ON A THEOREM OF GEL' FAND AND A NEW PROOF OF THE ORLICZ-PETTIS THEOREM (*)

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SOMMARIO. - *In questa nota si estende al caso di funzioni con valori in F-spazi un teorema di Gel'fand sulle funzioni debolmente continue definite su uno spazio topologico con valori in uno spazio di Banach. Usando questa estensione si dà una nuova dimostrazione del teorema di Orlicz-Pettis in F-spazi con base.*

SUMMARY. - *In this note a theorem of Gel'fand on the weak continuous functions defined on topological spaces with values in Banach spaces is extended to the case of functions with values in F-spaces. Using this extension we gave a new proof of Orlicz-Pettis theorem in F-spaces with basis.*

1. - Introduction

In this note G will denote a topological Hausdorf space, X a linear space, X^* the linear vector space of all continuous functionals on X and Γ a total subset of X^*. A function $f: G \to E$ is said to be weakly continuous iff $x^*(f(t))$ is continuous for each $x^* \in X^*$ and f is called Γ-continuous iff $x^*(f(t))$ is continuous for each $x^* \in \Gamma$. In this note we extend a theorem of Gel'fand [1] stating: if a function $f: G \to X$ defined on G with values in a separable Banach space $B = X$ is weakly continuous, then it is discontinuous on a subset of first category of G.

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In Sec. 2 we prove that if $X = E$ is a complete linear metric space (F-space) with basis $\{e_n\}$, and if $\Gamma \subset E^*$ is the total sequence of the coordinate functionals for $\{e_n\}$ and if $f: G \to E$ is Γ-continuous, then it is discontinuous on a set of first category of G.

Let $\sum_{M=1}^{\infty} x_n$ be a series of elements from E. It is called weakly subseries convergent iff for each subsequence $\pi = \{\pi(M)\}$ of naturals N, the series $\sum_{n\in\pi} x_n$ weakly converges to an element $x_\pi \in E$. It is called subseries Γ-convergent iff for each subsequence $\pi \subset N$ there exists $x_\pi \in E$ such that $\sum_{n\in\pi} x^*(x_n) = x^*(x_\pi), x^* \in \Gamma$. In Sec. 3 we give a new proof of Stiles extension of the Orlicz-Pettis theorem. This theorem says if the series $\sum_{n\in N} x_n$ with elements in a Banach space B is weakly subseries convergent, it is subseries convergent [3], [4]. This result has been extended to F-spaces with basis by Stiles [5] and to separable F-spaces with separating dual by Kalton [2].

2. - Gel'fand theorem

In this section G will denote a topological space, and E an F-space with basis $\{e_n\}$ and Γ the total system of E^* consisting of the coordinate functionals for $\{e_n\}$, i.e. $\Gamma = \{x^*_n : x^*_n(e_k) = \delta_{nk}, n, k \in N\}$. We prove the following extension of a result of Gel'fand [1].

THEOREM 2.1 - *Let $f: G \to E$ be Γ-continuous. Then f is discontinuous on a set of first category of G.*

Proof. Consider the following expansion of the function f by the basis $\{e_n\}: f(t) = \sum_{n=1}^{\infty} \alpha_n(t)\, e_n, t \in G$. Let $x^*_k \in \Gamma$ we have $x^*_k(f(t)) = \alpha_k(t), k \in N$. Hence $\{\alpha_k(t)\}$ are continuous functions on G. Therefore the partial sums $S_n(t) = \sum_{k=1}^{n} \alpha_k(t) e_k$ are continuous functions. Since $f(t)$ is a point-wise limit of $\{S_n(t)\}$, by Baire category theorem ([6] p. 13) $f(t)$ is discontinuous on a subset of first category of G.

3. - Orlicz-Pettis theorem

Let E be an F-space with dual space E^* separating the points of E. Denote by s_0 the set of all subsequences of the naturals N. Obviously, s_0 is a closed subset of the complete metric space s of all sequences of numbers endowed by the metric

$$\alpha(\xi,\eta) = \sum_{i=1}^{\infty} \frac{|\xi_i - \eta_i|}{2^i(1+|\xi_i - \eta_i|)}, \quad \xi = \{\xi_i\}, \eta = \{\eta_i\} \in s \tag{3.1}$$

Hence (s_0, d) is a complete metric space and therefore it is a set of second category. Let Γ be a total subset of E^* and let the series $\sum_{n=1}^{\infty} x_n$ be Γ-subseries convergent. So for each $\pi \in s_0$ there is $x_\pi \in E$ such that $\sum_{n \in \pi} x^*(x_n) = x^*(x_\pi), x^* \in \Gamma$. Define the function $\varphi : s_0 \to E$ by the following formula:

$$\varphi(\pi) = x_\pi = \sum_{n \in \pi} x_n \;, \quad \pi \in s_0 \tag{3.2}$$

We prove the following

LEMMA 3.1 - *The function φ defined by* (3.2) *is Γ-continuous.*

Proof. Let $\{\pi_k\} \in s_0, \pi_k \to \pi, k \to \infty$. We show that

$$x^*(\varphi(\pi_k)) \to x^*(\varphi(\pi)), k \to \infty, x^* \in \Gamma.$$

Indeed, since the series $\sum_{n \in \pi} x^*(x_n)$ is convergent for each subsequence $\pi \subset N, x^* \in \Gamma$, it is absolutely convergent, i.e. $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty, x^* \in \Gamma$. Hence for each $\varepsilon > 0$ and each $x^* \in \Gamma$ there exists $m = m(x^*, \varepsilon) \in N$ such that $\sum_{k=m+1}^{\infty} |x^*(x_n)| < \varepsilon/2$. Since $\pi_k \to \pi, k \to \infty$, there exists $r = r(x^*\varepsilon)$ such that $\pi_k(n) = \pi(n), k \geqslant r, n = 1, 2, \ldots, m$. This implies the inequality

$$|x^*(\varphi(\pi_k)) - x^*(\varphi(\pi))| =$$

$$= |\sum_{n \in \pi_k} x^*(X_n) - \sum_{n \in \pi} x^*(X_n)| \leqslant 2 \sum_{n=m+1}^{\infty} |x^*(x_n)| < \varepsilon, k \geqslant r.$$

This proves the lemma.

We now give a new proof of the extension of Stiles of the Orlicz-Pettis theorem.

THEOREM 3.1 - *Let E be an F-space with basis $\{e_n\}$ and Γ be the total system of the coordinate functionals for $\{e_n\}$. Let $\sum_{n=1}^{\infty} x_n$ be subseries Γ-convergent. Then it is subseries convergent.*

Proof. Since (s_0, d) is a set of second category, by Lemma 3.1 and Theorem 2.1 the function φ defined by (3.2) has at least one point of continuity. Using this fact we prove that $\|x_n\| \to 0, n \to \infty$. Assuming the contrary, there exists $\delta > 0$ and a subsequence $\pi \in N$ such that $\|x_n\| \geqslant \delta, n \in \pi$. Without loss of generality we can assume that $\|x_n\| \geqslant \delta, n \in N$. We show that φ has no points of continuity. Assuming that φ is continuous at $\pi_0 \in s_0$ and considering the sequence $\{\pi_k\} \in s_0$ defined by the formula $\pi_k(n) = \begin{cases} \pi_0(n), n \neq k \\ 0 \quad , n = k \end{cases}$ we get $\pi_k \to \pi_0$, $k \to \infty$. Hence $\varphi(\pi_k) \to \varphi(\pi_0)$, $k \to \infty$. We have

$$\| \varphi(\pi_k) - \varphi(\pi_0) \| = \| x_{\pi_0(k)} \| \to 0, k \to \infty .$$

This implies that there exists $m = m(\pi_0, \delta)$ such that $\| x_{\pi_0(n)} \| < \delta$, $n \geqslant m$ and this is a contradiction with the assumption $\| x_n \| \geqslant \delta$, $n \in N$. Hence every series satisfying the conditions of Theorem (3.1) has the property that its n^{th} element tends to 0 when $n \to \infty$. We now show that for each $\pi \in s_0$, the series $\sum_{n \in \pi} x_n$ is convergent. Assuming the contrary, there exists $\pi_0 \in s_0$ such that $\sum_{n \in \pi_0} x_n$ is not convergent. In this case there exists $\delta > 0$ and two subsequences $\{p_\alpha\}_\alpha \{q_k\} \subset N$ such that the sequence $\{y_k\}, y_k = \sum_{n \in \pi_0} x_n, p_k \leqslant n < q_k, q_k \leqslant p_{k+1}$ satisfies the property that $\| y_k \| \geqslant \delta, k \in N$. The series $\sum_{k=1} y_k$ is weakly subseries convergent and hence $\| y_k \| \to 0, k \to \infty$. This is a contradiction with the construction of $\{y_k\}$ and this proves the theorem.

Finally, we note that in the proof of Theorem 3.1 we used only the following property of the *F*-space *E*. Every Γ-cotinuous function $f: G \to E$ is continuous on at least one point of G.

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