\ \bar{\delta}_i &= \max \left\{ \varphi_{i,\sigma(i)}^{-1} \left(v_i - \frac{\bar{N}}{2} \bar{a}_i \right), \varphi_{i,\sigma(i)}^{-1} \left(v_i + \frac{\bar{N}}{2} \bar{a}_i \right) \right\}; \end{aligned}$$

then, since $[\bar{\gamma}_i, \bar{\delta}_i] \subset [\bar{\alpha}_i, \bar{\beta}_i]$, we obtain formula (23) again, from which

$$\sup_{i \in I_m} \sum_{j \in J_n \setminus J_m} |\varphi_{ij}(x_j)| \leq \sup_{i \in I_m} \sum_{j \in J \setminus J_m} |\varphi_{ij}(x_j)| \leq O_2,$$

and so $\|\bar{w}_{I_m}\|_{I_m} \leq O_3$, from (26).

Then, since the function $(\varphi^{(m,m)})^{-1}$ is continuous, from (25), we have $\|x_{J_m}\|_{J_m} \leq O_4$, and so $\|x\|_J \leq \max\{O_1, O_4\}$. Thus, we have

$$\begin{aligned} \left(\bar{\varphi}^{(l,n)}\right)^{-1} \left(\bar{E}_{M,N,a,v}^{(I)}\right) &\subset \prod_{j \in J} [-\max\{O_1, O_4\}, \max\{O_1, O_4\}] \\ &\subset \prod_{j \in J} [z_j - O, z_j + O]; \end{aligned} \quad (27)$$

finally, $\forall k \in \mathbf{N}$, $k \geq m$, from point 1 and formula (27), we obtain

$$\begin{aligned} \left(\bar{\varphi}^{(l,n)}\right)^{-1} \left(E_{M,N,a,v}^{(k,I)}\right) &\subset \left(\bar{\varphi}^{(l,n)}\right)^{-1} \left(E_{N,a,v}^{(k,I)}\right) \cap \left(\bar{\varphi}^{(l,n)}\right)^{-1} \left(\bar{E}_{M,N,a,v}^{(I)}\right) \\ &\subset E_{N,b,z}^{(k,J)} \cap \prod_{j \in J} [z_j - O, z_j + O] \subset E_{O,N,b,z}^{(k,J)}. \end{aligned}$$

In particular, if φ is (m, σ) -standard, $\forall N \in \mathbf{R}^+$, $\forall l, n, k \in \mathbf{N}$, $l, n, k \geq m$, from point 1, we have

$$\varphi^{-1} \left(E_{N,a,v}^{(k,I)}\right) = \left(\bar{\varphi}^{(l,n)}\right)^{-1} \left(E_{N,a,v}^{(k,I)}\right) = E_{N,b,z}^{(k,J)};$$

moreover, we have formulas (24) and (27) again, from which

$$\begin{aligned} \varphi^{-1} \left(E_{M,N,a,v}^{(k,I)}\right) &\subset E_{O,N,b,z}^{(k,J)}, \\ \left(\bar{\varphi}^{(l,n)}\right)^{-1} \left(E_{M,N,a,v}^{(k,I)}\right) &\subset E_{O,N,b,z}^{(k,J)}. \end{aligned}$$

□

PROPOSITION 4.4. *Let (S, Σ) be a measurable space, let \mathcal{I} be a π -system on S , and let μ_1 and μ_2 be two measures on (S, Σ) , σ -finite on \mathcal{I} ; if $\sigma(\mathcal{I}) = \Sigma$ and μ_1 and μ_2 coincide on \mathcal{I} , then μ_1 and μ_2 coincide on Σ .*

Proof. See, for example, Theorem 10.3 in Billingsley [8].

□

Now, we can prove the main result of our paper, that improves Theorem 47 in [5], and generalizes the change of variables' formula for the integration of a measurable function on \mathbf{R}^m with values in \mathbf{R} (see, for example, the Lang's book [11]).

THEOREM 4.5. (*Change of variables' formula*). Let $\varphi : U \subset E_J \longrightarrow E_I$ be a bijective, continuous and (m, σ) -general function, such that $\pi_{I, I \setminus I_m} \circ \bar{\varphi}$ is continuous and such that, for any $n \in \mathbf{N}$, $n \geq m$, the function $\bar{\varphi}^{(n, n)} : U \longrightarrow E_I$ is a diffeomorphism; moreover, suppose that there exists $\varepsilon = (\varepsilon_i : i \in I \setminus I_m) \in (\mathbf{R}^+)^{I \setminus I_m}$ such that $|\varphi_i^{(I, m)}(x_{J_m})| \leq \varepsilon_i$, for any $i \in I \setminus I_m$, for any $x_{J_m} \in U^{(m)}$, and such that $\prod_{i \in I \setminus I_m} (1 + 2\varepsilon_i) \in \mathbf{R}^+$; furthermore, suppose that the sequence $\left\{ (\bar{\varphi}^{(n, n)})^{-1} \right\}_{n \geq m}$ converges uniformly to φ^{-1} over the closed and bounded subsets of E_I , and the sequence $\left\{ \det J_{\bar{\varphi}^{(n, n)}} \right\}_{n \geq m}$ converges uniformly over the closed and bounded subsets of U ; finally, let $N \in [1, +\infty)$, let $a = (a_i : i \in I) \in [0, +\infty)^I$ such that $\prod_{i \in I : a_i \neq 0} a_i \in \mathbf{R}^+$, let $v \in E_I$, and let $b \in [0, +\infty)^J$ and $z \in E_J$ defined by Proposition 4.3. Then, for any $k \in \mathbf{N}$, $k \geq m$, for any $B \in \mathcal{B}^{(I)} \left(E_{N, a, v}^{(k, I)} \right)$ and for any measurable function $f : (\mathbf{R}^I, \mathcal{B}^{(I)}) \longrightarrow (\mathbf{R}, \mathcal{B})$ such that f^+ (or f^-) is $\lambda_{N, a, v}^{(k, I)}$ -integrable, one has

$$\int_B f d\lambda_{N, a, v}^{(k, I)} = \int_{\varphi^{-1}(B)} f(\varphi) \lim_{n \rightarrow +\infty} |\det J_{\bar{\varphi}^{(n, n)}}| d\lambda_{N, b, z}^{(k, J)}.$$

In particular, assume that, for any $x \in U$, there exists the function $J_\varphi(x) : E_J \longrightarrow E_I$; then, one has

$$\int_B f d\lambda_{N, a, v}^{(k, I)} = \int_{\varphi^{-1}(B)} f(\varphi) |\det J_\varphi| d\lambda_{N, b, z}^{(k, J)}.$$

Proof. The previous assumptions imply that $\bar{\varphi}$ is bijective, φ_{ij} is continuous, $\forall i \in I_m, \forall j \in J \setminus J_m$, and $\varphi^{(m, m)} : (U^{(m)}, \tau^{(m)}(U^{(m)})) \longrightarrow (\mathbf{R}^m, \tau^{(m)})$ is open; thus, $\forall M \in \mathbf{R}^+, \forall N \in [1, +\infty), \forall a = (a_i : i \in I) \in [0, +\infty)^I$ such that $\prod_{i \in I : a_i \neq 0} a_i \in \mathbf{R}^+$, and $\forall v \in E_I$, let $O \in \mathbf{R}^+$ and let b, z be the sequences

defined by Proposition 4.3. Then, $\forall n, k \in \mathbf{N}, n \geq k \geq m, \forall B = \prod_{i \in I} B_i \in$

$\mathcal{B}^{(I)} \left(E_{M, N, a, v}^{(k, I)} \right)$ and $\forall i \in I \setminus I_n$, we have $B_i \in \mathcal{B} \left(\left[v_i - \frac{N}{2} a_i, v_i + \frac{N}{2} a_i \right] \right)$; moreover, since $(\bar{\varphi}^{(n, n)})^{-1}(B) \subset E_{N, b, z}^{(k, J)}$, we have

$$\varphi_{i, \sigma(i)}^{-1}(B_i) \in \mathcal{B} \left(\left[z_{\sigma(i)} - \frac{N}{2} b_{\sigma(i)}, z_{\sigma(i)} + \frac{N}{2} b_{\sigma(i)} \right] \right),$$

from which

$$\begin{aligned}
\int_B d\lambda_{N,a,v}^{(k,I)} &= \int_{\prod_{p \in I} B_p} d \left(Leb^{(k)} \otimes \left(\bigotimes_{q \in I \setminus I_k} \frac{1}{N} Leb \Big|_{\mathcal{B}([v_q - \frac{N}{2} a_q, v_q + \frac{N}{2} a_q])} \right) \right) \\
&= \frac{1}{N^{n-k}} \int_{\prod_{p \in I_n} B_p \times \prod_{q \in I \setminus I_n} B_q} d \left(Leb^{(n)} \otimes \left(\bigotimes_{q \in I \setminus I_n} \frac{1}{N} Leb \Big|_{\mathcal{B}([v_q - \frac{N}{2} a_q, v_q + \frac{N}{2} a_q])} \right) \right) \\
&= \frac{1}{N^{n-k}} \int_{\prod_{p \in I_n} B_p} dLeb^{(n)} \cdot \int_{\prod_{q \in I \setminus I_n} B_q} d \left(\bigotimes_{q \in I \setminus I_n} \frac{1}{N} Leb \Big|_{\mathcal{B}([v_q - \frac{N}{2} a_q, v_q + \frac{N}{2} a_q])} \right). \quad (28)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\int_{\prod_{q \in I \setminus I_n} B_q} d \left(\bigotimes_{q \in I \setminus I_n} \frac{1}{N} Leb \Big|_{\mathcal{B}([v_q - \frac{N}{2} a_q, v_q + \frac{N}{2} a_q])} \right) = \int_{\prod_{q \in I \setminus I_n} B_q} d \left(\bigotimes_{q \in I \setminus I_n} \frac{1}{N} Leb \Big|_{\mathcal{B}(B_q)} \right) \\
&= \lim_{p \rightarrow +\infty} \int_{\prod_{q \in I_p \setminus I_n} B_q} d \left(\bigotimes_{q \in I_p \setminus I_n} \frac{1}{N} Leb \Big|_{\mathcal{B}(B_q)} \right) \quad (\text{by Theorem 2.1}) \\
&= \lim_{p \rightarrow +\infty} \int_{\prod_{q \in I_p \setminus I_n} \varphi_{q,\sigma(q)}^{-1}(B_q)} \prod_{q \in I_p \setminus I_n} |\varphi'_{q,\sigma(q)}| \cdot d \left(\bigotimes_{q \in I_p \setminus I_n} \frac{1}{N} Leb \Big|_{\mathcal{B}(\varphi_{q,\sigma(q)}^{-1}(B_q))} \right) \\
&\quad (\text{since, } \forall q \in I_p \setminus I_n, \varphi_{q,\sigma(q)} \text{ is a diffeomorphism, by Proposition 3.13}) \\
&= \int_{\prod_{q \in I \setminus I_n} \varphi_{q,\sigma(q)}^{-1}(B_q)} \prod_{q \in I \setminus I_n} |\varphi'_{q,\sigma(q)}| \cdot d \left(\bigotimes_{q \in I \setminus I_n} \frac{1}{N} Leb \Big|_{\mathcal{B}(\varphi_{q,\sigma(q)}^{-1}(B_q))} \right) \\
&\quad (\text{by Theorem 2.2}) \\
&= \int_{\prod_{q \in I \setminus I_n} \varphi_{q,\sigma(q)}^{-1}(B_q)} \prod_{q \in I \setminus I_n} |\varphi'_{q,\sigma(q)}| \cdot d \left(\bigotimes_{q \in I \setminus I_n} \frac{1}{N} Leb \Big|_{\mathcal{B}([z_{\sigma(q)} - \frac{N}{2} b_{\sigma(q)}, z_{\sigma(q)} + \frac{N}{2} b_{\sigma(q)}])} \right).
\end{aligned}$$

Moreover, from Proposition 3.13, $\varphi^{(n,n)}$ is a diffeomorphism, and so formula

(28) implies

$$\begin{aligned}
 \int_B d\lambda_{N,a,v}^{(k,I)} &= \frac{1}{N^{n-k}} \int_{(\varphi^{(n,n)})^{-1}\left(\prod_{p \in I_n} B_p\right)} |\det J_{\varphi^{(n,n)}}| dLeb^{(n)} \\
 &\cdot \int_{\prod_{q \in I \setminus I_n} \varphi_{q,\sigma(q)}^{-1}(B_q)} \prod_{q \in I \setminus I_n} |\varphi'_{q,\sigma(q)}| \cdot d \left(\bigotimes_{q \in I \setminus I_n} \frac{1}{N} Leb \Big|_{\mathcal{B}([z_{\sigma(q)} - \frac{N}{2} b_{\sigma(q)}, z_{\sigma(q)} + \frac{N}{2} b_{\sigma(q)}])} \right) \\
 &= \frac{1}{N^{n-k}} \int_{(\overline{\varphi}^{(n,n)})^{-1}(B)} |\det J_{\overline{\varphi}^{(n,n)}}| d \left(Leb^{(n)} \right. \\
 &\quad \left. \otimes \left(\bigotimes_{q \in I \setminus I_n} \frac{1}{N} Leb \Big|_{\mathcal{B}([z_{\sigma(q)} - \frac{N}{2} b_{\sigma(q)}, z_{\sigma(q)} + \frac{N}{2} b_{\sigma(q)}])} \right) \right) \\
 &= \int_{(\overline{\varphi}^{(n,n)})^{-1}(B)} |\det J_{\overline{\varphi}^{(n,n)}}| d \left(Leb^{(k)} \right. \\
 &\quad \left. \otimes \left(\bigotimes_{q \in I \setminus I_k} \frac{1}{N} Leb \Big|_{\mathcal{B}([z_{\sigma(q)} - \frac{N}{2} b_{\sigma(q)}, z_{\sigma(q)} + \frac{N}{2} b_{\sigma(q)}])} \right) \right) \\
 &\quad \left(\text{since } (\overline{\varphi}^{(n,n)})^{-1}(B) \subset E_{N,b,z}^{(k,J)} \right) \\
 &= \int_{(\overline{\varphi}^{(n,n)})^{-1}(B)} |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)}. \tag{29}
 \end{aligned}$$

Consider the measures μ_1 and μ_2 on $\Sigma \equiv \mathcal{B}^{(I)}(E_{M,N,a,v}^{(k,I)})$ defined by

$$\begin{aligned}
 \mu_1(B) &= \int_B d\lambda_{N,a,v}^{(k,I)}, \\
 \mu_2(B) &= \int_{(\overline{\varphi}^{(n,n)})^{-1}(B)} |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)};
 \end{aligned}$$

from (29), μ_1 and μ_2 coincide on the set

$$\mathcal{I} = \left\{ B \in \Sigma : B = \prod_{i \in I} B_i \right\};$$

moreover, we have $\mu_1 \left(E_{M,N,a,v}^{(k,I)} \right) = \mu_2 \left(E_{M,N,a,v}^{(k,I)} \right) < +\infty$, $E_{M,N,a,v}^{(k,I)} \in \mathcal{I}$, and so μ_1 and μ_2 are σ -finite on \mathcal{I} . Then, since \mathcal{I} is a π -system on $E_{M,N,a,v}^{(k,I)}$ such that $\sigma(\mathcal{I}) = \Sigma$, from Proposition 4.4, $\forall B \in \mathcal{B}^{(I)} \left(E_{M,N,a,v}^{(k,I)} \right)$, we have

$$\int_B d\lambda_{N,a,v}^{(k,I)} = \int_{(\overline{\varphi}^{(n,n)})^{-1}(B)} |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)}. \quad (30)$$

Moreover, since $E_{M,N,a,v}^{(k,I)}$ is closed and bounded, the sequence $\left\{ (\overline{\varphi}^{(n,n)})^{-1} \right\}_{n \geq k}$ converges uniformly to φ^{-1} over $E_{M,N,a,v}^{(k,I)}$; furthermore, since φ is continuous, $\varphi^{-1} \left(E_{M,N,a,v}^{(k,I)} \right)$ is closed; then, there exist $\bar{n} \in \mathbf{N}$, $\bar{n} \geq k$, and $\delta \in \mathbf{R}^+$ such that, $\forall i > \bar{n}$, $(\overline{\varphi}^{(i,i)})^{-1} \left(E_{M,N,a,v}^{(k,I)} \right) \subset \varphi^{-1} \left(E_{M,N,a,v}^{(k,I)} \right) + \overline{B_J(0, \delta)} \subset U$, from which

$$\begin{aligned} (\overline{\varphi}^{(n,n)})^{-1} (B) &\subset \bigcup_{h \geq k} (\overline{\varphi}^{(h,h)})^{-1} \left(E_{M,N,a,v}^{(k,I)} \right) \\ &\subset \left(\bigcup_{h=k}^{\bar{n}} (\overline{\varphi}^{(h,h)})^{-1} \left(E_{M,N,a,v}^{(k,I)} \right) \right) \cup \left(\varphi^{-1} \left(E_{M,N,a,v}^{(k,I)} \right) + \overline{B_J(0, \delta)} \right), \\ &\qquad \qquad \qquad \forall n \geq k; \end{aligned}$$

then, from Proposition 4.3, $\forall n \geq k$, we have

$$\begin{aligned} (\overline{\varphi}^{(n,n)})^{-1} (B) &\subset E_{O,N,b,z}^{(k,J)} \\ &\cap \left(\left(\bigcup_{h=k}^{\bar{n}} (\overline{\varphi}^{(h,h)})^{-1} \left(E_{M,N,a,v}^{(k,I)} \right) \right) \cup \left(\varphi^{-1} \left(E_{M,N,a,v}^{(k,I)} \right) + \overline{B_J(0, \delta)} \right) \right) \\ &\equiv E_{M,N,a,v}^{(k,I,\varphi,\delta)} \subset U, \quad (31) \end{aligned}$$

and so

$$\int_{E_{M,N,a,v}^{(k,I)}} 1_B d\lambda_{N,a,v}^{(k,I)} = \int_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} 1_B (\overline{\varphi}^{(n,n)}) |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)}. \quad (32)$$

Moreover, $\forall h \in \{k, \dots, \bar{n}\}$, $\varphi^{(h,h)}$ is continuous, since from Proposition 3.13 it is a diffeomorphism; then, since $\pi_{I,I \setminus I_m} \circ \overline{\varphi}$ is continuous, from Proposition 3.10, $\overline{\varphi}^{(h,h)}$ is continuous too, and so formula (31) implies that $E_{M,N,a,v}^{(k,I,\varphi,\delta)}$ is a closed

subset of U ; furthermore, we have $E_{M,N,a,v}^{(k,I,\varphi,\delta)} \subset E_{O,N,b,z}^{(k,J)}$, and so $E_{M,N,a,v}^{(k,I,\varphi,\delta)}$ is bounded.

From formula (32), if $\psi : (\mathbf{R}^I, \mathcal{B}(\mathbf{I})) \longrightarrow ([0, +\infty), \mathcal{B}([0, +\infty)))$ is a simple function such that $\psi(x) = 0, \forall x \notin E_{M,N,a,v}^{(k,I)}$, we have

$$\int_{E_{M,N,a,v}^{(k,I)}} \psi d\lambda_{N,a,v}^{(k,I)} = \int_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} \psi(\bar{\varphi}^{(n,n)}) |\det J_{\bar{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)}.$$

Then, if $l : (\mathbf{R}^I, \mathcal{B}(\mathbf{I})) \longrightarrow ([0, +\infty), \mathcal{B}([0, +\infty)))$ is a measurable function such that $l(x) = 0, \forall x \notin E_{M,N,a,v}^{(k,I)}$, and $\{\psi_i\}_{i \in \mathbf{N}}$ is a sequence of increasing positive simple functions over $(\mathbf{R}^I, \mathcal{B}(\mathbf{I}))$ such that $\lim_{i \rightarrow +\infty} \psi_i = l, \psi_i(x) = 0, \forall x \notin E_{M,N,a,v}^{(k,I)}, \forall i \in \mathbf{N}$, from Beppo Levi theorem we have

$$\begin{aligned} \int_{E_{M,N,a,v}^{(k,I)}} l d\lambda_{N,a,v}^{(k,I)} &= \lim_{i \rightarrow +\infty} \int_{E_{M,N,a,v}^{(k,I)}} \psi_i d\lambda_{N,a,v}^{(k,I)} \\ &= \lim_{i \rightarrow +\infty} \int_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} \psi_i(\bar{\varphi}^{(n,n)}) |\det J_{\bar{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)} \\ &= \int_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} l(\bar{\varphi}^{(n,n)}) |\det J_{\bar{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)}, \quad (33) \end{aligned}$$

from which

$$\int_{E_{M,N,a,v}^{(k,I)}} l d\lambda_{N,a,v}^{(k,I)} = \lim_{n \rightarrow +\infty} \int_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} l(\bar{\varphi}^{(n,n)}) |\det J_{\bar{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)}. \quad (34)$$

In particular, formula (34) is true if $l : \mathbf{R}^I \longrightarrow [0, +\infty)$ is $(\mathcal{B}(\mathbf{I}), \mathcal{B}([0, +\infty)))$ -measurable, $(\tau^I, \tau([0, +\infty)))$ -continuous and such that $l(\mathbf{R}^I) \subset [0, 1], l(x) = 0, \forall x \notin E_{M,N,a,v}^{(k,I)}$. In this case, let $\{f_n\}_{n \geq k}$ be the sequence of the measurable functions

$$f_n : \left(E_{M,N,a,v}^{(k,I,\varphi,\delta)}, \mathcal{B}^J \left(E_{M,N,a,v}^{(k,I,\varphi,\delta)} \right) \right) \longrightarrow ([0, +\infty), \mathcal{B}([0, +\infty)))$$

given by

$$f_n(x) = l(\bar{\varphi}^{(n,n)}(x)) |\det J_{\bar{\varphi}^{(n,n)}}(x)|, \forall x \in E_{M,N,a,v}^{(k,I,\varphi,\delta)}, \forall n \geq k;$$

since $E_{M,N,a,v}^{(k,I,\varphi,\delta)}$ is closed and bounded, the sequence $\{\det J_{\bar{\varphi}^{(n,n)}}\}_{n \geq k}$ converges uniformly over $E_{M,N,a,v}^{(k,I,\varphi,\delta)}$; then, there exists $\hat{n} \in \mathbf{N}, \hat{n} \geq k$, such that, $\forall x \in$

$E_{M,N,a,v}^{(k,I,\varphi,\delta)}$, $\forall n > \widehat{n}$, we have $|\det J_{\overline{\varphi}^{(n,n)}}(x)| \leq |\det J_{\overline{\varphi}^{(\widehat{n},\widehat{n})}}(x)| + 1$; thus, since $l(\mathbf{R}^I) \subset [0, 1]$, $\forall n \geq k$, we have $|f_n| \leq |\det J_{\overline{\varphi}^{(n,n)}}| \leq g$, where

$$g : \left(E_{M,N,a,v}^{(k,I,\varphi,\delta)}, \mathcal{B}^{(J)} \left(E_{M,N,a,v}^{(k,I,\varphi,\delta)} \right) \right) \longrightarrow ([0, +\infty), \mathcal{B}([0, +\infty)))$$

is the measurable function defined by

$$g(x) = \sum_{h=k}^{\widehat{n}} |\det J_{\overline{\varphi}^{(h,h)}}(x)| + |\det J_{\overline{\varphi}^{(\widehat{n},\widehat{n})}}(x)| + 1, \quad \forall x \in E_{M,N,a,v}^{(k,I,\varphi,\delta)}. \quad (35)$$

Moreover, $\forall h \in \{k, \dots, \widehat{n}\}$, we have

$$|\det J_{\overline{\varphi}^{(h,h)}}(x)| = |\det J_{\varphi^{(h,h)}}(x_{J_h})| \prod_{i \in I \setminus I_h} \left| \varphi'_{i,\sigma(i)}(x_{\sigma(i)}) \right|, \quad \forall x \in E_{M,N,a,v}^{(k,I,\varphi,\delta)}; \quad (36)$$

furthermore, from Proposition 3.23 and Proposition 3.13, $\forall h \in \{k, \dots, \widehat{n}\}$, $\forall i \in I \setminus I_h$, the functions $\det J_{\varphi^{(h,h)}}$ and $\varphi'_{i,\sigma(i)}$ are continuous; then, since the sets $\pi_{J,J_h} \left(E_{M,N,a,v}^{(k,I,\varphi,\delta)} \right)$ and $\pi_{J,\{\sigma(i)\}} \left(E_{M,N,a,v}^{(k,I,\varphi,\delta)} \right)$ are closed and bounded, from formulas (35) and (36), there exists $\beta \in \mathbf{R}^+$ such that $g(x) \leq \beta$, $\forall x \in E_{M,N,a,v}^{(k,I,\varphi,\delta)}$; thus, by definition of $E_{M,N,a,v}^{(k,I,\varphi,\delta)}$, we have

$$\begin{aligned} \int_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} g d\lambda_{N,b,z}^{(k,J)} &\leq \beta \lambda_{N,b,z}^{(k,J)} \left(E_{M,N,a,v}^{(k,I,\varphi,\delta)} \right) \leq \beta \lambda_{N,b,z}^{(k,J)} \left(E_{O,N,b,z}^{(k,J)} \right) \\ &= \beta \prod_{p \in J_k} \text{Leb}([z_p - O, z_p + O]) \prod_{q \in J \setminus J_k} \frac{1}{N} \text{Leb} \left(\left[z_q - \frac{N}{2} b_q, z_q + \frac{N}{2} b_q \right] \right) \\ &= \beta (2O)^k \prod_{q \in J \setminus J_k} b_q < +\infty. \end{aligned}$$

Moreover, since $\lim_{i \in I, i \rightarrow +\infty} \varepsilon_i = 0$, we have $\lim_{n \rightarrow +\infty} \overline{\varphi}^{(n,n)} = \varphi$, and so

$$\lim_{n \rightarrow +\infty} f_n(x) = l(\varphi(x)) \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}^{(n,n)}}(x)|, \quad \forall x \in E_{M,N,a,v}^{(k,I,\varphi,\delta)};$$

then, from the dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} l(\overline{\varphi}^{(n,n)}) |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)} \\ = \int_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} l(\varphi) \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)}; \end{aligned}$$

consequently, from (34), we have

$$\int_{E_{M,N,a,v}^{(k,I)}} l d\lambda_{N,a,v}^{(k,I)} = \int_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} l(\varphi) \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)}. \quad (37)$$

Let $B = \prod_{i \in I} B_i \in \mathcal{B}^{(I)} \left(E_{M,N,a,v}^{(k,I)} \right)$, where $B_i = (\alpha_i, \beta_i)$, $\forall i \in I$, and let $\delta_i = \frac{\beta_i - \alpha_i}{2}$, $\forall i \in I$; moreover, $\forall h \in \mathbf{N}^*$, $\forall t \in [0, 1]$, consider the set

$$A_{h,t} = \prod_{i \in I} \left(\alpha_i + \frac{t\delta_i}{h}, \beta_i - \frac{t\delta_i}{h} \right),$$

and consider the function $l_h : \mathbf{R}^I \rightarrow [0, +\infty)$ defined by

$$l_h(x) = \begin{cases} 1 & \text{if } x \in A_{h,1}^\circ \\ t & \text{if } x \in \partial A_{h,t} \\ 0 & \text{if } x \in \mathbf{R}^I \setminus \overline{A_{h,0}} \end{cases}.$$

Observe that, $\forall h \in \mathbf{N}^*$, $l_h : \mathbf{R}^I \rightarrow [0, +\infty)$ is a function such that $l_h(\mathbf{R}^I) \subset [0, 1]$, $l_h(x) = 0$, $\forall x \notin E_{M,N,a,v}^{(k,I)}$; moreover, $\forall t_1, t_2 \in [0, +\infty)$ such that $t_1 < t_2$, we have

$$l_h^{-1}((t_1, t_2)) = \begin{cases} \emptyset & \text{if } t_1 \geq 1 \\ A_{h,t_1}^\circ & \text{if } t_1 < 1 < t_2 \\ A_{h,t_1}^\circ \setminus \overline{A_{h,t_2}} & \text{if } t_1 < t_2 \leq 1 \end{cases},$$

$$l_h^{-1}([0, t_2)) = \begin{cases} \mathbf{R}^I & \text{if } t_2 > 1 \\ \mathbf{R}^I \setminus \overline{A_{h,t_2}} & \text{if } t_2 \leq 1 \end{cases};$$

thus, l_h is $(\mathcal{B}^{(I)}, \mathcal{B}([0, +\infty)))$ -measurable and $(\tau^{(I)}, \tau([0, +\infty)))$ -continuous. Then, since $\{l_h\}_{h \in \mathbf{N}^*}$ is an increasing positive sequence such that $\lim_{h \rightarrow +\infty} l_h =$

1_B , from Beppo Levi theorem and (37), we have

$$\begin{aligned}
\int_B d\lambda_{N,a,v}^{(k,I)} &= \int_{E_{M,N,a,v}^{(k,I)}} 1_B d\lambda_{N,a,v}^{(k,I)} = \lim_{h \rightarrow +\infty} \int_{E_{M,N,a,v}^{(k,I)}} l_h d\lambda_{N,a,v}^{(k,I)} \\
&= \lim_{h \rightarrow +\infty} \int_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} l_h(\varphi) \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)} \\
&= \int_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} 1_B(\varphi) \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)} \\
&= \int_{\varphi^{-1}(B)} \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)}. \quad (38)
\end{aligned}$$

Moreover, Proposition 4.4 implies that the previous formula (38) is true $\forall B \in \mathcal{B}^{(I)} \left(E_{M,N,a,v}^{(k,I)} \right)$. Consider the measures μ and ν on $\left(E_{N,a,v}^{(k,I)}, \mathcal{B}^{(I)} \left(E_{N,a,v}^{(k,I)} \right) \right)$ defined by

$$\begin{aligned}
\mu(B) &= \int_B d\lambda_{N,a,v}^{(k,I)}, \\
\nu(B) &= \int_{\varphi^{-1}(B)} \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)},
\end{aligned}$$

and set $B_l = B \cap E_{l,N,a,v}^{(k,I)}$, $\forall l \in \mathbf{N}^*$, $\forall B \in \mathcal{B}^{(I)} \left(E_{N,a,v}^{(k,I)} \right)$. Since $B_l \subset B_{l+1}$, $\varphi^{-1}(B_l) \subset \varphi^{-1}(B_{l+1})$, $\bigcup_{l \in \mathbf{N}^*} B_l = B$ and $\bigcup_{l \in \mathbf{N}^*} \varphi^{-1}(B_l) = \varphi^{-1}(B)$, from the continuity property of μ and ν and (38), we have

$$\begin{aligned}
\int_B d\lambda_{N,a,v}^{(k,I)} &= \lim_{l \rightarrow +\infty} \int_{B_l} d\lambda_{N,a,v}^{(k,I)} \\
&= \lim_{l \rightarrow +\infty} \int_{\varphi^{-1}(B_l)} \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)} \\
&= \int_{\varphi^{-1}(B)} \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)}. \quad (39)
\end{aligned}$$

Then, let $B \in \mathcal{B}^{(I)} \left(E_{N,a,v}^{\circ(k,I)} \right)$ and let $g : (\mathbf{R}^I, \mathcal{B}^{(I)}) \longrightarrow ([0, +\infty), \mathcal{B}([0, +\infty)))$

be a measurable function; $\forall x \notin E_{N,a,v}^{\circ(k,I)}$, we have $(g1_B)(x) = 0$; thus, by proceeding as in the proof of formula (33), formula (39) implies

$$\begin{aligned} \int_B g d\lambda_{N,a,v}^{(k,I)} &= \int_{\mathbf{R}^I} 1_B g d\lambda_{N,a,v}^{(k,I)} = \int_{\mathbf{R}^J} (1_B g)(\varphi) \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}(n,n)}| d\lambda_{N,b,z}^{(k,J)} \\ &= \int_{\varphi^{-1}(B)} g(\varphi) \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}(n,n)}| d\lambda_{N,b,z}^{(k,J)}. \end{aligned}$$

Then, for any measurable function $f : (\mathbf{R}^I, \mathcal{B}^{(I)}) \longrightarrow (\mathbf{R}, \mathcal{B})$ such that f^+ (or f^-) is $\lambda_{N,a,v}^{(k,I)}$ -integrable, we have

$$\begin{aligned} \int_B f d\lambda_{N,a,v}^{(k,I)} &= \int_B f^+ d\lambda_{N,a,v}^{(k,I)} - \int_B f^- d\lambda_{N,a,v}^{(k,I)} \\ &= \int_{\varphi^{-1}(B)} f^+(\varphi) \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}(n,n)}| d\lambda_{N,b,z}^{(k,J)} \\ &\quad - \int_{\varphi^{-1}(B)} f^-(\varphi) \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}(n,n)}| d\lambda_{N,b,z}^{(k,J)} \\ &= \int_{\varphi^{-1}(B)} f(\varphi) \lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}(n,n)}| d\lambda_{N,b,z}^{(k,J)}. \quad (40) \end{aligned}$$

In particular, assume that, $\forall x \in U$, there exists the function $J_\varphi(x) : E_J \longrightarrow E_I$; from Proposition 3.21, we have

$$\lim_{n \rightarrow +\infty} |\det J_{\overline{\varphi}(n,n)}(x)| = \left| \lim_{n \rightarrow +\infty} \det J_{\overline{\varphi}(n,n)}(x) \right| = |\det J_\varphi(x)|, \forall x \in U,$$

and so formula (40) implies

$$\int_B f d\lambda_{N,a,v}^{(k,I)} = \int_{\varphi^{-1}(B)} f(\varphi) |\det J_\varphi| d\lambda_{N,b,z}^{(k,J)}.$$

□

5. Problems for further study

A natural application of this paper, in the probabilistic framework, is the development of the theory of the infinite-dimensional continuous random vari-

ables, defined in the paper [4]. In particular, we can prove the formula of the density of such random variables composed with the (m, σ) -general functions, with further properties. Consequently, it is possible to introduce many random variables that generalize the well known continuous random vectors in \mathbf{R}^m (for example, the Beta random variables in E_I defined by the (m, σ) -general matrices), and to develop some theoretical results and some applications in the statistical inference. Moreover, we can define a convolution between the laws of two independent and infinite-dimensional continuous random variables, as in the finite case.

Furthermore, in the statistical mechanics, it is possible to describe the systems of smooth hard particles, by using the Boltzmann equation (see, for example, the paper [18]), or the more general Master kinetic equation, described in the papers [17] and [16]. In order to study the evolution of these systems, we can consider the model of countable particles, such that their joint infinite-dimensional density can be determined by composing a particular random variable with a (m, σ) -general function.

Finally, we can generalize the papers [2] and [3] (where we estimate the rate of convergence of some Markov chains on $[0, p]^k$ to a uniform random vector) by considering the recursion $\{X_n\}_{n \in \mathbf{N}}$ on $[0, p]^{\mathbf{N}^*}$ defined by

$$X_{n+1} = AX_n + B_n \pmod{p},$$

where $X_0 = x_0 \in E_I$, A is a bijective, linear, integer and (m, σ) -general function, $p \in \mathbf{R}^+$, and $\{B_n\}_{n \in \mathbf{N}}$ is a sequence of independent and identically distributed random variables with values on E_I . As noted above, it is possible to determine the density of the random variable AX_n , for any $n \in \mathbf{N}^*$; consequently, we expect to prove that, with some assumptions on the law of B_n , the sequence $\{X_n\}_{n \in \mathbf{N}}$ converges with geometric rate to a random variable with law

$\bigotimes_{i \in \mathbf{N}^*} \left(\frac{1}{p} \text{Leb} \Big|_{\mathcal{B}([0, p])} \right)$, that is the uniform random variable on $[0, p]^{\mathbf{N}^*}$. Moreover, we wish to quantify the rate of convergence in terms of A , p , m , and the law of B_n .

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Author’s address:

Claudio Asci
 Dipartimento di Matematica e Geoscienze
 Università degli Studi di Trieste
 Via Valerio 12/1, 34127 Trieste, Italy
 E-mail: casci@units.it

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On elliptic curves of bounded degree in a polarized Abelian variety

LUCIO GUERRA

ABSTRACT. For a polarized complex Abelian variety A we study the function $N_A(t)$ counting the number of elliptic curves in A with degree bounded by t . This extends our previous work in dimension two. We describe the collection of elliptic curves in the product $A = S \times F$ of an Abelian variety and an elliptic curve by means of an explicit parametrization, and in terms of the parametrization we express the degrees of elliptic curves relative to a split polarization. When this is applied to the self product $A = E^k$ of an elliptic curve, it turns out that an asymptotic estimate of the counting function $N_A(t)$ can be obtained from an asymptotic study of the degree form on the group of endomorphisms of the elliptic curve.

Keywords: Elliptic curve, Abelian variety, polarization.
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1. Introduction

Let A be a complex Abelian variety, of dimension $n > 1$, endowed with a polarization. With the expression ‘elliptic curve in an Abelian variety’ we mean a one-dimensional subtorus. Every algebraic curve in A has a degree with respect to the polarization, and the following finiteness theorem holds: for every integer $t \geq 1$ the collection of elliptic curves $E \subset A$ such that $\deg(E) \leq t$ is finite. In dimension $n = 2$ this was known to Bolza and Poincaré, and a modern account is in the paper of Kani [7]. For Jacobian varieties of arbitrary dimension the theorem was proved by Tamme and was brought to an effective form in another paper of Kani [6]. For an arbitrary Abelian variety A the theorem follows from a general result proved by Birkenhake and Lange in [1], to the effect that the collection of all Abelian subvarieties with bounded exponent in A is finite.

Denote by $N_A(t)$ the number of elliptic curves in A with degree bounded by t . In a previous paper [4], we presented an approach to the counting function $N_A(t)$ in dimension $n = 2$. In the most relevant situation, when the Abelian surface is the product $E \times E'$ of two elliptic curves, the approach was based on explicit coordinates in the Néron Severi group and an explicit Diophantine

equation for the collection of elliptic curves in the Abelian surface. We have to correct an expression given in that paper for the quantity δ that is required in the main theorem (as is explained in §3.2). This leaves the statement of the theorem formally unaltered, and the same is for its proof and its consequences.

The main aim of the present paper is to study the function $N_A(t)$ in arbitrary dimension. The problem of bounding this function is invariant under isogenies, and the most relevant case is when the Abelian variety A is the self product E^k of an elliptic curve, with a split polarization (the sum of pullback polarizations from the factors). An approach to the 3-dimensional counting function, still based on explicit Diophantine equations, was investigated in [3]. Here we present a different approach, which is based on parametrization rather than equations in coordinates.

We study the collection of elliptic curves in the product $S \times F$ of an Abelian variety and an elliptic curve. We show that the subcollection consisting of the elliptic curves which are not contained in $S \times \{0\}$ and are different from $\{0\} \times F$ is bijectively parametrized by a certain set of parameter data (Theorem 5.1) and that, with respect to a split polarization, the degrees of the corresponding elliptic curves in $S \times F$ can be expressed in terms of the parameter data (Theorem 5.2). When these results on parametrization are applied to the self product E^k of an elliptic curve, it turns out that, in this case, an estimate of the counting function $N_A(t)$ can be obtained from an asymptotic study of the degree form $f \mapsto \deg(f)$ on the group of endomorphisms of the elliptic curve (that is provided in Proposition 4.1). The tool for this is the same result from Number Theory, concerning the number of lattice points in a bounded region in the real plane, that was used in the previous work on the 2-dimensional counting function.

Here is the fundamental information that is needed in our asymptotic estimate of the counting function. Define:

- m the minimum of the degrees of the factors of E^k , the various copies of E , with respect to the given polarization;
- d the minimum degree of an isogeny $E \rightarrow E$;
- δ when the elliptic curve has complex multiplication, the (negative) discriminant of the degree form on the endomorphism group $\text{End}(E)$.

Assume moreover that $k \geq 2$. In terms of these data, we prove (in §6) the following main result:

THEOREM 1.1. *There is an asymptotic estimate*

$$N_{E^k}(t) = C t^r + O(t^i),$$

where $r = k$ if the curve admits no complex multiplication, and $r = 2k - 1$ if the curve has complex multiplication, the constant C being given by

$$\frac{2^k/(k+1)}{(\sqrt{d})^{k-1} m^k} \text{ for } r = k, \quad \frac{(2\pi)^{k-1}/k}{(\sqrt{-\delta})^{k-1} m^{2k-1}} \text{ for } r = 2k - 1,$$

the exponent i being

$$k - 1 \text{ for } r = k, \quad 2k - 3 + 2e \text{ for } r = 2k - 1,$$

where $e = 33/104 = 0.317\dots$.

Finally we show that the result above for the self product of an elliptic curve implies some result holding for an arbitrary polarized Abelian variety (Proposition 7.1).

2. Some preliminary material

2.1. Elliptic curves as homology classes

Let A be an Abelian variety, of dimension $n > 1$. Every curve $C \subset A$ determines a homology class $[C]$ in $H_2(A, \mathbb{Z})$. For elliptic curves (subgroups), the induced correspondence

$$\{\text{elliptic curves in } A\} \longrightarrow H_2(A, \mathbb{Z})$$

is injective and the homology classes $\gamma = [C]$ in $H_2(A, \mathbb{Z})$ corresponding to elliptic curves in A satisfy the following basic properties:

- γ is primitive (indivisible),
- $\gamma \cdot H > 0$ for some (every) ample divisor H .

These results are certainly well known (the last property is obvious), however a proof can be found in [3], §2.

In dimension $n = 2$, the homology classes of elliptic curves in the Abelian surface A belong to the Néron Severi group $NS(A) \hookrightarrow H_2(A, \mathbb{Z})$, and are characterized in that group by means of the two properties above together with the numerical condition (cf. [7], Theorem 1.1):

- $\gamma \cdot \gamma = 0$.

2.2. Degree with respect to a polarization

Let L in $NS(A)$ be an ample divisor class, representing a polarization of A . For every curve $C \subset A$ the degree with respect to the polarization is

$$\text{deg}(C) := C \cdot L.$$

Let A be a polarized Abelian variety (we usually omit an explicit reference to the polarization). The following is a classical result: for every integer $t \geq 1$

the collection of elliptic curves $E \subset A$ such that $\deg(E) \leq t$ is finite. It is a consequence of a general result proved by Birkenhake and Lange in [1], to the effect that the collection of all Abelian subvarieties with bounded exponent in A is finite.

Let us recall some definitions. The polarization defines a natural isogeny $\phi : A \rightarrow \widehat{A}$ to the dual variety. The order of $\ker(\phi)$ is the degree of the polarization and the exponent of $\ker(\phi)$ is called the exponent of the polarization on A . Clearly the exponent is a divisor of the degree. For an Abelian subvariety E of A one has the exponent and the degree of the induced polarization. If E is an elliptic curve in A we know that the degree of the curve is equal to the degree of the induced polarization. So elliptic curves with bounded degree have bounded exponent, and the theorem follows.

We define the function

$$N_A(t)$$

counting the number of elliptic curves in A with degree bounded by t .

2.3. Product Abelian surfaces

Consider an Abelian surface of the form $E \times E'$ where E, E' are elliptic curves. There is a natural isomorphism

$$\mathbb{Z}^2 \oplus \text{Hom}(E, E') \xrightarrow{\sim} \text{NS}(E \times E')$$

$$(a, b; f) \longmapsto (b-1)[E_h] + (a - \deg(f))[E'_v] + [\Gamma_{-f}],$$

where $E_h := E \times \{0\}$ and $E'_v := \{0\} \times E'$ are the ‘horizontal’ and the ‘vertical’ factor, and Γ_{-f} is the graph of the homomorphism $-f$. The intersection form on $\text{NS}(E \times E')$ is expressed as

$$D \cdot D' = ab' + ba' - (\deg(f + f') - \deg(f) - \deg(f'))$$

if the divisors D and D' arise as above from the data $(a, b; f)$ and $(a', b'; f')$.

This is a special case of the description of correspondences between two curves in terms of homomorphisms between the associated Jacobian varieties (cf. e.g. [2], Theorem 11.5.1) and also is a special case of a result of Kani ([8], Proposition 61) for the Néron Severi group of a product Abelian variety.

2.4. Elliptic curves in a product Abelian surface

Using the description of $\text{NS}(E \times E')$ in §2.3 above and the characterization of elliptic curves in an Abelian surface in §2.1, we can now describe the collection of elements $(a, b; f)$ in the group $\mathbb{Z}^2 \oplus \text{Hom}(E, E')$ such that the corresponding divisor class $[D]$ is the class of an elliptic curve in $E \times E'$.

Besides the condition of primitivity of the element $(a, b; f)$, the numerical condition $D \cdot D = 0$ becomes

$$ab = \deg(f)$$

and the positivity condition $D \cdot H > 0$ is equivalent to

$$a + b > 0$$

(using the ample divisor $H := E_h + E'_v$).

If on $E \times E'$ we choose a split polarization $L = mE_h + nE'_v$, where m, n are positive integers, then the degree of divisors with respect to the polarization is given by the linear function

$$\deg(D) = am + bn$$

if D corresponds to $(a, b; f)$.

When E and E' are not isogenous then clearly E_h and E'_v are the only elliptic curves in $E \times E'$. When E and E' are isogenous, the graphs of homomorphisms $E \rightarrow E'$ form an infinite collection of elliptic curves in $E \times E'$.

2.5. Reducibility

We will make use of the Poincaré reducibility theorem with respect to a polarization, in the following form.

If A is a polarized Abelian variety and B is an Abelian subvariety of A , there is a unique Abelian subvariety B' of A such that the sum homomorphism $B \times B' \rightarrow A$ is an isogeny and the pullback polarization on $B \times B'$ is the sum of the pullback polarizations from B and B' (cf. [2], Theorem 5.3.5 and Corollary 5.3.6).

2.6. A result from Number Theory

The following is a classical problem in Number Theory, originating from Gauss' circle problem. Given a compact convex subset K in \mathbb{R}^2 , estimate the number $N := \text{card}(\mathbb{Z}^2 \cap K)$ of integer vectors (or lattice points) belonging to the convex set. This number is naturally approximated by the area A of the subset, and then the question is to estimate the (error or) discrepancy $N - A$. The following estimate is due to Nosarzewska [9]. If K is a compact convex region in \mathbb{R}^2 of area A whose boundary is a Jordan curve of length L then

$$N \leq A + \frac{1}{2}L + 1.$$

We will apply this result through the following consequence. For every scale factor $t \in \mathbb{R}_{\geq 0}$ denote by $N(t)$ the number of lattice points in the deformed region $\sqrt{t}K$. Then

$$N(t) \leq At + \frac{L}{2}t^{1/2} + 1.$$

The inequality above is valid for arbitrary t . But in an asymptotic estimate

$$N(t) = At + O(t^e)$$

(an implicit inequality holding for $t \gg 0$) the exponent e may be lowered, and precisely one can take

$$e = 33/104 = 0.317\dots$$

This follows from a result of Huxley [5].

3. Summary of previous results, with correction

3.1. The homomorphism group and the degree form

For the basic theory of elliptic curves we refer to [10]. Let E, E' be elliptic curves. The homomorphism group $\text{Hom}(E, E')$ is a free abelian group of rank at most 2, and the degree map

$$\text{Hom}(E, E') \longrightarrow \mathbb{Z}$$

such that $f \mapsto \deg(f)$ is a quadratic form.

Assume now that the elliptic curves E, E' are isogenous, i.e. that the group $\text{Hom}(E, E')$ has rank > 0 . Denote by

d the minimum nonzero value of the degree form

and let $\varphi : E \rightarrow E'$ be an isogeny of minimum degree d .

If the group $\text{Hom}(E, E')$ has rank 1, one has the isomorphism $\mathbb{Z} \xrightarrow{\sim} \text{Hom}(E, E')$ given by $x \mapsto x\varphi$. For every $x \in \mathbb{Z}$ one has $\deg(x\varphi) = x^2d$ and this describes the degree form.

Assume now that $\text{Hom}(E, E')$ has rank 2. This happens if and only if E has complex multiplication, and the same is for E' . In this case there is an isomorphism $\mathbb{Z}^2 \xrightarrow{\sim} \text{Hom}(E, E')$ and the degree form is expressed as a binary quadratic form. So, when the elliptic curves have complex multiplication, we denote by

δ the discriminant of the degree form.

Explicit descriptions of the homomorphism group and the degree form, in presence of complex multiplication, are given in §3.3.

Remark that both d and δ only depend on the unordered pair E, E' . This is because the isomorphism $\text{Hom}(E, E') \longleftrightarrow \text{Hom}(E', E)$, sending a homomorphism f to the dual homomorphism \widehat{f} , preserves the degree forms.

3.2. Estimate for the counting function

Consider a product Abelian surface $E \times E'$ endowed with a split polarization. Assume that E, E' are isogenous elliptic curves. Together with the invariants d and δ introduced in §3.1 above, we also define

m the minimum of $\deg(E)$ and $\deg(E')$, the degrees with respect to the polarization.

THEOREM 3.1. *If E, E' are isogenous, there is an asymptotic estimate*

$$N_{E \times E'}(t) = C t^{r-1} + O(t^i),$$

with $r = 3$ when E, E' admit no complex multiplication and $r = 4$ when E, E' have complex multiplication, the constant C being given by

$$\frac{\pi}{4\sqrt{d} m^2} \text{ for } r = 3, \quad \frac{\pi}{3\sqrt{-\delta} m^3} \text{ for } r = 4,$$

the exponent i being

$$0 \text{ for } r = 3, \quad \frac{85}{52} = 1.634\dots \text{ for } r = 4.$$

The proof given in [4], §5.2, actually works without modification with the new definition of the quantity δ and independently of the order chosen for the factor curves. While the expression for δ given in [*ibid.*], §3.2, needs to be corrected, as is explained in the following section.

REMARK 3.2: In the statement of Theorem 3.1 the exponent which gives the order of growth of the asymptotic estimate has been written as $r - 1$ in order to remind the interpretation of r as the rank of the group $NS(E \times E')$. In higher dimensions such a purpose seems not to be meaningful any more. It must be noticed moreover that the estimate in Theorem 3.1 is slightly sharper than the estimate which is obtained from Theorem 1.1 in the special case $k = 2$ (in the earlier estimate the numerical part of the constant C is smaller and the exponent i is smaller in the case with no complex multiplication).

3.3. Computing the degree form

We use the representation of an elliptic curve E as the quotient \mathbb{C}/Λ where $\Lambda = \langle 1, \tau \rangle$ is the lattice in \mathbb{C} associated to a modulus τ for E , a complex number with positive imaginary part, that is determined up to the natural action of $SL(2, \mathbb{Z})$.

Let E and E' be elliptic curves, that we identify with \mathbb{C}/Λ and \mathbb{C}/Λ' with $\Lambda = \langle 1, \tau \rangle$ and $\Lambda' = \langle 1, \tau' \rangle$ for suitable moduli τ and τ' . There is the natural identification

$$Hom(E, E') \longleftrightarrow \{\alpha \in \mathbb{C} \text{ s.t. } \alpha\Lambda \subseteq \Lambda'\} =: \mathcal{H}.$$

Assume that there is an isogeny $E \rightarrow E'$. In this case, according to [4], Lemma 3.1, we can choose moduli τ and τ' such that

$$\tau' = \ell\tau \quad \text{and} \quad \ell = \frac{p}{q}$$

with p, q coprime positive integers. If the homomorphism group $\text{Hom}(E, E')$ has rank 1 the situation is clear (see §3.1).

Assume now that $\text{Hom}(E, E')$ has rank 2. Then E has complex multiplication, and the same is for E' . Therefore the modulus τ is algebraic of degree 2 over \mathbb{Q} . So, assume that τ satisfies the equation

$$\tau^2 + \frac{u}{w}\tau + \frac{v}{w} = 0$$

with u, v, w in \mathbb{Z} such that $w > 0$ and $(u, v, w) = (1)$ and moreover

$$u^2 - 4vw < 0$$

as τ is an imaginary complex number.

REMARK 3.3: In the previous paper [4] in Lemma 3.4 we made a wrong assertion (the error in the proof is the claim that certain three coefficients are always coprime). Although some (slightly different) statement of the same kind is nevertheless true, it turns out to be however unnecessary for the purposes of the paper. This is because the subsequent statements, Proposition 3.5 and Proposition 3.6, and their proofs, can be slightly modified so to provide general expressions for the degree form and its discriminant. The new statements are given just below.

From the pairs w, p and v, q , dividing in each pair by the greatest common divisor, we obtain coprime pairs

$$\bar{w}, \bar{p} \quad \text{and} \quad \bar{v}, \bar{q}.$$

Moreover, since p, q are coprime, we can write

$$u = pp' + qq'$$

for suitable integers p', q' .

PROPOSITION 3.4. *An explicit isomorphism $\mathbb{Z}^2 \xrightarrow{\sim} \mathcal{H}$ is given by*

$$(x, y) \longmapsto (xp + y\bar{p}\bar{q}q') + (y\bar{w}\bar{q})(\ell\tau).$$

Proof. The part of the proof which has to be adjusted is the analysis of the conditions for a complex number $\alpha = a + b(\ell\tau)$ to be an element of \mathcal{H} , namely the conditions that the rational numbers $b(p/q)(v/w)$ and $a(q/p) - b(u/w)$ be

integers. In particular, this requires that $bp(v/w)$ and $bp(u/w)$ are integers. Since u, v, w are coprime, it follows that $w \mid bp$ and hence that $\bar{w} \mid b$.

The full condition $(bp/w)(v/q) \in \mathbb{Z}$ means that $q \mid (bp/w)v$, that is $\bar{q} \mid (bp/w) = (b/\bar{w})\bar{p}$, and hence that $\bar{q} \mid (b/\bar{w})$ and $\bar{w}\bar{q} \mid b$. So the first condition above is satisfied if and only if one has $b = \bar{w}\bar{q}y$ with $y \in \mathbb{Z}$.

Then the second condition above requires that $aq - (bp/w)u$ belongs to $p\mathbb{Z}$, that is $aq - (\bar{p}\bar{q})yu \in p\mathbb{Z}$, that is $(\bar{p}\bar{q})yu = aq + a'p$ for some integer a' . Since p, q are coprime, the solutions are of the form $(a', a) = \bar{p}\bar{q}y(p', q') + x(-q, p)$ with $x \in \mathbb{Z}$. Thus $a = xp + y\bar{p}\bar{q}q'$, as in the statement. \square

PROPOSITION 3.5. *The degree of the homomorphism $f : E \rightarrow E'$ corresponding to $(x, y) \in \mathbb{Z}^2$ is given by*

$$\deg(f) = x^2(pq) + xy(\bar{p}\bar{q})(qq' - pp') + y^2(\bar{p}\bar{q})(-\bar{p}\bar{q}p'q' + \bar{v}\bar{w}).$$

The discriminant of the quadratic form $f \mapsto \deg(f)$ on $\text{Hom}(E, E')$ is equal to

$$\delta = (\bar{p}\bar{q})^2(u^2 - 4vw).$$

Proof. What is only to be adjusted is the computation of $\deg(f)$ as

$$\begin{vmatrix} a & -b\ell(v/w) \\ b & (a/\ell) - b(u/w) \end{vmatrix} = \begin{vmatrix} xp + y\bar{p}\bar{q}q' & -y\bar{p}\bar{v} \\ y\bar{w}\bar{q} & xq - y\bar{p}\bar{q}p' \end{vmatrix},$$

where we used the expressions for a, b given in the previous proposition: it leads to the expression given in the statement. It is also easy to calculate the discriminant δ of this quadratic form in x, y . \square

When the elliptic curves are isomorphic, the preceding formulas are simplified. In this case we have $p = q = 1$ and we can choose $p' = 0, q' = u$. Thus, in this particular case, the expressions given in the previous paper are indeed correct.

4. Homomorphisms with bounded degree

We present a result on the asymptotic behavior of the degree form

$$\text{Hom}(E, E') \longrightarrow \mathbb{Z}$$

that will be needed in the following. Define

$$\Phi(t)$$

to be the number of homomorphisms f having $\deg(f) \leq t$.

PROPOSITION 4.1. *Let E, E' be isogenous elliptic curves. The function $\Phi(t)$ admits the following asymptotic estimates:*

(i) *if E, E' are without complex multiplication then*

$$\Phi(t) = \frac{2}{\sqrt{d}}t^{1/2} + O(1)$$

where d is the minimum nonzero value of the degree form;

(ii) *if E, E' have complex multiplication then*

$$\Phi(t) = \frac{2\pi}{\sqrt{-\delta}}t + O(t^e)$$

where δ is the discriminant of the degree form and e is the exponent appearing in §2.6.

Proof. We have seen in §3.1 how the degree form $f \mapsto \deg(f)$ can be expressed in terms of coordinates. (i) In this case there is one coordinate x and the degree form is expressed as $x \mapsto x^2d$; the inequality $x^2d \leq t$ admits precisely $2 \left\lfloor \frac{1}{\sqrt{d}}t^{1/2} \right\rfloor + 1$ solutions. (ii) In this case, in terms of two coordinates, the degree form is expressed as a positive definite quadratic form $Q(x, y)$ with discriminant $\delta < 0$. Because of the result from Number Theory quoted in §2.6, the number of integer solutions of the inequality $Q(x, y) \leq t$ admits an estimate of the form $At + O(t^e)$ where A is the area of the ellipse $Q(x, y) \leq 1$ in \mathbb{R}^2 , that is given by $2\pi/\sqrt{-\delta}$. \square

5. Elliptic curves in a product Abelian variety

Let S be an Abelian variety, let F be an elliptic curve, and consider the product Abelian variety $A = S \times F$. We denote, for an arbitrary Abelian variety, with the symbol

$$\text{EC}(A)$$

the collection of homology classes $\gamma = [C]$ in $H_2(A, \mathbb{Z})$ corresponding to elliptic curves C in A . We now describe the collection $\text{EC}(A)$ for a product Abelian variety $A = S \times F$. Denote by $S_h := S \times \{0\}$ and by $F_v := \{0\} \times F$ the horizontal and the vertical factors in A .

If C is an elliptic curve in A , different from F_v , then $D = pr_1(C)$ is an elliptic curve in S , corresponding to an element $\gamma = [C]$ in the Néron Severi group $NS(D \times F)$. This group is described (see §2.3) by means of an isomorphism

$$\mathbb{Z}^2 \oplus \text{Hom}(D, F) \xrightarrow{\sim} NS(D \times F).$$

There is moreover the composite isomorphism

$$\mathbb{Z}^2 \oplus \text{Hom}(F, D) \xrightarrow{\sim} \text{NS}(F \times D) \xrightarrow{\sim} \text{NS}(D \times F)$$

where the right hand arrow is induced by the obvious isomorphism $j : F \times D \rightarrow D \times F$, and this composite isomorphism turns out to be

$$(u, v; g) \mapsto (u - \deg(g))[D_h] + (v - 1)[F_v] + [j_*\Gamma_{-g}].$$

In order to take into account at one time all possible elliptic curves D in S , we introduce the product $\mathbb{Z}^2 \times \text{Hom}'(F, S)$, where the superscript means nonzero homomorphisms, and the correspondence

$$C : \mathbb{Z}^2 \times \text{Hom}'(F, S) \rightarrow H_2(S \times F, \mathbb{Z})$$

$$C(u, v; g) := (u - \deg(g))[D(g)_h] + (v - 1)[F_v] + [j_*\Gamma_{-g}]$$

where by definition $D(g) = g(F)$ and $\deg(g)$ denotes the degree of the induced isogeny $F \rightarrow g(F)$. Here j denotes the obvious isomorphism $j : F \times S \rightarrow S \times F$.

So we define the set of “parameter data”

$$D(S \times F)$$

consisting of all elements $(u, v; g)$ in $\mathbb{Z}^2 \times \text{Hom}'(F, S)$ such that

$$(u, v; g) \text{ is primitive, } uv = \deg(g) \text{ and } u + v > 0.$$

Here the word primitive clearly refers to the module $\mathbb{Z}^2 \oplus \text{Hom}(F, S)$.

THEOREM 5.1. *There is a bijective correspondence*

$$D(S \times F) \longleftrightarrow \text{EC}(S \times F) \setminus \left(\text{EC}(S_h) \cup \{[F_v]\} \right)$$

induced by the correspondence C defined above.

Proof. Let C be an elliptic curve in $S \times F$ different from F_v . The projection of C into S is an elliptic curve D and the class of C in $\text{NS}(D \times F)$ is represented by a divisor of the form $(u - \deg(f))D_h + (v - 1)F_v + j_*\Gamma_{-f}$ where f is a homomorphism $F \rightarrow D$ and the conditions $(u, v; f)$ primitive and $uv = \deg(f)$ are satisfied. Note that $f = 0$ if and only if $C = D_h$ is contained in S_h (since $C \neq F_v$). Because the condition $uv = \deg(f) = 0$ admits two primitive solutions, $(1, 0; 0)$ and $(0, 1; 0)$, corresponding to the classes of D_h and F_v , respectively. If g denotes the composite homomorphism $F \rightarrow D \hookrightarrow S$ and if $f \neq 0$ then $D(g) = D$ and the class of C arises from the element $(u, v; g)$

belonging to $D(S \times F)$. This shows that the correspondence in the statement is surjective.

In order to prove that the correspondence is injective, consider the homomorphism $H_2(S \times F, \mathbb{Z}) \rightarrow H_2(S, \mathbb{Z})$ induced by the first projection map. It maps $[C(u, v; g)] \mapsto u[D(g)]$. If $(u, v; g)$ and $(u', v'; g')$ define the same class in $H_2(S \times F, \mathbb{Z})$ then $u[D(g)] = u'[D(g')]$. Since $g, g' \neq 0$ then $u, u' \neq 0$ and therefore $[D(g)] = [D(g')]$ and $u = u'$, as the class of an elliptic curve is primitive. And then $D(g) = D(g')$ since the homology class uniquely determines the elliptic curve. Furthermore, working with the homomorphism $H_2(S \times F, \mathbb{Z}) \rightarrow H_2(F, \mathbb{Z})$ induced by the second projection map, we also find that $v = v'$.

Let D be the elliptic curve $D(g) = D(g')$. The inclusion $D \hookrightarrow S$ induces injective homomorphisms $H_i(D, \mathbb{Z}) \rightarrow H_i(S, \mathbb{Z})$ for $i = 1, 2$. Therefore the homomorphism $H_2(D \times F, \mathbb{Z}) \rightarrow H_2(S \times F, \mathbb{Z})$ is injective too. Hence $(u, v; g)$ and $(u', v'; g')$ define the same class in $H_2(D \times F, \mathbb{Z})$ and it follows that $g = g'$ also holds. \square

Assume now that on $A = S \times F$ we are given a split polarization

$$L = \Theta_S + n\Theta_F$$

where Θ_S and Θ_F denote the pullbacks to A of a polarization on S and the principal polarization on F , respectively, and where n is a positive integer. If the polarization on S is represented by Θ then Θ_S is represented by $\Theta \times F$; similarly, Θ_F is represented by $S \times \{0\}$.

We also consider the particular case when $S = E_1 \times \cdots \times E_k$ is a product of elliptic curves, endowed with a split polarization

$$\Theta_S = m_1\Theta_1 + \cdots + m_k\Theta_k,$$

where Θ_i denotes the pullback to A of the principal polarization on the i th factor and the coefficients m_i are positive integers. Note that in this case a homomorphism $g : F \rightarrow S$ is given by a sequence h_1, \dots, h_k of homomorphisms $h_i : F \rightarrow E_i$.

THEOREM 5.2. *The degree function $D(S \times F) \rightarrow \mathbb{Z}$ is given by*

$$\deg C(u, v; g) = u D(g) \cdot \Theta + n v.$$

In the particular case $S = E_1 \times \cdots \times E_k$, one has the expression

$$\deg C(u, v; g) = \frac{m_1 \deg(h_1) + \cdots + m_k \deg(h_k)}{v} + n v.$$

Proof. We need the following intersection numbers:

$$\begin{aligned} D(g)_h \cdot L &= D(g) \cdot \Theta, \\ F_v \cdot L &= n, \\ j_* \Gamma_{-g} \cdot L &= \deg g^*(\Theta) + n = \deg(g) D(g) \cdot \Theta + n. \end{aligned}$$

Therefore for the intersection number $C(u, v; g) \cdot L$ we find the expression

$$(u - \deg(g)) D(g) \cdot \Theta + (v - 1)n + \deg(g) D(g) \cdot \Theta + n = u D(g) \cdot \Theta + n v.$$

In the particular case, we need the following intersection number:

$$\begin{aligned} D(g) \cdot \Theta &= m_1 \#g h_1^{-1}(0) + \cdots + m_k \#g h_k^{-1}(0) \\ &= m_1 \frac{\deg(h_1)}{\deg(g)} + \cdots + m_k \frac{\deg(h_k)}{\deg(g)}. \end{aligned}$$

Hence, because of the condition $uv = \deg(g) \neq 0$, we have for $\deg C(u, v; g)$ the expression

$$\begin{aligned} u \frac{m_1 \deg(h_1) + \cdots + m_k \deg(h_k)}{\deg(g)} + n v \\ = \frac{m_1 \deg(h_1) + \cdots + m_k \deg(h_k)}{v} + n v. \end{aligned}$$

□

6. On the number of elliptic curves

Let $A = E^k$, with $k \geq 2$, be the k th self product of an elliptic curve E , endowed with a split polarization $L = m_1 \Theta_1 + \cdots + m_k \Theta_k$, where Θ_i denotes the pullback to A of the principal polarization on the i th factor and the coefficients m_i are positive integers.

We keep the notation of §5, writing $A = E^{k-1} \times E$, and defining $(E^{k-1})_h := E^{k-1} \times \{0\}$ and $E_v := \{(0, \dots, 0)\} \times E$. Let moreover m be the minimum among the coefficients m_1, \dots, m_k .

The set $\text{EC}(E^k)$ is the disjoint union of $\text{EC}((E^{k-1})_h) \cup \{[E_v]\}$ and the complementary subset which, according to Proposition 5.1, is bijective to the set of parameter data $\text{D}(E^k)$. It follows that the number $N_{E^k}(t)$ is, for $t \gg 0$, the sum of

$$N_{E^{k-1}}(t) + 1$$

and the number of elements of the set

$$\left\{ (u, v; g) \text{ in } \text{D}(E^k) \text{ s.t. } \deg C(u, v; g) \leq t \right\}.$$

There is the following chain of injective maps

$$\begin{aligned} & \left\{ (u, v; g) \text{ prim. s.t. } u + v > 0, \ uv = \deg(g) \neq 0 \text{ and } \deg C(u, v; g) \leq t \right\} \\ & \quad \downarrow \\ & \left\{ (v; h_1, \dots, h_{k-1}) \text{ s.t. } 1 \leq m_k v \leq t, \right. \\ & \qquad \qquad \qquad \left. 1 \leq \frac{m_1 \deg(h_1) + \dots + m_{k-1} \deg(h_{k-1})}{v} \leq t \right\} \\ & \quad \downarrow \\ & \left\{ (v; h_1, \dots, h_{k-1}) \text{ s.t. } 1 \leq v \leq \frac{t}{m}, \ \deg(h_i) \leq v \frac{t}{m} \right\} \end{aligned}$$

so the number of elements of the set of parameter data in the top of the chain is bounded above by

$$\sum_{1 \leq v \leq \frac{t}{m}} \Phi \left(\frac{vt}{m} \right)^{k-1}$$

where $\Phi(t)$ is the function which counts endomorphisms of E having degree bounded by t . Let us denote the bounding function above with the symbol

$$\Phi_{E^k}(t).$$

LEMMA 6.1. *There is an asymptotic estimate*

$$\Phi_{E^k}(t) = C t^r + O(t^i)$$

with the same constant C and the same exponents r and i which are defined in the statement of Theorem 1.1.

Proof. This is obtained applying Proposition 4.1 for the function Φ which appears in the definition of Φ_{E^k} .

If the elliptic curve E admits no complex multiplication, for the bounding function we have the expression

$$\sum_{1 \leq v \leq \frac{t}{m}} \left\{ \frac{2}{\sqrt{d}} \left(\frac{vt}{m} \right)^{1/2} + O(1) \right\}^{k-1}$$

where the integer d is the minimum degree of an isogeny $E \rightarrow E$. The expression above can be written as

$$\left(\frac{2}{\sqrt{dm}} \right)^{k-1} \left(t^{(k-1)/2} \sum_{1 \leq v \leq \frac{t}{m}} v^{(k-1)/2} \right) + \sum_{1 \leq v \leq \frac{t}{m}} O\left((vt)^{(k-2)/2} \right).$$

Now, applying the estimate for a sum of powers of integers given in Remark 6.2 below, we substitute

$$\sum_{1 \leq v \leq \frac{t}{m}} v^{(k-1)/2} = \frac{2}{k+1} \left(\frac{t}{m}\right)^{(k+1)/2} + O\left(t^{(k-1)/2}\right)$$

and, writing $\sum_{1 \leq v \leq \frac{t}{m}} v^{(k-2)/2} = O\left(t^{k/2}\right)$, we substitute

$$\sum_{1 \leq v \leq \frac{t}{m}} O\left((vt)^{(k-2)/2}\right) = O\left(t^{(k-2)/2} \sum_{1 \leq v \leq \frac{t}{m}} v^{(k-2)/2}\right) = O\left(t^{k-1}\right)$$

and we end with the asymptotic estimate

$$\frac{2^k/(k+1)}{(\sqrt{d})^{k-1} m^k} t^k + O(t^{k-1}).$$

If the elliptic curve E has complex multiplication, the bounding function can be written as

$$\sum_{1 \leq v \leq \frac{t}{m}} \left\{ \frac{2\pi}{\sqrt{-\delta}} \left(\frac{vt}{m}\right) + O((vt)^e) \right\}^{k-1}$$

where the integer δ is the (negative) discriminant of the degree form on $\text{End}(E)$. The expression above can be written as

$$\begin{aligned} &= \sum_{1 \leq v \leq \frac{t}{m}} \left\{ \frac{(2\pi)^{k-1}}{(\sqrt{-\delta} m)^{k-1}} (vt)^{k-1} + O((vt)^{e+k-2}) \right\} \\ &= \frac{(2\pi)^{k-1}}{(\sqrt{-\delta} m)^{k-1}} \left(t^{k-1} \sum_{1 \leq v \leq \frac{t}{m}} v^{k-1} \right) + \sum_{1 \leq v \leq \frac{t}{m}} O((vt)^{k-2+e}). \end{aligned}$$

Here, using Remark 6.2 again, we substitute

$$\sum_{1 \leq v \leq \frac{t}{m}} v^{k-1} = \frac{1}{k} \left(\frac{t}{m}\right)^k + O\left(t^{k-1}\right)$$

and, writing $\sum_{1 \leq v \leq \frac{t}{m}} v^{k-2+e} = O\left(t^{k-1+e}\right)$, we substitute

$$\sum_{1 \leq v \leq \frac{t}{m}} O((vt)^{k-2+e}) = O\left(t^{k-2+e} \sum_{1 \leq v \leq \frac{t}{m}} v^{k-2+e}\right) = O\left(t^{2k-3+2e}\right)$$

and we end with the estimate

$$\frac{(2\pi)^{k-1}/k}{(\sqrt{-\delta})^{k-1} m^{2k-1}} t^{2k-1} + O(t^{2k-3+2\epsilon}). \quad \square$$

REMARK 6.2: (Communicated by the referee.) About partial sums of increasing functions. Let $f : [0, +\infty) \rightarrow [0, +\infty)$ be an increasing function. To estimate $\sum_{n=1}^t f(n)$ observe that in each interval $[n, n+1]$ it satisfies $f(n) \leq f(x) \leq f(n+1)$. It follows that for each positive integer t ,

$$\int_0^t f(x) dx \leq \sum_{n=1}^t f(n) \leq \int_1^{t+1} f(x) dx.$$

In the case $f(x) = x^p$ (p arbitrary positive real) we get

$$\frac{t^{p+1}}{p+1} \leq \sum_{n=1}^t n^p \leq \frac{(t+1)^{p+1}}{p+1} - \frac{1}{p+1}$$

from which it follows that

$$\sum_{n=1}^t n^p = \frac{t^{p+1}}{p+1} + O(t^p).$$

If t is a positive real number, an analogous estimate holds, which in the preceding proof is written in the form $\sum_{1 \leq n \leq t} n^p = \frac{t^{p+1}}{p+1} + O(t^p)$, where n is meant to be an integer ranging in the interval $[1, t]$.

We are now in a position to prove the result in the introduction.

Proof of Theorem 1.1. Remind that the function $N_{E^k}(t)$ is bounded above by

$$N_{E^{k-1}}(t) + 1 + \Phi_{E^k}(t).$$

We argue by induction on $k \geq 2$. The initial step $k = 2$ follows immediately from the estimate of $\Phi_{E^2}(t)$ given in Lemma 6.1 above.

When $k > 2$, if the statement holds for E^{k-1} then it holds for E^k too. In both cases, either with complex multiplication or not, by the inductive assumption we have

$$N_{E^{k-1}}(t) = O(t^{r'})$$

where, in both cases,

$$r' < r \quad \text{and} \quad r' \leq i.$$

From Lemma 6.1 we know that

$$\Phi_{E^k}(t) = Ct^r + O(t^i).$$

Hence the theorem follows. □

7. Arbitrary polarized Abelian varieties

7.1. Behavior under isogenies

Let A, B be polarized Abelian varieties and let $\varphi : B \rightarrow A$ be an isogeny, preserving the polarizations (the polarization on B is the pullback of the polarization on A), whose degree we call d . There is a one to one correspondence

$$\{\text{elliptic curves in } A\} \xrightarrow{\sim} \{\text{elliptic curves in } B\}.$$

Given $E \subset A$ the corresponding E^* in B is the connected component of 0 in the pre-image $\varphi^{-1}(E)$. The restricted isogeny $E^* \rightarrow E$ has degree $d_E \leq d$ (in fact a divisor of d), and the degree of E^* is given by

$$\deg(E^*) = d_E \deg(E)$$

(by the projection formula: $E^* \cdot \varphi^*L = \varphi_*E^* \cdot L = d_E E \cdot L$). Therefore:

$$\deg(E) \leq \deg(E^*) \leq d \deg(E).$$

It follows that the functions counting elliptic curves in A and in B are related by the following inequalities:

$$N_A(t) \leq N_B(dt) \quad \text{and} \quad N_B(t) \leq N_A(t).$$

7.2. On the counting function

Let A be a polarized Abelian variety, of dimension n . Let us say that a sequence E_1, \dots, E_i of elliptic curves in A is *independent* if the Abelian subvariety $E_1 + \dots + E_i$ has dimension i . If A contains i independent elliptic curves then, because of the reducibility theorem (§2.5), replacing the given elliptic curves without modifying the sequence of Abelian subvarieties $E_1 + \dots + E_j$, with $j = 1, \dots, i$, we can even obtain that, under the sum isogeny $E_1 \times \dots \times E_i \rightarrow E_1 + \dots + E_i \subseteq A$, the pullback polarization from A is a split polarization on $E_1 \times \dots \times E_i$.

Let k be the maximum number of independent elliptic curves in A . There is, as above, a special isogeny $E_1 \times \dots \times E_k \rightarrow E_1 + \dots + E_k \subseteq A$. Moreover, every elliptic curve in A is contained in $E_1 + \dots + E_k$. Hence, without loss of generality, we may assume that $A = E_1 + \dots + E_k$ and that, under the sum isogeny

$$E_1 \times \dots \times E_k \rightarrow A,$$

the pullback polarization from A is a split polarization on $E_1 \times \dots \times E_k$. Let d be the degree of such an isogeny. From the discussion in §7.1 above, we have

$$N_A(t) \leq N_{E_1 \times \dots \times E_k}(dt).$$

It follows that, in order to have a general estimate of the counting function $N_A(t)$, we can reduce to the particular case in which $A = E_1 \times \cdots \times E_k$ and the polarization splits.

PROPOSITION 7.1. *The function $N_A(t)$ can be given an asymptotic estimate of the form*

$$N_A(t) = Ct^r + O(t^i)$$

for some constant C and exponents r, i with $i < r$.

Proof. According to the preceding discussion, we only need to consider the case in which $A = E_1 \times \cdots \times E_k$ is a product of elliptic curves, with a split polarization.

If the factor elliptic curves are all isogenous, we can choose one elliptic curve E together with isogenies $E \rightarrow E_i$ and then construct a product isogeny $E^k \rightarrow A$, so that the pullback polarization on E^k is a split polarization again. If d is the degree of such an isogeny then, from the discussion in §7.1, we have

$$N_A(t) \leq N_{E^k}(dt)$$

and the statement follows from Theorem 1.1.

More generally, separating the collection E_1, \dots, E_k into (maximal) isogeny classes, and rearranging, we have an isomorphism

$$E_1 \times \cdots \times E_k \cong B_1 \times \cdots \times B_h,$$

each factor B_i being a maximal product of isogenous elliptic curves from the given collection. The split polarization on $E_1 \times \cdots \times E_k$ corresponds to a split polarization on $B_1 \times \cdots \times B_h$. Define $B := B_1 \times \cdots \times B_h$ and consider the isogeny $B \rightarrow A$. Let d be the degree of such an isogeny. From the discussion in §7.1, we have

$$N_A(t) \leq N_B(dt).$$

It is easy to see that $N_B(t) = N_{B_1}(t) + \cdots + N_{B_h}(t)$. This is because an elliptic curve in B , projecting non-trivially to different factors B_r and B_s , would therefore project onto non-isogenous elliptic curves E_i and E_j , which is impossible. From the discussion above, for a product of isogenous elliptic curves, we have

$$N_{B_\ell}(t) = C_\ell t^{r_\ell} + O(t^{i_\ell})$$

with $i_\ell < r_\ell$. It follows that $N_B(dt) = Ct^r + O(t^i)$, where $r = \max\{r_1, \dots, r_h\}$ and $i < r$. \square

REMARK 7.2: When $A = J(C)$ is the Jacobian of a curve of genus $g > 1$, there is an effective bound for the function $N_A(t)$ due to Kani (cf. [6], Theorem 4), which is asymptotically of order $O(t^{2g^2-2})$ (*ibid.*, p. 187). The asymptotic bound in the present paper (Theorem 1.1, Proposition 7.1) is instead of order $O(t^{2g-1})$.

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Author's address:

Lucio Guerra
Dipartimento di Matematica e Informatica
Università di Perugia
Via Vanvitelli 1, 06123 Perugia, Italia
E-mail: lucio.guerra@unipg.it

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Convexity, topology and nonlinear differential systems with nonlocal boundary conditions: a survey¹

JEAN MAWHIN AND KATARZYNA SZYMAŃSKA-DĘBOWSKA

ABSTRACT. *This paper is a survey of recent existence results for solutions of first and second order nonlinear differential systems with nonlocal boundary conditions using methods based upon convexity, topological degree and maximum-principle like techniques.*

Keywords: systems of differential equations, nonlocal boundary conditions, convexity, Leray-Schauder degree, maximum principle.
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1. Introduction

For a first order ordinary differential system

$$x' = f(t, x),$$

with $f : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ continuous, a natural generalization of the homogeneous two-point **boundary conditions (BC)**

$$A_1 x(a) = A_2 x(b),$$

where A_1, A_2 are $(n \times n)$ -matrices, is the m -point BC

$$\sum_{j=1}^m A_j x(t_j) = 0,$$

where $a = t_1 < t_2 < \dots < t_m = b$ and A_1, \dots, A_m are $(n \times n)$ -matrices. Such a multi-point boundary condition is itself a special case of the **nonlocal or**

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integral BC

$$\int_a^b d\mathcal{A}(s)x(s) = 0,$$

where $\mathcal{A} : [a, b] \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ has bounded variation and the integral is of Riemann–Stieltjes type.

Nonlocal boundary conditions of the type

$$\sum_{j=1}^m A_j x(t_j) + \int_0^1 B(s)x(s) ds = 0$$

for some $(n \times n)$ –matrix–valued function $B : [0, 1] \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, were first introduced for linear differential equations by Picone [91] in 1908, and already applied to physics by von Mises [111] in 1912. Using Riesz representation theorem, those conditions are themselves contained in the more general form

$$\int_0^1 d\mathcal{A}(s)x(s) = 0,$$

where $\mathcal{A} : [0, 1] \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is a $(n \times n)$ –matrix–valued functions with bounded variation. They were first introduced for linear systems in 1931 by Cioranescu [12] and by Smorgorshewsky [98] in 1940. A good survey of the linear theory is given by Krall in [57].

Nonlocal boundary conditions can be considered also for second order differential systems of the form

$$x'' = f(t, x, x'), \tag{1}$$

where $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous.

Without searching the maximum of generality, the most useful homogeneous **two–point BC** for system (1) are obtained by choosing two of the following expressions

$$\begin{aligned} x(a) &= Ax(b) + Bx'(b), & x'(a) &= Cx(b) + Dx'(b), \\ x(b) &= Ex(a) + Fx'(a), & x'(b) &= Gx(a) + Hx'(a), \end{aligned}$$

where A, B, C, D, E, F, G, H are $(n \times n)$ –matrices. The corresponding **nonlocal BC** are obtained by taking two of the following conditions

$$x(a) = \int_a^b d\mathcal{A}(s)x(s) + \int_a^b d\mathcal{B}(s)x'(s), \tag{2}$$

$$x'(a) = \int_a^b d\mathcal{C}(s)x(s) + \int_a^b d\mathcal{D}(s)x'(s), \quad (3)$$

$$x(b) = \int_a^b d\mathcal{E}(s)x(s) + \int_a^b d\mathcal{F}(s)x'(s), \quad (4)$$

$$x'(b) = \int_a^b d\mathcal{G}(s)x(s) + \int_a^b d\mathcal{H}(s)x'(s), \quad (5)$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H} : [a, b] \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ are $(n \times n)$ -matrix valued functions having bounded variation.

Choosing $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ constant, the conditions (2)–(3) reduce to the **initial type conditions**, and choosing $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ constant, the conditions (4)–(5) reduce to the **terminal type conditions** for (1). Choosing $\mathcal{A}, \mathcal{B}, \mathcal{E}, \mathcal{F}$ constant, the conditions (2)–(4) become the **Dirichlet conditions**, and choosing $\mathcal{C}, \mathcal{D}, \mathcal{G}, \mathcal{H}$ constant, the conditions (3)–(5) become the **Neumann conditions**. **Mixed conditions** are obtained by choosing $\mathcal{A}, \mathcal{B}, \mathcal{G}, \mathcal{H}$ constant in (2)–(5) or $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$ constant in (3)–(4). The **periodic conditions** can be obtained by taking \mathcal{A} such that $\int_a^b d\mathcal{A}(s)x(s) = x(b)$, \mathcal{B} constant, \mathcal{C} constant and \mathcal{D} such that $\int_a^b d\mathcal{D}(s)x'(s) = x'(b)$. It suffices, for example, to take $\mathcal{A}(s) = h(s)I_n$ with $h(a) = 0$ and $h(s) = 1$ for $s \in (a, b]$.

The first paper dealing with nonlinear differential equations with integral boundary conditions seems to be due to Birkhoff and Kellogg [7] in 1922, as an application of their famous extension of Brouwer's fixed point theorem to some function spaces. Interesting surveys of the nonlinear theory have been given by Whyburn [112], Conti [13], Krall [57], Ma [66] and Ntouyas [87]. They mostly deal with scalar problems and cover the period 1908–2005.

In this survey, we concentrate on differential systems of first and second order (excluding specific results for scalar equations and for higher-order equations), and on methods based upon convexity, topological degree and maximum principle-like techniques to obtain pointwise estimates for the possible solutions. Because of the nonlocal character of the boundary conditions, those methods are more delicate to use than for two-point boundary value problems. We first deal with first order systems, and then with second order systems, discuss the sharpness of the obtained existence conditions and compare them with some well known classical ones for standard two-point boundary conditions like the initial, terminal, periodic, Dirichlet, mixed and Neumann ones. Let us mention also that the nonlocal boundary conditions presented are not by far the most general ones to which the methods apply, but have been chosen

in order to associate a minimum of technical complication with a maximum of significancy.

Various other methods have been used to study nonlocal boundary value problems and various other classes of conditions have been imposed to the nonlinearities to obtain existence and multiplicity results. Let us mention iteration methods for Lipschitzian nonlinearities with sufficiently small coefficients [9, 15, 23, 76, 80, 81], topological methods for nonlinearities satisfying suitable growth and/or sign conditions [2, 4, 5, 8, 11, 14, 43, 44, 50, 52, 53, 55, 56, 65, 68, 71, 72, 77, 82, 83, 90, 95, 96, 97, 102, 105, 106, 107, 108, 109], maximal principle type arguments for monotone nonlinearities [17, 18, 22, 23, 40], fixed point theorems and index on cones for positive solutions [3, 16, 20, 24, 25, 26, 35, 36, 37, 38, 39, 41, 42, 45, 46, 47, 48, 49, 51, 61, 62, 63, 64, 94, 99, 113, 114, 115, 117], variational methods for potential nonlinearities [1, 21, 27, 28, 29, 33, 67, 84, 85, 86, 110, 116]. Those methods and results will not be considered here.

2. First order systems

2.1. Boundary conditions

Let us consider a first order system of ordinary differential equations

$$x' = f(t, x) \tag{6}$$

with $f : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ continuous. We choose $[0, 1]$ for the independent variable without loss of generality.

The homogeneous **two point BC** have the form

$$Ax(0) = Bx(1)$$

where A and B are $(n \times n)$ -matrices.

Notice that for $f \equiv 0$ in (6) the BVP becomes

$$x' = 0 \Leftrightarrow x(t) = c \in \mathbb{R}^n, (A - B)c = 0.$$

Two cases are possible. If

$$\det(A - B) \neq 0,$$

0 is the unique solution and we say that the BC is **non-resonant**. If

$$\det(A - B) = 0,$$

the problem has infinitely any solutions and the BC is called **resonant**.

Standard examples of “two-point boundary conditions” for (6) are given by the **initial value condition** on $[0, 1]$ $x(0) = 0$ ($A = I_n, B = 0_n$, non-resonant), the **terminal value condition** on $[0, 1]$ $x(1) = 0$ ($A = 0_n, B = I_n$, non-resonant), the **anti-periodic BC** $x(0) + x(1) = 0$ ($A = -B = I_n$, non-resonant), and the **periodic BC** $x(0) = x(1)$ ($A = B = I_n$, resonant).

Given $0 = t_1 < t_2 < \dots < t_m = 1$, one can consider also the **m-point BC**

$$\sum_{j=1}^m A_j x(t_j) = 0. \tag{7}$$

For $f \equiv 0$ in (6), the solutions are $x(t) = c$ with c such that $(\sum_{j=1}^m A_j)c = 0$. Again, if $\det(\sum_{j=1}^m A_j) \neq 0$, the BC (7) is called **non-resonant**, and if this determinant is equal to zero, the BC is called **resonant**.

2.2. Nonlocal initial or terminal type BC

For simplicity of exposition and of the statements, we restrict ourself to the special but representative cases of the **nonlocal initial type condition**

$$x(0) = \int_0^1 x(s) dh(s) \tag{8}$$

and of the **nonlocal terminal type condition**

$$x(1) = \int_0^1 x(s) dh(s), \tag{9}$$

where

(h0) $h : [0, 1] \rightarrow \mathbb{R}$ is non-decreasing.

Recall that, for any continuous functions $x : [0, 1] \rightarrow \mathbb{R}^n$, the corresponding Riemann–Stieltjes integrals always exist. Without loss of generality, we can assume that

$$h(0) = 0.$$

We first discuss the situation where

$$(h1) \int_0^1 dh(s) = h(1) < 1.$$

This is a non-resonant situation because each problem

$$x' = 0, x(0) = \int_0^1 x(s) dh(s), \quad x' = 0, x(1) = \int_0^1 x(s) dh(s)$$

has the solution $x(t) = c$ with c verifying the equation

$$c = h(1)c,$$

which has only the trivial solution. This case contains of course the initial and terminal null conditions.

Then, we consider the case where

$$(h2) \quad \int_0^1 dh(s) = h(1) = 1.$$

In this situation, which is clearly a resonant one, the second members of (8) and (9) can be seen as some average of the values of $x(s)$ on the interval $[0, 1]$.

In order to prevent the right-hand member in (8) to be $x(0)$, which would reduce (8) to an identity, we must prevent in (8) h to have the form

$$h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \in (0, 1] \end{cases}, \quad (10)$$

which corresponds to assume that

$$(h3) \quad \text{there exists } \tau \in (0, 1) \text{ such that } h(\tau) < 1.$$

Similarly, in order to prevent (9) to become an identity, we exclude in (9) h of the form

$$h(x) = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1 \end{cases}, \quad (11)$$

which corresponds to assume that

$$(h4) \quad \text{there exists } \tau \in (0, 1) \text{ such that } h(\tau) > 0.$$

EXAMPLE 2.1: For h given by (11), we have $\int_0^1 x(s) dh(s) = x(1)$, and (8) reduces to the periodic BC $x(0) = x(1)$.

For h given by (10), we have $\int_0^1 x(s) dh(s) = x(0)$, and (9) reduces again to the periodic BC.

EXAMPLE 2.2: For

$$h(x) = \begin{cases} 0 & \text{if } x \in [0, \alpha) \\ \gamma & \text{if } x \in [\alpha, 1) \\ 1 & \text{if } x = 1 \end{cases}$$

where $\alpha, \gamma \in (0, 1)$, we have

$$\int_0^1 x(s) dh(s) = \gamma x(\alpha) + (1 - \gamma)x(1),$$

and (8) reduces to the **three-point BC** $x(0) = \gamma x(\alpha) + (1 - \gamma)x(1)$.

EXAMPLE 2.3: For

$$h(x) = \begin{cases} 0 & \text{if } x = 0 \\ \gamma & \text{if } x \in (0, \alpha] \\ 1 & \text{if } x \in (\alpha, 1] \end{cases}$$

where $\alpha, \gamma \in (0, 1)$, we have

$$\int_0^1 x(s) dh(s) = \gamma x(0) + (1 - \gamma)x(\alpha),$$

and (8) reduces to the **three-point BC** $x(1) = (1 - \gamma)x(\alpha) + \gamma x(0)$.

2.3. Linear nonlocal initial or terminal type BVP

Let C^0 be the space $C([0, 1], \mathbb{R}^n)$ of all continuous mappings from $[0, 1]$ into \mathbb{R}^n with the uniform norm $\|x\| = \max \{\|x_1\|_\infty, \dots, \|x_n\|_\infty\}$.

The following results are useful to reduce our problems to a fixed point form.

LEMMA 2.4. *If conditions (h0), (h1) or (h0), (h2), (h3) hold, then, for each $z \in C^0$, the linear nonlocal initial value problem*

$$x' + x = z(t), \quad x(0) = \int_0^1 x(s) dh(s) \tag{12}$$

has the unique solution

$$x(t) = \left(1 - \int_0^1 e^{-s} dh(s)\right)^{-1} \int_0^1 \int_0^u e^{-t-u+s} z(s) ds dh(u) + \int_0^t e^{-(t-s)} z(s) ds. \tag{13}$$

Proof. By the variation of constants formula, the initial value problem

$$x' + x = z(t), \quad x(0) = c$$

has the unique solution

$$x(t) = ce^{-t} + \int_0^t e^{-(t-s)} z(s) ds \quad (c \in \mathbb{R}^n).$$

It satisfies the boundary condition (8) if and only if c satisfies the linear algebraic system

$$c = c \int_0^1 e^{-t} dh(t) + \int_0^1 \int_0^t e^{-(t-s)} z(s) ds dh(t),$$

which has the unique solution

$$c = \left(1 - \int_0^1 e^{-s} dh(s) \right)^{-1} \int_0^1 \int_0^u e^{-(u-s)} z(s) ds dh(u)$$

if $\int_0^1 e^{-s} dh(s) \neq 1$. This is trivially the case if conditions (h0), (h1) hold. If conditions (h0), (h2), (h3) hold, we have

$$\begin{aligned} \int_0^1 e^{-s} dh(s) &= \int_0^1 d[e^{-s}h(s)] + \int_0^\tau e^{-s}h(s) ds + \int_\tau^1 e^{-s}h(s) ds \\ &\leq e^{-1} + h(\tau) \int_0^\tau e^{-s} ds + \int_\tau^1 e^{-s} ds \\ &= e^{-1} + h(\tau)(1 - e^{-\tau}) + (e^{-\tau} - e^{-1}) \\ &= (1 - e^{-\tau})h(\tau) + e^{-\tau} < 1 \end{aligned}$$

and the result follows. \square

Let us denote by $K_1 : C^0 \rightarrow C^0$ the linear operator mapping z into x given by (13). Notice that each $K_1 z$ is of class C^1 .

COROLLARY 2.5. *There exists $C_1 > 0$ such that, for each $z \in C^0$, one has*

$$\|K_1 z\| \leq C_1 \|z\|, \quad \|(K_1 z)'\| \leq (C_1 + 1) \|z\|,$$

and K_1 is a compact operator.

Proof. Follows easily from (12) and (13) and Arzelà-Ascoli's theorem. \square

LEMMA 2.6. *If conditions (h0), (h1) or (h0), (h2), (h4) hold, then, for each $z \in C^0$, the linear nonlocal terminal value problem*

$$x' - x = z(t), \quad x(1) = \int_0^1 x(s) dh(s) \quad (14)$$

has the unique solution

$$\begin{aligned} x(t) &= \left(1 - \int_0^1 e^{s-1} dh(s) \right)^{-1} \int_0^1 \int_1^u e^{t-1+u-s} z(s) ds dh(u) \\ &\quad + \int_1^t e^{t-s} z(s) ds. \quad (15) \end{aligned}$$

Proof. Let $z \in C^0$. By the variation of constants formula, the terminal value problem

$$x' - x = z(t), \quad x(1) = c$$

has the unique solution

$$x(t) = e^{t-1}c + \int_1^t e^{t-s}z(s) ds \quad (c \in \mathbb{R}^n).$$

It satisfies the boundary condition (9) if c verifies the linear algebraic system

$$c - \left(\int_0^1 e^{t-1} dh(t) \right) c = \int_0^1 \int_1^t e^{t-s}z(s) ds dh(t).$$

which has the unique solution

$$c = \left(1 - \int_0^1 e^{s-1} dh(s) \right)^{-1} \int_0^1 \int_1^u e^{u-s}z(s) ds dh(u),$$

if $1 \neq \int_0^1 e^{s-1} dh(s)$. This is trivially the case if conditions (h0), (h1) hold. In the second case, we have

$$\begin{aligned} \int_0^1 e^{s-1} dh(s) &= \int_0^1 d[e^{s-1}h(s)] - \int_0^\tau e^{s-1}h(s) ds - \int_\tau^1 e^{s-1}h(s) ds \\ &\leq 1 - h(\tau) \int_\tau^1 e^{s-1} ds = 1 - h(\tau)(1 - e^{\tau-1}) < 1 \end{aligned}$$

and the result follows. \square

Let us denote by $K_2 : C^0 \rightarrow C^0$ the linear operator mapping z into x given by (15). Notice that each K_2z is of class C^1 .

COROLLARY 2.7. *There exists $C_2 > 0$ such that, for each $z \in C^0$, one has*

$$\|K_2z\| \leq C_2\|z\|, \quad \|(K_2z)'\| \leq (C_2 + 1)\|z\|,$$

and K_2 is a compact operator.

Proof. Follows easily from (14) and (15) and Arzelà-Ascoli's theorem. \square

2.4. Fixed point formulation of nonlinear nonlocal initial or terminal type BVP

Let now $f : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous and define the mapping $N_1 : C^0 \rightarrow C^0$ by

$$N_1x = f(\cdot, x(\cdot)) - x(\cdot), \quad \forall x \in C^0$$

and the mapping $N_2 : C^0 \rightarrow C^0$ by

$$N_2x = f(\cdot, x(\cdot)) + x(\cdot), \quad \forall x \in C^0.$$

It is easy to show that N_1 and N_2 are continuous on C^0 and take bounded sets of C^0 into bounded sets of C^0 . Under the conditions of Lemma 2.4,

$$G_1 := K_1N_1 : C^0 \rightarrow C^0$$

is compact on bounded subsets of C^0 , and the **nonlinear nonlocal initial type problem**

$$x' = f(t, x), \quad x(0) = \int_0^1 x(s) dh(s) \quad (16)$$

is equivalent to the fixed point problem in C^0

$$x = G_1x. \quad (17)$$

Similarly, under the conditions of Lemma 2.6,

$$G_2 := K_2N_2 : C^0 \rightarrow C^0$$

is compact on bounded subsets of C^0 , and the **nonlinear nonlocal terminal type problem**

$$x' = f(t, x), \quad x(1) = \int_0^1 x(s) dh(s) \quad (18)$$

is equivalent to the fixed point problem in C^0

$$x = G_2x. \quad (19)$$

We apply to the equations (17) and (19) the following existence result, which follows easily from Leray-Schauder continuation theorem [60, 69].

PROPOSITION 2.8. *Let X be a real normed space, $\Omega \subset X$ be an open bounded neighborhood of 0, and $T : \bar{\Omega} \rightarrow X$ be a compact operator. If $x \neq \lambda Tx$ for every $(x, \lambda) \in \partial\Omega \times (0, 1)$, then T has at least a fixed point in $\bar{\Omega}$.*

2.5. Some classical results for periodic BC

Let $\langle \cdot | \cdot \rangle$ denote the classical inner product in \mathbb{R}^n , $|\cdot|$ the corresponding Euclidian norm, and $B_R \subset \mathbb{R}^n$ the closed ball of center 0 and radius $R > 0$.

A classical existence theorem for the periodic BVP associated to (6) is the following one.

THEOREM 2.9. *If there exists $R > 0$ such that either*

$$\langle u|f(t, u) \rangle \geq 0, \forall (t, u) \in [0, 1] \times \partial B_R, \quad (20)$$

or

$$\langle u|f(t, u) \rangle \leq 0, \forall (t, u) \in [0, 1] \times \partial B_R, \quad (21)$$

then the problem

$$x' = f(t, x), \quad x(0) = x(1) \quad (22)$$

has at least one solution taking values in \overline{B}_R .

Notice that the two statements in Theorem 2.9 are equivalent : each one implies the other one through the change of variables $\tau = 1 - t$. The full statement can be seen as a nonlinear version of the following linear elementary result

PROPOSITION 2.10. *For each $\lambda \in \mathbb{R} \setminus \{0\}$ and each $e \in C^0$, the problem*

$$x' = \lambda x + e(t), \quad x(0) = x(1)$$

has a solution.

Quite strangely, it is difficult to locate the first appearance of Theorem 2.9 in the literature. It is a special case (not directly mentioned !) of Theorem 3.2 in Krasnosel'skii's monograph [58] of 1966. On the other hand, it is explicitly mentioned by Gustafson and Schmitt in 1974 [30] (with strict inequalities in (20) or (21)) as a special case of the following theorem.

Let C be an open convex neighborhood of 0 in \mathbb{R}^n . It is well known that $\forall u \in \partial C, \exists \nu(u) \in \mathbb{R}^n \setminus \{0\} : \langle \nu(u)|u \rangle > 0$ and $C \subset \{v \in \mathbb{R}^n : \langle \nu(u)|v - u \rangle < 0\}$. The mapping $\nu : \partial C \rightarrow \mathbb{R}^n \setminus \{0\}$ is called an **outer normal field** on ∂C .

THEOREM 2.11. *If there exists a bounded convex open neighborhood C of 0 in \mathbb{R}^n , and an outer normal field ν on ∂C such that either*

$$\langle \nu(u)|f(t, u) \rangle > 0, \forall (t, u) \in [0, 1] \times \partial C$$

or

$$\langle \nu(u)|f(t, u) \rangle < 0, \forall (t, u) \in [0, 1] \times \partial C,$$

then the problem (22) has at least one solution taking values in C .

Notice that Krasnosel'skii's monograph is not quoted by Gustafson and Schmitt. In [69], the connexion between Krasnosel'skii's results and Gustafson-Schmitt's ones is made explicit, the Gustafson-Schmitt's theorem is extended to the case of weak inequalities and Krasnosel'skii's theorem is shown to be a special case of this extension of Gustafson-Schmitt's theorem.

2.6. Nonlocal initial type BVP

The following theorem essentially comes from [73]. The special case of a global initial value problem can be found in [70].

THEOREM 2.12. *If $h : [0, 1] \rightarrow \mathbb{R}$ satisfies conditions (h0), $h(1)$ or conditions (h0), $h(2)$, (h3), and if there exists an open, bounded, convex neighborhood C of 0 in \mathbb{R}^n and an outer normal field ν on ∂C such that*

$$\langle \nu(u) | f(t, u) \rangle \leq 0, \quad \forall (t, u) \in [0, 1] \times \partial C, \quad (23)$$

then the problem (16) has at least one solution taking values in \overline{C} for all $t \in [0, 1]$.

Proof. Let us consider the equation (17) and define the open bounded neighborhood Ω of 0 in C^0 by

$$\Omega = \{x \in C^0 : x(t) \in C, \forall t \in [0, 1]\}. \quad (24)$$

Notice that

$$\begin{aligned} \overline{\Omega} &= \{x \in C^0 : x([0, 1]) \subset \overline{C}\}, \\ \partial\Omega &= \{x \in \overline{\Omega} : \exists t_0 \in [0, 1] : x(t_0) \in \partial C\}. \end{aligned} \quad (25)$$

By the discussion above G_1 is compact on $\overline{\Omega}$. According to Proposition 2.8, a solution of (17) in $\overline{\Omega}$, i.e. a solution of (16) such that $x(t) \in \overline{C}$ for all $t \in [0, 1]$ will exist, if we can show that, for each $\lambda \in (0, 1)$, no possible solution of the problem

$$x' + x = \lambda[f(t, x) + x], \quad x(0) = \int_0^1 x(s) dh(s), \quad (26)$$

belongs to $\partial\Omega$. Let $\lambda \in (0, 1)$ and $x(t) \in \partial\Omega$ be a possible solution to (26). Then $x(t) \in \overline{C}$ for all $t \in [0, 1]$ and there is some $t_0 \in [0, 1]$ such that $x(t_0) \in \partial C$. Therefore, for all $t \in [0, 1]$,

$$\xi_{t_0}(t) := \langle \nu(x(t_0)) | x(t) \rangle \leq \langle \nu(x(t_0)) | x(t_0) \rangle = \xi_{t_0}(t_0), \quad \forall t \in [0, 1],$$

which means that the real function $\xi_{t_0} : [0, 1] \rightarrow \mathbb{R}$ reaches its maximum at t_0 . If $t_0 \in (0, 1]$,

$$\begin{aligned} 0 &\leq \xi'_{t_0}(t_0) = \langle \nu(x(t_0)) | x'(t_0) \rangle \\ &= -(1 - \lambda) \langle \nu(x(t_0)) | x(t_0) \rangle + \lambda \langle \nu(x(t_0)) | f(t_0, x(t_0)) \rangle < 0, \end{aligned}$$

a contradiction. If $t_0 = 0$ and conditions (h0), (h1) hold, then

$$\begin{aligned}\xi_0(0) &= \langle \nu(x(0)), x(0) \rangle = \int_0^1 \langle \nu(x(0)), x(s) \rangle dh(s) \\ &\leq \max_{[0,1]} \langle \nu(x(0)), x(s) \rangle \int_0^1 dh(s) < \max_{[0,1]} \langle \nu(x(0)), x(s) \rangle \\ &= \max_{s \in [0,1]} \xi_0(s),\end{aligned}$$

a contradiction. If $t_0 = 0$ and conditions (h0), (h2), (h3) hold, it remains only to consider the case where 0 is the only value of t at which $x(t) \in \partial C$, i.e. the case where ξ_0 reaches its maximum only at 0. Then

$$\xi_0(s) < \xi_0(0), \forall s \in (0, 1],$$

and hence, using the boundary condition and assumptions (h2) and (h3),

$$\begin{aligned}\xi_0(0) &= \left\langle \nu(x(0)) \left| \int_0^1 x(s) dh(s) \right. \right\rangle = \int_0^1 \langle \nu(x(0)) | x(s) \rangle dh(s) \\ &= \int_0^1 \langle \nu(x(0)) | x(s) \rangle dh(s) = \int_0^1 \xi_0(s) dh(s) \\ &= \int_0^\tau \xi_0(s) dh(s) + \int_\tau^1 \xi_0(s) dh(s) \\ &\leq \xi_0(0)h(\tau) + (\max_{[\tau,1]} \xi_0)[1 - h(\tau)] < \xi_0(0),\end{aligned}$$

a contradiction. Consequently the assumptions of Proposition 2.8 are satisfied for G_1 on $\bar{\Omega}$, and the conclusion follows. \square

COROLLARY 2.13. *If $h : [0, 1] \rightarrow \mathbb{R}$ satisfies conditions (h0), h(1) or conditions (h0), (h2), (h3), and if there exists $R > 0$ such that*

$$\langle u | f(t, u) \rangle \leq 0, \forall (t, u) \in [0, 1] \times \partial B_R, \quad (27)$$

then the problem (16) has at least one solution taking values in \bar{B}_R .

Proof. Take $C = B_R$ and, for each $u \in \partial B_R$, $\nu(u) = u$. \square

COROLLARY 2.14. *If $h : [0, 1] \rightarrow \mathbb{R}$ satisfies conditions (h0), h(1) or conditions (h0), (h2), (h3), and if there exists $R_j > 0$ ($1 \leq j \leq n$) such that*

$$u_i f_i(t, u) \leq 0, \forall (t, u) \in [0, 1] \times \prod_{j=1}^n [-R_j, R_j] : |u_i| = R_i \ (1 \leq i \leq n), \quad (28)$$

then the problem (16) has at least one solution taking values in $\Pi_{j=1}^n [-R_j, R_j]$.

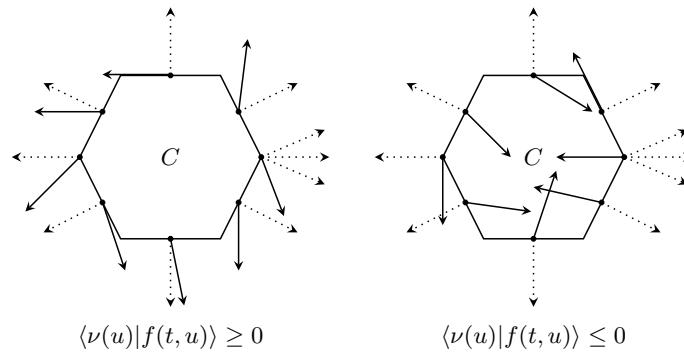


Figure 1: The case when C is a hexagon.

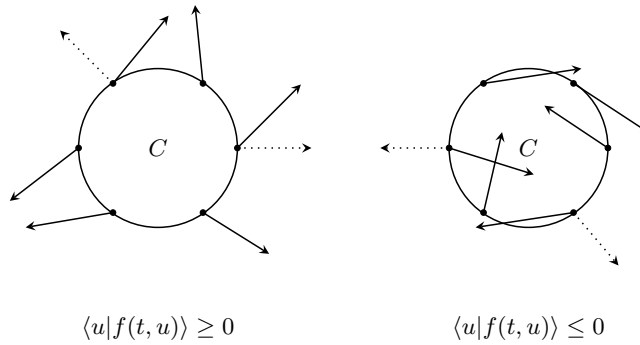


Figure 2: The case when C is a ball.

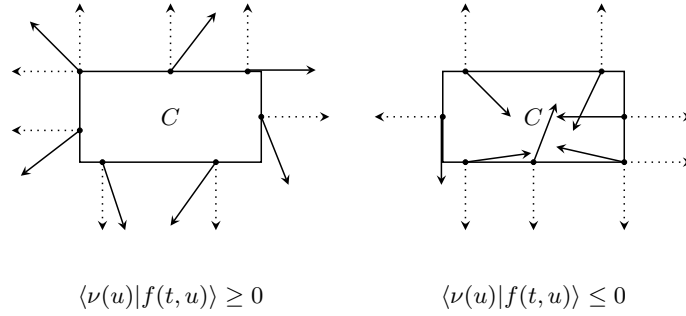
Proof. Take $C = \prod_{j=1}^n (-R_j, R_j)$ and, for each $u \in \prod_{j=1}^n [-R_j, R_j]$ and $|u_i| = R_i$, $\nu(u) = u_i e^i$, where $e^i = (0, \dots, 0, 1, 0, \dots, 0)$ is the i^{th} element of the canonical basis of \mathbb{R}^n ($i = 1, \dots, n$). \square

2.7. Nonlocal terminal type BVP

The following theorem essentially comes from [73].

THEOREM 2.15. *If $h : [0, 1] \rightarrow \mathbb{R}$ satisfies conditions (h0), h(1), or conditions (h0), (h2), (h4), and if there exists an open, bounded, convex neighborhood C of 0 in \mathbb{R}^n and an outer normal field ν on ∂C such that*

$$\langle \nu(u) | f(t, u) \rangle \geq 0, \quad \forall (t, u) \in [0, 1] \times \partial C, \tag{29}$$


 Figure 3: The case when C is a rectangle.

then the problem (18) has at least one solution taking values in \bar{C} .

Proof. Let us consider the equation (19) and let Ω the open bounded neighborhood of 0 in C^0 defined by (24). By the discussion above G_2 is compact on $\bar{\Omega}$. According to Proposition 2.8, a solution of (19) in $\bar{\Omega}$, i.e. a solution of (16) such that $x(t) \in \bar{C}$ for all $t \in [0, 1]$ will exist if we can show that for each $\lambda \in (0, 1)$, no possible solution of the problem

$$x' - x = \lambda[f(t, x) - x], \quad x(1) = \int_0^1 x(s) dh(s), \quad (30)$$

belongs to $\partial\Omega$. Let $\lambda \in (0, 1)$ and $x(t) \in \partial\Omega$ be a possible solution to (26). Then $x(t) \in \bar{C}$ for all $t \in [0, 1]$ and there is some $t_0 \in [0, 1]$ such that $x(t_0) \in \partial C$. Therefore, for all $t \in [0, 1]$,

$$\xi_{t_0}(t) := \langle \nu(x(t_0)) | x(t) \rangle \leq \langle \nu(x(t_0)) | x(t_0) \rangle = \xi_{t_0}(t_0), \quad \forall t \in [0, 1],$$

which means that the real function $\xi_{t_0} : [0, 1] \rightarrow \mathbb{R}$ reaches its maximum at t_0 . If $t_0 \in [0, 1)$,

$$\begin{aligned} 0 &\geq \xi'_{t_0}(t_0) = \langle \nu(x(t_0)) | x'(t_0) \rangle \\ &= (1 - \lambda) \langle \nu(x(t_0)) | x(t_0) \rangle + \lambda \langle \nu(x(t_0)) | f(t_0, x(t_0)) \rangle > 0, \end{aligned}$$

a contradiction. If $t_0 = 1$, and conditions (h0), (h1) holds, then, because of the boundary condition,

$$\begin{aligned} \xi_1(1) &= \langle \nu(x(1)) | x(1) \rangle = \int_0^1 \langle \nu(x(1)) | x(s) \rangle dh(s) \\ &\leq \max_{[0,1]} \langle \nu(x(1)) | x(s) \rangle \int_0^1 dh(s) < \max_{[0,1]} \langle \nu(x(1)) | x(s) \rangle \\ &= \max_{s \in [0,1]} \xi_1(s), \end{aligned}$$

a contradiction. If $t_0 = 1$, and conditions (h0), (h2), (h4) hold, it remains only to consider the case where 1 is the only value of t at which $x(t) \in \partial C$, i.e. the case where ξ_1 reaches its maximum only at 1. Then

$$\xi_1(s) < \xi_1(1), \forall s \in [0, 1),$$

and hence, using the boundary condition and Assumptions (h3), (h4),

$$\begin{aligned} \xi_1(1) &= \left\langle \nu(x(1)) \left| \int_0^1 x(s) dh(s) \right. \right\rangle = \int_0^1 \langle \nu(x(1)) | x(s) \rangle dh(s) \\ &= \int_0^1 \langle \nu(x(1)) | x(s) \rangle dh(s) = \int_0^1 \xi_1(s) dh(s) \\ &= \int_0^\tau \xi_1(s) dh(s) + \int_\tau^1 \xi_1(s) dh(s) \\ &\leq (\max_{[0, \tau]} \xi_1) h(\tau) + \xi_1(1) [1 - h(\tau)] < \xi_1(1), \end{aligned}$$

a contradiction. Consequently the assumptions of Proposition 2.8 are satisfied for G_2 and $\bar{\Omega}$, and the conclusion follows. \square

COROLLARY 2.16. *If $h : [0, 1] \rightarrow \mathbb{R}$ satisfies conditions (h0), h(1) or conditions (h0), h(2), h(4), and if there exists $R > 0$ such that*

$$\langle u | f(t, u) \rangle \geq 0, \forall (t, u) \in [0, 1] \times \partial B_R, \quad (31)$$

then the problem (18) has at least one solution taking values in \bar{B}_R .

Proof. Take $C = B_R$ and, for each $u \in \partial B_R$, $\nu(u) = u$. \square

COROLLARY 2.17. *If $h : [0, 1] \rightarrow \mathbb{R}$ satisfies conditions (h0), h(1) or conditions (h0), (h2), (h4), and if there exists $R_j > 0$ ($1 \leq j \leq n$) such that*

$$u_i f_i(t, u) \geq 0, \forall (t, u) \in [0, 1] \times \prod_{j=1}^n [-R_j, R_j] : |u_i| = R_i \ (1 \leq i \leq n), \quad (32)$$

then the problem (18) has at least one solution taking values in $\prod_{j=1}^n [-R_j, R_j]$.

Proof. Take $C = \prod_{j=1}^n (-R_j, R_j)$ and, for each $u \in \prod_{j=1}^n [-R_j, R_j]$ and $|u_i| = R_i$, $\nu(u) = u_i e^i$, where $e^i = (0, \dots, 0, 1, 0, \dots, 0)$ is the i^{th} element of the canonical basis of \mathbb{R}^n ($i = 1, \dots, n$). \square

2.8. Lower and upper solutions for nonlocal initial or terminal BVP

Corollaries 2.14 and 2.17 can be generalized by extending the classical concepts of lower and upper solutions to our nonlocal boundary value problems.

DEFINITION 2.18. We say that $\alpha \in C^1([0, 1], \mathbb{R}^n)$ is a **lower solution** and $\beta \in C^1([0, 1], \mathbb{R}^n)$ an **upper solution** for problem (16), if

$$\alpha_i(t) \leq \beta_i(t) \quad (1 \leq i \leq n)$$

and, for each $i \in \{1, \dots, n\}$,

$$\begin{aligned} \alpha'_i(t) &\leq f(t, u_1, \dots, u_{i-1}, \alpha_i(t), u_{i+1}(t), \dots, u_n(t)), \\ \beta'_i(t) &\geq f(t, u_1, \dots, u_{i-1}, \beta_i(t), u_{i+1}(t), \dots, u_n(t)), \end{aligned} \quad (33)$$

whenever $\alpha_j(t) \leq u_j \leq \beta_j(t)$, $t \in [0, 1]$, $j \in \{1, \dots, n\} \setminus \{i\}$,

$$\alpha_i(0) \leq \int_0^1 \alpha_i(s) dh(s), \quad \beta_i(0) \geq \int_0^1 \beta_i(s) dh(s).$$

DEFINITION 2.19. We say that $\alpha \in C^1([0, 1], \mathbb{R}^n)$ is a **lower solution** and $\beta \in C^1([0, 1], \mathbb{R}^n)$ an **upper solution** for problem (18), if

$$\alpha_i(t) \leq \beta_i(t) \quad (1 \leq i \leq n)$$

and, for each $i \in \{1, \dots, n\}$,

$$\begin{aligned} \alpha'_i(t) &\geq f(t, u_1, \dots, u_{i-1}, \alpha_i(t), u_{i+1}(t), \dots, u_n(t)), \\ \beta'_i(t) &\leq f(t, u_1, \dots, u_{i-1}, \beta_i(t), u_{i+1}(t), \dots, u_n(t)), \end{aligned} \quad (34)$$

whenever $\alpha_j(t) \leq u_j \leq \beta_j(t)$, $t \in [0, 1]$, $j \in \{1, \dots, n\} \setminus \{i\}$,

$$\alpha_i(1) \geq \int_0^1 \alpha_i(s) dh(s), \quad \beta_i(1) \leq \int_0^1 \beta_i(s) dh(s).$$

For the initial value problem and a scalar equation, the concept and the corresponding theorem was introduced by Peano [88] in 1895, rediscovered by Perron [89] in 1912, and extended to systems by Müller in 1927 [79]. The case of periodic solutions was first considered by Moretto [78] in the scalar case, by Knobloch [54] in 1962 for locally Lipschitzian systems, and generalized to continuous systems in 1974 [69].

THEOREM 2.20. If conditions (h0), (h1) or conditions (h0), (h2), (h3) hold, and if a couple of lower and upper solutions α, β exists for (16), then the problem (16) has a solution x such that $\alpha_i(t) \leq x_i(t) \leq \beta_i(t)$ for all $t \in [0, 1]$ and all $i \in \{1, \dots, n\}$.

Proof. For each $i \in \{1, \dots, n\}$, define the continuous and bounded function $\gamma_i : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\gamma_i(t, u_i) := \begin{cases} \alpha_i(t) & \text{if } u_i < \alpha_i(t) \\ u_i & \text{if } \alpha_i(t) \leq u_i \leq \beta_i(t) \\ \beta_i(t) & \text{if } u_i > \beta_i(t) \end{cases} \quad (35)$$

and consider the modified problem

$$\begin{aligned} x'_i &= -[x_i - \gamma_i(t, x_i)] + f_i(t, \gamma_1(t, x_1), \dots, \gamma_n(t, x_n)) := g_i(t, x) \quad (1 \leq i \leq n) \\ x(0) &= \int_0^1 x(s) dh(s). \end{aligned} \quad (36)$$

As each γ_i and $f_i(\cdot, \gamma_1(\cdot, \cdot), \dots, \gamma_n(\cdot, \cdot))$ are bounded, for each $i \in \{1, \dots, n\}$, there exists $R_i > 0$ such that $g_i(t, u) \geq 0$ for all $u \in \prod_{j=1}^n [-R_j, R_j]$ verifying $u_i = -R_i$ and such that $g_i(t, u) \leq 0$ for all $u \in \prod_{j=1}^n [-R_j, R_j]$ verifying $u_i = R_i$. Using Corollary 2.14, we have a solution x to (36) such that $x(t) \in \prod_{j=1}^n [-R_j, R_j]$ for all $t \in [0, 1]$. We now show that $\alpha_i(t) \leq x_i(t) \leq \beta_i(t)$ for all $t \in [0, 1]$ and all $i \in \{1, \dots, n\}$, so that x is a solution to (16). Fix some $i \in \{1, \dots, n\}$ and assume that there is some $\tau \in [0, 1]$ such that $x_i(\tau) < \alpha_i(\tau)$. Then $x_i - \alpha_i$ reaches a negative minimum at some $t_0 \in [0, 1]$. If $t_0 \in (0, 1]$, then $x'_i(\tau) - \alpha'_i(\tau) \leq 0$, and hence

$$\begin{aligned} \alpha'_i(\tau) &\geq x'_i(\tau) = -[x_i(\tau) - \alpha_i(\tau)] \\ &\quad + f_i(\tau, \gamma_1(\tau, x_1(\tau)), \dots, \alpha_i(\tau), \dots, \gamma_n(\tau, x_n(\tau))) \\ &> f_i(\tau, \gamma_1(\tau, x_1(\tau)), \dots, \alpha_i(\tau), \dots, \gamma_n(\tau, x_n(\tau))), \end{aligned}$$

a contradiction with the definition (33) of lower solution for (16). If $t_0 = 0$, then, using the previous contradiction, we can assume that

$$x_i(0) - \alpha_i(0) < x_i(t) - \alpha_i(t) \quad \forall t \in (0, 1]$$

and hence, integrating over $[0, 1]$ and using the boundary conditions for x_i and α_i ,

$$[x_i(0) - \alpha_i(0)]h(1) \leq \int_0^1 x_i(t) dh(t) - \int_0^1 \alpha_i(t) dh(t) \leq x_i(0) - \alpha_i(0)$$

so that

$$[1 - h(1)][x_i(0) - \alpha_i(0)] \geq 0,$$

and, using (h1),

$$x_i(0) - \alpha_i(0) \geq 0,$$

a contradiction. We leave to the reader the proof in the case where conditions (h0), (h2), (h3) hold. The reasoning is similar to show that $x_i(t) \leq \beta_i(t)$ for all $t \in [0, 1]$ and all $i \in \{1, \dots, n\}$. Hence, the solution x to (36) is also a solution to problem (16). \square

A similar proof provides the corresponding result for the nonlocal terminal type BVP.

THEOREM 2.21. *If conditions (h0), (h1) or conditions (h0), (h2), (h4) hold, and if a couple of lower and upper solutions α, β exists for (18), then the problem (18) has a solution x such that $\alpha_i(t) \leq x_i(t) \leq \beta_i(t)$ for all $t \in [0, 1]$ and all $i \in \{1, \dots, n\}$.*

Extensions of Knobloch’s theorem to some multipoint boundary value problems have been given by Ponomarev [92, 93].

2.9. Periodic vs resonant nonlocal initial or terminal type BC

The special case of Theorem 2.12 with h given by (11) (which satisfies assumptions (h0), (h2), (h3)), together with the special case of Theorem 2.15 with h given by (10) (which satisfies assumptions (h0), (h2), (h4)) provide a proof of the generalized version of Theorem 2.11 with non-strict inequalities in the assumptions, and of its consequence Theorem 2.9, for periodic boundary conditions.

Comparing the statement of Theorem 2.9 for the periodic problem, with the statements of the corresponding Corollaries 2.13 and 2.16 we see that the sense of the inequality in conditions (27) and (31) depends upon the nonlocal boundary condition.

On the other hand, as it is easily verified by direct computation, the system

$$x' = \lambda x + e(t),$$

with each of the three-point boundary conditions

$$x'(0) = \frac{1}{2}[x(1/2) + x(1)], \quad x(1) = \frac{1}{2}[x(1/2) + x(0)],$$

has a solution for each $e \in C^0$ and each $\lambda \in \mathbb{R} \setminus \{0\}$. This is a consequence of the fact that the only real eigenvalue of $\frac{d}{dt}$ with each boundary condition is 0.

Hence a natural question is to know whether the conclusion of the above corollaries still holds when the sense of the corresponding inequality upon f is reversed.

We show by some counterexamples that the answer is negative in general, which of course implies that the same negative answer holds for Theorems 2.12 and 2.15. In this sense, the existence conditions given in Theorems 2.12 and 2.15 are sharp.

The construction of our counterexamples depends upon the study of some associated complex eigenvalue problem and of the corresponding Fredholm alternative for some special three-point boundary conditions. The results are taken from [74].

2.10. Nonlocal initial or terminal type eigenvalue problems

We first consider the three-point eigenvalue problem

$$z'(t) = \lambda z(t), \quad z(0) = \frac{1}{2}[z(1/2) + z(1)], \quad (37)$$

where $\lambda \in \mathbb{C}$, $z : [0, 1] \rightarrow \mathbb{C}$. The boundary condition is a special case of the one in Corollary 2.13 with

$$h(s) = \begin{cases} 0 & \text{if } s \in [0, \frac{1}{2}) \\ \frac{1}{2} & \text{if } s \in [\frac{1}{2}, 1) \\ 1 & \text{if } s = 1. \end{cases}$$

PROPOSITION 2.22. *The problem (37) has the eigenvalues*

$$\lambda_{IC,1,k} = 2k(2\pi i), \quad \lambda_{IC,2,k} = \log 4 + (2k + 1)(2\pi i) \quad (k \in \mathbb{Z}).$$

They are located in the right part of the complex plane.

Proof. The eigenvalue problem (37) has a nontrivial solution if and only if $\lambda \in \mathbb{C}$ is such that

$$1 = \frac{1}{2}e^{\lambda/2} + \frac{1}{2}e^{\lambda}. \quad (38)$$

Setting $\mu := e^{\lambda/2}$, the equation (38) becomes the equation in μ

$$\frac{1}{2}\mu^2 + \frac{1}{2}\mu - 1 = 0$$

with solutions $\mu_{IC,1} = 1$ and $\mu_{IC,2} = -2$. The equation $e^{\lambda/2} = \mu_{IC,1}$ is satisfied for $\frac{\lambda}{2} = 2k\pi i$ ($k \in \mathbb{Z}$), which gives the eigenvalues $\lambda_{IC,1,k}$ ($k \in \mathbb{Z}$). The equation $e^{\lambda/2} = \mu_{IC,2} = -2$ is satisfied for $\frac{\lambda}{2} = \log 2 + (2k + 1)(\pi i)$ ($k \in \mathbb{Z}$), which gives the eigenvalues $\lambda_{IC,2,k}$ ($k \in \mathbb{Z}$). \square

Similarly, we consider the three-point eigenvalue problem

$$z'(t) = \lambda z(t), \quad z(1) = \frac{1}{2}[z(0) + z(1/2)], \quad (39)$$

where $\lambda \in \mathbb{C}$, $z : [0, 1] \rightarrow \mathbb{C}$. The boundary condition is a special case of the one in Corollary 2.16 with

$$h(s) = \begin{cases} 0 & \text{if } s = 0 \\ \frac{1}{2} & \text{if } s \in (0, \frac{1}{2}] \\ 1 & \text{if } s \in (\frac{1}{2}, 1]. \end{cases}$$

PROPOSITION 2.23. *The problem (39) has the eigenvalues*

$$\lambda_{TC,1,k} = 2k(2\pi i), \quad \lambda_{TC,2,k} = -\log 4 + (2k + 1)(2\pi i) \quad (k \in \mathbb{Z}).$$

They are located in the left part of the complex plane.

Proof. The eigenvalue problem (39) has a nontrivial solution if and only if $\lambda \in \mathbb{C}$ is such that

$$e^\lambda = \frac{1}{2} + \frac{1}{2}e^{\lambda/2}. \quad (40)$$

Setting $\mu := e^{\lambda/2}$, the equation (40) becomes the equation in μ

$$\mu^2 - \frac{1}{2}\mu - \frac{1}{2} = 0,$$

with solutions $\mu_{TC,1} = 1$, $\mu_{TC,2} = -\frac{1}{2}$. The equation $e^{\lambda/2} = \mu_{TC,1} = 1$ is satisfied for $\frac{\lambda}{2} = 2k\pi i$ ($k \in \mathbb{Z}$), which gives the eigenvalues $\lambda_{TC,1,k}$ ($k \in \mathbb{Z}$). The equation $e^{\lambda/2} = \mu_{TC,2} = -\frac{1}{2}$ is satisfied for $\frac{\lambda}{2} = -\log 2 + (2k+1)\pi i$ ($k \in \mathbb{Z}$), which gives the eigenvalues $\lambda_{TC,2,k}$ ($k \in \mathbb{Z}$). \square

REMARK 2.24: The eigenvalues of the periodic boundary conditions

$$z' = \lambda z, \quad z(0) = z(1)$$

are, as easily seen, $\lambda_{P,k} = k(2\pi i)$ ($k \in \mathbb{Z}$). In the case of the problem (39), half of the eigenvalues of the periodic problem move to the line $\Re z = -\log 4$, and, in the case of the problem (37), the same half moves to the line $\Re z = \log 4$. In each case, the symmetry of the spectrum with respect to the imaginary axis is lost (see Fig. 3).

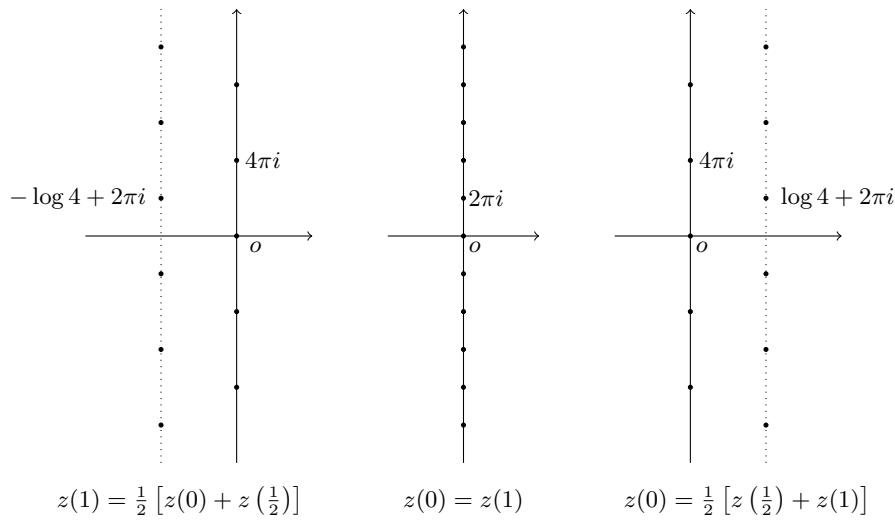


Figure 4: Eigenvalues.

2.11. Fredholm alternative for some linear nonlocal initial or terminal type BVP

To construct our counterexamples, we use of the Fredholm alternative for the corresponding forced eigenvalue problems.

PROPOSITION 2.25. *λ is an eigenvalue of (37) (resp. (39)) if and only if there exists $e \in C([0, 1], \mathbb{C})$ such that the nonhomogeneous problem (41) (resp. (42)) has no solution.*

Proof. It is shown in Lemmas 2.4 and 2.6 (or by direct verification) that the non-homogeneous problems

$$z' + z = e(t), \quad z(0) = \frac{1}{2}z(1/2) + \frac{1}{2}z(1)$$

and

$$z' - z = e(t), \quad z(1) = \frac{1}{2}z(1/2) + \frac{1}{2}z(0)$$

have a unique solution $z = K_1 e$ and $z = K_2 e$ for every $e \in C([0, 1], \mathbb{C})$, with K_1 and K_2 compact in $C([0, 1], \mathbb{C})$. Consequently, each problem

$$z' - \lambda z = e(t), \quad z(0) = \frac{1}{2}z(1/2) + \frac{1}{2}z(1), \tag{41}$$

and

$$z' - \lambda z = e(t), \quad z(1) = \frac{1}{2}z(1/2) + \frac{1}{2}z(0), \quad (42)$$

can be written equivalently

$$z = (\lambda + 1)K_1z + K_1e, \quad z = (\lambda - 1)K_2z + K_2e,$$

and the Fredholm alternative follows from Riesz theory of linear compact operators. \square

2.12. Counterexamples to Corollaries 2.13 and 2.16 under opposite vector fields sign conditions

We finalize the construction of our counterexamples.

In the case of a three-point boundary condition of initial type, we apply Proposition 2.25 to the eigenvalue $\lambda_{IC,2,0} = \log 4 + 2\pi i$ of (37). Using Proposition 2.25, let $e \in C([0, 1], \mathbb{C})$ be such that the problem

$$z'(t) = (\log 4 + 2\pi i)z(t) + e(t), \quad z(0) = \frac{1}{2}z(1/2) + \frac{1}{2}z(1)$$

has no solution. Setting $z(t) = x_1(t) + ix_2(t)$, $e(t) = e_1(t) + ie_2(t)$, the equivalent planar real problem

$$x' = f(t, x), \quad x(0) = \frac{1}{2}x(1/2) + \frac{1}{2}x(1) \quad (43)$$

with

$$f(t, u) := ((\log 4)u_1 - 2\pi u_2 + e_1(t), 2\pi u_1 + (\log 4)u_2 + e_2(t)),$$

is such that

$$\begin{aligned} \langle u|f(t, u) \rangle &= (\log 4)(u_1^2 + u_2^2) + u_1e_1(t) + u_2e_2(t) \\ &\geq (\log 4)|u|^2 - |e(t)||u| > 0 \end{aligned}$$

when $|u| \geq R$ for some sufficiently large R , and has no solution.

Applying Proposition 2.25 to the case of the eigenvalue $\lambda_{TC,2,0} = -\log 4 + 2\pi i$ of (39), let $e \in C([0, 1], \mathbb{C})$ be such that the problem

$$z'(t) = (-\log 4 + 2\pi i)z(t) + e(t), \quad z(1) = \frac{1}{2}z(0) + \frac{1}{2}z(1/2)$$

has no solution. Setting $z(t) = x_1(t) + ix_2(t)$, $e(t) = e_1(t) + ie_2(t)$, the equivalent planar real problem

$$x' = f(t, x), \quad x(1) = \frac{1}{2}x(0) + \frac{1}{2}x(1/2) \quad (44)$$

with

$$f(t, u) := (-\log 4)u_1 - 2\pi u_2 + e_1(t), 2\pi u_1 - (\log 4)u_2 + e_2(t),$$

is such that

$$\begin{aligned} \langle u|f(t, u) \rangle &= -(\log 4)(u_1^2 + u_2^2) + u_1 e_1(t) + u_2 e_2(t) \\ &\leq -(\log 4)|u|^2 + |e(t)||u| < 0, \end{aligned} \quad (45)$$

when $|u| \geq R$ for some sufficiently large R , and has no solution.

REMARK 2.26: The symmetry-breaking for the spectra of the three-point BVP of terminal or initial type, explains the difference in the existence conditions for the nonlinear problems with the three-point boundary conditions and with the periodic conditions. The presence of the complex spectrum in the left or the right half plane influences like a ghost the existence of solutions of the real nonlinear systems. Of course, extra conditions upon f could provide existence results with the sign conditions of the counterexamples.

REMARK 2.27: Our counterexamples do not cover the case of n odd. For $n = 3$, if one adds the equations

$$x'_3 = (\log 4)x_3 + \frac{\log 4}{4}(x_1 + x_2), \quad x_3(0) = \frac{1}{2}[x_3(1/2) + x_3(1)],$$

or

$$x'_3 = -(\log 4)x_3 + \frac{\log 4}{4}(x_1 + x_2), \quad x_3(1) = \frac{1}{2}[x_3(0) + x_3(1/2)]$$

to (43) or to (44) respectively, the corresponding boundary value problems have no solutions and the nonlinear parts verify the opposite sign conditions to Corollaries 2.13 and 2.16 respectively. The counterexamples for $n = 2$ and $n = 3$ easily provide counterexamples in any dimension $n \geq 2$.

REMARK 2.28: As easily seen, the periodic problem

$$z' = 2\pi iz + e^{2\pi it}, \quad z(0) = z(1). \quad (46)$$

has no solution. Letting $z = x_1 + ix_2$, the equivalent real planar problem

$$x' = f(t, x), \quad x(0) = x(1)$$

with

$$f_1(t, x_1, x_2) = -2\pi x_2 + \cos(2\pi t), \quad f_2(t, x_1, x_2) = 2\pi x_1 + \sin(2\pi t),$$

has no solution, and is such that

$$\langle x|f(t, x) \rangle = \cos(2\pi t)x_1 + \sin(2\pi t)x_2$$

For $x = R[\cos(2\pi\theta), \sin(2\pi\theta)] \in \partial B_R$ ($\theta \in [0, 1]$), we have

$$\begin{aligned} \langle x|f(t, x) \rangle &= R[\cos(2\pi t)\cos(2\pi\theta) + \sin(2\pi t)\sin(2\pi\theta)] \\ &= R\cos[2\pi(t - \theta)] \quad (t, \theta \in [0, 1]), \end{aligned}$$

which implies that, for each $t \in [0, 1]$, $\langle x|f(t, x) \rangle$ takes both positive and negative values on ∂B_R . Hence, the assumptions of the existence theorems for periodic problems given above are sharp.

2.13. An easy extension of Theorems 2.12 and 2.15

Let $g : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous. By replacing f by g in the equivalent formulation as a fixed point problem and in the proofs, it is immediate to prove the following extensions of Theorems 2.12 and 2.15.

THEOREM 2.29. *If $h : [0, 1] \rightarrow \mathbb{R}$ satisfies conditions (h0), (h1), or conditions $h(0)$, $h(2)$, $h(3)$, and if there exists an open, bounded, convex neighborhood C of 0 in \mathbb{R}^n and an outer normal field ν on ∂C such that*

$$\langle \nu(u)|g(t, v, u) \rangle \leq 0, \quad \forall (t, v, u) \in [0, 1] \times \bar{C} \times \partial C, \quad (47)$$

then the problem

$$x'(t) = g\left(t, \int_0^t x(s) ds, x(t)\right) \quad (t \in [0, 1]), \quad x(0) = \int_0^1 dh(s)x(s) \quad (48)$$

has at least one solution taking values in \bar{C} .

Proof. The main difference in the proof is that the nonlinear mapping $N_1 : C^0 \rightarrow C^0$ occurring in the fixed point formulation is now defined by

$$N_1x(t) = g\left(t, \int_0^t x, x(t)\right) - x(t) \quad (t \in [0, 1]),$$

and its value at $t \in [0, 1]$ depends not only on $x(t)$ but on all values of $x(s)$ for $s \in [0, t]$. It is easily checked that it does not modify the compactness properties of the operator K_1N_1 . All the other arguments of the proof remain valid *mutatis mutandis* because of the uniformity of assumption (47) with respect to v . \square

THEOREM 2.30. *If $h : [0, 1] \rightarrow \mathbb{R}$ satisfies conditions (h0), (h1), or conditions $h(0)$, $h(2)$, $h(4)$, and if there exists an open, bounded, convex neighborhood C of 0 in \mathbb{R}^n and an outer normal field ν on ∂C such that*

$$\langle \nu(u)|g(t, v, u) \rangle \geq 0, \quad \forall (t, v, u) \in [0, 1] \times \bar{C} \times \partial C,$$

then the problem

$$x'(t) = g\left(t, \int_0^t x(s) ds, x(t)\right) \quad (t \in [0, 1]), \quad x(1) = \int_0^1 x(s) dh(s) \quad (49)$$

has at least one solution taking values in \overline{C} .

Proof. Similar to Theorem 2.15 using the remarks in the proof of Theorem 2.29. \square

Of course, the following extensions, where the value of $x'(t)$ depends this time upon the values of $x(s)$ for $s \in [t, 1]$, are obtained in a similar way.

THEOREM 2.31. *If $h : [0, 1] \rightarrow \mathbb{R}$ satisfies conditions (h0), (h1), or conditions h(0), h(2), h(3), and if there exists an open, bounded, convex neighborhood C of 0 in \mathbb{R}^n and an outer normal field ν on ∂C such that*

$$\langle \nu(u) | g(t, v, u) \rangle \leq 0, \quad \forall (t, v, u) \in [0, 1] \times \overline{C} \times \partial C,$$

then the problem

$$x'(t) = g\left(t, \int_1^t x(s) ds, x(t)\right) \quad (t \in [0, 1]), \quad x(0) = \int_0^1 dh(s)x(s)$$

has at least one solution taking values in \overline{C} .

THEOREM 2.32. *If $h : [0, 1] \rightarrow \mathbb{R}$ satisfies conditions (h0), (h1), or conditions h(0), h(2), h(4), and if there exists an open, bounded, convex neighborhood C of 0 in \mathbb{R}^n and an outer normal field ν on ∂C such that*

$$\langle \nu(u) | g(t, v, u) \rangle \geq 0, \quad \forall (t, v, u) \in [0, 1] \times \overline{C} \times \partial C,$$

then the problem

$$x'(t) = g\left(t, \int_1^t x(s) ds, x(t)\right) \quad (t \in [0, 1]), \quad x(1) = \int_0^1 x(s) dh(s)$$

has at least one solution taking values in \overline{C} .

3. Second order systems

3.1. Boundary conditions

We now consider the case of second order (1) where $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous. Again there is no loss of generality in taking the independent variable in $[0, 1]$.

We consider in what follows the following particular nonlocal conditions:
the **Dirichlet type nonlocal conditions**

$$x(0) = 0, \quad x(1) = \int_0^1 x(s) dh(s), \quad (50)$$

the **mixed type nonlocal conditions**

$$x'(0) = 0, \quad x(1) = \int_0^1 x(s) dh(s), \quad (51)$$

the **nonlocal conditions of initial type**

$$x(0) = 0, \quad x'(0) = \int_0^1 x'(s) dh(s),$$

the **mixed type nonlocal conditions**

$$x(0) = 0, \quad x'(1) = \int_0^1 x'(s) dh(s),$$

and the **nonlocal conditions of terminal type**

$$x(1) = 0, \quad x'(1) = \int_0^1 x'(s) dh(s).$$

Neumann type nonlocal conditions are considered in [73, 102, 108] using other continuation theorems.

3.2. Some nonlocal BVP for linear second order systems

We start with the Dirichlet type nonlocal BC.

LEMMA 3.1. *If conditions (h0), (h1) or (h0), (h2), (h4) hold, then, for each $z \in C^0$, the linear nonlocal Dirichlet type problem*

$$x'' - x = z(t), \quad x(0) = 0, \quad x(1) = \int_0^1 x(s) dh(s) \quad (52)$$

has the unique solution

$$x(t) = \left(\sinh 1 - \int_0^1 \sinh s dh(s) \right)^{-1} \int_0^1 \int_0^u \sinh(u-s) z(s) ds dh(u) \sinh t + \int_0^t \sinh(t-s) z(s) ds. \quad (53)$$

Proof. By the variation of constants formula for each $c \in \mathbb{R}^n$, the initial value problem

$$x'' - x = z(t), \quad x(0) = 0, \quad x'(0) = c,$$

has the unique solution

$$x(t) = c \sinh t + \int_0^t \sinh(t-s)z(s) ds.$$

It satisfies the boundary condition (50) if and only if c satisfies the linear algebraic system

$$c \sinh 1 = c \int_0^1 \sinh t dh(t) + \int_0^1 \int_0^t \sinh(t-s)z(s) ds dh(t),$$

which has the unique solution

$$c = \left(\sinh 1 - \int_0^1 \sinh s dh(s) \right)^{-1} \int_0^1 \int_0^u \sinh(u-s)z(s) ds dh(u).$$

if $\int_0^1 \sinh s dh(s) \neq \sinh 1$. Following the reasoning of the corresponding Lemma for first order systems, and noticing that \sinh reaches its maximum on $[0, 1]$ at 1, this is the case if conditions (h0), (h1) or conditions (h0), (h2), (h4) hold. The result follows. \square

Let C^1 be the Banach space of mappings $x : [0, 1] \rightarrow \mathbb{R}^n$ of class C^1 with the norm

$$\|x\| := \max\{\|x_1\|_\infty, \dots, \|x_n\|_\infty, \|x'_1\|_\infty, \dots, \|x'_n\|_\infty\}.$$

Like in the first order case, formula (53) defines a compact linear mapping

$$K_1 : C^0 \rightarrow C^1, \quad z \mapsto x.$$

In a similar way, one can prove the corresponding results for the mixed type nonlocal BC.

LEMMA 3.2. *If conditions (h0), (h1) or (h0), (h2), (h4) hold, then, for each $z \in C^0$, the linear nonlocal mixed type problem*

$$x'' - x = z(t), \quad x'(0) = 0, \quad x(1) = \int_0^1 x(s) dh(s) \quad (54)$$

has a unique solution x and the corresponding linear mapping

$$K_2 : C^0 \rightarrow C^1, \quad z \mapsto x$$

is compact.

3.3. Fixed point formulation of nonlinear nonlocal BVP of the second order

Let now $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous and define the mapping $N : C^1 \rightarrow C^0$ by

$$Nx = f(\cdot, x(\cdot), x'(\cdot)) - x(\cdot).$$

It is easy to show that N is continuous on C^1 and take bounded sets of C^1 into bounded sets of C^0 . Under the conditions of Lemma 3.1,

$$G_1 := K_1 N : C^1 \rightarrow C^1$$

is compact on bounded subsets of C^1 , and the **nonlinear nonlocal Dirichlet type problem**

$$x'' = f(t, x, x'), \quad x(0) = 0, \quad x(1) = \int_0^1 x(s) dh(s) \tag{55}$$

is equivalent to the fixed point problem in C^1

$$x = G_1 x. \tag{56}$$

Similarly, under the conditions of Lemma 3.2,

$$G_2 := K_2 N : C^1 \rightarrow C^1$$

is compact on bounded subsets of C^1 , and the **nonlinear nonlocal mixed type problem**

$$x'' = f(t, x, x'), \quad x'(0) = 0, \quad x(1) = \int_0^1 x(s) dh(s) \tag{57}$$

is equivalent to the fixed point problem in C^1

$$x = G_2 x. \tag{58}$$

We want to apply to the equations (56) and (58) the Leray–Schauder existence result given in Proposition 2.8, where now $X = C^1$, so that the a priori estimates are requested not only upon x but also upon x' .

3.4. Bernstein–Hartman lemma

In order to obtain the a priori estimates on x' requested by Proposition 2.8 when an a priori estimate on x is known, we use the following lemma, a special case of a more general result of Hartman [31, 32] for functions with values in \mathbb{R}^n . For $n = 1$, the result, without condition (59), was proved by Bernstein in 1912 [6].

LEMMA 3.3. Assume that $x \in C^2([0, 1], \mathbb{R}^n)$ satisfies the following inequalities

$$|x(t)| \leq R,$$

and

$$|x''(t)| \leq \gamma |x'(t)|^2 + K$$

for all $t \in [0, 1]$ and some $R > 0$, $K \geq 0$ and $\gamma \geq 0$ such that

$$\gamma R < 1. \quad (59)$$

Then, there exists $M = M(R, \gamma, K)$ such that for all $t \in [0, 1]$,

$$|x'(t)| \leq M.$$

REMARK 3.4: For $n \geq 2$, the condition (59) is sharp, as shown by the example of the sequence of functions, introduced by Heinz [34],

$$x_n : [0, 2\pi] \rightarrow \mathbb{R}^2, \quad t \mapsto (\cos nt, \sin nt) \quad (n \in \mathbb{N}),$$

for which, with $\langle \cdot | \cdot \rangle$ the usual inner product and $|\cdot|$ the Euclidian norm in \mathbb{R}^2 ,

$$|x_n(t)| = 1, \quad |x_n''(t)| = |x_n'(t)|^2 = n^2,$$

so that that the conclusion of Lemma 3.3 does not hold for $\gamma R = 1$ and $T = 2\pi$, as $|x_n'(t)| = n$ can be arbitrary large.

3.5. Some nonlocal nonlinear BVP of Dirichlet or mixed type

We first show that conditions with respect to u on the vector field $f(t, u, v)$ similar to those introduced for first order systems also lead to the existence of solutions for second order systems.

THEOREM 3.5. Assume that conditions (h0), (h1) or conditions (h0), (h2), (h4) hold, and that there exists an open, bounded, convex neighborhood C of 0 in \mathbb{R}^n , and an outer normal vector field ν on ∂C , such that

$$\langle \nu(u) | f(t, u, v) \rangle \geq 0, \quad \text{whenever } u \in \partial C \text{ and } \langle v | \nu(u) \rangle = 0$$

and

$$|f(t, u, v)| \leq \gamma |v|^2 + K,$$

for some $\gamma \geq 0$ such that $\gamma R < 1$, $K \geq 0$, and $R = \max_{u \in \overline{C}} |u|$. Then the problem (55) has at least one solution such that $x(t) \in \overline{C}$ for all $t \in [0, 1]$.

Proof. Let us consider the equation (56) and define first the open bounded neighborhood Ω_1 of 0 in C^0 by

$$\Omega_1 = \{x \in C^0 : x(t) \in C, \forall t \in [0, 1]\}. \quad (60)$$

As first step in applying Proposition 2.8, we show that for each $\lambda \in (0, 1)$, no possible solution of the problem

$$x'' - x = \lambda[f(t, x, x') - x], \quad x(0) = 0, \quad x(1) = \int_0^1 x(s) dh(s), \quad (61)$$

belongs to $\partial\Omega_1$. Let $\lambda \in (0, 1)$ and $x(t) \in \partial\Omega_1$ be a possible solution to (61). Then $x(t) \in \bar{C}$ for all $t \in [0, 1]$ and there is some $t_0 \in [0, 1]$ such that $x(t_0) \in \partial C$. Therefore, for all $t \in [0, 1]$,

$$\xi_{t_0}(t) := \langle \nu(x(t_0)) | x(t) \rangle \leq \langle \nu(x(t_0)) | x(t_0) \rangle = \xi_{t_0}(t_0), \quad \forall t \in [0, 1],$$

which means that the real function $\xi_{t_0} : [0, 1] \rightarrow \mathbb{R}$ reaches its maximum at t_0 . Because of the first boundary condition, we cannot have $t_0 = 0$. If $t_0 \in (0, 1)$, $0 = \xi'_{t_0}(t_0) = \langle \nu(x(t_0)) | x'(t_0) \rangle$ and

$$\begin{aligned} 0 &\geq \xi''_{t_0}(t_0) = \langle \nu(x(t_0)) | x''(t_0) \rangle \\ &= (1 - \lambda) \langle \nu(x(t_0)) | x(t_0) \rangle + \lambda \langle \nu(x(t_0)) | f(t_0, x(t_0), x'(t_0)) \rangle > 0, \end{aligned}$$

a contradiction. Finally, if $t_0 = 1$, we use the second boundary condition like in the nonlocal terminal like problem for first order systems to obtain the contradiction.

Now, as $x(t) \in \bar{C}$ for all $t \in [0, 1]$, we have, for all $t \in [0, 1]$,

$$\begin{aligned} |x''(t)| &= |(1 - \lambda)x(t) + \lambda f(t, x(t), x'(t))| \\ &\leq R + \gamma|x'(t)|^2 + K = \gamma|x'(t)|^2 + (R + K) \end{aligned}$$

and Lemma 3.3 implies the existence of $M > 0$ depending only upon R, γ, K such that $|x'(t)| \leq M$ for all $t \in [0, 1]$. If we set

$$\Omega_2 := \{x \in C^1 : |x'(t)| < M + 1, \forall t \in [0, 1]\},$$

and $\Omega = \Omega_1 \cap \Omega_2$, all the assumptions of Proposition 2.8 are satisfied and the conclusion follows. \square

In a similar way, we can prove the following existence result for the problem (57).

THEOREM 3.6. *Assume that conditions (h0),(h1) or conditions (h0),(h2),(h4) hold, and that there exists an open, bounded, convex neighborhood C of 0 in \mathbb{R}^n , and an outer normal vector field ν on ∂C , such that*

$$\langle \nu(u) | f(t, u, v) \rangle \geq 0, \quad \text{whenever } u \in \partial C, \langle v | \nu(u) \rangle = 0$$

and

$$|f(t, u, v)| \leq \gamma|v|^2 + K,$$

for some $\gamma \geq 0$ such that $\gamma R < 1$, $K \geq 0$, and $R = \max_{u \in \overline{C}} |u|$. Then the problem (57) has at least one solution such that $x(t) \in \overline{C}$ for all $t \in [0, 1]$.

The choice of $C = B_R$ provide the corresponding special cases.

COROLLARY 3.7. *Assume that conditions (h0), (h1) or conditions (h0), (h2), (h4) hold, and that there exists $R > 0$ such that*

$$\langle u|f(t, u, v) \rangle \geq 0, \quad \text{whenever } u \in \partial B_R \text{ and } \langle v|u \rangle = 0$$

and

$$|f(t, u, v)| \leq \gamma|v|^2 + K,$$

for some $\gamma \geq 0$ such that $\gamma R < 1$, and $K \geq 0$. Then the problem (55) has at least one solution such that $x(t) \in \overline{B}_R$ for all $t \in [0, 1]$.

Special cases of Corollary 3.7 can be found in [100].

COROLLARY 3.8. *Assume that conditions (h0), (h1) or conditions (h0), (h2), (h4) hold, and that there exists $R > 0$ such that*

$$\langle u|f(t, u, v) \rangle \geq 0, \quad \text{whenever } u \in \partial B_R \text{ and } \langle v|u \rangle = 0$$

and

$$|f(t, u, v)| \leq \gamma|v|^2 + K,$$

for some $\gamma \geq 0$ such that $\gamma R < 1$, and $K \geq 0$. Then the problem (57) has at least one solution such that $x(t) \in \overline{B}_R$ for all $t \in [0, 1]$.

The assumptions in Corollaries 3.7 and 3.8 can be slightly improved by taking in account the curvature of the ball, in contrast with the general case of convex sets which may have flat curvature almost everywhere (polyhedra). The corresponding conditions were first obtained by Hartman [31, 32] for Dirichlet boundary conditions, and extended by various authors to other classical two-point BC like mixed, Neumann or periodic, and to some four-point boundary conditions and component-wise Bernstein-Nagumo conditions by Calábek [10].

THEOREM 3.9. *Assume that conditions (h0),(h1) or conditions (h0),(h2),(h4) hold, and that there exists $R > 0$ such that*

$$|v|^2 + \langle u|f(t, u, v) \rangle \geq 0, \quad \text{whenever } u \in \partial B_R \text{ and } \langle v|u \rangle = 0$$

and

$$|f(t, u, v)| \leq \gamma|v|^2 + K,$$

for some $\gamma \geq 0$ such that $\gamma R < 1$, and $K \geq 0$. Then the problem (55) has at least one solution such that $x(t) \in \overline{B}_R$ for all $t \in [0, 1]$.

Proof. Following the lines of the proof of Theorem 3.5, we define first the open bounded neighborhood Ω_1 of 0 in C^0 by

$$\Omega_1 = \{x \in C^0 : |x(t)| < R, \forall t \in [0, 1]\}.$$

As first step in applying Proposition 2.8, we show that for each $\lambda \in (0, 1)$, no possible solution of the problem

$$x'' - x = \lambda[f(t, x, x') - x], \quad x(0) = 0, \quad x(1) = \int_0^1 x(s) dh(s), \quad (62)$$

belongs to $\partial\Omega_1$. Let $\lambda \in (0, 1)$ and $x(t) \in \partial\Omega_1$ be a possible solution to (62). Then $|x(t)|^2 \leq R^2$ for all $t \in [0, 1]$ and there is some $t_0 \in [0, 1]$ such that $|x(t_0)|^2 = R^2$. Therefore the function $\xi(t) := |x(t)|^2/2$ reaches its maximum at t_0 . Because of the first boundary condition, we cannot have $t_0 = 0$. If $t_0 \in (0, 1)$, $0 = \xi'(t_0) = \langle x(t_0)|x'(t_0) \rangle$ and

$$\begin{aligned} 0 &\geq \xi''_{t_0}(t_0) = |x'(t_0)|^2 + \langle x(t_0)|x''(t_0) \rangle \\ &\geq \lambda|x'(t_0)|^2 + (1 - \lambda)|x(t_0)|^2 + \lambda\langle x(t_0)|f(t_0, x(t_0), x'(t_0)) \rangle > 0, \end{aligned}$$

a contradiction. Finally, if $t_0 = 1$, we use the second boundary condition and its consequence

$$|x(1)| \leq \int_0^1 |x(s)| dh(s),$$

and the nonlocal terminal type problem for first order systems to obtain the contradiction. The remaining part of the proof is exactly similar to that of Theorem 3.5. \square

In a similar way, we prove the corresponding result for the mixed case.

THEOREM 3.10. *Assume that conditions (h0), (h1) or conditions (h0), (h2), (h4) hold, and that there exists $R > 0$ such that*

$$|v|^2 + \langle u|f(t, u, v) \rangle \geq 0, \quad \text{whenever } u \in \partial B_R \text{ and } \langle v|u \rangle = 0$$

and

$$|f(t, u, v)| \leq \gamma|v|^2 + K,$$

for some $\gamma \geq 0$ such that $\gamma R < 1$, and $K \geq 0$. Then the problem (57) has at least one solution such that $x(t) \in \overline{B}_R$ for all $t \in [0, 1]$.

3.6. Other nonlocal nonlinear BVP of mixed type

Let us consider the nonlocal BVP of initial type

$$x'' = f(t, x, x'), \quad x(0) = 0, \quad x'(0) = \int_0^1 x'(s) dh(s). \quad (63)$$

The following existence theorem is given in [75]. This time the vector field condition similar to the one for first order systems is made on $f(t, u, v)$ with respect to v .

THEOREM 3.11. *Assume that h satisfies conditions (h0), (h1), or conditions (h0), (h2), (h3) and that there exists an open, bounded, convex neighborhood C of 0 in \mathbb{R}^n and an outer normal field ν to ∂C such that*

$$\langle \nu(v) | f(t, u, v) \rangle \leq 0, \quad \forall (t, u, v) \in [0, 1] \times \overline{C} \times \partial C. \quad (64)$$

Then the problem (63) has at least one solution x such that $x(t) \in \overline{C}$ and $x'(t) \in \overline{C}$ for all $t \in [0, 1]$.

Proof. We set $y = x'$, so that, using the first boundary condition $x(0) = 0$,

$$x(t) = \int_0^t x'(s) ds = \int_0^t y(s) ds,$$

and the problem (63) can be written, in terms of y ,

$$y'(t) = f\left(t, \int_0^t y(s) ds, y(t)\right), \quad y(0) = \int_0^1 y(s) dh(s). \quad (65)$$

The result follows then from Theorem 2.29 and the fact that, by the convexity of C , $\int_0^t y(s) ds \in \overline{C}$ for all $t \in [0, 1]$. \square

A similar result, with a similar proof using Theorem 2.30, holds for the following nonlinear BVP of mixed type.

THEOREM 3.12. *Assume that h satisfies conditions (h0), (h1), or conditions (h0), (h2), (h4) and that there exists an open, bounded, convex neighborhood C of 0 in \mathbb{R}^n and an outer normal field ν to ∂C such that*

$$\langle \nu(v) | f(t, u, v) \rangle \geq 0, \quad \forall (t, u, v) \in [0, 1] \times \overline{C} \times \partial C. \quad (66)$$

Then the problem

$$x'' = f(t, x, x'), \quad x(0) = 0, \quad x'(1) = \int_0^1 x'(s) dh(s)$$

has at least one solution x such that $x(t) \in \overline{C}$ and $x'(t) \in \overline{C}$ for all $t \in [0, 1]$.

Finally, using Theorems 2.31 and 2.32 and the fact that $x(t) = \int_1^t x'(s) ds = \int_1^t y(s) ds$, we obtain in a similar way the following results.

THEOREM 3.13. *Assume that h satisfies conditions (h0), (h1), or conditions (h0), (h2), (h3) and that there exists an open, bounded, convex neighborhood C of 0 in \mathbb{R}^n and an outer normal field ν to ∂C such that condition (64) holds. Then the problem*

$$x'' = f(t, x, x'), \quad x'(0) = \int_0^1 x'(s) dh(s), \quad x(1) = 0$$

has at least one solution x such that $x(t) \in \bar{C}$ and $x'(t) \in \bar{C}$ for all $t \in [0, 1]$.

THEOREM 3.14. *Assume that h satisfies conditions (h0), (h1), or conditions (h0), (h2), (h4) and that there exists an open, bounded, convex neighborhood C of 0 in \mathbb{R}^n and an outer normal field ν to ∂C such that the condition (66) holds. Then the problem*

$$x'' = f(t, x, x'), \quad x(1) = 0, \quad x'(1) = \int_0^1 x'(s) dh(s)$$

has at least one solution x such that $x(t) \in \bar{C}$ and $x'(t) \in \bar{C}$ for all $t \in [0, 1]$.

Special cases of those results when $C = B_R$ as well as results for other similar nonlocal boundary conditions can be found in [59, 101, 103, 104].

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Authors' addresses:

Jean Mawhin
Institut de Recherche en Mathématique et Physique
Université Catholique de Louvain
Chemin du Cyclotron, 2, B-1348 Louvain-la-Neuve, Belgium
E-mail: jean.mawhin@uclouvain.be

Katarzyna Szymańska-Dębowska
Institute of Mathematics
Lodz University of Technology
90-924 Łódź, ul. Wólczajska 215, Poland
E-mail: grampa@zbiorcza.net.lodz.pl

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On upper and lower bounds for finite group-actions on bounded surfaces, handlebodies, closed handles and finite graphs¹

BRUNO P. ZIMMERMANN

ABSTRACT. In the present paper, partly a survey, we discuss upper and lower bounds for finite group-actions on bounded surfaces, 3-dimensional handlebodies and closed handles, handlebodies in arbitrary dimensions and finite graphs (the common feature of these objects is that they have all free fundamental group).

Keywords: upper bounds for finite group action, handlebody in arbitrary dimension, surface with boundary, finite graph.
MS Classification 2010: 57M60, 57S17, 57S25.

1. Introduction

We consider finite group-actions of large order on various low-dimensional manifolds with free fundamental group, and also on higher-dimensional analogues and on finite graphs. All group-actions in the present paper will be faithful and smooth. The manifolds we consider are the following:

Section 1.1: 3-dimensional handlebodies and the closely related case of surfaces with nonempty boundary;

Section 1.2: closed 3-dimensional handles, i.e. connected sums $\#_g(S^1 \times S^2)$ of g copies of $S^1 \times S^2$, considering first arbitrary actions and then free actions (the case of free actions is in close analogy with the results in section 1.1);

Section 1.3: handlebodies in arbitrary dimensions;

Section 1.4: finite graphs, considering also finite group-actions on finite graphs embedded in spheres.

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1.1. Finite group-actions on 3-dimensional handlebodies and bounded surfaces

In analogy with the classical Hurwitz-bound $84(g-1)$ for the order of a finite, orientation-preserving group-action on a closed, orientable surface of genus $g \geq 2$, an upper bound for the order of a finite group of orientation-preserving diffeomorphisms of a 3-dimension handlebody V_g of genus $g \geq 2$ is $12(g-1)$ ([18],[5, Theorem 7.2]). More generally, the following holds.

THEOREM 1.1 ([8]). *Let $m_{hb}(g)$ denote the maximum order of a finite, orientation-preserving group-action on a 3-dimensional handlebody of genus $g > 1$.*

i) There are the upper and lower bounds

$$4(g+1) \leq m_{hb}(g) \leq 12(g-1),$$

and both $4(g+1)$ and $12(g-1)$ occur for infinitely many genera g .

ii) If g is odd then $m_{hb}(g) = 8(g-1)$ or $m_{hb}(g) = 12(g-1)$, and both cases occur for infinitely many values of g .

iii) The possible values of $m_{hb}(g)$ are of the form $\frac{4n}{n-2}(g-1)$, for an integer $n \geq 3$, and infinitely many values of n occur resp. do not occur. Moreover if a value of n occurs then it occurs for infinitely many g .

For finite group-actions on bounded surfaces (compact with nonempty boundary, orientable or not), exactly the same results hold (and with the same proofs), using the setting in [6] (see [8, Section 3]). Note that, by taking the product with an interval (twisted if the surface is nonorientable), every finite group acting on a bounded surface of *algebraic genus* g (defined as the rank of the free fundamental group) admits also an orientation-preserving action on a handlebody of genus g .

THEOREM 1.2. *Let $m_{bs}(g)$ denote the maximum order of a finite, possibly orientation-reversing group-action on a bounded, orientable or nonorientable surface of algebraic genus $g > 1$.*

i) $m_{bs}(g) \leq m_{hb}(g)$.

ii) All statements of Theorem 1.1 remain true for $m_{bs}(g)$.

iii) There are values of g such that $m_{bs}(g)$ is strictly smaller than $m_{hb}(g)$.

Part iii) of Theorem 1.2 is proved in [1] by computational methods; the two smallest values of g such that $m_{bs}(g) < m_{hb}(g)$ are $g = 161$ and $g = 3761$.

1.2. Finite group-actions on closed handles

After the classical cases of 3-dimensional handlebodies and bounded surfaces, we consider actions of finite groups G on closed 3-dimensional analogues of handlebodies, the connected sums $H_g = \#_g(S^1 \times S^2)$ of g copies of $S^1 \times S^2$ (similar as $V_g = \#_g^\partial(S^1 \times D^2)$ is the boundary-connected sum of g copies of $S^1 \times D^2$; so H_g is the double of V_g along its boundary). We will call H_g a *closed handle* or just a *handle* of genus g

Since H_g admits S^1 -actions (see [9]), it admits finite cyclic group-actions of arbitrarily large order acting trivially on the fundamental group. Let G_0 denote the normal subgroup of all elements of G acting trivially on the fundamental group (up to inner automorphisms); by [14, Proposition 2], G_0 is cyclic, the quotient H_g/G_0 is again a closed handle of the same genus g and the factor group G/G_0 acts faithfully on the fundamental group of the quotient $H_g/G_0 \cong H_g$. Hence one is led to consider actions of finite groups G on H_g which act faithfully on the fundamental group, i.e. induce an injection into the outer automorphism group $\text{Out } F_g$ of the fundamental group of H_g , the free group F_g of rank g .

THEOREM 1.3 ([14]). *Let $m_{ch}(g)$ denote the maximum order of a finite, orientation-preserving group-action on a closed handle H_g of genus $g > 1$ which induces a faithful action on the fundamental group.*

- i) For $g \geq 15$, there is the quadratic upper bound $m_{ch}(g) \leq 24g(g-1)$.*
- ii) For all g , there are the quadratic lower bounds $2g^2 \leq m_{ch}(g)$ if g is even, and $(g+1)^2 \leq m_{ch}(g)$ if g is odd.*

We don't know the exact value of $m_{ch}(g)$ at present but believe that for large g it coincides with the lower bounds $2g^2$ resp. $(g+1)^2$ of the second part of Theorem 1.3; for small values of g there are group-actions of larger orders, e.g. $m_{ch}(2) = 12$, $m_{ch}(3) = 48$ and $m_{ch}(4) = 192$.

Next we consider the case of *free* actions of finite groups on closed handles H_g which is in strong analogy with the cases of arbitrary (i.e., not necessarily free) actions on handlebodies and bounded surfaces (where free means that every nontrivial element has empty fixed point set).

THEOREM 1.4. *Let $m_{chf}(g)$ denote the maximum order of a free, orientation-preserving finite group-action on a closed handle H_g of genus $g > 1$.*

- i) For all $g > 1$,*

$$2(g+1) \leq m_{chf}(g) \leq 6(g-1)$$

and both $2(g+1)$ and $6(g-1)$ occur for infinitely many genera g .

- ii) If g is odd then $m_{\text{ch},f}(g) = 4(g-1)$ or $m_{\text{ch},f}(g) = 6(g-1)$, and both cases occur for infinitely many g .
- iii) The possible values of $m_{\text{ch},f}(g)$ are of the form $\frac{2n}{n-2}(g-1)$, for an integer $n \geq 3$, and infinitely many values of n occur resp. do not occur.

We note that exactly the same results hold for finite orientation-preserving group-actions on bounded, orientable surfaces of algebraic genus g .

The proof of Theorem 1.4 combines methods of the handlebody case (Theorem 1.1) with those for closed handles (Theorem 1.3); since it is shorter and less technical, as an illustration of the methods we give the proof in section 2.

1.3. Finite group-actions on handlebodies in arbitrary dimensions

A closed handle H_g is the boundary of a 4-dimensional handlebody, in particular the upper bounds of Theorem 1.3 i) hold also for finite group-actions on 4-dimensional handlebodies. More generally, an orientable handlebody V_g^d of dimension d and genus g is defined as a regular neighbourhood of a finite graph, with free fundamental group of rank g , embedded in the sphere S^d ; alternatively, it is obtained from the closed disk D^d of dimension d by attaching along its boundary g copies of a handle $D^{d-1} \times [0, 1]$ in an orientable way, or as the boundary-connected sum $\#_g^{\partial}(S^1 \times D^{d-1})$ of g copies of $S^1 \times D^{d-1}$.

After Thurston and Perelman, finite group-actions in dimension 3 are geometric; this is no longer true in higher dimensions, so in order to generalize Theorem 1.3 one has to consider some kind of standard actions also in higher dimensions. A natural way to proceed is to uniformize handlebodies V_g^d by Schottky groups (free groups of Möbius transformations of D^d acting by isometries on its interior, the Poincaré-model of hyperbolic space \mathbb{H}^d); this realizes the interior of a handlebody V_g^d as a hyperbolic manifold, and we will consider finite groups of isometries of such hyperbolic (Schottky type) handlebodies.

By [15], every finite subgroup of the outer automorphism group $\text{Out } F_g$ of a free group $F_g \cong \pi_1(V_g^d)$ can be realized by the action of a group of isometries of a hyperbolic handlebody V_g^d (in the sense of the Nielsen realization problem), for a sufficiently large dimension d .

THEOREM 1.5 ([7]). *Let G be a finite group of isometries of a hyperbolic handlebody V_g^d , of dimension $d \geq 3$ and genus $g > 1$, which acts faithfully on the fundamental group.*

- i) *The order of G is bounded by a polynomial of degree $d/2$ in g if d is even, and of degree $(d+1)/2$ if d is odd.*

- ii) The degree $d/2$ is best possible in even dimensions whereas in odd dimensions the optimal degree is either $(d-1)/2$ or $(d+1)/2$.*

So in odd dimensions the optimal degree remains open at present; note that, for $d = 3$, the bound $(d+1)/2 = 2$ is not best possible since it gives a quadratic bound instead of the actual linear bound $12(g-1)$, so maybe for all odd dimensions the optimal degree is $(d-1)/2$.

1.4. Finite group-actions on finite graphs

Let G be a finite group of automorphisms of a finite graph $\tilde{\Gamma}$ of rank $g > 1$ (defined as the rank of its free fundamental group), allowing closed and multiple edges. Note that, without changing the rank of a graph, we can delete all *free edges*, i.e. nonclosed edges with a vertex of valence 1 (an isolated vertex). By possibly subdividing edges, we can also assume that G acts *without inversions* (of edges), i.e. no element acts on an edge as a reflection in its midpoint. We say that a finite graph is *hyperbolic* if it has rank $g > 1$ and no free edges. In the following, all finite group-actions on graphs will be faithful and without inversions.

By [15], each finite subgroup G of the outer automorphism group $\text{Out } F_g$ of a free group F_g can be induced by an action of G on a finite graph of rank g (this is again a version of the Nielsen realization problem). Conversely, if G acts on a hyperbolic graph then G induces an injection into $\text{Out } F_g$ ([16, Lemma 1]). By [13], for $g \geq 3$ the largest possible order of a finite subgroup of $\text{Out } F_g$ is $2^g g!$; in particular, there is no linear or polynomial bound in g for the order of G . In strong analogy with Theorems 1.1 and 1.4, the following holds (proved in section 3):

THEOREM 1.6. *i) The maximal order of a finite group G acting with trivial edge stabilizers and without inversions on a finite hyperbolic graph of rank g is equal to $6(g-1)$ or $4(g-1)$, and both cases occur for infinitely many values of g .*

- ii) Let G be a finite group acting without inversions on a finite hyperbolic graph. If c denotes the order of an edge stabilizer of the action of G then $|G| \leq 6c(g-1)$.*

- iii) Equality $|G| = 6c(g-1)$ is obtained only for $c = 1, 2, 4, 8$ and 16 . There are infinitely many values of c such that the second largest possibility $|G| = 4c(g-1)$ is obtained.*

Finally, Theorem 1.5 has the following application to finite group-actions on finite graphs embedded in spheres.

THEOREM 1.7 ([19]). *Let G be a finite subgroup of the orthogonal group $O(d+1)$ acting on a pair (S^d, Γ) , for a finite hyperbolic graph Γ of genus $g > 1$ embedded in S^d . Then the order of G is bounded above by a polynomial of degree $d/2$ in g if d is even and of degree $(d+1)/2$ if d is odd. The degree $d/2$ is best possible in even dimensions whereas in odd dimensions the optimal degree is either $(d-1)/2$ or $(d+1)/2$.*

For the case of the 3-sphere, by Theorem 1.1 an upper bound for the order of G is $12(g-1)$. The finitely many (hyperbolic) graphs Γ in S^3 for which this upper bound is attained are classified in [12], and the possible genera are $g = 2, 3, 4, 5, 6, 9, 11, 17, 25, 97, 121, 241$ and 601 . By Theorem 1.3, an upper bound for the case of S^4 is $24g(g-1)$, for $g \geq 15$.

2. Finite group-actions on closed handles and the proof of Theorem 1.4

We briefly recall some concepts from [14]. Let G be a finite group acting on a handle H_g . By the equivariant sphere theorem, there is an equivariant decomposition of H_g into 0-handles S^3 connected by 1-handles $S^2 \times [-1, 1]$, and an associated finite graph $\tilde{\Gamma}$ with an action of g . In the language of [14], this induces on the quotient orbifold H_g/G the structure of a *closed handle-orbifold*, i.e. H_g/G decomposes into 0-handle orbifolds S^3/G_v connected by 1-handle orbifolds $(S^2/G_e) \times [-1, 1]$ (in the case of a free action, H_g/G is a 3-manifold and one may just use the classical decomposition into prime manifolds). This defines a finite graph of finite groups (Γ, \mathcal{G}) associated to the G -action, with underlying graph $\Gamma = \tilde{\Gamma}/G$; by subdividing edges, we assume here that G acts without inversions of edges on Γ . The vertices of Γ correspond to the 0-handle orbifolds, the edges to the 1-handle orbifolds. The vertex groups G_v of (Γ, \mathcal{G}) are the stabilizers in G of the 0-handles S^3 of H_g and isomorphic to finite subgroups of the orthogonal group $SO(4)$, the edge groups G_e are stabilizers of 1-handles of H_g and isomorphic to finite subgroups of $SO(3)$. The fundamental group $\pi_1(\Gamma, \mathcal{G})$ of the graph of groups (Γ, \mathcal{G}) is defined as the iterated free product with amalgamation and HNN-extension of the vertex groups along the edge groups (starting with a maximal tree), and is isomorphic to the orbifold fundamental group of the quotient orbifold H_g/G . There is a surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \rightarrow G$, and H_g is the orbifold covering of H_g/G associated to the kernel of ϕ (isomorphic to the free group F_g). Conversely, if $\phi : \pi_1(\Gamma, \mathcal{G}) \rightarrow G$ is a surjection with torsionfree kernel onto a finite group G then its kernel is the fundamental group of a graph of groups with trivial vertex and edge groups (a free group) defining a handlebody which is a regular orbifold covering of the handle-orbifold associated to the graph of groups (Γ, \mathcal{G}) .

We will assume in the following that the graph of groups (Γ, \mathcal{G}) has no *trivial*

edges, i.e. edges with two different vertices such that the edge group coincides with one of the two vertex groups (by contracting such edges).

We denote by

$$\chi(\Gamma, \mathcal{G}) = \sum \frac{1}{|G_v|} - \sum \frac{1}{|G_e|}$$

the Euler characteristic of the graph of groups (Γ, \mathcal{G}) (the sum is taken over all vertex groups G_v resp. edge groups G_e of (Γ, \mathcal{G})); then

$$g - 1 = -\chi(\Gamma, \mathcal{G}) |G|$$

(see [10, 11, 17] for the general theory of graphs of groups, groups acting on trees and groups acting on finite graphs).

REMARK 2.1: The approach to finite group-actions on 3-dimensional handlebodies (Theorem 1.1) is analogous, using the equivariant Dehn lemma/loop theorem instead of the equivariant sphere theorem. The 0-handles are disks D^3 connected by 1-handles $D^2 \times [-1, 1]$, the vertex groups of the graph of groups (Γ, \mathcal{G}) are finite subgroups of $SO(3)$ and the edge groups finite subgroups of $SO(2)$ (i.e., cyclic groups). In the case of maximal order $12(g - 1)$, $\pi_1(\Gamma, \mathcal{G})$ is one of the following four products with amalgamation ([5, 15]):

$$\mathbb{D}_2 *_{\mathbb{Z}_2} \mathbb{S}_3, \quad \mathbb{D}_3 *_{\mathbb{Z}_3} \mathbb{A}_4, \quad \mathbb{D}_4 *_{\mathbb{Z}_4} \mathbb{S}_4, \quad \mathbb{D}_5 *_{\mathbb{Z}_5} \mathbb{A}_5$$

where \mathbb{D}_n denotes the dihedral group of order $2n$, \mathbb{A}_4 and \mathbb{A}_5 the alternating groups of orders 12 and 60, and \mathbb{S}_4 the symmetric group of order 24.

REMARK 2.2: For the case of finite group-actions on bounded surfaces (Theorem 1.2), one decomposes the action along properly embedded arcs; the 0-handles are disks D^2 connected by 1-handles $D^1 \times [-1, 1]$, the vertex groups of (Γ, \mathcal{G}) are finite subgroups of $O(2)$ (cyclic or dihedral) and the edge groups subgroups of $O(1) \cong \mathbb{Z}_2$ (i.e., of order two generated by a reflection of D^1 , or trivial). In the case of maximal order $12(g - 1)$, $\pi_1(\Gamma, \mathcal{G})$ is the free product with amalgamation $\mathbb{D}_2 *_{\mathbb{Z}_2} \mathbb{D}_3$ (the first of the four groups in part i).

Proof of Theorem 1.4. i) Suppose that G acts freely on a closed handle H_g , $g > 1$. Since there are no orientation-preserving free actions of a finite group on S^2 , the edge groups of the associated graph of groups (Γ, \mathcal{G}) are all trivial. It is easy to see then that the minimum positive value for $-\chi(\Gamma, \mathcal{G})$ is realized exactly by the graph of groups (Γ, \mathcal{G}) with exactly one edge and vertex groups \mathbb{Z}_2 and \mathbb{Z}_3 , with $\pi_1(\Gamma, \mathcal{G}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ and $-\chi(\Gamma, \mathcal{G}) = 1 - 1/2 - 1/3 = 1/6$ (we will say that (Γ, \mathcal{G}) is of type (2,3) in the following), and hence $|G| = 6(g - 1)$ is the largest possible order.

Let M be the 3-manifold which is the connected sum of two lens spaces with fundamental groups \mathbb{Z}_2 and \mathbb{Z}_3 , with $\pi_1(M) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Let ϕ be a surjection

from $\mathbb{Z}_2 * \mathbb{Z}_3$ to a cyclic or dihedral group G of order 6. The regular covering of M associated to the kernel F_2 of ϕ (torsionfree, hence free) is a closed handle of genus 2 on which G acts as the group of covering transformations, and this realizes the largest possible order $|G| = 6(g - 1)$. By factorizing $\mathbb{Z}_2 * \mathbb{Z}_3$ by characteristic subgroups of F_2 of arbitrary large finite indices, one obtains examples for the maximal order $|G| = 6(g - 1)$ for arbitrarily large values of g (see also the remark after the proof for explicit examples realizing the maximum order).

Concerning the lower bound $2(g + 1)$, we consider a graph of groups (Γ, \mathcal{G}) with exactly one edge, of type $(2, n)$, with $\pi_1(\Gamma, \mathcal{G}) \cong \mathbb{Z}_2 * \mathbb{Z}_n$ and $-\chi(\Gamma, \mathcal{G}) = 1 - 1/2 - 1/n = (n - 2)/2n$. Let M be the connected sum of two lens spaces, with $\pi_1(M) \cong \mathbb{Z}_2 * \mathbb{Z}_n$, and let $\phi : \mathbb{Z}_2 * \mathbb{Z}_n \rightarrow \mathbb{D}_n$ be a surjection onto the dihedral group of order $2n$. The covering of M associated to the kernel of ϕ is a closed handle H_g of genus g with a free G -action; also, $g - 1 = (n - 2)|G|/2n = n - 2$, hence $n = g + 1$ and $|G| = 2(g + 1)$. (Note that, if n is even, there is also a surjection of $\mathbb{Z}_2 * \mathbb{Z}_n$ onto the cyclic group \mathbb{Z}_n which gives an order $|G| = n = 2g < 2(g + 1)$.)

It remains to show that $|G| = 2(g + 1)$ is the largest possible order for infinitely many values of g . It is easy to see that the graphs of groups (Γ, \mathcal{G}) with trivial edge groups and with a possible surjection of $\pi_1(\Gamma, \mathcal{G})$ onto a group of order $|G| > 2(g + 1)$ have exactly one edge, of type $(2, n)$, $(3, 3)$, $(3, 4)$ or $(3, 5)$.

We exclude first the case of a graph of groups (Γ, \mathcal{G}) of type $(2, n)$. A finite quotient G of $\pi_1(\Gamma, \mathcal{G})$ with torsionfree kernel has order xn , for some positive integer x , hence $|G| = xn = 2n(g - 1)/(n - 2)$ and $x(n - 2) = 2(g - 1)$. Suppose that $g - 1$ is a prime number. As seen above, the cases $x = 1$ and $x = 2$ give orders $2g$ and $2(g + 1)$, so we can assume that $x > 2$ and hence $n = 3$ or $n = 4$.

Let $n = 3$, so G has order $6(g - 1)$. Suppose in addition that $g > 7$; then 6 and $g - 1$ are coprime and, by a result of Schur-Zassenhaus, G is a semidirect product of \mathbb{Z}_{g-1} and a group \bar{G} of order 6. We can also assume that 3 does not divide $g - 2$ (or, equivalently, that $g - 1$ is one of the infinitely many primes congruent to 2 mod 3, by a result of Dirichlet). Then an element of order 3 in \bar{G} acts trivially on \mathbb{Z}_{g-1} by conjugation, the element of order 2 acts trivially or dihedrally, and this implies easily that G cannot be generated by two elements of orders 2 and 3.

Now let $n = 4$; then G has order $4(g - 1)$ and is a semidirect product of \mathbb{Z}_{g-1} and \mathbb{Z}_4 (since $g - 1$ is prime). Suppose that $g - 1$ is one of the infinitely many primes congruent to 11 mod 12, so 4 does not divide $g - 2$ (and, as before, 3 does not divide $g - 2$). Then the element of order two in \mathbb{Z}_4 acts trivially on \mathbb{Z}_{g-1} , is the unique element of order two in G and the square of every element of order 4, so clearly G cannot be generated by two elements of orders 2 and 4.

It remains to exclude the types $(3, 3)$, $(3, 4)$ and $(3, 5)$. If (Γ, \mathcal{G}) is of type

(3,3) then $|G| = 3(g - 1)$. As before, since 3 does not divide $g - 2$, there is a unique subgroup of order 3 in G and G is not generated by two elements of order 3. In the cases (3,4) and (3,5) one has $|G| = 12(g - 1)/5$ and $|G| = 15(g - 1)/7$; since $g = 8$ is already excluded, also these two cases are not possible.

We have shown that $m_{chf}(g) = 2(g + 1)$ for infinitely many genera g , and this concludes the proof of part i) of Theorem 1.4.

ii) For an odd integer g , we consider the semidirect product $G = (\mathbb{Z}_{(g-1)/2} \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_2$, of order $4(g - 1)$. Denoting by x a generator of $\mathbb{Z}_{(g-1)/2}$, by y a generator of \mathbb{Z}_4 and by t a generator of \mathbb{Z}_2 , the actions of the semidirect product are given by $xyx^{-1} = x^{-1}$, $txt^{-1} = x^{-1}$ and $tyt^{-1} = xy$. There is a surjection with torsionfree kernel $\phi : \mathbb{Z}_2 * \mathbb{Z}_4 \rightarrow G$ which maps a generator of \mathbb{Z}_2 to t and a generator of \mathbb{Z}_4 to y . As before, ϕ defines a free action of G on a closed handle H_g of genus g , so $m_{chf}(g) \geq 4(g - 1)$; this leaves the possibilities $m_{chf}(g) = 4(g - 1)$ and $m_{chf}(g) = 6(g - 1)$.

Suppose that $g = 2p + 1$, for a prime $p > 12$. We show that there is no surjection ϕ of $\mathbb{Z}_2 * \mathbb{Z}_3$ onto a group G of order $6(g - 1) = 12p$, and hence $m_{chf}(g) = 4(g - 1)$. By the Sylow theorems, such a group G has a normal subgroup \mathbb{Z}_p , and the factor group is the alternating group \mathbb{A}_4 (since this is the only group of order 12 generated by two elements of orders 2 and 3). Again by the theorem of Schur-Zassenhaus, G is a semidirect product $\mathbb{Z}_p \rtimes \mathbb{A}_4$. If the action of \mathbb{A}_4 on \mathbb{Z}_p is trivial then clearly such a surjection ϕ does not exist. Suppose that the action of \mathbb{A}_4 on \mathbb{Z}_p is nontrivial; since the automorphism group of \mathbb{Z}_p is cyclic, the action of \mathbb{A}_4 factors through a nontrivial action of the factor group \mathbb{Z}_3 of \mathbb{A}_4 , and the subgroup \mathbb{D}_2 of \mathbb{A}_4 acts trivially. By the Sylow theorems, up to conjugation we can assume that a surjection ϕ maps the factor \mathbb{Z}_3 of $\mathbb{Z}_2 * \mathbb{Z}_3$ to \mathbb{A}_4 . Since any involution in \mathbb{A}_4 acts trivially on \mathbb{Z}_p , every element of order 2 in G is in \mathbb{A}_4 , and hence ϕ is not surjective.

Finally, any surjection $\phi : \mathbb{Z}_2 * \mathbb{Z}_3 \rightarrow \mathbb{A}_4$ defines a free action of \mathbb{A}_4 on a closed handle of genus 3. Factorizing by characteristic subgroups of arbitrary large indices of the kernel F_3 of ϕ , one obtains $m_{chf}(g) = 6(g - 1)$ for infinitely many odd values of g .

iii) Suppose that $n - 2$ is prime, and let $g = n - 1$ and $|G| = 2n$. Then it follows easily as above that $m_{chf}(g) = 2n(g - 1)/(n - 2)$, hence infinitely many values of n occur. We will show that also infinitely many values of n do not occur.

Let n be congruent to 2 mod 8, $n \neq 2$, and suppose that there exists g such that $m_{chf}(g) = 2n(g - 1)/(n - 2)$. Then 8 divides $(n - 2)m_{chf}(g) = 2n(g - 1)$ and also $4(g - 1)$, so $g - 1$ is even and g is odd. By ii), $m_{chf}(g) = 2n(g - 1)/(n - 2) \geq 4(g - 1)$, and this gives the contradiction $n \leq 4$.

This concludes the proof of Theorem 1.4. □

REMARK 2.3: We give an explicit construction realizing the maximum order

$|G| = 6(g-1)$ for infinitely many values of g . Let $g = p+1$, for a prime p such that 6 divides $p-1$. We shall define a surjection ϕ of $\mathbb{Z}_2 * \mathbb{Z}_3$ onto a semidirect product $G = \mathbb{Z}_p \rtimes \mathbb{Z}_6$; this defines an action of G on a closed handle of genus $g = p+1$ of maximal possible order $|G| = 6(g-1)$, hence $m_{chf}(g) = 6(g-1)$. Writing \mathbb{Z}_p additively and \mathbb{Z}_6 multiplicatively, suppose that a generator t of \mathbb{Z}_6 acts by conjugation on \mathbb{Z}_p by an automorphism of order 6, in particular t^3 acts dihedrally on \mathbb{Z}_p ; let α be the automorphism of order 3 of \mathbb{Z}_p induced by t^2 . Fixing a generator a of \mathbb{Z}_p , one has $\alpha(a) = a + b$ for some $b \neq 0$ in \mathbb{Z}_p ; then $\alpha^3(a) = a$ implies $b + \alpha(b) + \alpha^2(b) = 0$. Considering the factors of $\mathbb{Z}_2 * \mathbb{Z}_3$, let ϕ map a generator of \mathbb{Z}_2 to $bt^3 = t^3(-b)$ and a generator of \mathbb{Z}_3 to bt^2 . Since also b generates \mathbb{Z}_p , clearly ϕ is a surjection.

3. Finite group-actions on finite graphs and the proof of Theorem 1.6

Let G be a finite group acting without inversions on a finite, hyperbolic graph $\tilde{\Gamma}$ of rank g . Considering the quotient graph $\Gamma = \tilde{\Gamma}/G$, we associate to each vertex group and edge group of Γ the stabilizer in G of a preimage in $\tilde{\Gamma}$ (starting with a lift of a maximal tree in Γ to $\tilde{\Gamma}$); this defines a finite graph of finite groups (Γ, \mathcal{G}) and a surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \rightarrow G$, injective on vertex groups, with kernel F_g . Conversely, by the theory of groups acting on trees and graphs of groups (see [10, 11, 17]), such a surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \rightarrow G$ defines an action of G on a finite graph $\tilde{\Gamma}$ of rank $g = -\chi(\Gamma, \mathcal{G}) |G| + 1$; the action of G on $\tilde{\Gamma}$ is faithful if and only if every finite normal subgroup of $\pi_1(\Gamma, \mathcal{G})$ is trivial (since a finite normal subgroup must be contained in all edge groups; see [7, Lemma 1]).

Proof of Theorem 1.6. i) Let G be a finite group which acts with trivial edge stabilizers and without inversions on a finite graph $\tilde{\Gamma}$, of rank $g > 1$. Then the associated graph of groups (Γ, \mathcal{G}) has trivial edge groups, and clearly $-\chi(\Gamma, \mathcal{G}) = 1/6$ is the smallest positive value which can be obtained for the Euler characteristic $\chi(\Gamma, \mathcal{G})$ (realized by the graph of groups with one edge and edge groups \mathbb{Z}_2 and \mathbb{Z}_3). Hence $|G| \leq 6(g-1)$ and, as in the proof of Theorem 1.4, the upper bound $6(g-1)$ is obtained for infinitely many values of g .

For an integer $m > 1$, choose a surjection $\phi : \mathbb{Z}_2 * \mathbb{D}_2 \rightarrow G$ where G is the dihedral group \mathbb{D}_{2m} or the group $\mathbb{Z}_2 \times \mathbb{D}_m$, of order $4m$. Then ϕ defines an action of G on a finite graph of rank $g = m+1$, hence $|G| = 4m = 4(g-1)$. On the other hand, if $g-1$ is a prime such that 3 does not divide $g-2$ then it follows as in the proof of Theorem 1.4 that there does not exist a surjection of $\mathbb{Z}_2 * \mathbb{Z}_3$ onto any group of order $6(g-1)$, so $4(g-1)$ is the maximal possible order for infinitely many g .

ii) As before, the action of G on $\tilde{\Gamma}$ is associated to a surjection with torsion-free kernel $\phi : \pi_1(\Gamma, \mathcal{G}) \rightarrow G$, for a finite graph of finite groups (Γ, \mathcal{G}) . We can

assume that (Γ, \mathcal{G}) has no trivial edges, i.e. edges with two different vertices such that the edge group coincides with one of the vertex groups (by contracting such an edge). Since the action of G on $\tilde{\Gamma}$ is faithful, every finite normal subgroup of $\pi_1(\Gamma, \mathcal{G})$ is trivial. Let e be an edge of Γ with an edge group of order c ; let $\chi = \chi(\Gamma, \mathcal{G})$ denote the Euler-characteristic of (Γ, \mathcal{G}) and n the order of G .

Suppose first that e is a closed edge (a loop). If e is the only edge of (Γ, \mathcal{G}) then

$$-\chi \geq \frac{1}{c} - \frac{1}{2c} = \frac{1}{2c}, \quad g - 1 = -\chi n \geq \frac{n}{2c}, \quad n \leq 2c(g - 1)$$

(since every finite normal subgroup of $\pi_1(\Gamma, \mathcal{G})$ is trivial, the edge group of e cannot coincide with the vertex group).

If e is closed and not the only edge then

$$-\chi \geq \frac{1}{c}, \quad g - 1 = -\chi n \geq \frac{n}{c}, \quad n \leq c(g - 1).$$

Suppose that e is not closed. If e is the only edge of (Γ, \mathcal{G}) then both vertices of e are isolated and

$$-\chi \geq \frac{1}{c} - \frac{1}{2c} - \frac{1}{3c} = \frac{1}{6c}, \quad g - 1 = -\chi n \geq \frac{n}{6c}, \quad n \leq 6c(g - 1).$$

If e is not closed, not the only edge and has exactly one isolated vertex then

$$-\chi \geq \frac{1}{c} - \frac{1}{2c} = \frac{1}{2c}, \quad g - 1 = -\chi n \geq \frac{n}{2c}, \quad n \leq 2c(g - 1).$$

Finally, if e is not closed, not the only edge and has no isolated vertex then

$$-\chi \geq \frac{1}{c}, \quad g - 1 = -\chi n \geq \frac{n}{c}, \quad n \leq c(g - 1).$$

Concluding, in all cases we have $|G| \leq 6c(g - 1)$, proving ii).

iii) By [3] and [4], there are only finitely many free products with amalgamation of two finite groups, without nontrivial finite normal subgroups, such that the amalgamated subgroup has indices 2 and 3 in the two factors (and the same holds also for indices 3 and 3). These *effective (2,3)-amalgams* are classified in [3], there are exactly seven such amalgams (described below), and the amalgamated subgroups have order 1, 2, 4, 8 or 16. It follows then from the proof of ii) that equality $|G| = 6c(g - 1)$ can be obtained only for these values of c .

On the other hand, by [2] there are infinitely many effective (2,4)-amalgams, and hence $|G| = 4c(g - 1)$ is obtained for infinitely many values of c .

This concludes the proof of Theorem 1.6. □

REMARK 3.1: The seven effective $(2, 3)$ -amalgams, with amalgamated subgroups of orders $c = 1, 2, 4, 8$ or 16 , are the following:

$$\begin{aligned} \mathbb{Z}_2 * \mathbb{Z}_3, \quad \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{D}_3, \quad \mathbb{D}_2 *_{\mathbb{Z}_2} \mathbb{D}_3, \quad \mathbb{D}_4 *_{\mathbb{D}_2} \mathbb{D}_6, \quad \mathbb{D}_8 *_{\mathbb{D}_4} \mathbb{S}_4, \quad \tilde{\mathbb{D}}_8 *_{\mathbb{D}_4} \mathbb{S}_4, \\ \mathbb{K}_{32} *_{(\mathbb{D}_4 \times \mathbb{Z}_2)} (\mathbb{S}_4 \times \mathbb{Z}_2), \end{aligned}$$

where $\tilde{\mathbb{D}}_8$ denotes the quasidihedral group of order 16 and K_{32} a group of order 32.

REMARK 3.2: Finally, we describe the two families which realize the largest possible orders for all g . As noted before, the largest order of a finite group G of automorphisms of a finite graph of rank $g > 2$ without free edges (or equivalently, of a finite subgroup G of $\text{Out } F_g$) is $2^g g!$, and this is realized by the automorphism group $(\mathbb{Z}_2)^g \rtimes \mathbb{S}_g$ of a finite graph with one vertex and g closed edges (a bouquet of g circles or a multiple closed edge), subdividing edges to avoid inversions. Considering the quotient graph/graph of groups, this action is associated to a surjection

$$\phi : ((\mathbb{Z}_2)^g \rtimes \mathbb{S}_g) *_{((\mathbb{Z}_2)^{g-1} \rtimes \mathbb{S}_{g-1})} ((\mathbb{Z}_2)^g \rtimes \mathbb{S}_{g-1}) \rightarrow (\mathbb{Z}_2)^g \rtimes \mathbb{S}_g$$

For $g \geq 3$, this realizes the unique action of maximal possible order $2^g g!$ (the unique finite subgroup of $\text{Out } F_g$ of maximal order, up to conjugation).

The second family of large orders is given by the automorphism groups $\mathbb{S}_{g+1} \times \mathbb{Z}_2$ of the graphs with two vertices and $g+1$ connecting edges (a multiple nonclosed edge, subdividing edges again to avoid inversions), associated to the surjections

$$\phi : \mathbb{S}_{g+1} *_{\mathbb{S}_g} (\mathbb{S}_g \times \mathbb{Z}_2) \rightarrow \mathbb{S}_{g+1} \times \mathbb{Z}_2$$

These realize the largest possible order for $g = 2$, and again for $g = 3$.

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Author's address:

Bruno P. Zimmermann
Università degli Studi di Trieste
Dipartimento di Matematica e Geoscienze
34127 Trieste, Italy
E-mail: zimmer@units.it

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