ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO THE N-TH ORDER NONLINEAR DIFFERENTIAL EQUATION UNDER FORCING (*)

by JAN ANDRES and VLADIMÍR VLČEK (in Olomouc) (**)

SOMMARIO. - *Si studia il comportamento asintotico delle soluzioni di una classe di equazioni differenziali ordinarie di ordine n, sotto varie condizioni relativamente alla nonlinearità ed al termine forzante.*

SUMMARY. - *We investigate the asymptotic behaviour of the solutions to some n-th order ordinary differential equations under various conditions on the restoring and the forcing terms.*

Introduction.

Consider

(1) $$x^{(n)} + Lx + h(x) = p(t) \,,$$

where $Lx := \sum_{j=1}^{n-1} a_j x^{(n-j)}$ is a linear differential operator with positive constant coefficients and the functions $h(x)$ and $p(t)$ are either continuously differentiable or with bounded derivatives everywhere. Let the characteristic polynomial associated with the differential operator

$$y^{(n-1)} + \sum_{j=1}^{n-1} a_j y^{(n-j-1)}$$

(*) Pervenuto in Redazione il 2 luglio 1990.

(**) Indirizzo degli Autori: Department of Mathematical Analysis and Numer. Math. – Faculty of Science – Palacký University – Videňská 15 – 771 46 Olomouc (Czechoslovakia).

be stable, i.e. $H_j > 0$ $(j = 1, \dots, n-1)$, where

$$H_1 = a_1, H_k = \begin{vmatrix} a_1 & a_3 & \dots a_{a_{2k-1}} \\ 1 & a_2 & \dots a_{a_{2k-2}} \\ 0 & a_1 & \dots a_{a_{2k-3}} \\ 0 & 1 & \dots a_{a_{2k-4}} \\ \vdots & & \\ 0 & 0 & \dots a_k \end{vmatrix} \quad (k > 1, a_m = 0 \text{ for } m > n-1).$$

If the restoring and forcing terms are bounded, then it is well-known (for a rather long time – see e.g. [1]-[3],[17]) that the derivatives of all solutions of (1) are uniformly ultimately bounded up to the n-th order including.

If it is moreover

$$\liminf_{|x| \to \infty} h(x) \operatorname{sgn} \ x > 0 \ ,$$

then the authors of [1] and [2] have shown that the same is also true for all solutions of (1), i.e. that (1) is dissipative in the sense of Levinson.

The natural question arises with this respect, namely how the restoring term of the type, for example, $h(x) := \sin x$ or $h(x) := -\text{arctg}\ x$ can affect the asymptotic behaviour of solutions.

Using the method developed by Voráček in [4, pp. 384-392] for $n = 3$, we are able to receive the explicit asymptotic formula (not derived in [1]-[3]) for the derivatives of all solutions of (1), provided all the roots of the polynomials

$$(2) \qquad \lambda^{n-p} + \sum_{j=1}^{n-p} a_j \lambda^{n-j-p} \ (p = 1, \dots, n-1)$$

are negative single, which allows us to ensure the Lagrange-like stability of (1), when the distances between the zero points of $h(x)$ are sufficiently large. This has been performed for $n = 3$, besides another, in [5].

In the latter case (i.e., for example, $h(x) := -\text{arctg}\ x$), we apply the second Liapunov method in order to guarantee the existence of a bounded solution, using the arguments from [6] for $n = 3$, because a family of solutions of (1) tending to infinity exists, according to [7].

All these results can be still precised with respect to the oscillatory behaviour of solutions. Since all the ideas, approaches and results have been transformed from concrete to the general case, we restrict ourselves to perform only such parts of the proofs which are significant in such a point of view.

1. Oscillatory restoring term.

LEMMA 1. *Let all the roots of the polynomials* (2) *be negative single and*

(3) $\limsup_{|x|\to\infty} |h(x)| < \infty \; (\Rightarrow \exists H - const.: |h(x)| < H \, for \, all \, x)$,

(4) $\limsup_{t\to\infty} |p(t)| < \infty \; (\Rightarrow \exists P - const.: |p(t)| < H \, for \, all \, t > 0)$.

Then the derivatives $x^{(j)}(t)$ $(j = 1, \ldots, n-1)$ of all solutions $x(t)$ of (1) *are uniformly ultimately bounded in the following way:*

(5) $$\limsup_{t\to\infty} |x^{(j)}(t)| < j(H+P)/a_{n-j} := D^j \; (j = 1, \ldots, n-1)$$

$$\left(\Rightarrow \limsup_{t\to\infty} |x^{(n)}(t)| < H + P + \sum_{j=1}^{n-1} a_j D^j \right) .$$

Proof. Substituting $y := x'$ into (1), we get for every solution $x(t)$ (evidently defined on the whole positive half-line, cf. [8, pp.16-17]) of (1) the identity

(6) $$y^{(n-1)} + \sum_{j=1}^{n-1} a_j y^{(n-j-1)} = p(t) - h(x(t)) .$$

The characteristic equation associated to the homogeneous equation

(7) $$y^{(n-1)} + \sum_{j=1}^{n-1} a_j y^{(n-j-1)} = 0$$

reads

$$\lambda^{n-1} + \sum_{j=1}^{n-1} a_j \lambda^{n-j-1} = 0 \ .$$

A solution $y(t)$ of (6) takes the form

$$y(t) = \bar{y}(t) + y_p(t) \ ,$$

where $\bar{y}(t)$ satisfies (7) and

$$y_p(t) = \int_0^t \tilde{y}(t-s)[p(s) - h(x(s))]ds \ .$$

By $\tilde{y}(t)$ we denote the solution of (7) satisfying the following initial conditions

$$\tilde{y}(0) = \tilde{y}'(0) = \ldots = \tilde{y}^{(n-3)}(0) = 0, \ \tilde{y}^{(n-2)}(0) = 1 \ .$$

Let $\exp(\lambda_j t)$, $j = 1, \ldots, n-1$, be the fundamental system of solutions of (7). If $C_j, j = 1, \ldots, n-1$, are suitable real constants such that

$$\tilde{y}(t) = \sum_{j=1}^{n-1} C_j e^{\lambda_j t} \ ,$$

we arrive at the system

$$\sum_{j=1}^{n-1} C_j \lambda_j^k = \delta_{k,n-2} \ , \ k = 0, \ldots, n-2 \ ,$$

where $\delta_{k,n-2}$ is the Kronecker symbol. This leads us to the solution $y_p(t)$ expressed as follows

$$y_p(t) = \sum_{j=1}^{n-1} (-1)^{n+j} \frac{V_j}{V} \int_0^t e^{\lambda_j(t-s)}[p(s) - h(x(s))]ds \ ,$$

where V_j is the subdeterminant associated to the element λ_j^{n-j-1}, $j = 1, \ldots, n-1$, of the Vandermond determinant

$$V = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & & \lambda_{n-1} \\ \vdots & & & \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \ldots & \lambda_{n-1}^{n-2} \end{vmatrix} \ .$$

Using (3), (4), we come to the inequality

$$|y_p(t)| \le \frac{H+P}{|V|}\int_0^t \left|\sum_{j=1}^{n-1}(-1)^{n+j}V_j e^{\lambda_j(t-s)}\right| ds$$
$$\le \frac{H+P}{|V|}\left|\sum_{j=1}^{n-1}\frac{(-1)^{n+j}V}{\lambda_j}(1-e^{\lambda_j t})\right|$$

for $t > 0$.

Since $\bar{y}(t)$ tends to zero as $t \to \infty$ and

$$\limsup_{t\to\infty}|y_p(t)| \le \frac{H+P}{|V|}\left|\sum_{j=1}^{n-1}\frac{(-1)^{n+j}V_j}{\lambda_j}\right| = \frac{H+P}{(-1)^{n-1}\prod_{j=1}^{n-1}\lambda_j} = \frac{H+P}{a_{n-1}},$$

we obtain that

$$\limsup_{t\to\infty}|x'(t)| \le \frac{H+P}{a_{n-1}}. \tag{8}$$

Now, substituting $z := x''$ into (1), we get for every solution $x(t)$ of (1) the identity

$$z^{(n-2)} + \sum_{j=1}^{n-1} a_j z^{n-j-2} = p(t) - h(x(t)) - a_{n-1}y(t),$$

where $y(t) = x'(t)$. Repeating the same manner as above to the equation

$$\bar{z}^{(n-2)} + \sum_{j=1}^{n-2} a_j \bar{z}^{n-j-2} = 0$$

and realizing that the negative single roots $\mu_1, \dots, \mu_{n-2}$ of its associated characteristic equation

$$\mu^{n-2} + \sum_{j=1}^{n-2} a_j \mu^{n-j-2} = 0$$

satisfy

$$a_{n-2} = (-1)^n \prod_{j=1}^{n-2} \mu_j ,$$

we obtain by means of (8) that

$$\limsup_{t\to\infty} |x''(t)| \le \frac{2(H+P)}{a_{n-2}} ,$$

etc. The last substitution $u := x^{(n-1)}$ into (1) leads us to the final estimate

$$\limsup_{t\to\infty} |x^{(n-1)}(t)| \le \frac{(n-1)(H+P)}{a_1} .$$

For real roots of (2), the assumption of Lemma 1 can be "weaken" in the following way.

LEMMA 2. *If all the roots of the polynomials* (2) *are real and if the polynomial*

$$\lambda^{n-1} + \sum_{j=1}^{n-2} a_j \lambda^{n-j-1}$$

is stable, then all the roots of the polynomials (2) *are negative.*

Proof. By the hypothesis, all the roots $\lambda_1, \dots, \lambda_{n-1}$ of the polynomial (2) for $p = 1$, namely

$$P_{n-1}(\lambda) = \lambda^{n-1} + \sum_{j=1}^{n-2} a_j \lambda^{n-j-1} = \prod_{j=1}^{n-1} (\lambda - \lambda_j) ,$$

are negative, i.e. $\lambda_j < 0$ for all $j = 1, \dots, n-1$.

Letting $b > \max\{\lambda_1, \dots, \lambda_{n-1}\} := m$, we have

$$(9) \qquad P'_{n-1}(b) = P_{n-1}(b) \sum_{j=1}^{n-2} \frac{1}{b - \lambda_j} \neq 0 ;$$

thus, because of $m < 0$

$$P'_{n-1}(b) \neq 0 \text{ for all } b \geq 0 ,$$

and consequently

$$(10) \qquad P_{n-1}(b) - P_{n-1}(0) \neq 0 \text{ for all } b > 0 \ ,$$

where $P_{n-1}(0) = a_{n-1} \neq 0$.

Hence, defining the "shifted" polynomial $Q_{n-2}(\lambda)$ as follows

$$\lambda Q_{n-2}(\lambda) = P_{n-1}(\lambda) - P_{n-1}(0) \ ,$$

we can give the following conclusion.

If a would be a real root of $Q_{n-2}(\lambda)$, then $P_{n-1}(a) - P_{n-1}(0) = 0$, and therefore a cannot be positive with respect to (10).

Moreover, derivating the relation containing $Q_{n-2}(\lambda)$, we obtain

$$Q_{n-2}(\lambda) + \lambda Q'_{n-2}(\lambda) = P'_{n-1}(\lambda)$$

which for $\lambda = 0$ turns out to

$$Q_{n-2}(0) = P'_{n-1}(0) \ .$$

Applying (9), $Q_{n-2}(0) \neq 0$, and consequently $a = 0$ cannot be the root of $Q_{n-2}(\lambda)$ as well.

So, all real roots of $Q_{n-2}(\lambda)$ must be negative.

It can be shown in the same way that all the polynomials (2) taken successively for $p = 1, \ldots, n-1$ have all their roots negative.

THEOREM 1. *Let the assumptions of Lemma 1 be fulfilled and*

$$\limsup_{t\to\infty} \left| \int_0^t p(s)\,ds \right| < \infty (\Rightarrow \exists P_0 - \text{const.}:$$

$$\left| \int_0^t p(s)\,ds \right| < P_0 \ \textit{for all}\ t > t_0 \geq 0) \ .$$

Let such zero points $\bar{x}_k$ ($k = 0, \pm 2, \pm 4, \ldots$) of $h(x)$ exist separately that

$$h(x) \text{ sgn } (x - \bar{x}_k) \geq 0 \ \textit{for}\ x \in \langle \bar{x}_{k-1}, \bar{x}_{k+1} \rangle \ ,$$

where $\bar{x}_{k-1} < \bar{x}_k < \bar{x}_{k+1}$ are the roots of $h(x)$ with ("d" denotes the distance)

$$d(\bar{x}_k, \bar{x}_{k\pm 1}) > 2\frac{H+P}{a_{n-1}} \sum_{j=1}^{n-2} a_j \frac{n-j-1}{a_{j+1}} + \frac{P_0}{a_{n-1}}$$

for $k = 0, \pm 2, \pm 4, \ldots$, respectively.

Then all solutions of (1) are bounded.

Proof- can be done quite analogously to [5], where the case $n = 3$ has been of an interest.

COROLLARY 1. *Let the assumptions of Theorem 1 be fulfilled and let all the roots $\bar{x}$ of $h(x) \in C^1(R^1)$ be isolated. If*

$$\limsup_{t\to\infty} |p(t)| > 0 \text{ and } \limsup_{t\to\infty} |p'(t)| < \infty, \tag{11}$$

then every solution $x(t)$ of (1) oscillates around a suitable zero point $\bar{x}$ of $h(x)$ with a finite amplitude everywhere, namely

$$\limsup_{t\to\infty} |x(t) - \bar{x}| > 0 = \liminf_{t\to\infty} |x(t) - \bar{x}| .$$

Proof- can be done quite analogously to [5].

THEOREM 2. *Let all the roots of the polynomial*

$$\lambda^{n-1} + \sum_{j=1}^{n-2} a_j \lambda^{n-j-1}$$

be negative single, and

$$p(t) \equiv p(t+\theta) (\Rightarrow \exists P - const.: |p(t)| < P \; for \; all \; t > 0). \tag{12}$$

$$\int_0^\theta p(t)\,dt = 0 . \tag{13}$$

If a zero point $\bar{x}$ of $h(x)$ exists such that

(14) $0 < h(x)\operatorname{sgn}(x-\bar{x}) < H$ *for* $x \in \langle \bar{x} - \theta D', \bar{x} + \theta D' \rangle \setminus \{\bar{x}\}$,

where $D' := (H+P)/a_{n-1}$ *with a suitable constant* H, *then equation* (1) *admits a* θ*-periodic solution.*

Proof. Instead of (1), consider the equation

$$x^{(n)} + Lx + h^*(x) = p(t) \ , \tag{1*}$$

where

$$h^*(x) := \begin{cases} h(x+\bar{x}) \ for \ |x| \le \theta D' \\ h(\bar{x} + \theta D' \operatorname{sgn} x) \ for \ |x| \ge \theta D' \end{cases} .$$

It is clear that if equation (1*) admits a θ-periodic solution $x^*(t)$ such that

$$||x^*(t)|| := \max_{t \in \langle 0, \theta \rangle} |x^*(t)| \le \theta D' \ ,$$

then $(x^*(t) + \bar{x})$ is a θ-periodic solution of (1) as well.

Furthermore, consider the enlarged equation

$$x^{(n)} + Lx + \mu h^*(x) + (1-\mu)ax = \mu p(t), \ \mu \in \langle 0, 1 \rangle \ , \tag{1^*_μ}$$

where a is a sufficiently small positive constant such that the characteristic polynomial

$$\lambda^n + \sum_{j=1}^{n-2} a_j \lambda^{n-1} + a \tag{15}$$

associated to (1^*_0) is stable. Note that the desired coefficient a can be always found, because (for more detail see e.g. [9]) H_j, $j = 1, \ldots, n-1$ can be certainly regarded as the initial values of the polynomials $H_j(a)$, $j = 1, \ldots, n-1$, associated to the Hurwitz condition with respect to (15), where $H_n(a) := a > 0$, i.e. $H_j(0) = H_j > 0$, $j = 1, \ldots, n-1$ and consequently

$$H_j(a) > 0 \text{ for } j = 1, \ldots, n-1 \ (H_n(a) > 0) \ .$$

Since the characteristic values related to (15) are different from $2k\pi\sqrt{-1}/\theta$ for all integers k, we can apply the well-known (cf. e.g. [9]) Leray-Schauder alternative, consisting in verification the uniform a priori estimates of all solutions of (1^*_μ) together with their derivatives up to $(n-1)$-th order including, independently of $\mu \in \langle 0,1\rangle$.

Hence, let $x(t)$ be a θ-periodic solution of (1^*_μ), $\mu \in \langle 0,1\rangle$. Substituting $x(t)$ into (1^*_μ) and applying Lemma 1 jointly with the result of [2] for $p = j-1$ (for more detail see the proof of Lemma 1 and the introducing text of our paper), we have

$$(16)\qquad \limsup_{t\to\infty} |x^{(j)}(t)| < D^j_a \text{ for } j = 1,\dots,n-1\ ,$$

where D^j_a are suitable constants with $D^1_a := (H + a||x(t)|| + P)/a_{n-1}$, according to (12), (14) and consequently $||x^{(j)}(t)|| < D^j_a$ for $j = 1,\dots, n-1$.

Integrating still the obtained identity from 0 to θ, we arrive at the relation [cf.(13)]

$$\int_0^\theta [\mu h^*(x(t)) + (1-\mu)ax(t)]dt = 0$$

which leads evidently to a contradiction for

$$\min_{t\in\langle 0,\theta\rangle} |x(t)| > 0\ ,$$

when multiplying it by sgn $x(t)$, because of (14). Thus,

$$\min_{t\in\langle 0,\theta\rangle} |x(t)| = 0\ .$$

If $t_0 \in \langle 0,\theta\rangle$ is the point with $x(t_0) = 0$, one obtains

$$|x(t)| \le \left|\int_0^t x'(s)ds\right| \le \int_0^\theta |x'(s)|ds < \theta(H + a||x(t)|| + P)/a_{n-1}$$

by means of (16), i.e. $||x(t)|| < \theta(H+P)/(a_{n-1} - a\theta)$ or

$$||x(t)|| \le \theta(H+P)/a_{n-1} + |\sigma(a)| := \theta D' + |\sigma(a)|$$

for sufficiently small a.

Hence, we have not only

$$\limsup_{t\to\infty} |x(t)| \le \theta D' + |\sigma(a)|,$$

but also

$$\limsup_{t\to\infty} |x^{(j)}(t)| < D^j, \ i.e.\ \|x^{(j)}(t)\| < D^j,$$

where D^j, $j = 1, \dots, n-1$ are suitable constants, independently of $\mu \in \langle 0, 1\rangle$. Therefore equation (1*) admits a θ-periodic solution $x^*(t)$ such that $\|x'(t)\| \le \theta D'$. This completes the proof.

REMARK 1. Assumption (14) from Theorem 2 can be replaced by

$$(14') \quad -H < h(x)\operatorname{sgn}(x - \bar{x}) < 0 \ \text{for}\ x \in \langle \bar{x} - \theta D', \bar{x} + \theta D'\rangle \setminus \{x\},$$

when such a negative (absolutely) small enough constant a exists that all characteristic values related to (15) are different from $2k\pi\sqrt{-1}/\theta$ for all integers k. Condition (14) or (14′) can be even weaken to

$$h(x)\operatorname{sgn} x \ge 0 \ \text{or}\ h(x)\operatorname{sgn} x \le 0$$

on the appropriate intervals, respectively, when all the zero points of $h(x)$ for these intervals are isolated.

2. Investigation of Lagrange-like nonstable equation.

THEOREM 3. *Let the polynomial*

$$\lambda^{n-1} + \sum_{j=1}^{n-1} a_j \lambda^{n-j-1}$$

be stable and let (3), (4) *be satisfied. If it is furthermore*

$$(17) \qquad \lim_{|x|\to\infty} h(x)\operatorname{sgn} x < - \sup_{t\in(0,\infty)} |p(t)|,$$

then equation (1) *admits a bounded solution together with its derivatives up to the* $(n-1)$*-th order including.*

Proof. We apply the technique which has been pointed out in [10] and employed practically in [6] for $n = 3$, consisting of the construction of two continuously differentiable autonomous Liapunov functions $V(X_1)$ and $W(X)$ (representing, according to Voráček [7], the sufficient conditions for the existence of the so called D'-divergent solutions of (1)) such that

$$\text{(i)} \qquad \begin{cases} \lim\limits_{\|X_1\|\to\infty} V(X_1) = \infty\,,\ X_1 := (x', \dots, x^{(n-1)})\,, \\ \text{and} \\ V'_{(1)}(X_1) < 0 \text{ for } \|X_1\| \geq R_2\,, \end{cases}$$

(ii)

$$\begin{cases} \lim_{|x_1|\to\infty} W(X) = \infty \text{ for } \|X_1\| \leq R_3\,,\ X := (x, x', \dots, x^{(n-1)})\,, \\ \text{and} \\ W'_{(1)}(X) \geq w \text{ for } |x_1| \geq R_1\,,\ \|X_1\| \leq R_3\,, \end{cases}$$

where $R_1, R_2 < R_3$ are suitable positive constants which may be large enough, while $V'_{(1)}(X_1)$ and $W'_{(1)}(x)$ denote the time-derivatives of the functions $V(X_1)$ and $W(X)$ with respect to (1), w is a positive constant.

This is enough, jointly with (cf. [6], [10])

$$\limsup_{|x|\to\infty} h(x) < -|p(0)|\,,$$

to guarantee the existence of a bounded solution of (1).

Since the last relation follows trivially from (17), it is sufficient to verify (i), (ii).

Because of the Hurwitz structure of the coefficients a_j ($j = 1, \dots, n-1$) a positive definite quadratic form $V(X_1)$ exists according to the well-known Liapunov theorem (cf. e.g.[3, p.287]) such that its time-derivative with respect to (1), where $h(x) \equiv 0 \equiv p(t)$, reads

$$-\sum_{j=1}^{n-1} x^{(j)^2}\,.$$

Hence, taking into account (1) in general, one obtains

$$V'_{(1)}(X_1) := \langle (x'', \dots, x^{(n-1)}, p(t) - h(x) - Lx),\ \text{grad } V(X_1)\rangle =$$

$$= -\sum_{j=1}^{n-1} x^{(j)^2} - \frac{\partial V(X_1)}{\partial x^{(n-1)}}[h(x) - p(t)] ,$$

where $\partial V(X_1)/\partial x^{(n-1)}$ is a linear form of $x', \dots, x^{(n-1)}$. Because of (3), (4) we have $V'_{(1)}(X_1) < 0$ for sufficiently large values of $||X_1||$ uniformly with respect to $t \in \langle 0, \infty \rangle$. Thus, (i) is fulfilled.

Defining the Liapunov function $W(X)$ as in [7] (but for the another goal), namely

$$2W(X) := 2a_{n-2} \int_0^X h(s)ds + [x^{(n-1)} + \sum_{j=1}^{n-1} a_j x^{(n-j-1)}]^2 ,$$

we obtain

$$W'_{(1)}(X) := \langle (X_1, p(t) - h(x) - Lx), \text{ grad } W(X) \rangle =$$

$$= [p(t) - h(x)] \left\{ a_{n-1}x + \left[x^{(n-1)} + \sum_{j=1}^{n-2} a_j x^{(n-j-1)} \right] \right\} .$$

Because of (17) we have $W'_{(1)}(X) \geq w$ for sufficiently large values of $|x|$ and $||X_1||$ uniformly bounded by a suitable positive constant.

Thus, (ii) is fulfilled in view of (17).

This completes the proof.

COROLLARY 2. *Let the assumptions of Theorem* 1 *be fulfilled and let all the roots* $\bar{x}$ *of* $h(x) \in C^1(R^1)$ *be isolated. If* (11) *is satisfied, then a solution* $x(t)$ *of* (1) *exists oscillating around a suitable zero point* $\bar{x}$ *of* $h(x)$ *with a finite amplitude everywhere (in the same way as in Corollary* 1).

Proof- can be done quite analogously to [5].

REMARK 2. A natural question arises, whether sufficient conditions can be found such that the bounded solution $x(t)$ of (1) from Theorem 3 tends to the origin as $t \to \infty$, provided $h(0) = 0$ and $h'(x) < 0$ for all x. Assuming

$$\int_0^\infty |p(t)|dt < \infty ,$$

moreover, we have given an affirmative answer to this problem for $n = 3$ in [11] and for $n = 4$ in [12], where even $x(t) \in L_2\langle 0, \infty)$ jointly with its derivatives up to the $(n-1)$-th order including. However, for $n = 5$ some additional growth restrictions have to be imposed on $h'(x)$ (see [12]).

3. Concluding remarks.

Although the second present author has shown explicitly in [13]-[15] that the asymptotic formula (5) remains valid for $n = 3, 4, 5$ even without the additional restrictions concerning the roots of (2) being negative single (it is sufficient in order (2) to be stable for $p = 1$, only) it seems to be difficult to verify (5) in general. Already for $n = 6$, e.g., the Hurwitz polynomial $\lambda^5 + \lambda^4 + 4\lambda^3 + 3\lambda^2 + 3.5\lambda + 1$ does not namely imply the same stable property for $\lambda^4 + \lambda^3 + 4\lambda^2 + 3\lambda + 3.5$. Nevertheless, the necessary and sufficient conditions for the roots of (2), $p = 1, \ldots, n-1$, to be real (single) negative (cf. Lemma 2) can be expressed explicitly in terms of the Sturm functions; see e.g. [19].

Voráček has recently obtained in [16] the dissipativity result concerning a more general equation than (1); see also [18]. This approach consists in the transformation of the studied equation to the equivalent system with the matrix A related to its linear part and in the application of the well-known improved version of the Gelfand-Shilov inequality

$$||e^{At}|| \leq e^{\rho t} \sum_{k=0}^{n-2} \frac{(2t||A||)^k}{k!} \quad \text{for } t \geq 0 ,$$

where ρ is the maximum of the real parts of the eigenvalues of A. Let us note, with this respect, that the best upper estimates for ρ seem to be those in [20]. In spite of the important fact that no additional restrictions concerning the eigenvalues of the stable matrix A are required there, the appropriate ultimate estimates for the derivatives of solutions might be less accurate than in (5) which can be easily checked for $n = 3$.

The study of the nonautonomous case, when

$$\limsup_{|x| \to \infty} h(x) \operatorname{sgn} x = 0$$

remains yet as an open problem.

Acknowledgements.

The authors wish to express their thanks to Professor Jan Voráček for his critical comments and many valuable suggestions improving significantly the text. We are also grateful to Professor Alex Turull (University of Miami, Fla.) for suggesting the idea of the proof of Lemma 2.

REFERENCES

[1] SEDZIWY S., *Asymptotic properties of solutions of nonlinear differential equations of the higher order*, Zeszyty Nauk. Univ. Jagiel. *131* (1966), 69-80.

[2] VORÁČEK J., *Note on the paper [1] of S. Sedziwy*, Acta UPO *33* (1971), 157-161.

[3] BARBASHIN E.A. and TABUEVA V.A., *Dynamical systems with cylindrical phase space*, Nauka, Moscow 1969 (in Russian).

[4] REISSIG R., SANSONE G. and CONTI R., *Nichtlineare Differentialgleichungen höherer Ordnung*, Cremonese, Roma 1969.

[5] ANDRES J., *Boundedness of solutions of the third order differential equation with the oscillatory restoring and forcing terms*, Czech. Math. J. *36*, 1 (1986), 1-6.

[6] ANDRES J., *Boundedness results to the equation* $x''' + ax'' + g(x)x' + h(x) = p(t)$ *without the hypothesis* $h(x)\,\mathrm{sgn}\,x \geq 0$ *for* $|x| > R$, Atti Accad. Naz. Lincei *80*, 5 (1986), 533-539.

[7] VORÁČEK J., *Über D'-divergente Lösungen der Differentialgleichung* $x^{(n)} = f(x, x', \ldots, x^{(n-1)}; t)$, Acta UPO *41* (1973), 83-89.

[8] NEMYCKII V.V. and STEPANOV V.V., *Qualitative theory of differential equations*, GITL, Moscow 1949 (in Russian).

[9] REISSIG R., *On the existence of periodic solutions of a certain non-autonomous differential equation*, Ann. Mat. Pura Appl. (4), *85* (1970), 235-240.

[10] ANDRES J., *A useful proposition to nonlinear differential systems with a solution of the prescribed asymptotic properties*, Acta UPO 85, Math. *25* (1986), 157-164.

[11] ANDRES J., *Dichotomies for solutions of a certain third order nonlinear differential equation which is not from the class D*, Fasc. Math. *17* (1987), 55-62.

[12] ANDRES J. and VLČEK V.,, *On the existence of square integrable solutions and their derivatives to fourth and fifth order differential equations*, Acta UPO *94*, Math. *28* (1989), 65-86.

[13] VLČEK V., *Note on a certain nonlinear differential equation of the third order*, Acta UPO *91*, Math. *27* (1988), 263-272.

[14] VLČEK V., *On the boundedness of solutions of certain fourth-order nonlinear differential equation*, Acta UPO *91*, Math. *27* (1988), 273-288.

[15] VLČEK V., *Boundedness of solutions of certain fifth-order nonlinear differential equation*, Acta UPO *94*, Math. *28* (1989), 87-121.