

UNCERTAINTY RISK IN MORTALITY TRENDS: SCENARIO BASED  
MODELS AND DYNAMIC MODELLING

A THESIS SUBMITTED TO  
DEPARTMENT OF DEAMS "B.deFinetti"  
IN  
UNIVERSITY OF TRIESTE

BY

HATİCE ANAR

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE  
DEGREE OF  
DOCTOR OF PHILOSOPHY  
IN  
INSURANCE AND FINANCE: MATHEMATICS AND MANAGEMENT

APRIL 2015



I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name: Hatice ANAR

Signature:



# ABSTRACT

## UNCERTAINTY RISK IN MORTALITY TRENDS: SCENARIO BASED MODELS AND DYNAMIC MODELLING

Anar, Hatice

Ph.D., Insurance and Finance: Mathematics and Management

Supervisor: Prof. Ermanno Pitacco

April 2015, 96 pages

The change in mortality trends experienced over the last decades leads to the use of projected mortality tables in order to avoid underestimation of the future liabilities and costs in long term insurance products such as life annuities and pension funds. Although the projected mortality tables aim to capture the dynamic structure of mortality in the future, the future mortality trend itself is random and systematic deviations from the projected mortality might take place. Being a non-pooling risk, the impact of this “uncertainty risk” on the insurance portfolios can be dramatic due to the fact that the severity resulting from it increases as the size of the portfolio. For this reason, a proper modelling of uncertainty risk in mortality trends is required.

In this work the uncertainty risk modelling in mortality trends has been studied. In this aspect, the two stochastic models in the literature, scenario based and dynamic models have been adopted and assessed their level of capturing the uncertainty in mortality trends. One of the models, the static model, has been extended to the continuous case with the allowance of the multiple cohorts in the

portfolio. As defining the model, two approximation methods has been adopted to define the distribution of total number of deaths in the portfolio. Bayesian inferential procedure has been used in updating the random variables representing the uncertainty risk to the experience in the portfolio.

**Key words:** Stochastic mortality, Systematic deviations, Process risk, Uncertainty risk, Longevity risk, Bayesian inference, Life annuities.

*To my family*



# ACKNOWLEDGEMENTS

First of all, I would like to express my sincere gratitude to my supervisor Prof. Ermanno Pitacco for his guidance, support, valuable comments and directional ideas during all stages of my Ph.D. thesis.

My sincere admiration and gratitude is to Jale Körezliođlu, wife of lovely person Hayri Körezliođlu. I thank to her for her kindness, support and friendship.

I would like to express my deepest gratitude to Ali İhsan Karaođlan, Nazan Karaođlan and their sons Ali and Emre for supporting me throughout my life.

Last but not least, my greatest thanks go to my family, to my admirable father Bekir, my lovely mother Ayşe, my brothers, Ertuđrul and Ersun, for their unconditional love, endless support and patience throughout my life.



# TABLE OF CONTENTS

PLAGIARISM .....	ii
ABSTRACT .....	iii
DEDICATION .....	v
ACKNOWLEDGEMENTS .....	vi
TABLE OF CONTENTS .....	vii
1 INTRODUCTION .....	2
2 PRELIMINARIES .....	6
2.1 Life tables .....	6
2.1.1 Cohort tables and period tables .....	7
2.1.2 Probabilistic approach in life tables .....	9
2.1.3 The random lifetime .....	11
2.2 Summarizing a life table .....	12
2.3 Mortality dynamics and mortality trends .....	13
2.4 Mortality in continuous time .....	16
2.5 Mortality laws .....	19
3 MODELLING UNCERTAINTY RISK .....	23
3.1 Static model .....	24
3.1.1 The model .....	24
3.1.2 Numerical results .....	32

3.2	Dynamic model . . . . .	53
3.2.1	The model . . . . .	54
3.2.2	Life expectancy . . . . .	66
3.2.3	Numerical results . . . . .	67
4	CONCLUSION . . . . .	75
A	KOLMOGOROV-TYPE APPROXIMATION . . . . .	77
B	STATIC MODEL FIGURES . . . . .	83

# LIST OF FIGURES

2.1	Survival curve . . . . .	8
2.2	Curve of deaths . . . . .	8
3.1	Poisson approximation: expected lifetime · multiple cohorts $n_{x_0,j-1} = 1000, d_{x,j} = n_{x,j-1}q_{x,j}$ . . . . .	34
3.2	Poisson approximation: random fluctuations · multiple cohorts $n_{x_0,j-1} = 1000, d_{x,j} = n_{x,j-1}q_{x,j}$ . . . . .	34
3.3	Poisson approximation: uncertainty risk · multiple cohorts $n_{x_0,j-1} = 1000, d_{x,j} = n_{x,j-1}q_{x,j}$ . . . . .	35
3.4	Poisson approximation: variance of the lifetime · multiple cohorts $n_{x_0,j-1} = 1000, d_{x,j} = n_{x,j-1}q_{x,j}$ . . . . .	35
3.5	Poisson approximation: Lexis point · multiple cohorts $n_{x_0,j-1} = 1000, d_{x,j} = n_{x,j-1}q_{x,j}$ . . . . .	36
3.6	Poisson approximation: expected lifetime · multiple cohorts $n_{x_0,j-1} = 1000, d_{x,j} = 0.75 n_{x,j-1}q_{x,j}$ . . . . .	37
3.7	Poisson approximation: random fluctuation · multiple cohorts $n_{x_0,j-1} = 1000, d_{x,j} = 0.75 n_{x,j-1}q_{x,j}$ . . . . .	37
3.8	Poisson approximation: uncertainty risk · multiple cohorts $n_{x_0,j-1} = 1000, d_{x,j} = 0.75 n_{x,j-1}q_{x,j}$ . . . . .	38
3.9	Poisson approximation: variance of the lifetime · multiple cohorts $n_{x_0,j-1} = 1000, d_{x,j} = 0.75 n_{x,j-1}q_{x,j}$ . . . . .	38
3.10	Poisson approximation: Lexis point · multiple cohorts $n_{x_0,j-1} = 1000, d_{x,j} = 0.75 n_{x,j-1}q_{x,j}$ . . . . .	39
3.11	Poisson approximation: expected lifetime · multiple cohorts $n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1}q_{x,j}$ . . . . .	39
3.12	Poisson approximation: random fluctuations · multiple cohorts $n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1}q_{x,j}$ . . . . .	40

3.13	Poisson approximation: uncertainty risk · multiple cohorts	
	$n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ . . . . .	40
3.14	Poisson approximation: variance of the lifetime · multiple cohorts	
	$n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ . . . . .	41
3.15	Poisson approximation: Lexis point · multiple cohorts	
	$n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ . . . . .	41
3.16	Poisson approximation: expected lifetime · multiple cohorts	
	$n_{x_0,j-1} = 100, 1000, d_{x,j} = n_{x,j-1} q_{x,j}$ . . . . .	42
3.17	Poisson approximation: random fluctuations · multiple cohorts	
	$n_{x_0,j-1} = 100, 1000, d_{x,j} = n_{x,j-1} q_{x,j}$ . . . . .	43
3.18	Poisson approximation: uncertainty risk · multiple cohorts	
	$n_{x_0,j-1} = 100, 1000, d_{x,j} = n_{x,j-1} q_{x,j}$ . . . . .	43
3.19	Poisson approximation: variance of the lifetime · multiple cohorts	
	$n_{x_0,j-1} = 100, 1000, d_{x,j} = n_{x,j-1} q_{x,j}$ . . . . .	44
3.20	Poisson approximation: Lexis point · multiple cohorts	
	$n_{x_0,j-1} = 100, 1000, d_{x,j} = n_{x,j-1} q_{x,j}$ . . . . .	44
3.21	Poisson vs Kolmogorov appr: expected lifetime	
	$n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ . . . . .	45
3.22	Poisson vs Kolmogorov appr: random fluctuations	
	$n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ . . . . .	45
3.23	Poisson vs Kolmogorov appr: uncertainty risk	
	$n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ . . . . .	46
3.24	Poisson vs Kolmogorov appr: variance of the lifetime	
	$n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ . . . . .	46
3.25	Poisson vs Kolmogorov appr: Lexis point	
	$n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ . . . . .	47
3.26	Poisson approximation: expected lifetime · one cohort	
	$d_{x,j} = n_{x,j-1} q_{x,j}$ . . . . .	48

3.27	Poisson approximation: random fluctuations · one cohort:	
	$d_{x,j} = n_{x,j-1}q_{x,j}$ . . . . .	49
3.28	Poisson approximation: uncertainty risk · one cohort	
	$d_{x,j} = n_{x,j-1}q_{x,j}$ . . . . .	49
3.29	Poisson approximation: variance of the lifetime · one cohort	
	$d_{x,j} = n_{x,j-1}q_{x,j}$ . . . . .	50
3.30	Poisson approximation: Lexis points · one cohort	
	$d_{x,j} = n_{x,j-1}q_{x,j}$ . . . . .	50
3.31	Poisson approximation: expected lifetime · one cohort	
	$d_{x,j} = 0.75 n_{x,j-1}q_{x,j}$ . . . . .	51
3.32	Poisson approximation: random fluctuation risk · one cohort	
	$d_{x,j} = 0.75 n_{x,j-1}q_{x,j}$ . . . . .	51
3.33	Poisson approximation: uncertainty risk · one cohort	
	$d_{x,j} = 0.75 n_{x,j-1}q_{x,j}$ . . . . .	52
3.34	Poisson approximation: variance of the lifetime · one cohort	
	$d_{x,j} = 0.75 n_{x,j-1}q_{x,j}$ . . . . .	52
3.35	Poisson approximation: Lexis points	
	$d_{x,j} = 0.75 n_{x,j-1}q_{x,j}$ . . . . .	53
3.36	Expected systematic deviation in mortality – one cohort:	
	$\bar{\alpha} = \bar{\beta}, \bar{\beta} = 100; d_{x,s} = n_{x,s-1} q_{x,s}^*$ . . . . .	68
3.37	Expected systematic deviation in mortality – multiple cohorts:	
	$\bar{\alpha} = \bar{\beta}, \bar{\beta} = 100; d_{x,s} = n_{x,s-1} q_{x,s}^*$ . . . . .	69
3.38	Expected systematic deviation in mortality – one cohort:	
	$\bar{\alpha} = \bar{\beta}, \bar{\beta} = 100; d_{x,s} = 0.75 n_{x,s-1} q_{x,s}^*$ . . . . .	69
3.39	Expected systematic deviation in mortality – multiple cohorts:	
	$\bar{\alpha} = \bar{\beta}, \bar{\beta} = 100; d_{x,s} = 0.75 n_{x,s-1} q_{x,s}^*$ . . . . .	70
3.40	Expected systematic deviation in mortality - one cohort:	
	$\bar{\alpha} = \bar{\beta}, \bar{\beta} = 100; d_{x,s} = 1.25 n_{x,s-1} q_{x,s}^*$ . . . . .	70

3.41	Expected systematic deviation in mortality – multiple cohorts:	
	$\bar{\alpha} = \bar{\beta}, \bar{\beta} = 100; d_{x,s} = 1.25 n_{x,s-1} q_{x,s}^*$	71
3.42	Expected systematic deviation in mortality – one cohort:	
	$\bar{\alpha} = \bar{\beta}, n_{65,s} = 1000; d_{x,s} = n_{x,s-1} q_{x,s}^*$	72
3.43	Expected systematic deviation in mortality – multiple cohorts:	
	$\bar{\alpha} = \bar{\beta}, n_{65,s} = 1000; d_{x,s} = n_{x,s-1} q_{x,s}^*$	72
3.44	Expected systematic deviation in mortality – one cohort:	
	$\bar{\alpha} = \bar{\beta}, n_{65,s} = 1000; d_{x,s} = 0.75 n_{x,s-1} q_{x,s}^*$	73
3.45	Expected systematic deviation in mortality – multiple cohorts:	
	$\bar{\alpha} = \bar{\beta}, n_{65,s} = 1000; d_{x,s} = 0.75 n_{x,s-1} q_{x,s}^*$	73
3.46	Expected systematic deviation in mortality – one cohort:	
	$\bar{\alpha} = \bar{\beta}, n_{65,s} = 1000; d_{x,s} = 1.25 n_{x,s-1} q_{x,s}^*$	74
3.47	Expected systematic deviation in mortality – multiple cohorts:	
	$\bar{\alpha} = \bar{\beta}, n_{65,s} = 1000; d_{x,s} = 1.25 n_{x,s-1} q_{x,s}^*$	74
B.1	Poisson approximation: the lifetime quantities · multiple cohorts	
	$n_{x_0,j-1} = 100, d_{x,j} = n_{x,j-1} q_{x,j}$	84
B.2	Poisson approximation: lifetime quantities · multiple cohorts	
	$n_{x_0,j-1} = 100, d_{x,j} = 0.75 n_{x,j-1} q_{x,j}$	85
B.3	Poisson approximation: lifetime quantities · multiple cohorts	
	$n_{x_0,j-1} = 100, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$	86
B.4	Poisson approximation: lifetime quantities · multiple cohorts	
	$d_{x,j} = 0.75 n_{x,j-1} q_{x,j}$	87
B.5	Poisson approximation: lifetime quantities · multiple cohorts	
	$d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$	88
B.6	Kolmogorov-type approximation: lifetime quantities	
	$n_{x_0,j-1} = 100, d_{x,j} = n_{x,j-1} q_{x,j}$	89
B.7	Kolmogorov-type approximation: lifetime quantities	
	$n_{x_0,j-1} = 100, d_{x,j} = 0.75 n_{x,j-1} q_{x,j}$	90

B.8	Kolmogorov-type approximation: lifetime quantities	
	$n_{x_0,j-1} = 100, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$	91
B.9	Poisson vs Kolmogorov appr: lifetime quantities:	
	$n_{x_0,j-1} = 1000, d_{x,j} = n_{x,j-1} q_{x,j}$	92
B.10	Poisson vs Kolmogorov appr: lifetime quantities:	
	$n_{x_0,j-1} = 1000, d_{x,j} = 0.75 n_{x,j-1} q_{x,j}$	93
B.11	Poisson approximation: the lifetime quantities · one cohort:	
	$d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$	94

# CHAPTER 1

## INTRODUCTION

In life insurance contracts, the insurer takes risks originating from various causes related to financial aspects (e.g. investment yield, inflation, etc), policyholders' behaviour (lapses, surrenders, etc.), demographic aspects (e.g. lifetimes of policyholders, disability) and expenses. As a demographic risk, mortality risk constitutes an important source of risk underlying in life insurance portfolios. Mortality risk profile of an individual, called individual mortality, is affected by a number of risk factors in respect to the aggregate mortality, the general mortality level in the relevant population. These risk factors might be listed as age, gender, health condition, profession, smoking habits, etc. In actuarial studies, the age-pattern of mortality is described by a life table, which provides for each age  $x$ , between 0 and a maximum attainable age,  $\omega$ , the probability  $q_x$  of dying in a year, i.e. before age  $x + 1$ , or the expected number  $l_x$  of people alive at that age in a national cohort, say 100000 individuals. Construction of a life table from mortality experience in a given cohort assumes that the mortality pattern doesn't change in the future, i.e. a 'static' (non changing) mortality pattern follows in the future.

The mortality trends experienced over the last decades have proved a decreasing structure in aggregate mortality, in particular at adult and old ages (e.g. see Benjamin and Soliman, 1993; Macdonald, 1997; Macdonald et al.,1998). The effects of this change in the aggregate mortality structure can be seen as an increase in the concentration around the mode of the curves of deaths, the movement of lexis point towards older ages and a higher level and larger dispersion of accidental deaths at young ages. As far as insurance covers and annuities are concerned,

these changes affects the expected present values, key quantities in pricing and reserving, hence an appropriate mortality projection is required in order to avoid underestimation of the future liabilities and costs.

The projected life tables aim to capture the dynamic structure of mortality in the future, and consider the mortality as function of the calendar year as well as the age. In this sense, the traditional approaches rely on the extrapolation of mortality trends observed in the past, yielding a deterministic or stochastic age-pattern of mortality for each calendar year, depending on how the data points (mortality) are considered within the extrapolation procedure. When the observed mortality is considered as “simply numbers”, the extrapolation yields a point estimate of future mortality, without any statistical feature. However, if the observed mortality is considered as “outcomes of random variables”, then the extrapolation procedure relies on a statistical framework and a stochastic model for mortality projection is needed. In this framework, the Lee-Carter projection model (see Lee and Carter 1992) is one of the leading stochastic mortality models in the literature. In stochastic framework, the results of projection procedures consists in both point and interval estimates of future mortality rates, namely, random fluctuations around the expected mortality projection, known as “process risk”, are allowed. Besides traditional approaches for mortality projection, some parametric models, i.e. mortality laws, have also been proposed to express age-pattern of mortality. Among these, Gompertz, Makeham, Weibull and Heligman-Pollard represent the age-pattern of mortality at old ages, while Thiele or Heligman-Pollard can project mortality over the whole life span.

Whatever kind of projection model is adopted, the future mortality trend itself is random and systematic deviations from the projected mortality might take place. The risk of systematic deviations, called “uncertainty risk”, constitutes an important component of mortality risk in insurance contracts due to the fact that being a non-pooling risk, the severity resulting from it increases as the size of the portfolio. As far as the unanticipated mortality improvements, especially

at old ages, are concerned, the uncertainty risk is usually referred to “longevity risk”. Researches in uncertainty risk, in particular longevity risk, can be found in recent literature with applications to life annuities and longevity-linked securities. For example, CMI (2002) and CMI (2006) propose a scenarios base approach, i.e. adopting a number of scenarios representing alternatives to the best estimate (projected) one. Olivieri (2001) adopts a similar scenario approach with a suggestion of a probability distribution on the scenarios, which provides the unconditional valuations to the portfolio results. In their work, Olivieri and Pitacco (2002) further define a Bayesian inferential model to update the probability distribution on the scenario set. An application of the scenario based model to Solvency requirements can be found in Olivieri and Pitacco (2003). An additional to the scenario based model, Olivieri and Pitacco (2009),(2011), introduced a dynamic approach to uncertainty risk using Bayesian inference.

In this work, the uncertainty risk modelling in mortality trends has been studied. In this aspect, two stochastic models in the literature, scenario based and dynamic models proposed by Olivieri and Pitacco (2002), (2009) (2011), have been adopted and assessed their level of capturing the uncertainty risk in mortality trends. The basic idea behind these two models is to consider a set of possible future scenarios on mortality and to define a distribution on it. Then the distribution on the scenario set is updated to the experience in the portfolio via Bayesian inferential procedure. Being originally defined in the original paper under discrete set of scenarios and allowance of one cohort in the portfolio, the scenario-based model has been extended by defining the scenario set on real plane, allowing multiple cohorts in this work. As defining the model, Poisson and Kolmogorov type of approximation methods has been adopted to define the distribution of total number of deaths in the portfolio. On the other hand, the dynamic model has been adopted as it is in the original model, and a brief definition of life expectancy under the model has been provided.

The outline of the work is as follows: Chp. 2 provides some preliminaries to

lifetime modelling. In this chapter is also discussed the mortality dynamics and mortality modelling i.e deterministic and stochastic models. Chp. 3 defines two stochastic models for the uncertainty risk, scenario-based model and dynamic model. Numerical results of analysis of the models are also provided in the relevant sections. Some final remarks in Chp. 4 concludes the work.

## CHAPTER 2

# PRELIMINARIES

As being belong to demographic risk part of the insurance contracts, mortality risk constitutes an important source of risk in life insurance portfolios. Individual mortality is affected by a number of risk factors such as age, gender, health condition, profession, smoking habits, etc. This chapter aims to provide preliminary information on lifetime modelling, i.e modelling the age-pattern of mortality. The tools and the notations used to described them will be introduced. Furthermore mortality dynamics and mortality projection will be discussed.

### 2.1 Life tables

A life table is simply defined as a set of sequences which is represented as in Table 2.1. The two main elements of a life table are the age, denoted by  $x$ , and the estimated number of people alive at age  $x$  in a properly defined population, denoted by  $l_x$ . Note that, in the Table 2.1, the numbers  $l_0, l_1, l_2, \dots$  constitute a decreasing sequence with  $l_{109} \approx 0$ , meaning 108 represents the maximum attainable age. This age, denoted by  $\omega$ , implies that  $l_\omega > 0$  whilst  $l_{\omega+1} = 0$ .

The other two sequences in the life table,  $d_x$  and  $q_x$ , represent the number of deaths between age  $x$  and  $x + 1$  and the probability of an individual aged  $x$  dying within one year, respectively. As the definitions of these sequences indicate, the  $d_x$ s are calculated as

$$d_x = l_x - l_{x+1} \tag{2.1.1}$$

**Table 2.1:** A life table

$x$	$l_x$	$d_x$	$1000 q_x$
0	100000	879	8.788
1	99121	46	0.461
2	99076	33	0.332
...	...	...	...
50	93016	426	4.582
51	92590	459	4.961
...	...	...	...
110+	1	1	1000.000

with

$$\sum_{x=0}^{\omega} d_x = l_0, \quad (2.1.2)$$

and then, the probability  $q_x$  can be expressed as

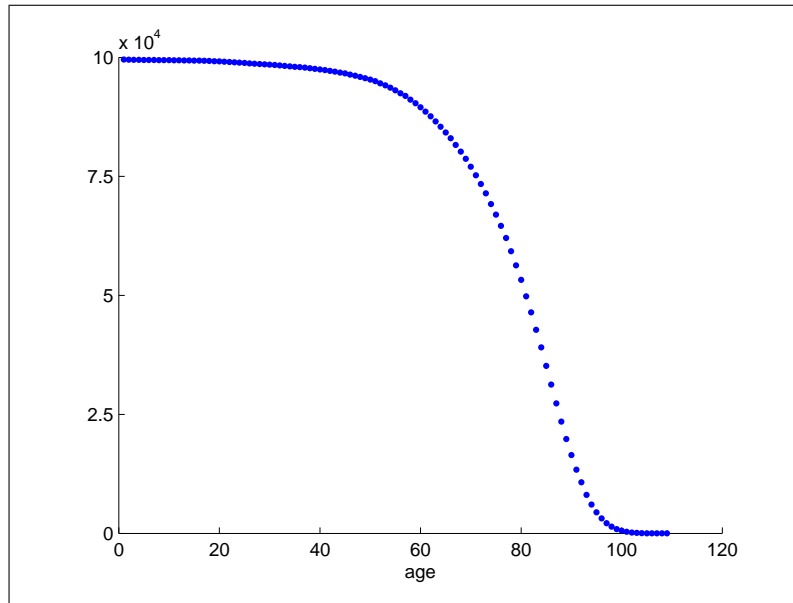
$$q_x = \frac{d_x}{l_x} \quad (2.1.3)$$

Note that, these sequences  $d_x$  and  $q_x$  are strictly related to  $l_x$ .

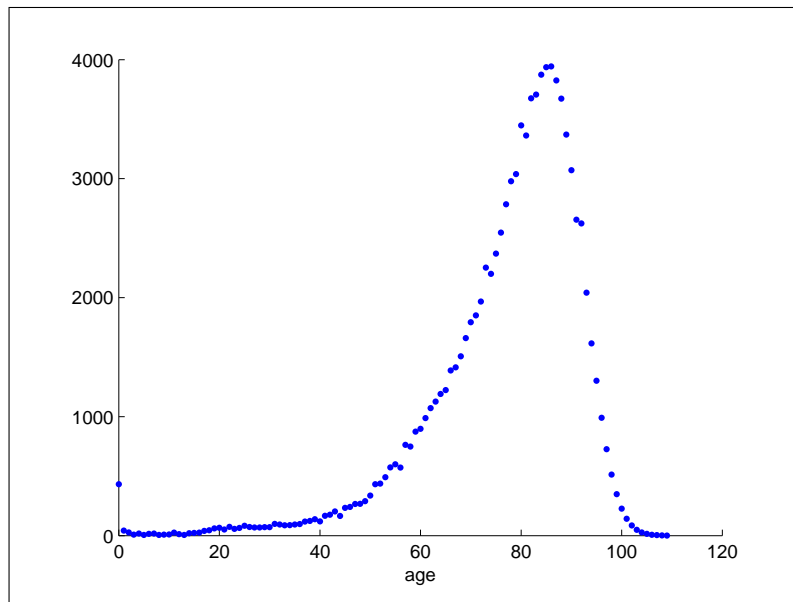
The plots of  $l_x$ 's and the  $d_x$ 's against  $x$  are called the *survival curve* and the *curve of deaths* respectively. Figs.2.1 and 2.2 show how the survival curve and curve of deaths look like. Some features observed in most of the mortality tables and seen in the graphs are results of *a)* the infant mortality, *b)* the mortality hump at young-adult ages, *c)* the age of maximum mortality, at old ages (called *Lexis point*)

### 2.1.1 Cohort tables and period tables

Assume that the sequence  $l_0, l_1, \dots, l_{\omega}$  is obtained observing the actual number of individuals alive at ages  $1, 2, \dots, \omega$  out of a given initial *cohort* consisting of  $l_0$  newborns. Namely the observation is *by year of birth*. The life table constructed



**Figure 2.1:**  $l_x$  in the Belgium male population - 2012 (source: Human Mortality Database)



**Figure 2.2:**  $d_x$  in the Belgium male population - 2012 (source: Human Mortality Database)

from those observations is called *cohort life table*. Assuming  $\omega$  is the maximum attainable age,  $\omega + 1$  years are needed to construct a cohort table.

Differently, assume that the probability of deaths at many different ages  $x$ , i.e. the sequence  $q_0, q_1, \dots, q_\omega$ , is obtained observing a sample group throughout a given period, for example one year. This observation is *by year of death*. Then a life table can be derived via formula

$$l_{x+1} = l_x(1 - q_x) \quad (2.1.4)$$

for  $x = 0, 1, \dots, \omega - 1$  with an assigned value to  $l_0$  and  $\omega$  satisfying  $l_\omega > 0$  whilst  $l_{\omega+1} = 0$ . The life table derived  $l_0, l_1, \dots, l_\omega$  is called a *period* or *cross-sectional life table*.

Construction of a life table from mortality experience in a given period is based on the assumption of a 'static' (non changing) mortality pattern in future, namely, the assumption of that the mortality pattern doesn't change in the future. This assumption might be valid and used for the products having short or medium duration, e.g. the term insurance and the endowment insurance, however for the products with longer durations, e.g. the life annuities and the pension funds, the static life tables should be avoided due to the changes in mortality trends in long term. For this aspect, the life annuities and the pension funds require *projected life tables*.

### 2.1.2 Probabilistic approach in life tables

Starting from the sequence  $l_0, l_1, \dots, l_\omega$  in a life table, a set of probabilities can be defined to be used in life insurance calculations.

Let  $p_x$  denote the probability of an individual age  $x$  being alive at age  $x + 1$ . Clearly, it emerges from the life table as

$$p_x = 1 - q_x \quad (2.1.5)$$

which yields

$$p_x = \frac{l_{x+1}}{l_x} \quad (2.1.6)$$

Additionally, let  ${}_h p_x$  define the probability of an individual age  $x$  being alive at age  $x + h$ . The probability  ${}_h p_x$  can also be expressed in terms of one year probabilities,

$${}_h p_x = p_x p_{x+1} \cdots p_{x+h-1} \quad (2.1.7)$$

Using (2.1.6), the following relation is then obtained

$${}_h p_x = \frac{l_{x+1}}{l_x} \frac{l_{x+2}}{l_{x+1}} \cdots \frac{l_{x+h}}{l_{x+h-1}} = \frac{l_{x+h}}{l_x} \quad (2.1.8)$$

noting that  ${}_0 p_x = 1$  and  ${}_1 p_x = p_x$ . Another relation, widely used, related to one year probabilities  ${}_h p_x$  is given by

$${}_{h+x} p_x = {}_h p_x {}_k p_{x+h} \quad (2.1.9)$$

Similarly, let  ${}_h q_x$  define the probability that an individual age  $x$  dies within next  $h$  years. Using relation (2.1.8), it is found

$${}_h q_x = 1 - {}_h p_x = \frac{l_x - l_{x+h}}{l_x} \quad (2.1.10)$$

noting that  ${}_0 q_x = 1$  and  ${}_1 q_x = q_x$ .

Furthermore, the probability of an individual aged  $x$  dying between age  $x + h$  and  $x + h + k$ , denoted by  ${}_{h|k} q_x$ , is widely used in actuarial calculations. It is usually called as 'deferred' probability and can be expressed in terms of two probabilities, being alive at age  $x + h$  and then dying before age  $x + h + k$ , which is

$${}_{h|k} q_x = {}_h p_x {}_k q_{x+h} = \frac{l_{x+h} - l_{x+h+k}}{l_x} \quad (2.1.11)$$

In case  $k = 1$ , the deferred probability corresponds to the probability of an individual aged  $x$  dying within one year at  $h$  years from now, i.e.

$${}_{h|1} q_x = {}_h p_x q_{x+h} = \frac{l_{x+h} - l_{x+h+1}}{l_x} = \frac{d_{x+h}}{l_x} \quad (2.1.12)$$

For a newborn, namely  $x = 0$ , the deferred probability provides many useful conclusions. Considering

$${}_{h|1} q_0 = \frac{d_h}{l_0} \quad (2.1.13)$$

the following relation emerges:

$$\sum_{h=0}^{\omega} {}_h|1q_0 = \frac{1}{l_0} \sum_{h=0}^{\omega} d_h = 1 \quad (2.1.14)$$

This summation results from the fact that a newborn will die at a time in the future (unknown from now), i.e. a sure event, and summing up the probabilities of happening this event in periods  $1, 2, \dots, \omega$  equals to 1. In other words, the probabilities  ${}_h|1q_0$  constitutes the probability distribution of the lifetime of a newborn. Furthermore, for an integer  $k$ , the summation yields

$$\sum_{h=k}^{\omega} {}_h|1q_0 = \frac{1}{l_0} \sum_{h=0}^{\omega} d_h = \frac{l_k}{l_0} = {}_k p_0 \quad (2.1.15)$$

Finally, it is worth to introduce a quantity which enables an easy mathematical modelling of the lifetimes. Being called *mortality odds*, it is expressed as

$$\phi_x = \frac{q_x}{1 - q_x} \quad (2.1.16)$$

Following from  $0 < q_x < 1$ , the odds is a positive quantity,  $\phi_x > 0$ .

### 2.1.3 The random lifetime

A formal setting in lifetime modelling can be established via defining a random variable for the *remaining lifetime* of an individual aged  $x$ . In the literature, this random variable is denoted as  $T_x$  with real valued outcomes between 0 and  $\omega - x$ . Particularly, the random variable  $T_0$  defines the *total lifetime* of a newborn. The relation between these two random variables can be given by

$$T_x = T_0 - x | T_0 > x \quad (2.1.17)$$

Assuming a life table is available, the probabilities regarding the remaining lifetime, similarly total lifetime, can be calculated

$$\mathbb{P}[T_x > h] = {}_h p_x = \frac{l_{x+h}}{l_x} \quad (2.1.18)$$

$$\mathbb{P}[T_x \leq h] = {}_h q_x = 1 - {}_h p_x = \frac{l_x - l_{x+h}}{l_x} \quad (2.1.19)$$

$$\mathbb{P}[h < T_x \leq h + x] = {}_{h|k} q_x = \frac{l_{x+h} - l_{x+h+k}}{l_x} \quad (2.1.20)$$

The integer part of the remaining lifetime, called the *curtate remaining lifetime* and denoted by  $K_x$ , is expressed as

$$K_x = \lfloor T_x \rfloor \quad (2.1.21)$$

yielding

$$\begin{aligned} 0 < T_x < 1 &\Leftrightarrow K_x = 0 \\ 1 \leq T_x < 2 &\Leftrightarrow K_x = 1 \\ 2 \leq T_x < 3 &\Leftrightarrow K_x = 2 \\ \dots &\qquad \qquad \dots \end{aligned} \quad (2.1.22)$$

Similarly, it is also possible to define *curtate total lifetime* in the same way.

## 2.2 Summarizing a life table

In actuarial calculations, some indices, also called *markers*, are defined to summarize life tables, so as the lifetime probability distribution. The expected value of the random lifetime, i.e life expectancy, is a typical marker. Other markers which can be adopted are the *Lexis point*, the (old) age with the highest mortality, the *variance* of the probability distribution of the total lifetime.

Considering the curtate total lifetime  $K_0$ , the expected value of the the random variable  $K_0 + \frac{1}{2}$  is given by

$$\mathbb{E}[K_0 + \frac{1}{2}] = \sum_{h=0}^{\omega} \left( h + \frac{1}{2} \right) {}_{h|1} q_0. \quad (2.2.1)$$

This quantity (2.2.1) is called *expected totallifetime*, or *life expectancy at birth*, and denoted by  $\mathring{e}_0$ .

Referring to an individual aged  $x$ , the *expected remaining lifetime, or life expectancy at age  $x$* , denoted by  $\dot{e}_x$ , is defined as follows

$$\dot{e}_x = \mathbb{E}\left[K_x + \frac{1}{2}\right] = \sum_{h=0}^{\omega-x} \left(h + \frac{1}{2}\right) {}_h|1q_x, \quad (2.2.2)$$

## 2.3 Mortality dynamics and mortality trends

Over the last decades, the mortality experienced in many countries has shown some aspects whose effects might be clearly seen in the shapes of the survival curves and the curve of deaths, especially at adult and old ages. These aspects observed can be summarized as follows:

1. an increase in concentration of deaths around the mode (at old ages) of the curve of deaths, called as *rectangularization*;
2. the mode of the curve of deaths (lexis point) moves towards older ages, due to the rectangularization. This aspect is called the *expansion* of the survival function;
3. higher level and larger dispersion of accidental deaths at young ages (the so-called mortality hump).

The mortality trends experienced also affect the quantities such as the life expectancy and mortality rates, which is observed as

4. an increase in life expectancy (at birth as well as at old ages);
5. a decrease in the infant mortality, and in mortality rates particularly at adult and old ages.

All these aspects mentioned above have proved the existence of mortality dynamics and motivated the realistic mortality modelling approaches. Considering the dynamic structure of mortality, the "static" mortality is far from capturing the mortality pattern when long periods of time are referred to. That's why, the

period life tables should be used only for the products having short or medium durations (such as 5 to 10 years), e.g. the term insurance, endowment insurance. For the products having longer durations, e.g. life annuities, pension funds, the use of "projected" life tables is required.

The projected life tables are constructed in order to capture the dynamic structure of mortality in the future, and considered as an estimate to the future mortality trend. The basic idea behind projection is to consider the mortality as function of the calendar year  $t$  as well as the age. Focusing on one year mortality rates,  $q_x(t)$  represents the mortality of an individual aged  $x$  at year  $t$  dying in one year. The table 2.2 shows the matrix of one year mortality rates. What the table tells is the following:

1. the columns  $q_0(t), q_1(t), \dots, q_x(t), \dots$  correspond to *period life tables*, referring to the people living in a given calendar year  $t$ ;
2. the diagonals  $q_0(t), q_1(t+1), \dots, q_x(t+x), \dots$  correspond to *cohort life tables*, referring to the cohort born in year  $t$ ;
3. the rows  $\dots, q_x(t-1), q_x(t), \dots, q_x(t+1), \dots$  yield the *mortality profiles*, referring to the mortality trend for a given age  $x$ ;

**Table 2.2:** Annual mortality rates in dynamic context

	...	$t - 1$	$t$	$t + 1$	...
0	...	$q_0(t - 1)$	$q_0(t)$	$q_0(t + 1)$	...
1	...	$q_1(t - 1)$	$q_1(t)$	$q_1(t + 1)$	...
...	...	...	...	...	...
$x$	...	$q_x(t - 1)$	$q_x(t)$	$q_x(t + 1)$	...
$x + 1$	...	$q_{x+1}(t - 1)$	$q_{x+1}(t)$	$q_{x+1}(t + 1)$	...
...	...	...	...	...	...
$\omega$	...	$q_\omega(t - 1)$	$q_\omega(t)$	$q_\omega(t + 1)$	...

The matrix in Table 2.2 contains the elements referring to the past, current and future years. Let  $t'$  denote the year for which the most recent period life table is available. The probabilities  $q_x(t)$  for  $t > t'$  referring to the future are estimated using a projection procedure. For a given year  $t'$  and the given maximum year  $t^*$  (time horizon), the *projected life table* is a submatrix of the whole matrix in Table 2.2

$$\{q_x(t) : \quad x = 0, 1, \dots, \omega; \quad t = t' + 1, t' + 2, \dots, t^*\}$$

Here we believe that the appropriate use of one year probabilities in a dynamic context should be noted. The one year probabilities concerning the lifetime of a person age  $x$  in a year, say  $t$ , are derived from the diagonal

$$q_x(t), q_{x+1}(t+1), q_{x+2}(t+2), \dots \quad (2.3.1)$$

i.e. the relevant cohort table. Hence, the probability of a person age  $x$  in year  $t$  being alive at age  $x+h$  is calculated as

$${}_h p_x(t) = (1 - q_x(t)) (1 - q_{x+1}(t+1)) \dots (1 - q_{x+h-1}(t+h-1)) \quad (2.3.2)$$

from which the deferred probabilities is found as

$${}_{h|1} q_x(t) = {}_h p_x(t) q_{x+h}(t+h) \quad (2.3.3)$$

Moreover, the (cohort) life expectancy for an individual aged  $x$  in year  $t$  is given by

$$\dot{e}_x(t) = \sum_{h=0}^{\omega-x} \left( h + \frac{1}{2} \right) {}_{h|1} q_x(t). \quad (2.3.4)$$

Among the approaches adopted to mortality projection, is the graduation-extrapolation method mostly known. This method relies on the assumption that the observed trend in the past continues in the future years. Hence, the observed mortality rates over time constitutes the input data in graduation phase of the method. The method is mainly restricted to the recent observations in order to avoid the inclusion of the mortality effects which are not valid anymore. An

important point to be noted is how the data points will be interpreted in the estimation: *a)* if the data points are considered as "simply numbers", then the extrapolation procedure doesn't allow any statistical feature of the data, consequently, a point estimate of the future mortality is obtained; *b)* if the data points are considered as "outcomes of random variables", then the extrapolation procedure relies on a statistical framework, and interval estimates as well as the point estimates of the future mortality are obtained.

## 2.4 Mortality in continuous time

So far we have defined mortality in discrete context has been defined. Whereas, in order to calculate the probabilities and quantities at ages and durations in real numbers, such as regarding the remaining/total lifetimes, a continuous time mortality modelling is needed.

The main tool in order to define mortality in continuous context is the *survival function*,  $S(t)$ , for  $t \geq 0$ . It is defined as

$$S(t) = \mathbb{P}[T_0 > t] \quad (2.4.1)$$

where  $T_0$  is the total lifetime of a newborn, which has already been defined in Sect. 2.1.3.

The probabilities related to random lifetimes can also be evaluated in terms of survival function. Using the relation (2.1.17), the distribution of the remaining lifetime is obtained by

$$\mathbb{P}[T_x > h] = \mathbb{P}[T_0 > x + h | T_0 > x] = \frac{\mathbb{P}[T_0 > x + h]}{\mathbb{P}[T_0 > x]} \quad (2.4.2)$$

which yields the following probabilities

$${}_h p_x = \frac{S(x+h)}{S(x)} \quad (2.4.3)$$

$${}_h q_x = \frac{S(x) - S(x+h)}{S(x)} \quad (2.4.4)$$

$${}_{h|k}q_x = \frac{S(x+h) - S(x+h+k)}{S(x)} \quad (2.4.5)$$

Other functions used in defining age-continuous modelling are the *probability density function* (pdf) and the *distribution function* (df) of the random lifetime. Let  $f_0(x)$  and  $F_0(x)$  denote the pdf and the distribution function the random lifetime of a newborn, i.e.  $T_0$ , respectively. Using the properties of pdf and df, the following relations, frequently used in actuarial calculations, are obtained:

$$F_0(x) = \mathbb{P}[T_0 \leq x] = {}_xq_0 \quad (2.4.6)$$

$$F_0(x) = 1 - S(x) \quad (2.4.7)$$

Under the assumption of that  $f_0(x)$  is a continuous function, we have the pdf in terms of survival function as

$$f_0(x) = \frac{dF_0(x)}{dx} = -\frac{dS(x)}{dx} \quad (2.4.8)$$

The graph of the pdf  $f_0(x)$  is usually called the *curve of deaths*, and when compared to the Figure ??, the similarity of the two curves can be clearly seen. Hence we can declare that the equation (2.4.8) certainly defines the relation between the curves of deaths and the survival curve.

Analogous relations can also be obtained for the remaining lifetime  $T_x, x > 0$ :

$$F_x(t) = \mathbb{P}[T_x \leq t] = \mathbb{P}[T_0 \leq x+t | T_0 > x] \quad (2.4.9)$$

$$= \frac{\mathbb{P}[x < T_0 \leq x+t]}{\mathbb{P}[T_0 \leq x]} = \frac{F_0(x+t) - F_0(x)}{S(x)} \quad (2.4.10)$$

$$f_x(t) = \frac{dF_x(t)}{dt} = \frac{\frac{dF_0(x+t)}{dt}}{S(x)} = \frac{f_0(x+t)}{S(x)} \quad (2.4.11)$$

It is worth to point out that all the probabilities involved in actuarial calculations can be calculated once the pdf or the df of a remaining/total random lifetime is given, i.e.

$${}_tP_x = 1 - F_x(t) = \int_t^\infty f_x(u) du = \frac{1}{S(x)} \int_t^\infty f_0(x+u) du \quad (2.4.12)$$

An important function defined in continuous context is the *force of mortality*, and notated as  $\mu_x$ . Known also as the *mortality intensity* or the *hazard function*, the force of mortality is defined for all  $x \geq 0$  as follows:

$$\mu_x = \lim_{t \rightarrow 0} \frac{{}_tq_x}{t} \quad (2.4.13)$$

A number of parametric formulas (namely mortality laws) have been proposed in the literature regarding to actuarial and demographic studies. Some important mortality laws are summarized in Sect. 2.5.

Using the relations derived previously, the force of mortality can be written in terms of survival function as follows:

Substitution of  ${}_tq_x$  yields

$$\mu_x = \lim_{t \rightarrow 0} \frac{S(x) - S(x+t)}{t S(x)} \quad (2.4.14)$$

which is simplified as

$$\mu_x = \frac{-\frac{dS(x)}{dx}}{S(x)} \quad (2.4.15)$$

Then using the relation (2.4.8), the force of mortality becomes

$$\mu_x = \frac{f_0(x)}{S(x)} \quad (2.4.16)$$

Hence, once the survival function  $S(x)$  has been assigned, the force of mortality can be derived.

The relation between the survival function and the force of mortality also enables that, once  $\mu_x$  has been assigned, for example via a mortality, the survival function can be derived. Provided that  $\mu_x$  is known, the relation (2.4.15) is a differential equation and solving it with respect to  $S(x)$  with the boundary condition  $S(0) = 1$  gives

$$S(x) = e^{-\int_0^x \mu_t dt} \quad (2.4.17)$$

Once the survival function has been obtained, all survival and death probabilities, seen in this chapter, can then be derived via the following equation:

$$q_x = 1 - p_x = 1 - \frac{S(x+1)}{S(x)} = 1 - e^{-\int_x^{x+1} \mu_t dt} \quad (2.4.18)$$

## 2.5 Mortality laws

Mortality tables, constructed empirically, represent mortality over the whole human lifespan. The attempts to describe age-pattern of mortality in mathematical terms led the researchers to analytical formulas, i.e. mortality laws, which summarize empirical mortality tables by a small number of parameters without sacrificing much information. The mortality laws might refer to the mortality rates, the odds or the force of mortality, and might be in age-discrete or age-continuous context. Avoiding going into too much in detail, a summary of well known and widely used mortality laws will be provided in the following sections.

### The Heligman-Pollard law

The formula, proposed by Heligman and Pollard in 1980 to fit Australian mortality rates, aims to represent the age-pattern of mortality over whole lifespan. The first Heligman-Pollard law, representing the odds, was defined in the form

$$\phi_x = A^{(x+B)^C} + D e^{-(\ln x - \ln F)^2} + G H^x \quad (2.5.1)$$

while the second Heligman-Pollard law, focusing on the mortality rates, was defined as

$$q_x = A^{(x+B)^C} + D e^{-(\ln x - \ln F)^2} + \frac{G H^x}{1 + G H^x} \quad (2.5.2)$$

Both laws give mortality at high ages as

$$q_x \approx \frac{G H^x}{1 + G H^x} \quad (2.5.3)$$

which can be used in calculations related to life annuities and pensions, e.g. for  $x \geq 65$

### The Gompertz law

In 1825, Gompertz proposed a two parameter formula in terms of the force of mortality defining as

$$\mu_x = B c^x \quad (2.5.4)$$

with  $B, c > 0$ . An equivalent notation used for the Gormpertz law is

$$\mu_x = \alpha e^{\beta x} \quad (2.5.5)$$

The Gompertz law is used to represents the age progression of mortality at the old ages, namely the senescent mortality.

## The Makeham laws

A generalization of the Gompertz law was first provided in 1867, and defined as

$$\mu_x = A + B c^x \quad (2.5.6)$$

where the age independent term  $A \geq 0$  represents non-senescent mortality, e.g. caused from accidents. An equivalent notation to the first Makeham law is

$$\mu_x = \gamma + \alpha e^{\beta x} \quad (2.5.7)$$

In 1890 was the second Makeham law proposed in 1890 in the following form:

$$\mu_x = A + H x + B c^x \quad (2.5.8)$$

which is a further generalization of the Gompertz law.

## The Thiele law

Proposed in 1871, the Thiele law represents the age-pattern of mortality over the whole life span.

$$\mu_x = A e^{Bx} + C e^{-D(x-E)^2} + F G^x \quad (2.5.9)$$

where all the parameters are positive real numbers. Each term in the formula represents a specific period of life span. The first term, decreasing as the age increases, stands for the infant mortality. The second term, having a "Gaussian" shape, is for the mortality hump at young-adult ages. Finally, the third term represents the senescent mortality.

## The Perks laws

Two mortality laws were proposed by Perk in 1932. The first Perks law is

$$\mu_x = \frac{\alpha e^{\beta x} + \gamma}{\delta e^{\beta x} + 1} \quad (2.5.10)$$

while the second Perks law has the following more general structure

$$\mu_x = \frac{\alpha e^{\beta x} + \gamma}{\delta e^{\beta x} + \epsilon e^{-\beta x} + 1} \quad (2.5.11)$$

The Perks' laws differ from the previous models as having an important role in representing the mortality pattern at very old ages, e.g.  $x \geq 80$ . Recent statistical evidences show that the force of mortality is slowly increasing at very old ages, such as approaching a rather flat shape, which leads to the rejection of the exponential increase implied by the other models. The Perks' law, having logistic shape in their graphs, captures better this unexponential mortality structure at very old ages.

## The Weibull law

The Weibull law, proposed in 1951 in the context of the reliability theory, is given by

$$\mu_x = Ax^B \quad (2.5.12)$$

or, in equivalent terms:

$$\mu_x = \frac{\alpha}{\beta} \left( \frac{x}{\beta} \right)^{\alpha-1} \quad (2.5.13)$$

The corresponding pdf of random lifetime of newborn,  $T_0$  is

$$f_0(x) = \frac{\alpha}{\beta} \left( \frac{x}{\beta} \right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha} \quad (2.5.14)$$

with the survival function given by

$$S(x) = e^{-\left(\frac{x}{\beta}\right)^\alpha} \quad (2.5.15)$$

Whilst the Weibull law doesn't represent whole life span, due to the specific features of infant and young-adult mortality, it provides a reasonable representation of mortality at adult and old ages. One advantage of the Weibull law is its capability of expressing the statistical quantities of the distribution of the random lifetime of a newborn. The mode (at adult ages), that is the Lexis point, the expected value and the variance of a newborn having Weibull distributed lifetime are the following:

$$\text{Mod}[T_0] = \beta \left( \frac{\alpha - 1}{\alpha} \right)^{\frac{1}{\alpha}} ; \alpha > 1 \quad (2.5.16)$$

$$\mathbb{E}[T_0] = \beta \Gamma \left( \frac{1}{\alpha} + 1 \right) \quad (2.5.17)$$

$$\text{Var}[T_0] = \beta^2 \left[ \Gamma \left( \frac{2}{\alpha} + 1 \right) + \left( \Gamma \left( \frac{1}{\alpha} + 1 \right) \right)^2 \right] \quad (2.5.18)$$

## CHAPTER 3

# MODELLING UNCERTAINTY RISK

This chapter is dedicated to the uncertainty risk modelling in mortality trends. The uncertainty risk originates from the risk of the systematic deviations in estimated mortality, i.e. projected mortality trends. In this aspect, the two stochastic models introduced to the literature by Olivieri (2001), and Olivieri and Pitacco (2009) are the topic of this work. The basic idea behind these models is to consider a set of possible future scenarios on mortality and define a distribution on it. Then the distribution on the scenario set is updated to the experience via Bayesian inferential. In this chapter, these two models will be defined and given in details.

The first model, called as "static" model, takes a definite set of scenarios on future mortality trend and defines a distribution it. As the scenario set, Olivieri (2001) and Olivieri and Pitacco (2003) defined three projected survival functions (obtained from Heligman-Pollard law with different parameters), whilst Olivieri and Pitacco (2002) adopted Weibull model representing the probability distribution of the random life time. They set a number of parameter values, each represents a different mortality scenario. In this work, the latter, i.e. Weibull static model is adopted. While the original model was defined in discrete set of scenarios and worked on one cohort in the portfolio, it is improved in that work to the scenario set on real plane, allowing multiple cohorts in the portfolio. By time, the distribution on the scenario set is updated to the experience via Bayesian inference.

The second model, called as "dynamic" model, was proposed by Olivieri and Pitacco (2009). In their model, they focused on the annual number of deaths

in a given cohort, allowing the random mortality rate. A Poisson model was adopted in modelling the annual number of death with Gamma distributed random parameter, namely random mortality rate. The random mortality itself was considered as time depended. A Bayesian inferential procedure was defined for updating the random mortality rates to the experience in the portfolio, i.e. the parameters defining the mortality process are updated during the inferential procedure. In this work, the dynamic model is adopted as it is in the original model, studying both one and multi-cohort cases.

### 3.1 Static model

Adopting the static model represented by Weibul lifetime distribution, the static model is extended to the parameter set on real plane, i.e. a continuous modelling, allowing the multiple cohorts in the portfolio. The prior distribution defined on the parameter (scenario) set is updated to the experience, i.e. the number of deaths, in the portfolio. Assuming the binomial distribution representing the number of death in the cohorts, the multiple cohorts lead us to adopting some approximation methods for the total number of deaths, to be used in posterior distribution calculations (Sect. 3.1.1). The results of some numerical analysis on the model are found in Sect. 3.1.2.

#### 3.1.1 The model

Assume that the probability distribution of the total lifetime of newborns in year  $y$  is represented by the Weibull model,

$$f(\tau | a(y), b(y)) = \frac{a(y)}{b(y)} \left( \frac{\tau}{b(y)} \right)^{a(y)-1} \exp \left( - \left( \frac{\tau}{b(y)} \right)^{a(y)} \right) \quad (3.1.1)$$

with the Survival function

$$S(\tau | a(y), b(y)) = \exp \left( - \left( \frac{\tau}{b(y)} \right)^{a(y)} \right) \quad (3.1.2)$$

In order to deal with more than one cohort in the model, some assumptions should be adopted. Assume that the mortality of two generation born in consecutive years are linked to each other in a given way, such that

$$a(y) = \Gamma a(y - 1) \quad (3.1.3a)$$

$$b(y) = \Delta b(y - 1) \quad (3.1.3b)$$

with  $\Gamma, \Delta$  are two random variables which can express the hypotheses of rectangularization and expansion in mortality. For simplicity, as building the model, a constant link between generations will be assumed. This simplifies the equations (3.1.3) to

$$a(y) = \gamma a(y - 1) \quad (3.1.4a)$$

$$b(y) = \delta b(y - 1) \quad (3.1.4b)$$

with  $\gamma, \delta > 0$ .

In the model, we refer to a portfolio which starts at time  $t_0$  and requires age  $x_0$  at entry while  $\omega$  represents the maximum attainable age, assumed to be known and common to all the cohorts. Further  $t, t = 0, 1, \dots$  denotes the number of years since the initial (starting) time  $t_0$ .

The model will be set for more than one cohort in the portfolio, from which one cohort case is straight forward to get. It is assumed that one new cohort (generation) enters the portfolio at the beginning of each unit period  $(t - 1, t)$ ,  $t = 0, 1, 2, \dots$ , namely, at time  $t = 0$  the first generation all aged  $x_0$  enters, at time  $t = 1$  the second generation (again all aged  $x_0$ ) enters and so on. In general terms, the generation entering to the portfolio at time  $i - 1$  is the  $i$ -th generation. Recalling that all individuals in these cohorts entered to portfolio at their age  $x_0$ , the birth year of these generations can be listed as  $y_0, y_0 + 1, y_0 + 2, \dots$  where  $y_0 = t_0 - x_0$  defines the birth year of the first generation.

Equation (3.1.1) defines the total lifetime in a generation in terms of the birth year. However, defining the lifetime model in terms of generation number

(or order) provides more efficiency in the calculations. Hence, let's define the parameters of the Weibull model of the first generation (cohort from now on), the one entered to the portfolio at time  $t_0$ , as

$$\begin{aligned} a(y_0) &= a(t_0 - x_0) := a \\ b(y_0) &= b(t_0 - x_0) := b \end{aligned}$$

Using the link defined between cohorts in equations (3.1.4), the Weibull parameters for the  $i$ -th cohort,  $i = 1, 2, \dots$  are obtained as follows:

$$\begin{aligned} a(y_0 + i - 1) &= \gamma^{i-1} a \\ b(y_0 + i - 1) &= \delta^{i-1} b \end{aligned}$$

The total lifetime distribution of  $i$ -th cohort,  $i = 1, 2, \dots$ , in terms of the parameters of the first cohort  $a$  and  $b$  can then be written as

$$f^{(i)}(\tau|a, b) = \frac{\gamma^{i-1} a}{\delta^{i-1} b} \left( \frac{\tau}{\delta^{i-1} b} \right)^{\gamma^{i-1} a - 1} \exp \left( - \left( \frac{\tau}{\delta^{i-1} b} \right)^{\gamma^{i-1} a} \right) \quad (3.1.7)$$

with the survival function

$$S^{(i)}(\tau|a, b) = \exp \left( - \left( \frac{\tau}{\delta^{i-1} b} \right)^{\gamma^{i-1} a} \right) \quad (3.1.8)$$

Note that, conditional on  $a$  and  $b$ , the expected value and the variance of the random lifetime  $T$  having pdf  $f(\tau|a, b)$ , are respectively given by

$$\mathbb{E}[T|a, b] = \int_0^\infty \tau f(\tau|a, b) d\tau \quad (3.1.9)$$

$$\mathbb{V}\text{ar}[T|a, b] = \int_0^\infty (\tau - \mathbb{E}[T])^2 f(\tau|a, b) d\tau \quad (3.1.10)$$

On the other hand, the unconditional expected value and the variance are respectively

$$\begin{aligned} \mathbb{E}[T] &= \int_0^\infty \tau f(\tau) d\tau \\ &= \int_0^\infty \iint \tau f(\tau|a, b) h(a, b) da db d\tau \end{aligned} \quad (3.1.11)$$

$$\begin{aligned}\mathbb{V}\text{ar}[T] &= \int_0^\infty (\tau - \mathbb{E}[T])^2 f(\tau) d\tau \\ &= \int_0^\infty \iint (\tau - \mathbb{E}[T])^2 f(\tau|a, b) h(a, b) da db d\tau\end{aligned}\quad (3.1.12)$$

where  $h(a, b)$  is the distribution function defined on the parameters  $a$  and  $b$  (will be specified in next paragraphs). As a well known relation, the following result holds for the variance:

$$\mathbb{V}\text{ar}[T] = \mathbb{E}[\mathbb{V}\text{ar}[T|a, b]] + \mathbb{V}\text{ar}[E[T|a, b]] \quad (3.1.13)$$

where the first term on the right-hand side represents random fluctuations around the expected values, whilst the second one expresses systematic deviations from the expected ones, namely represents the systematic risk.

What is defined until now is the total lifetime distribution of the individuals in cohorts, each refers to a specific birth year, regardless referring to whether or not being alive at a specific time during the portfolio. Although  $i$ -th cohort,  $i = 1, 2, \dots$ , refers to a specific birth year, the individuals in the cohort get older during the portfolio. Hence, the total lifetime of an individual in the  $i$ -th cohort,  $i = 1, 2, \dots$ , given that alive at aged  $x$  is defined as

$$f_x^{(i)}(\tau|a, b) = \frac{f^{(i)}(\tau|a, b)}{S^{(i)}(x|a, b)} \quad (3.1.14)$$

Referring to a generic time interval  $(t-1, t)$ , the total number of individuals at the beginning of period and the total number of deaths at the end are respectively given by the equations

$$n_{t-1} = \sum_{i=1}^t n_{x_0+t-i, t-1} \quad (3.1.15)$$

$$d_t = \sum_{i=1}^t d_{x_0+t-i, t} \quad (3.1.16)$$

where  $n_{x_0+t-i, t-1}$  is the number of survivals in the  $i$ -th cohort at the beginning of period and  $d_{x_0+t-i, t}$  is the number of deaths in the same cohort at the end of the period.

Assume that the (conditional) number of deaths in cohort  $i$ ,  $i = 1, 2, \dots, t$ , has binomial distribution, i.e.

$$[D_{x_0+t-i,t}|a, b] \sim \text{Bin}(n_{x_0+t-i,t-1}, q_{x_0+t-i,t}) \quad (3.1.17)$$

where

$$q_{x_0+t-i,t} = 1 - S_{x_0+t-i}^{(i)}(x_0 + t - i + 1|a, b) \quad (3.1.18)$$

is the probability of dying of individuals in the  $i$ -th cohort during the period  $[t - 1, t]$ . So the probability of the total number of deaths  $D_t$  is given by

$$\begin{aligned} \mathbb{P}[D_t = d_t|a, b] &= \sum_{j_1=0}^{d_t} \mathbb{P}[D_{x_0+t-1,t} = j_1|a, b] \sum_{j_2=0}^{d_t-j_1} \mathbb{P}[D_{x_0+t-2,t} = j_2|a, b] \\ &\quad \dots \sum_{j_t=0}^{d_t-\sum_{i=1}^{t-1} j_i} \mathbb{P}[D_{x_0,t} = j_t|a, b] \end{aligned} \quad (3.1.19)$$

As said at the beginning of the section, the static model assumes a distribution on the scenario set and then updates it to the experience in the portfolio. In the model, each parameter values  $a$  and  $b$  of the total lifetime, equation (3.1.7), defines a scenario for the future mortality and the distribution on the parameter set is updated via a Bayesian inference from starting time on.

At time  $t = 0$ , starting with one cohort having the Weibull lifetime distribution  $f^{(1)}(\tau|a, b)$ , assume that the parameters are random variables,  $\tilde{a}$  and  $\tilde{b}$ , with joint prior distribution  $h(a, b)$ . At time  $t = 1$ , after getting the observation on total number of deaths  $D_1 = d_1$  from the first period, the (posterior) distribution of parameters is obtained by

$$h(a, b|d_1) \propto h(a, b)l_1(a, b|d_1) \quad (3.1.20)$$

where  $l_1(a, b|d_1)$  is the likelihood of the observation  $D_1 = d_1$ . Moving to the end of the second period, i.e. time  $t = 2$ , the second period observation  $D_2 = d_2$  gets available. Then the posterior distribution of the parameters is obtained by

$$h(a, b|d_1, d_2) \propto h(a, b|d_1)l_2(a, b|d_1, d_2) \quad (3.1.21)$$

where  $h(a, b|d_1)$  is the prior distribution of the parameters at time 1, which have already been calculated via equation (3.1.20), and  $l_2(a, b|d_1, d_2)$  is the likelihood of the observation in the second period  $D_2 = d_2$ . Substitution of (3.1.20) into (3.1.21) yields

$$h(a, b|d_1, d_2) \propto h(a, b)l_1(a, b|d_1)l_2(a, b|d_1, d_2) \quad (3.1.22)$$

Following the same steps until the period  $t$ , the posterior distribution of parameters is defined by

$$h(a, b|d_1, \dots, d_t) \propto h(a, b) \prod_{j=1}^t l_j(a, b|d_1, \dots, d_j) \quad (3.1.23)$$

The (posterior) predictive pdf  $f_x^{(i)}(\tau|d_1, \dots, d_t)$  of an individual in cohort  $i$  aged  $x$  is then calculated as

$$f_x^{(i)}(\tau|d_1, \dots, d_t) = \frac{\iint f_x^{(i)}(\tau|a, b)h(a, b) \prod_{j=1}^t l_j(a, b|d_1, \dots, d_j) da db}{\iint h(a, b) \prod_{j=1}^t l_j(a, b|d_1, \dots, d_j) da db} \quad (3.1.24)$$

In Bayesian inferential model defined above, in equation (3.1.23), the initial (time 0) prior distribution of parameters,  $h(a, b)$ , and the distribution of the total number of deaths  $D_t$ , need to be specified. Let's assume a uniform prior distribution on the set  $R = [a_l, a_r] \times [b_l, b_r]$  i.e.

$$h(a, b) = \begin{cases} \frac{1}{(a_r - a_l)(b_r - b_l)} & \text{if } (a, b) \in R \\ 0 & \text{otherwise} \end{cases} \quad (3.1.25)$$

Uniform prior distribution of parameters originates from a serious lack of information about mortality evaluation.

The equation (3.1.19) defines the true distribution of the total number of deaths. However this distribution is too complicated to work with, which leads us using some approximated distributions. Assuming that the number of deaths in each cohorts has binomial distribution, given in (3.1.17), the total number

of deaths is simply sum of the binomial distributions, i.e. (3.1.16). Butler and Stephen (1993) examined two approximation methods to the distribution of sum of binomial distributions in their research paper. These methods enable that the total number of deaths  $D_t$  might approximated using

1. Pearson-family type approximation, and
2. Kolmogorov-type approximation.

Additional to these two, another approximation method can be defined by using Poisson distribution as an approximation to the binomial distribution of the number deaths in the cohorts  $D_{x_0+t-i,t}$ ,  $i = 1, 2, \dots$ :

3. Poisson approximation

In the following are described the approximations and how the likelihood is calculated under each one is provided.

### **Pearson-family type approximation**

Pearson-family type approximation is based on the technique used to approximate sums of continuous random variables, which means to find the first four moments of the sum, and then to fit a Pearson curve. Suppose that the total number of deaths  $D_t^*$  is a continuous random variable with a distribution in the Pearson family, and suppose  $D_t^*$  and  $D_t$  have the same first four moments or cumulants. Then  $\mathbb{P}(D_t \leq d)$ , where  $d$  is an integer, is approximated by  $\mathbb{P}(D_t^* \leq d+0.5)$  which can be calculated by fitting Pearson curves.

Using the first four cumulants of a binomial distribution, i.e.  $\kappa_1 = np$ ,  $\kappa_2 = npq$ ,  $\kappa_3 = npq(q-p)$  and  $\kappa_4 = npq(1-6pq)$  with index  $n$  and  $q$  of the binomial distribution, the cumulants of the total number of deaths  $D_t$  are given as

$$\begin{aligned}
 {}_t\kappa_1 &= \sum_{i=1}^t n_{x_0+t-i,t-1} q_{x_0+t-i,t} \\
 {}_t\kappa_2 &= \sum_{i=1}^t n_{x_0+t-i,t-1} q_{x_0+t-i,t} (1 - q_{x_0+t-i,t}) \\
 {}_t\kappa_3 &= \sum_{i=1}^t n_{x_0+t-i,t-1} q_{x_0+t-i,t} (1 - 2q_{x_0+t-i,t}) \\
 {}_t\kappa_4 &= \sum_{i=1}^t n_{x_0+t-i,t-1} q_{x_0+t-i,t} (1 - 6q_{x_0+t-i,t}(1 - q_{x_0+t-i,t}))
 \end{aligned}$$

From the cumulants, the skewness and the kurtosis are found as  $\sqrt{(\beta_1)} = \kappa_3/\kappa_2^{3/2}$  and  $\beta_2 = \kappa_4/\kappa_2^2$ , respectively. Then the first four moments calculated are used to fit a Pearson curve.

Although the approximation defined seems very easy and applicable, Pearson curve fitting might not always result in positive, which is the case for our portfolio. As said before, the total number of deaths in the portfolio is defined as the sum of the binomial distributions and the terms in the summation increases over time since a new cohort joins to the portfolio at the beginning of each time interval. Except the first time interval (total number of deaths has binomial distribution because there is just one cohort), no Pearson curve which fits to the total number of deaths has been found.

### **Kolmogorov-type approximation**

In their paper, Butler and Stephens (1993) provided an approximation method based on the idea of taking multiples of differences of probabilities aiming to match the moments of the true and approximating distribution. The method allows any discrete distribution to be approximated by any other 'more easily calculated' distribution with the only requirement of the existence of the moment of the true distribution, at least up to some order. The Kolmogorov approximation is summarized in Appendix A together with its algorithm (reference to the paper of Butler and Stephen). Furthermore, in the appendix are provided the evaluations needed to calculate the moments of the sum of the random variables (true distribution).

### **Poisson approximation**

Lets approximate the number of deaths in a cohort with Poisson distribution. Namely, in period  $[j - 1, j], j = 1, 2, \dots$ , let the number of deaths in cohort  $i, i = 1, \dots, j$  have Poisson distribution i.e.

$$[D_{x_0+j-i,j}|a, b] \sim \text{Poi}(n_{x_0+j-i,j-1} q_{x_0+j-i,j}|a, b) \quad (3.1.26)$$

where  $n_{x_0+j-i,j-1}$  is the number of survivals in the  $i$ -th cohort at the beginning of period and  $q_{x_0+j-i,j}|_{a,b}$  is the death probability for the individuals in the cohort  $i$  in period  $[j-1, j]$ , which is defined from the equation (2.4.18) as

$$q_{x_0+j-i,j}|_{a,b} = 1 - S_{x_0+j-i}^{(i)}(x_0 + j - i + 1|a, b) \quad (3.1.27)$$

Then the total number of deaths  $D_j$  is Poisson distributed as well, i.e.

$$[D_j|a, b] \sim \text{Poi} \left( \sum_{i=1}^j n_{x_0+j-i,j-1} q_{x_0+j-i,j}|_{a,b} \right) \quad (3.1.28)$$

So the likelihood function in posterior distribution (3.1.23) is calculated as the following:

$$\begin{aligned} \prod_{j=1}^t l_j(a, b|d_1, \dots, d_j) &= \prod_{j=1}^t \mathbb{P}(D_j = d_j|a, b, d_1, \dots, d_{j-1}) \\ &= \prod_{j=1}^t \frac{\Lambda^{d_j} e^{-\Lambda}}{d_j!} \end{aligned} \quad (3.1.29)$$

where  $\Lambda = \sum_{i=1}^j n_{x_0+j-i,j-1} q_{x_0+j-i,j}|_{a,b,d_1, \dots, d_{j-1}}$

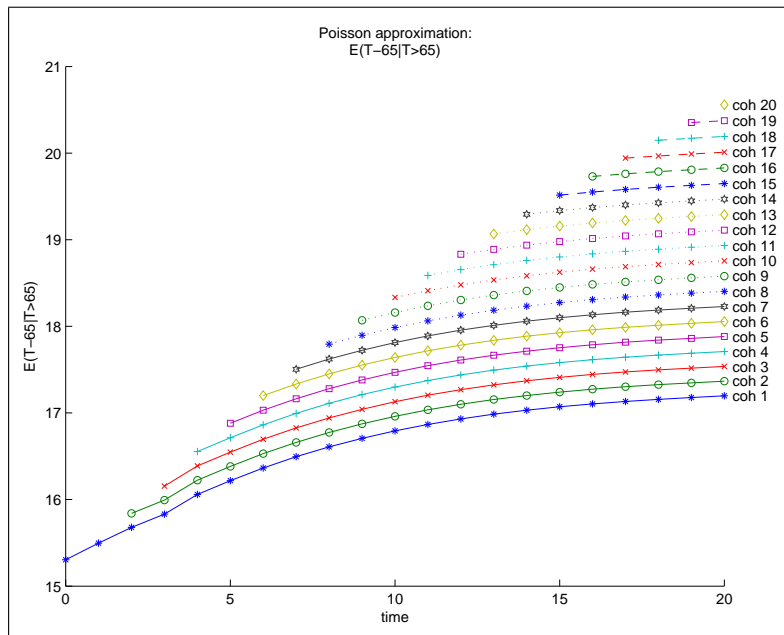
### 3.1.2 Numerical results

In this section, we will provide the numerical analysis of the static model. The analysis have been performed for both Poisson and Kolmogorov approximations. Considering that Kolmogorov approximation is an approximation method to the true distribution of the sum of the random variables having Binomial distribution, this approximation is not applicable to the one cohort case. Hence, only the multiple cohorts has been analysed for this approximation type. On the other hand, the Poisson approximation is studied under both one cohort and multiple cohorts.

The analysis are based on the calculation of the posterior quantities  $\mathbb{E}(T - 65|T > 65)$ ,  $\text{Var}(T - 65|T > 65)$ ,  $\mathbb{E}(\text{Var}(T - 65|T > 65; a, b))$ ,  $\text{Var}(\mathbb{E}(T - 65|T > 65; a, b))$  and  $\text{Mode}(T)$ . The inputs used in the calculations are  $x_0 = 65$ ,  $\omega = 115$ ,

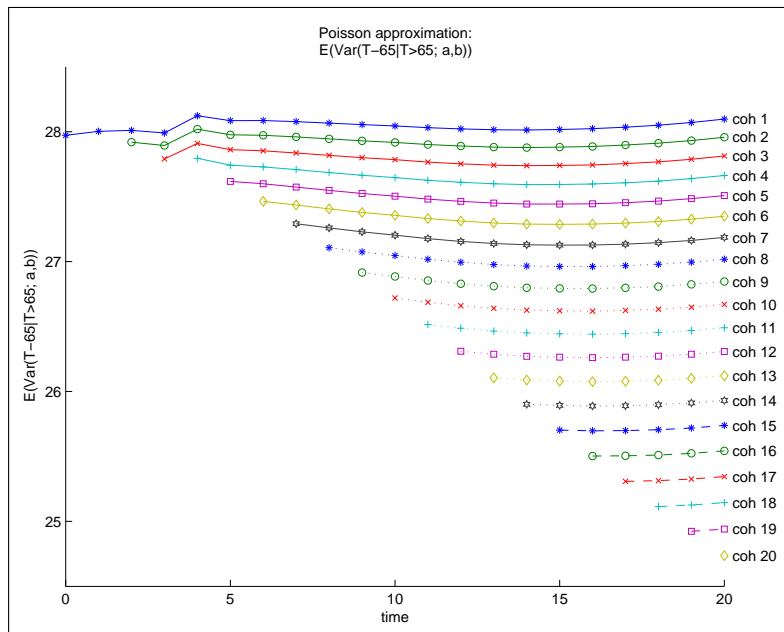
$\gamma = 1.008$ ,  $\delta = 1.002$ . The values  $\gamma$  and  $\delta$ , representing the link between cohorts who have been born in consecutive years, are based on expert judgment, such as they are believed to express the hypotheses of rectangularization and expansion in mortality. Besides, the scenario set, i.e. the set of Weibull parameters  $(a, b)$ , is defined on the set  $[15, 19] \times [80, 85] \subset \mathbb{R}$ . Furthermore, in multiple cohorts, it is assumed that number of individuals entering the portfolio at the beginning of each period are constant for each year, namely, at each period, a fix number of people, all age  $x_0 = 65$ , enters to the portfolio. In order to see the effect of the cohort sizes, two cases are studied, which are  $n_{x_0, j-1} = 100$  and  $n_{x_0, j-1} = 1000$  where  $j = 1, 2, \dots$ .

Starting with the Poisson approximation, in Figs 3.1 – 3.5, it is assumed that the annual number of deaths observed in the cohorts in each year is equal to expected number of deaths in the cohorts, i.e.  $d_{x,j} = n_{x,j-1}q_{x,j}$ . Moreover, in these figures is assumed  $n_{x_0, j-1} = 1000$ . The graphical results of the expected values and modes of the lifetimes, Figs 3.1 and 3.5 prove the effect of expansion in each cohort by the experienced observations. The quantity  $\mathbb{E}(\text{Var}(T - 65|T > 65; a, b))$ , being the component of the variance representing the random fluctuations, follows a stable behaviour for each cohort during the portfolio even though shows some fluctuations at the beginning. The variances and its components  $\text{Var}(\mathbb{E}(T - 65|T > 65; a, b))$ , representing the uncertainty risk, consistently reducing by time. This might be considered as the effect of learning process in inference procedure and it proves that the static model captures the uncertainty risk pretty well. Similar results are found when  $n_{x_0, j-1} = 100$  is assumed (See Fig. B.1 in Appendix B). While some fluctuations (larger with respect to the case  $n_{x_0, j-1} = 1000$ ) are observed at beginning periods, resulting from the smaller size of the individuals entering to the portfolio, the quantities get stable for all cohorts in the next periods. The size effect on the fluctuations happened during the beginning periods can be seen better when the mode of the random life time is concerned (Fig B.1e)



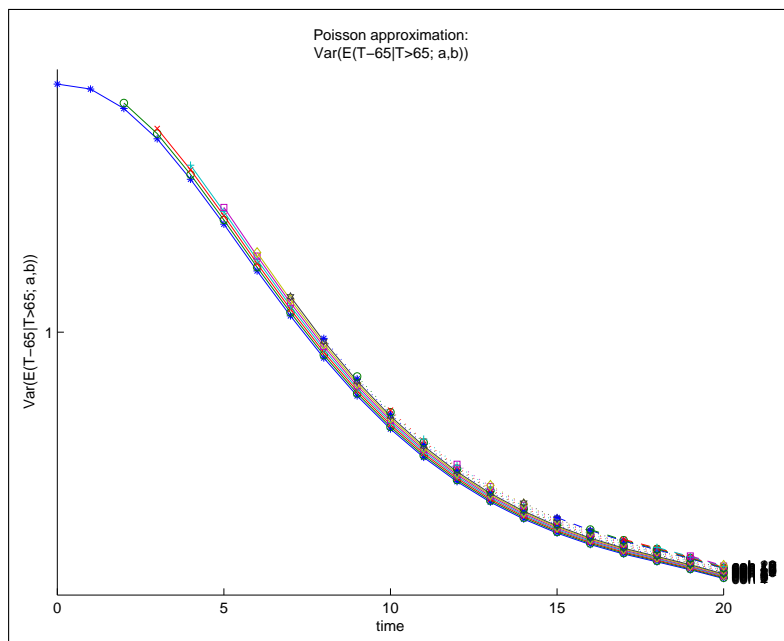
**Figure 3.1:** Expected lifetimes under Poisson approximation:

$n_{x_0, j-1} = 1000$ ,  $d_{x, j} = n_{x, j-1}q_{x, j}$ , multiple cohorts



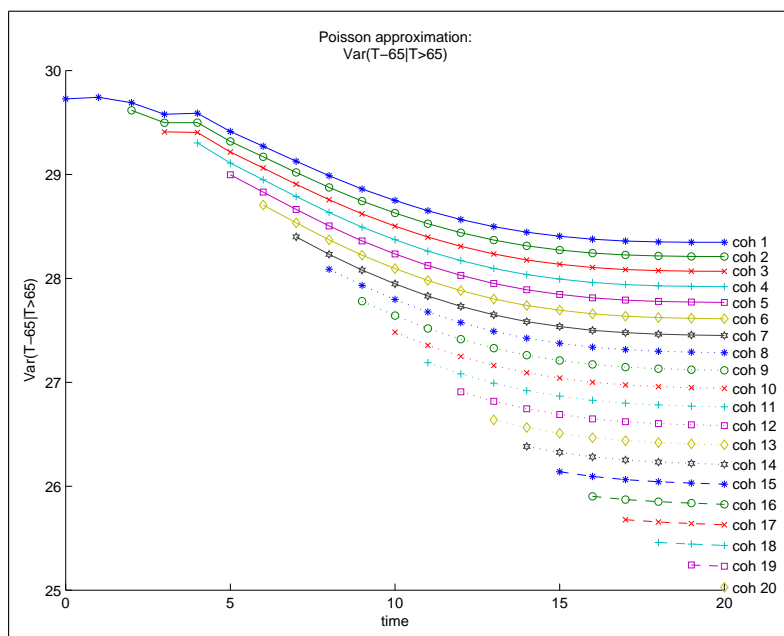
**Figure 3.2:** Random fluctuations under Poisson approximation:

$n_{x_0, j-1} = 1000$ ,  $d_{x, j} = n_{x, j-1}q_{x, j}$ , multiple cohorts



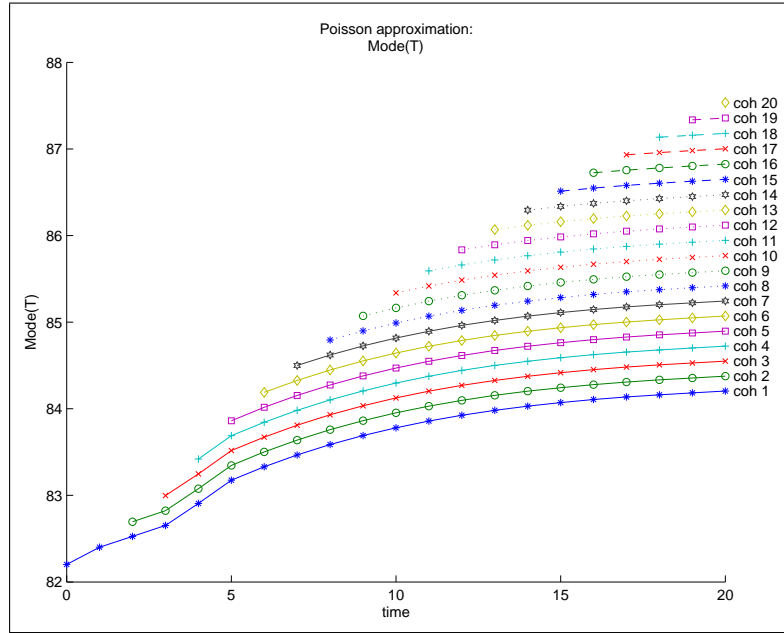
**Figure 3.3:** Uncertainty risk under Poisson approximation:

$n_{x_0,j-1} = 1000$ ,  $d_{x,j} = n_{x,j-1}q_{x,j}$ , multiple cohorts



**Figure 3.4:** Variance of the lifetime under Poisson approximation:

$n_{x_0,j-1} = 1000$ ,  $d_{x,j} = n_{x,j-1}q_{x,j}$ , multiple cohorts

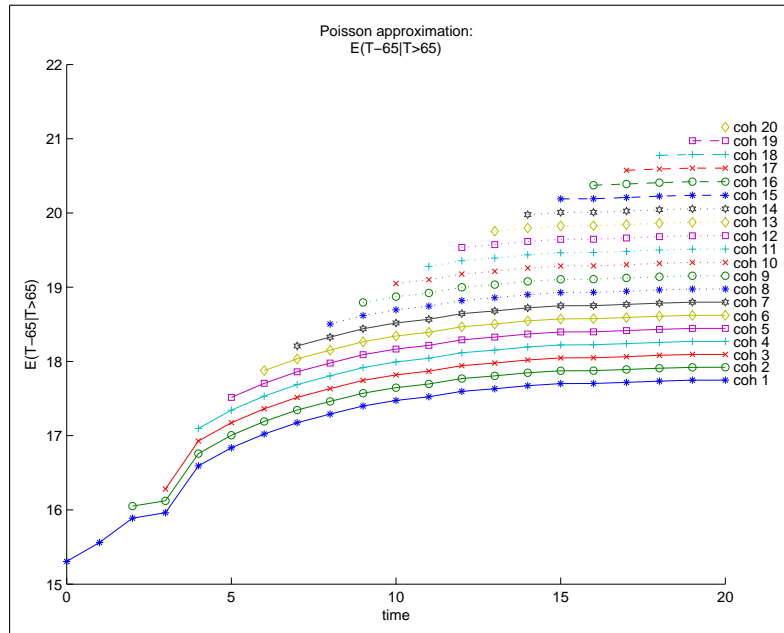


**Figure 3.5:** Lexis points under Poisson approximation:

$$n_{x_0, j-1} = 1000, d_{x, j} = n_{x, j-1} q_{x, j}, \text{ multiple cohorts}$$

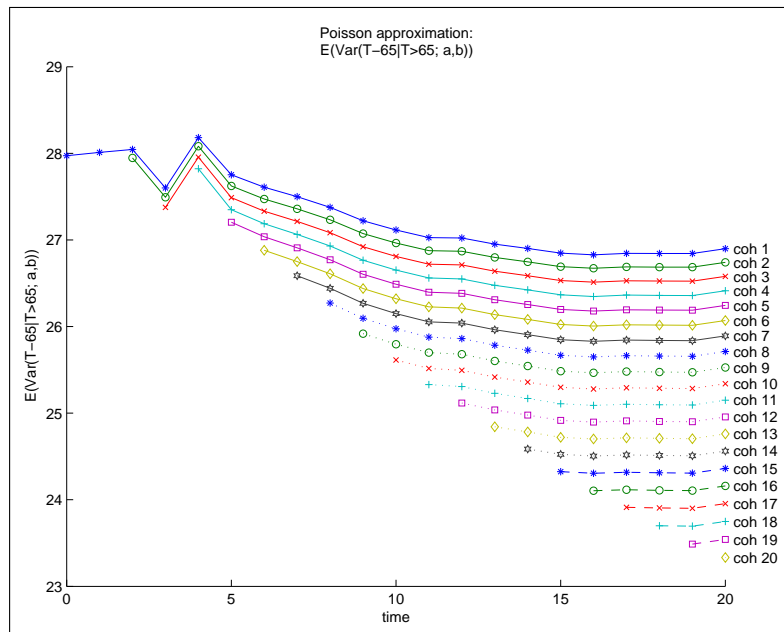
In Figs. 3.6 – 3.10 and Figs. 3.11 – 3.15 is assumed respectively that the annual number of deaths observed in the cohorts in each year is 25% lower than the expected number of deaths, i.e.  $d_{x, j} = 0.75 n_{x, j-1} q_{x, j}$ , and 25% higher than the expected number of deaths, i.e.  $d_{x, j} = 1.25 n_{x, j-1} q_{x, j}$ . While some fluctuations in the quantity values occur at the beginning periods, especially when the mode and  $\mathbb{E}(\text{Var}(T - 65 | T > 65; a, b))$  values are concerned, the quantities get stable in next periods, which might be considered the effect of learning process in inferential as well as the fact that the size of the portfolio increases due to the new enters over time. To give a special attention to the quantity  $\text{Var}(\mathbb{E}(T - 65 | T > 65; a, b))$ , representing the uncertainty risk, under the observation of 25% higher number of deaths than the expected ones, quantity fluctuates more at the beginning periods respect to the 25% lower observations case.

The Figs 3.16 – 3.20 provides a comparison on the size of the cohorts entering to the portfolio. Assuming that expected number of deaths is observed in the



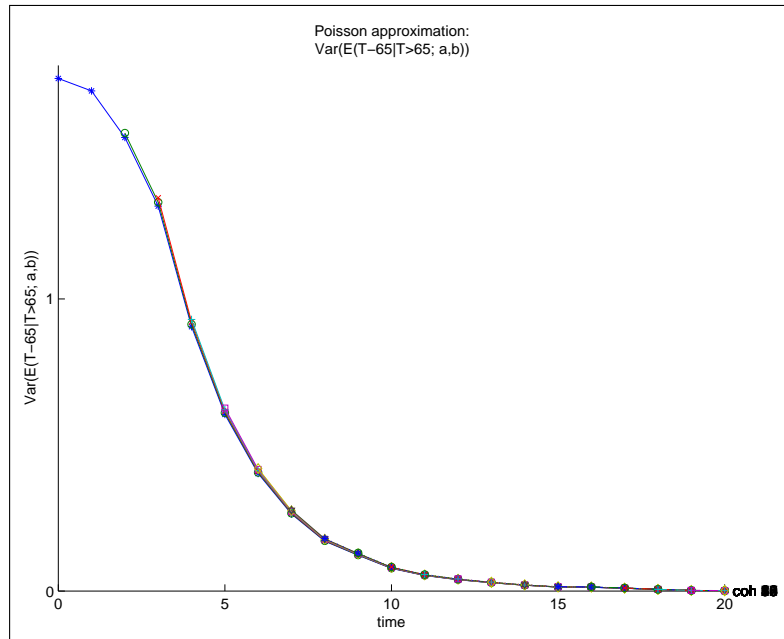
**Figure 3.6:** Expected lifetime under Poisson approximation:

$n_{x_0,j-1} = 1000$ ,  $d_{x,j} = 0.75 n_{x,j-1} q_{x,j}$ , multiple cohorts



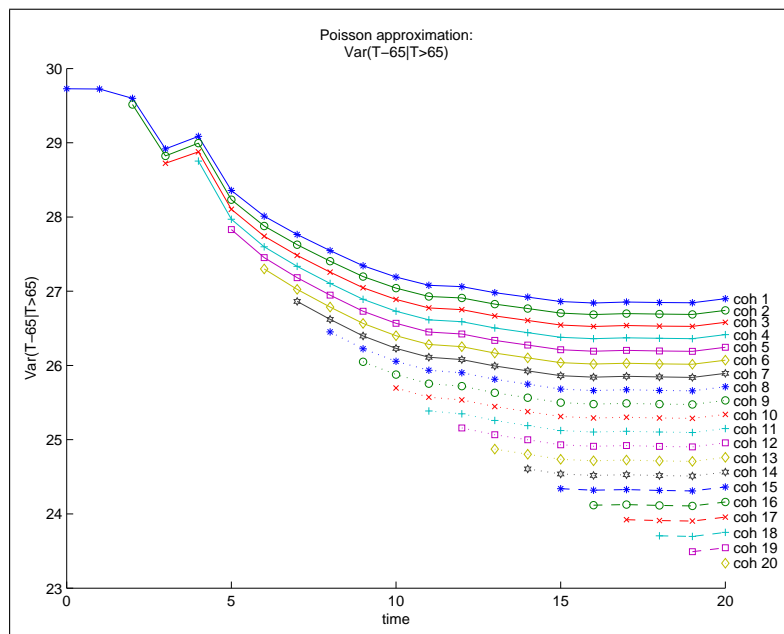
**Figure 3.7:** Random fluctuations under Poisson approximation:

$n_{x_0,j-1} = 1000$ ,  $d_{x,j} = 0.75 n_{x,j-1} q_{x,j}$ , multiple cohorts



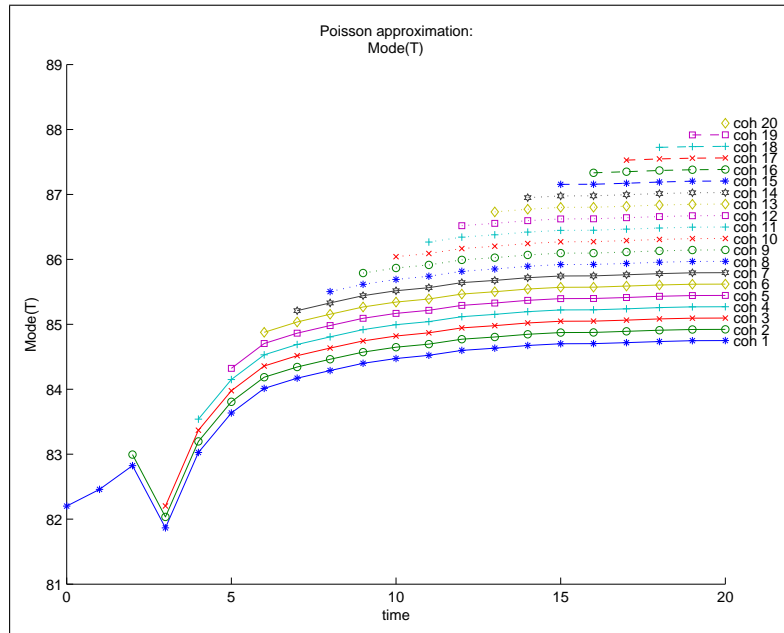
**Figure 3.8:** Uncertainty risk under Poisson approximation:

$n_{x_0,j-1} = 1000, d_{x,j} = 0.75 n_{x,j-1} q_{x,j}$ , multiple cohorts



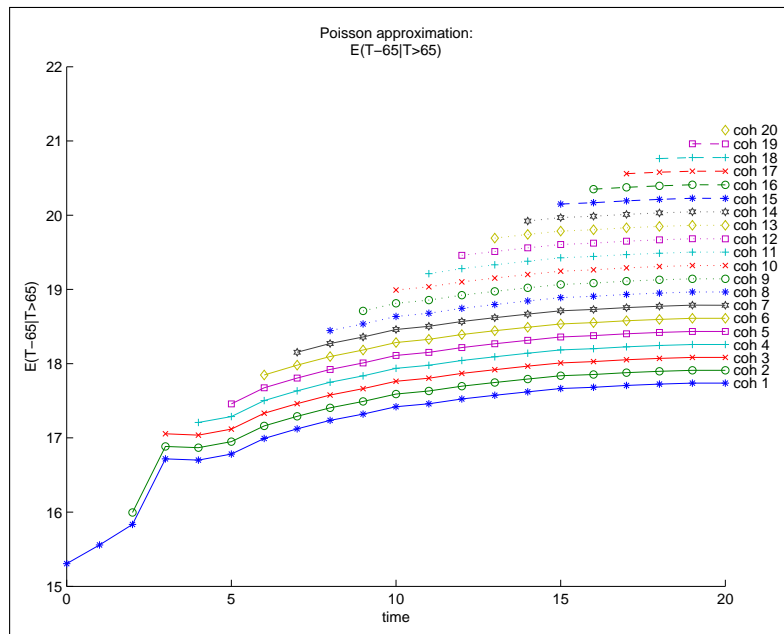
**Figure 3.9:** Variance of the lifetime under Poisson approximation:

$n_{x_0,j-1} = 1000, d_{x,j} = 0.75 n_{x,j-1} q_{x,j}$ , multiple cohorts



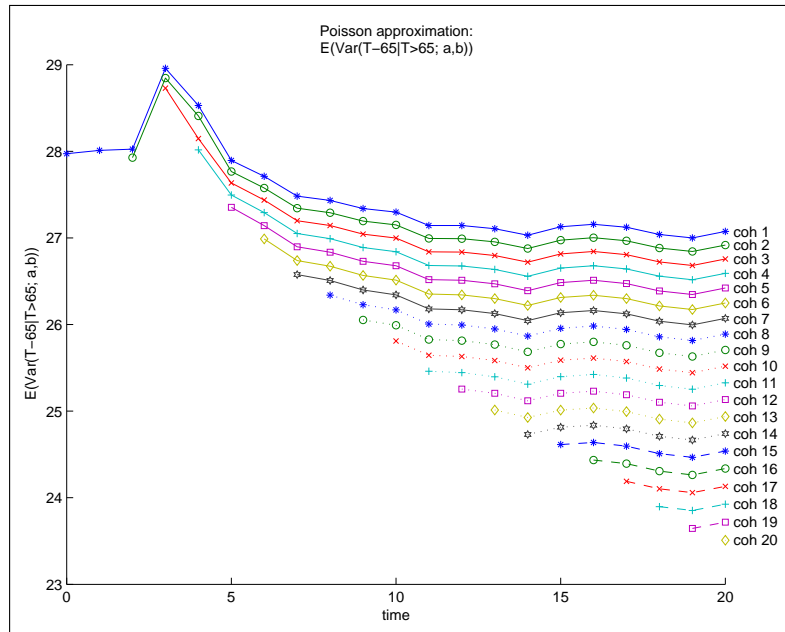
**Figure 3.10:** Lexis point under Poisson approximation:

$n_{x_0,j-1} = 1000$ ,  $d_{x,j} = 0.75 n_{x,j-1} q_{x,j}$ , multiple cohorts



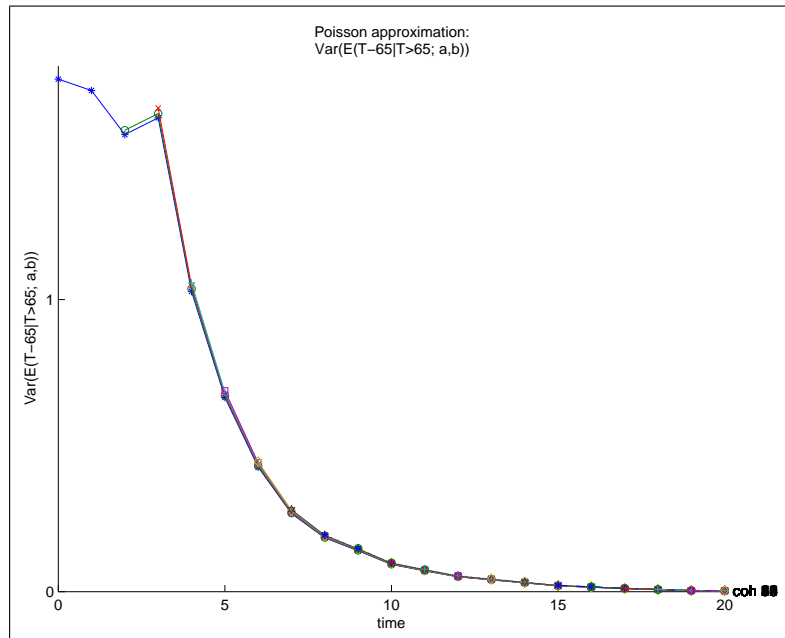
**Figure 3.11:** Expected lifetime under Poisson approximation:

$n_{x_0,j-1} = 1000$ ,  $d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ , multiple cohorts



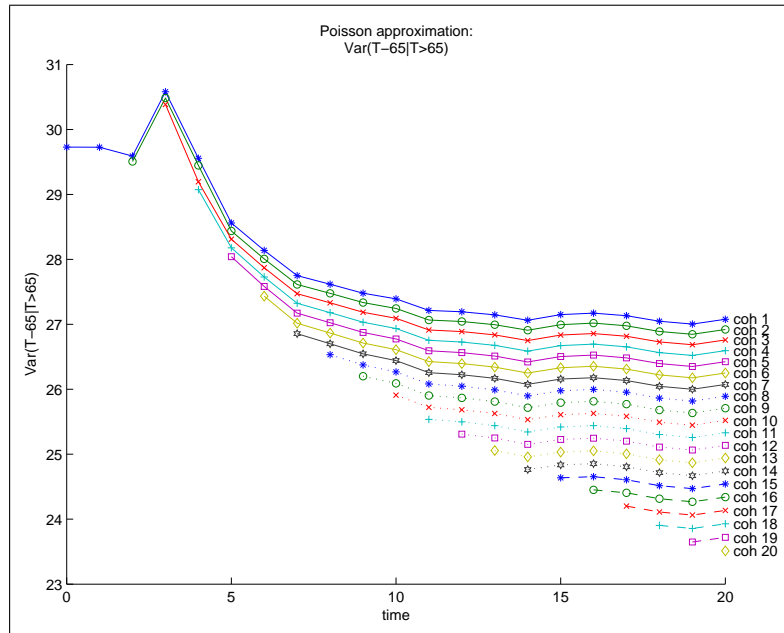
**Figure 3.12:** Random fluctuations under Poisson approximation:

$n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ , multiple cohorts



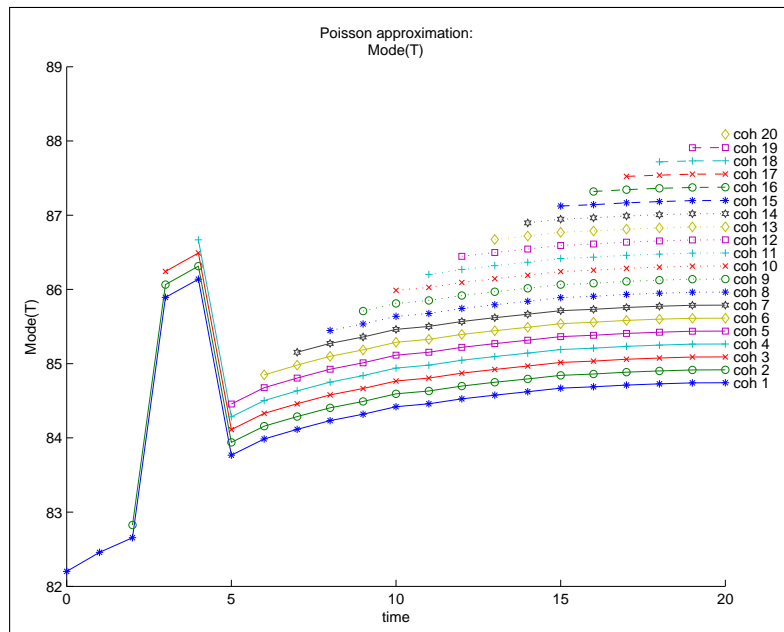
**Figure 3.13:** Uncertainty risk under Poisson approximation:

$n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ , multiple cohorts



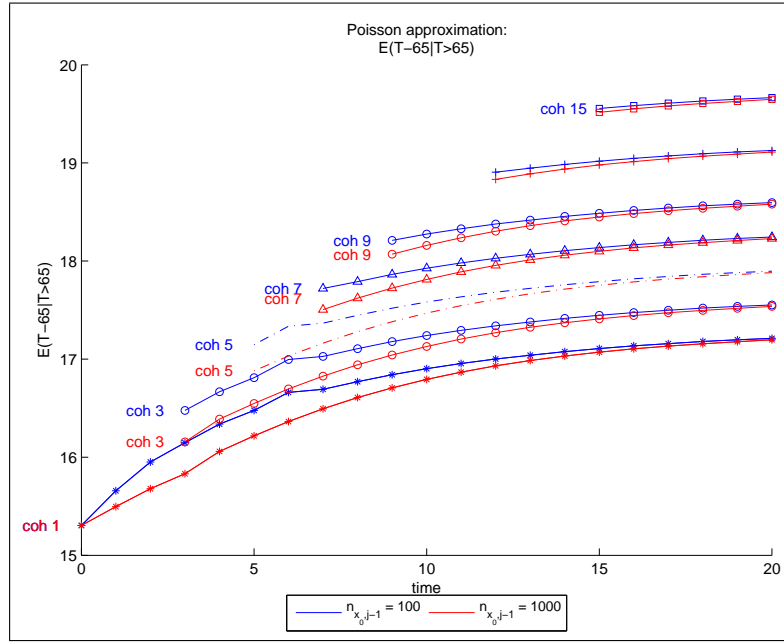
**Figure 3.14:** Variance of the lifetime under Poisson approximation:

$n_{x_0,j-1} = 1000$ ,  $d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ , multiple cohorts



**Figure 3.15:** Lexis point under Poisson approximation:

$n_{x_0,j-1} = 1000$ ,  $d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ , multiple cohorts

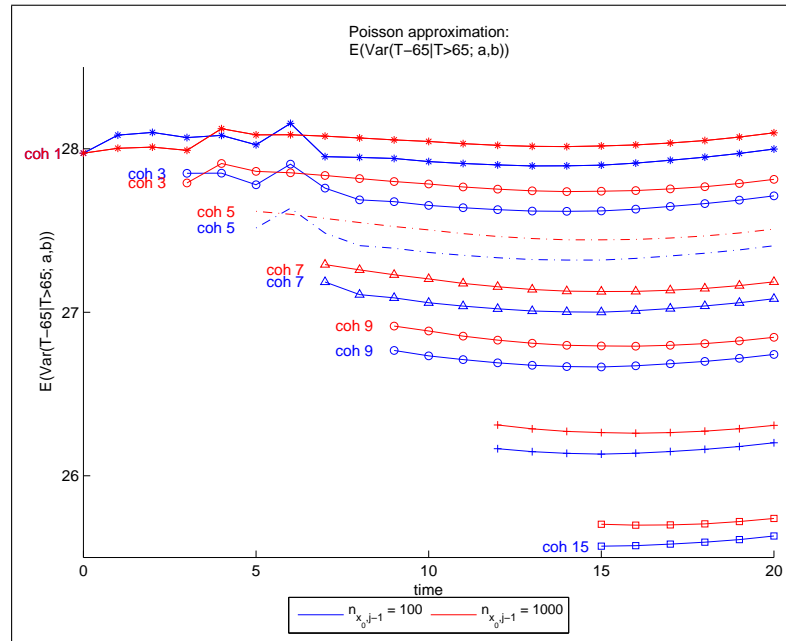


**Figure 3.16:** Expected lifetime under Poisson approximation:

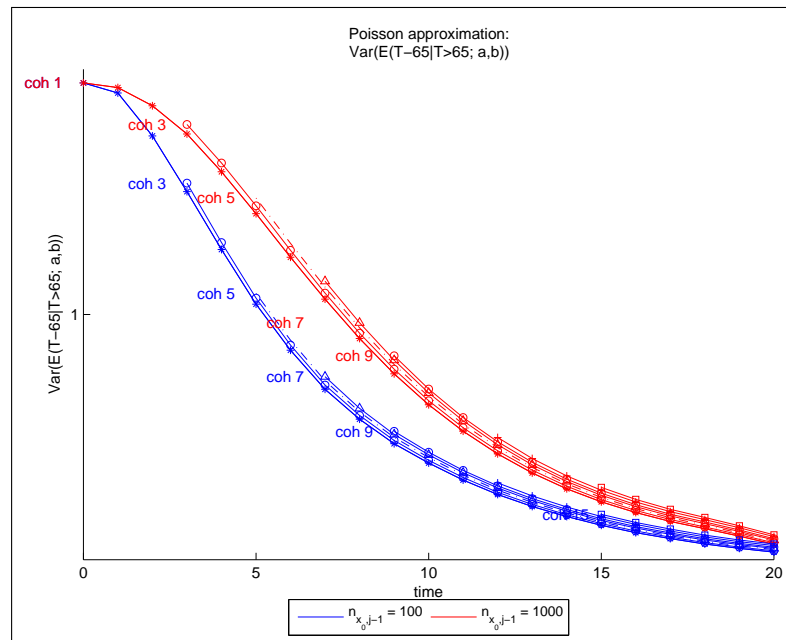
$n_{x_0, j-1} = 100, 1000$ ,  $d_{x, j} = n_{x, j-1}q_{x, j}$ , multiple cohorts

portfolio, the results for  $n_{x_0, j-1} = 100$ ,  $n_{x_0, j-1} = 1000$  show similar pattern in all quantities, with higher values when higher number of individuals joins to portfolio. Considering that the results regarding to the observations 25% lower and higher number of deaths than expected ones are quite similar, we provided them in Fig. B.4 in Appendix B.

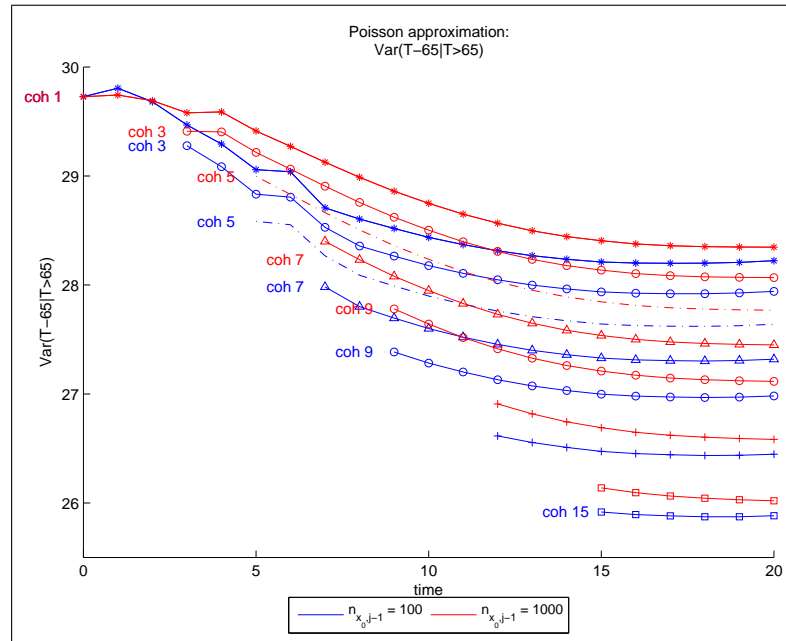
The graphical analysis of the Kolmogorov approximation are given in Appendix B (See Figs. B.6 – B.8) considering that they have quite similar, almost exact, results as the Poisson approximation, which can be clearly seen in the Figs. 3.21 – 3.25. As described in Appendix A, the Kolmogorov approximation is an approximation method to the true distribution of the total number of deaths in the sense that the moments of the approximated and the true distributions match. From this definition, we can conclude that Poisson distribution also approximates to the true distribution of the total number of deaths as much as Kolmogorov approximation does.



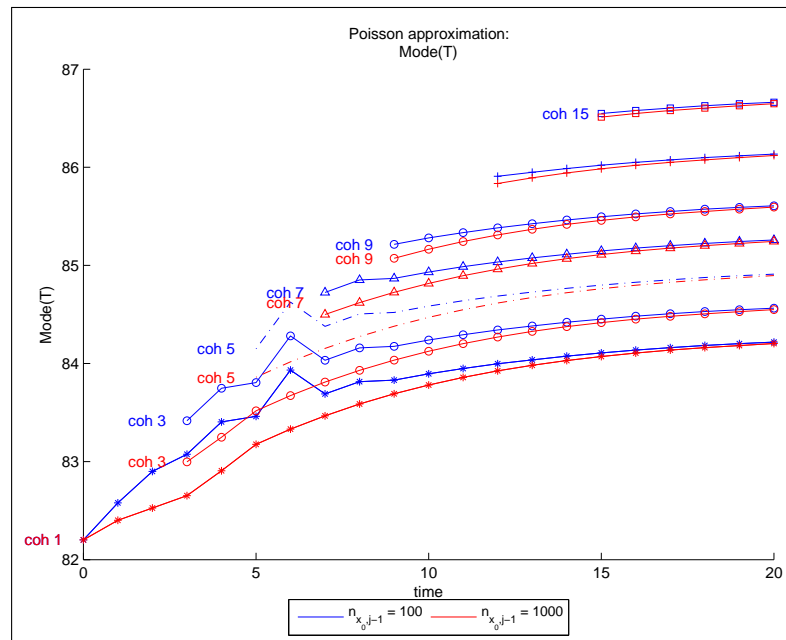
**Figure 3.17:** Random fluctuations under Poisson approximation:  
 $n_{x_0, j-1} = 100, 1000, d_{x,j} = n_{x,j-1}q_{x,j}$ , multiple cohorts



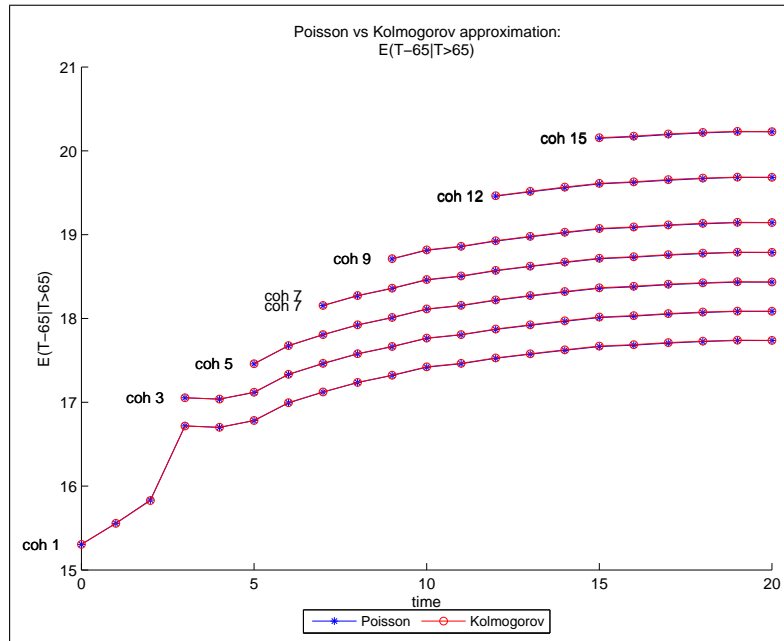
**Figure 3.18:** Uncertainty risk under Poisson approximation:  
 $n_{x_0, j-1} = 100, 1000, d_{x,j} = n_{x,j-1}q_{x,j}$ , multiple cohorts



**Figure 3.19:** Variance of the lifetime under Poisson approximation:  
 $n_{x_0, j-1} = 100, 1000$ ,  $d_{x, j} = n_{x, j-1}q_{x, j}$ , multiple cohorts

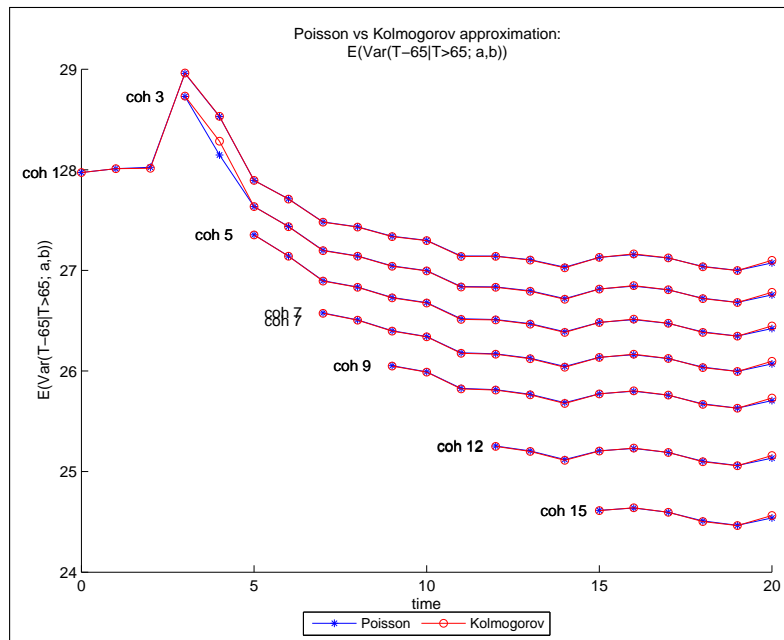


**Figure 3.20:** Lexis points under Poisson approximation:  
 $n_{x_0, j-1} = 100, 1000$ ,  $d_{x, j} = n_{x, j-1}q_{x, j}$ , multiple cohorts



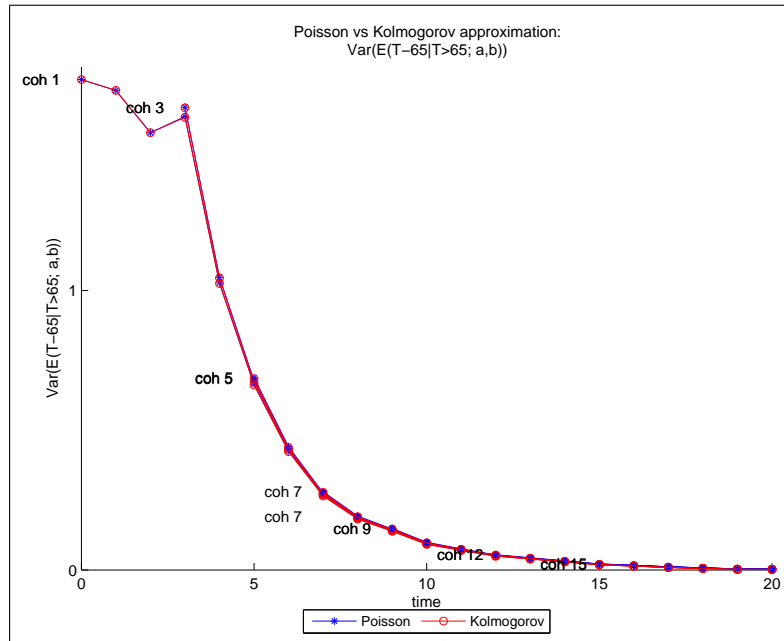
**Figure 3.21:** Expected lifetime: Poisson vs Kolmogorov appr.

$$n_{x_0, j-1} = 1000, d_{x, j} = 1.25 n_{x, j-1} q_{x, j}$$



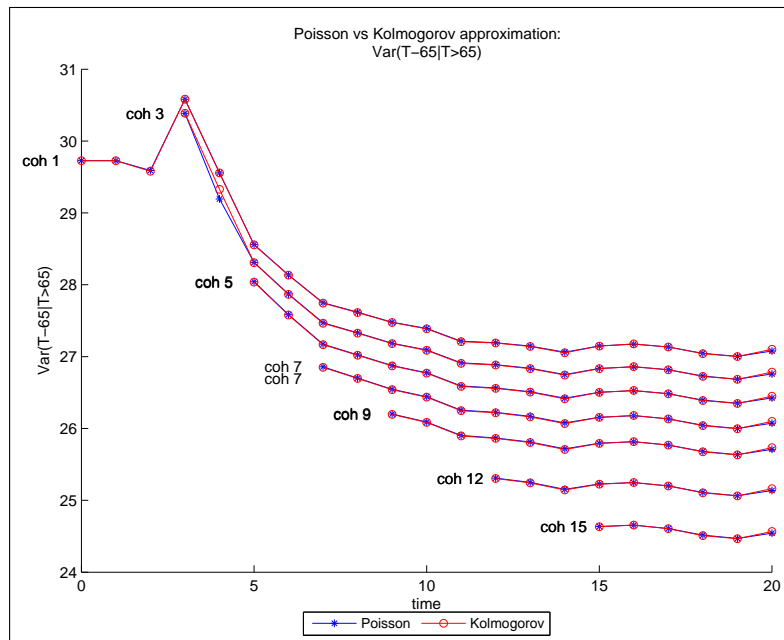
**Figure 3.22:** Random fluctuations: Poisson vs Kolmogorov appr.

$$n_{x_0, j-1} = 1000, d_{x, j} = 1.25 n_{x, j-1} q_{x, j}$$



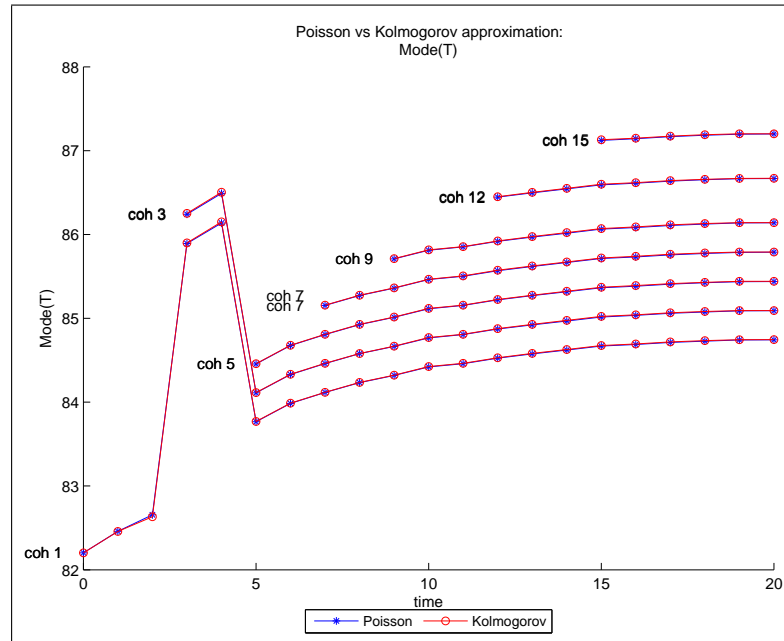
**Figure 3.23:** Uncertainty risk: Poisson vs Kolmogorov appr.

$$n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$$



**Figure 3.24:** Variance of the lifetime: Poisson vs Kolmogorov appr.

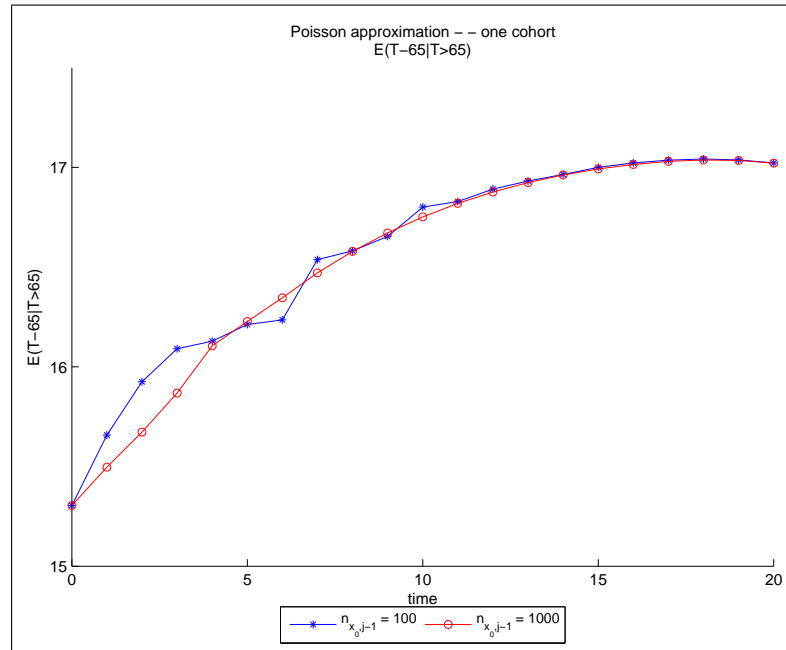
$$n_{x_0,j-1} = 1000, d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$$



**Figure 3.25:** Lexis points: Poisson vs Kolmogorov aprx.

$$n_{x_0, j-1} = 1000, d_{x, j} = 1.25 n_{x, j-1} q_{x, j}$$

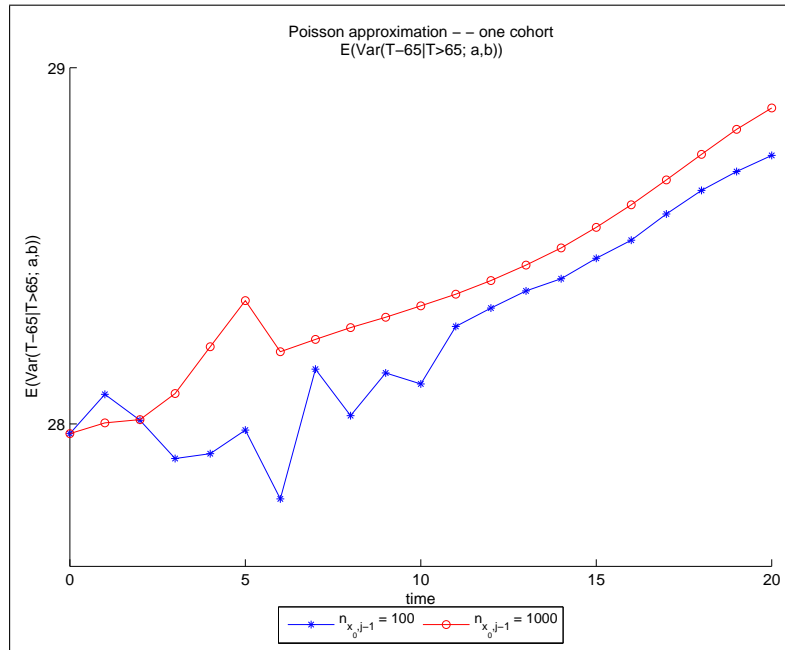
One cohort case has been studied under Poisson approximation. In Figs. 3.26 – 3.30 is assumed that the annual number of deaths observed in the cohort in each year is equal to expected number of deaths. Aiming to make a comparison between different cohort sizes, the results for  $n_{x_0, j-1} = 100$ ,  $n_{x_0, j-1} = 1000$  cases are provided in the same figure. While the expected lifetime in the cohort and uncertainty risk have similar results as multiple cohorts, in the other quantities' results are observed more fluctuations at the beginning periods. An important finding to be noted is the quantity  $\mathbb{E}(\text{Var}(T - 65|T > 65; a, b))$  representing random fluctuations in mortality trends. It is increasing by time with some observed fluctuations. This quantity shows stable behaviour when multiple cohorts present in the portfolio (see Fig. 3.17). Furthermore, when 25% lower or higher number of deaths in the cohort than the expected is assumed, the fluctuations in the quantities can be seen more clearly (e.g. see Figs. 3.31 – 3.35 for 25% reduction in mortality, and Fig. B.11 for 25% increase in mortality than expected). An



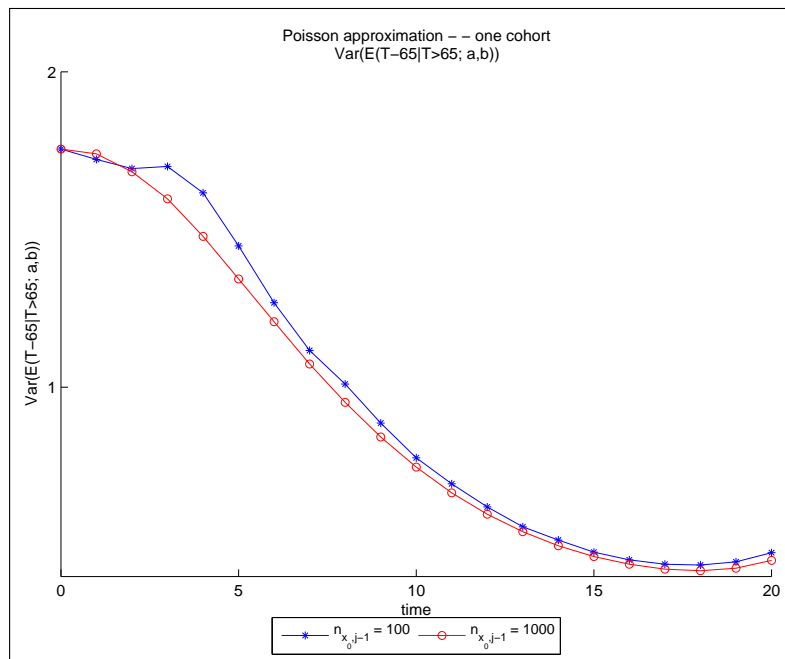
**Figure 3.26:** Expected lifetime under Poisson approximation:

$$n_{x_0, j-1} = 100, 1000, d_{x, j} = n_{x, j-1}q_{x, j}, \text{ one cohort}$$

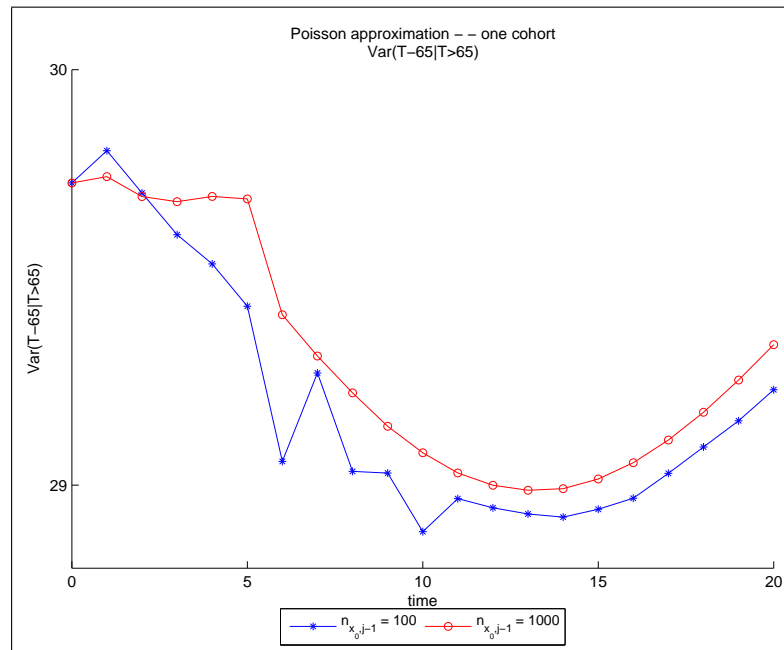
important point to be noted is that the effect of cohort's size in the quantities representing random fluctuations and uncertainty risk in mortality trends. The random fluctuation and systematic risk in mortality have lower values most of the time when the portfolio has higher number of insureds. However in a portfolio allowing multiple cohorts, the uncertainty risk decreases when higher number of the new enters presents.



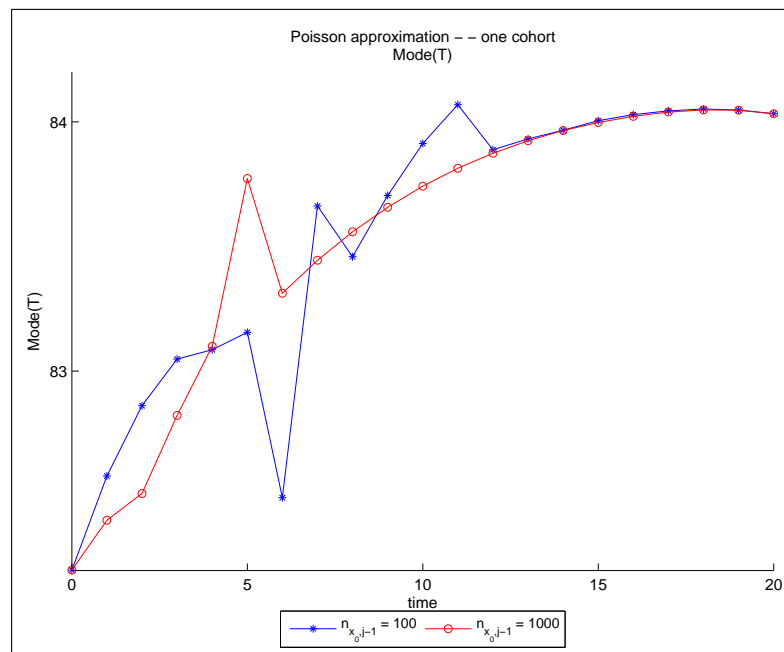
**Figure 3.27:** Random fluctuations under Poisson approximation:  
 $n_{x_0, j-1} = 100, 1000$ ,  $d_{x, j} = n_{x, j-1}q_{x, j}$ , one cohort



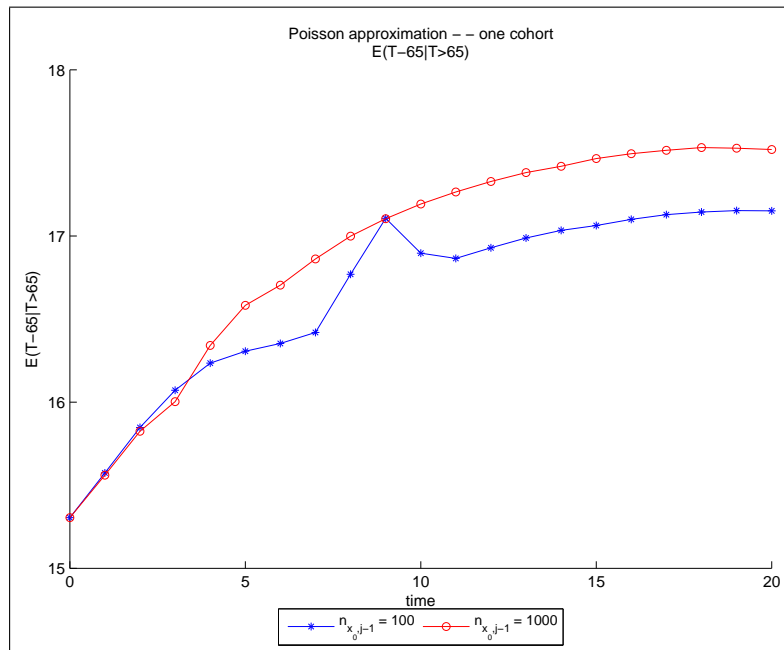
**Figure 3.28:** Uncertainty risk under Poisson approximation:  
 $n_{x_0, j-1} = 100, 1000$ ,  $d_{x, j} = n_{x, j-1}q_{x, j}$ , one cohort



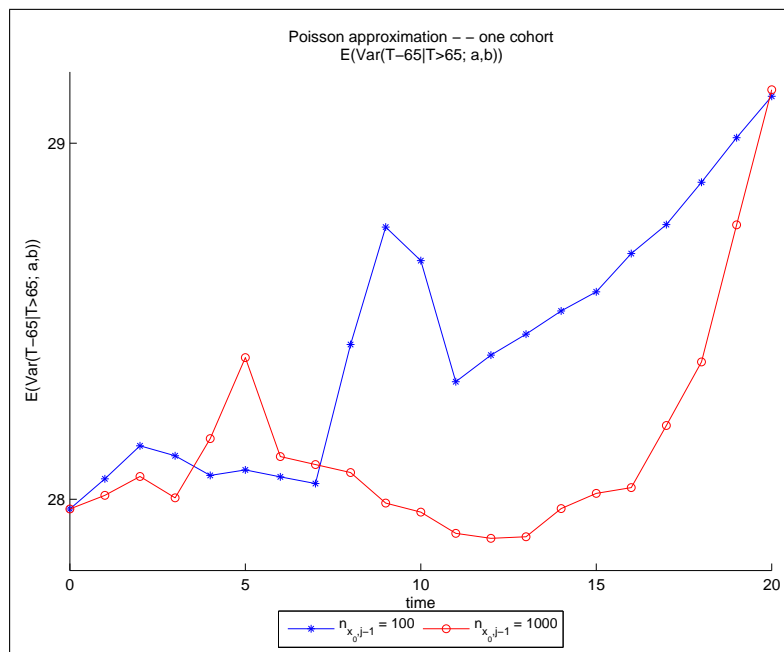
**Figure 3.29:** Variance of the lifetime under Poisson approximation:  
 $n_{x_0, j-1} = 100, 1000$ ,  $d_{x, j} = n_{x, j-1}q_{x, j}$ , one cohort



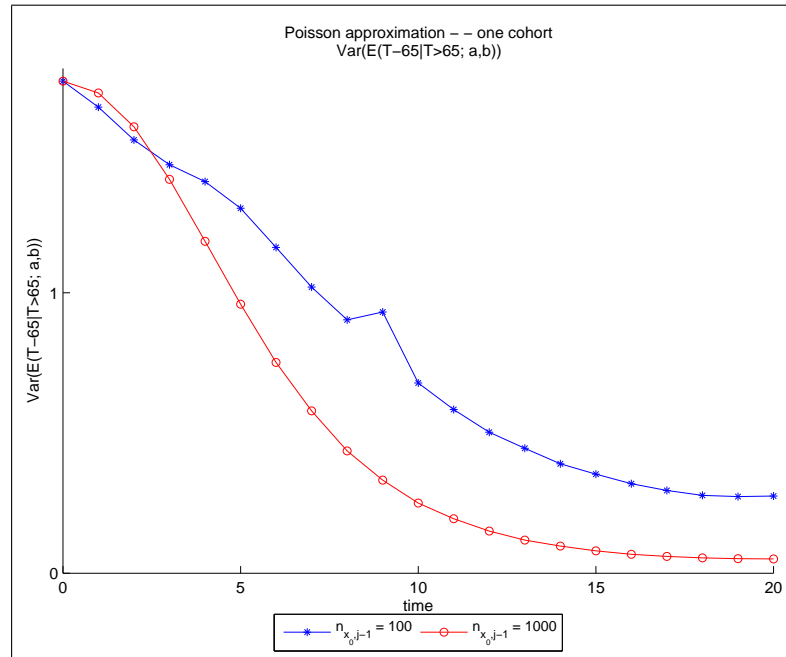
**Figure 3.30:** Lexis points under Poisson approximation:  
 $n_{x_0, j-1} = 100, 1000$ ,  $d_{x, j} = n_{x, j-1}q_{x, j}$ , one cohort



**Figure 3.31:** Expected lifetime under Poisson approximation:  
 $n_{x_0, j-1} = 100, 1000$ ,  $d_{x, j} = 0.75 n_{x, j-1} q_{x, j}$ , one cohort

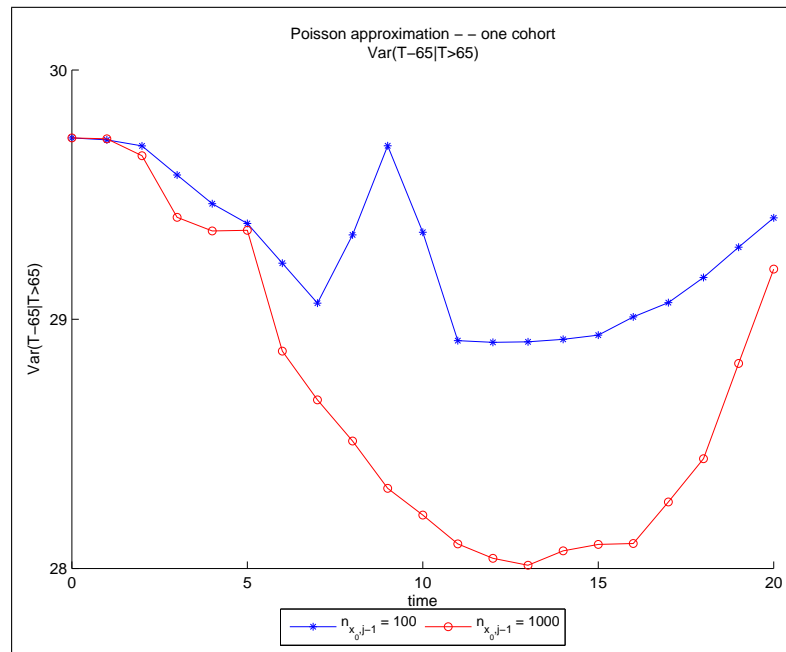


**Figure 3.32:** Random fluctuations under Poisson approximation:  
 $n_{x_0, j-1} = 100, 1000$ ,  $d_{x, j} = 0.75 n_{x, j-1} q_{x, j}$ , one cohort



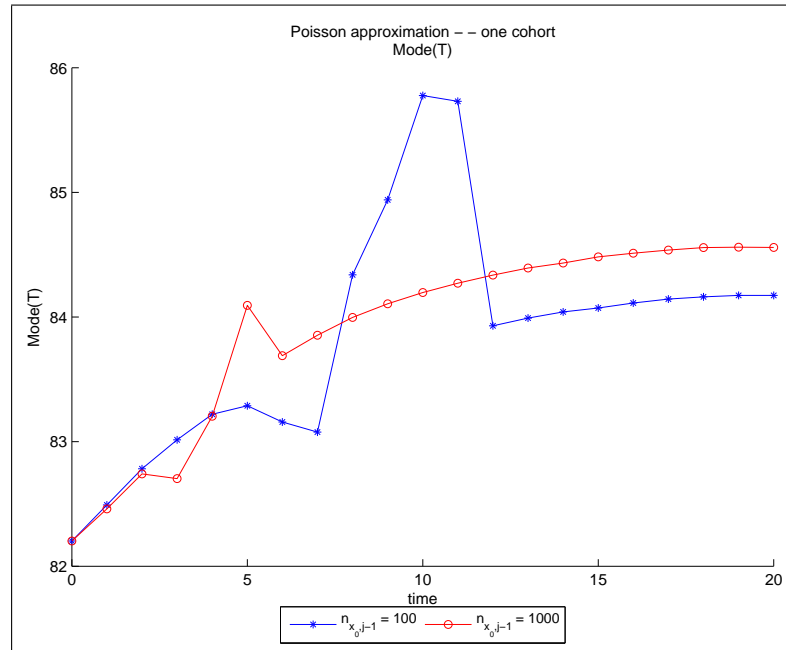
**Figure 3.33:** Uncertainty risk under Poisson approximation:

$n_{x_0,j-1} = 100, 1000$ ,  $d_{x,j} = 0.75 n_{x,j-1} q_{x,j}$ , one cohort



**Figure 3.34:** Variance of the lifetime under Poisson approximation:

$n_{x_0,j-1} = 100, 1000$ ,  $d_{x,j} = 0.75 n_{x,j-1} q_{x,j}$ , one cohort



**Figure 3.35:** Lexis point under Poisson approximation:  
 $n_{x_0, j-1} = 100, 1000$ ,  $d_{x, j} = 0.75 n_{x, j-1} q_{x, j}$ , one cohort

## 3.2 Dynamic model

Defined on the annual number of deaths in a given cohort, the dynamic model adopts Poisson model with a time dependent random parameter. For the random parameter, namely random mortality rate, is considered a multiplicative model, which is widely used in actuarial mathematics. The components of the multiplicative model are the best estimate mortality rate and a random variable representing the deviations in the mortality rate. Hence, the model requires that a (projected) life table providing a best estimate assessment of future mortality is available in terms of death probabilities as an input. The component representing the deviations provides the randomness and is assumed to have Gamma distribution. Its parameters are updated to the experience in the portfolio via a Bayesian inferential procedure Sect. 3.2.1). The numerical results to dynamic model are given in Sect. 3.2.3.

### 3.2.1 The model

In the dynamic model, the same portfolio as defined in Static model will be addressed. Nevertheless, we recall and define the portfolio here again, sticking to the terminology used for the model in the original paper [?]. The portfolio starts at time  $t_0$  with the required age at entry  $x_0$ , and  $t$ ,  $t = 0, 1, \dots$  denotes the number of years since the initial (starting) time  $t_0$ . The random number of deaths in year  $(t - 1, t)$  is denoted as  $D_t$ , and in case of more than one cohort with ages ranging between  $x_0$  and  $\omega$ , defined by  $D_t = \sum_{x=x_0}^{\min\{x_0+t-1, \omega\}} D_{x,t}$  with  $D_{x,t}$  representing the random number of deaths for those aged  $x$  at time  $t - 1$ . In case of only one cohort  $D_t$  is simply given by  $D_{x,t}$  with  $x = x_0 + t - 1$ , the only current age of annuitants. Similarly if we define the random number of survivors at time  $t$  by  $N_t$ , then, in case of more than one cohort,  $N_t = \sum_{x=x_0}^{\min\{x_0+t-1, \omega\}} N_{x,t}$  with  $N_{x,t}$  representing the random number survivors at time  $t$ . And again if only one cohort is present  $N_t$  is simply given by  $N_{x,t}$ . The realized values of these random variables,  $D_t$ ,  $D_{x,t}$ ,  $N_t$ , and  $N_{x,t}$ , will be denoted with small letters, i.e.  $d_t$ ,  $d_{x,t}$ ,  $n_t$ , and  $n_{x,t}$ . The lives both within the cohorts, and among cohorts are assumed to be “homogeneous and independent”.

#### One cohort

Assume that the number of deaths in year  $(t - 1, t)$  for those aged  $x$  at the time  $t - 1$  have Binomial distribution, i.e.

$$[D_{x,t} | q_{x,t}; n_{x,t-1}] \sim \text{Bin}(n_{x,t-1}, q_{x,t}) \quad (3.2.1)$$

where  $n_{x,t-1}$  is the number of livings aged  $x$  observed at time  $t - 1$ , and  $q_{x,t}$  is the assumed (possibly, the best estimate) mortality rate for those aged  $x$  at the beginning of the year  $(t - 1, t)$ . Provided that  $n_{x,t-1}$  is large enough and  $q_{x,t}$  is low, the product  $n_{x,t-1}q_{x,t}$  is stable and the Poisson approximation is accepted as the model, i.e.

$$[D_{x,t} | q_{x,t}; n_{x,t-1}] \sim \text{Poi}(n_{x,t-1} q_{x,t}) \quad (3.2.2)$$

with probability mass function

$$f(d_{x,t}|q_{x,t}; n_{x,t-1}) = \frac{(n_{x,t-1} q_{x,t})^{d_{x,t}} e^{-n_{x,t-1} q_{x,t}}}{d_{x,t}!}$$

The uncertainty risk is represented by the random mortality rate,  $Q_{x,t}$ , and defined as

$$Q_{x,t} = q_{x,t}^* Z_{x,t} \quad (3.2.3)$$

where  $Z_{x,t}$  is a (positive) random adjustment to the best-estimate mortality rate  $q_{x,t}^*$ . Under the Gamma distribution assumption for  $Z_{x,t}$ , i.e.

$$Z_{x,t} \sim \text{Gamma}(\alpha_{x,t}, \beta_{x,t}) \quad (3.2.4)$$

we have the probability distribution of the random mortality  $Q_{x,t}$  as

$$Q_{x,t} \sim \text{Gamma}\left(\alpha_{x,t}, \frac{\beta_{x,t}}{q_{x,t}^*}\right) \quad (3.2.5)$$

with the probability density function

$$h(q_{x,t}) = \frac{\left(\frac{\beta_{x,t}}{q_{x,t}^*}\right)^{\alpha_{x,t}}}{\Gamma(\alpha_{x,t})} q_{x,t}^{\alpha_{x,t}-1} e^{-\left(\frac{\beta_{x,t}}{q_{x,t}^*}\right)q_{x,t}}$$

The Gamma distribution of random mortality enables us to find the unconditional (predictive) distribution of the number of deaths in a year as Negative Binomial, namely

$$[D_{x,t}|n_{x,t-1}] \sim \text{NBin}\left(\alpha_{x,t}, \frac{\theta_{x,t}}{\theta_{x,t} + 1}\right), \quad (3.2.6)$$

where  $\theta_{x,t} = \frac{\beta_{x,t}}{n_{x,t-1} q_{x,t}^*}$ . The calculations for the conclusion (3.2.6) are given in the following:

*Proof.* The unconditional distribution, namely the predictive distribution, func-

tion of the number of deaths is calculated as

$$\begin{aligned}
f(d_{x,t}|n_{x,t-1}) &= \int_0^1 f(d_{x,t}|q_{x,t}; n_{x,t-1}) f(q_{x,t}) \, d(q_{x,t}) \\
&= \int_0^1 \frac{(n_{x,t-1}q_{x,t})^{d_{x,t}} e^{-n_{x,t-1}q_{x,t}}}{d_{x,t}!} \frac{\left(\frac{\beta_{x,t}}{q_{x,t}^*}\right)^{\alpha_{x,t}}}{\Gamma(\alpha_{x,t})} (q_{x,t})^{\alpha_{x,t}-1} e^{-\left(\frac{\beta_{x,t}}{q_{x,t}^*}\right)q_{x,t}} \, dq_{x,t} \\
&= \int_0^1 \frac{(n_{x,t-1}q_{x,t})^{d_{x,t}} \left(\frac{\beta_{x,t}}{q_{x,t}^*}\right)^{\alpha_{x,t}} (q_{x,t})^{\alpha_{x,t}-1}}{d_{x,t}! \Gamma(\alpha_{x,t})} e^{-\underbrace{\left(\frac{\beta_{x,t}}{q_{x,t}^*} + n_{x,t-1}\right)q_{x,t}}_y} \, dq_{x,t} \\
&= \int_0^1 \frac{(n_{x,t-1})^{d_{x,t}} \left(\frac{\beta_{x,t}}{q_{x,t}^*}\right)^{\alpha_{x,t}} \left(\frac{\beta_{x,t}}{q_{x,t}^*} + n_{x,t-1}\right)^{-\alpha_{x,t}-d_{x,t}}}{d_{x,t}! \Gamma(\alpha_{x,t})} y^{\alpha_{x,t}+d_{x,t}-1} e^{-y} \, dy \\
&= \frac{\left(\frac{\beta_{x,t}}{q_{x,t}^*}\right)^{\alpha_{x,t}} \left(\frac{q_{x,t}^*}{\beta_{x,t} + n_{x,t-1}q_{x,t}^*}\right)^{d_{x,t}}}{d_{x,t}! \Gamma(\alpha_{x,t})} \underbrace{\int_0^1 y^{\alpha_{x,t}+d_{x,t}-1} e^{-y} \, dy}_{\Gamma(\alpha_{x,t}+d_{x,t})}
\end{aligned}$$

Defining  $\theta_{x,t} = \frac{\beta_{x,t}}{n_{x,t-1}q_{x,t}^*}$ , we get

$$f(d_{x,t}|n_{x,t-1}) = \frac{\Gamma(\alpha_{x,t} + d_{x,t})}{\Gamma(\alpha_{x,t})d_{x,t}!} \left(\frac{\theta_{x,t}}{1 + \theta_{x,t}}\right)^{\alpha_{x,t}} \left(1 - \frac{\theta_{x,t}}{1 + \theta_{x,t}}\right)^{d_{x,t}}$$

which means

$$[D_{x,t}|n_{x,t-1}] \sim \text{NBin}\left(\alpha_{x,t}, \frac{\theta_{x,t}}{1 + \theta_{x,t}}\right)$$

where  $\theta_{x,t} = \frac{\beta_{x,t}}{n_{x,t-1}q_{x,t}^*}$ . □

From the distributions (3.2.5) and (3.2.6) it follows that

$$\mathbb{E}[Q_{x,t}] = \frac{\alpha_{x,t}}{\beta_{x,t}} q_{x,t}^* \tag{3.2.7}$$

$$\mathbb{E}[D_{x,t}|n_{x,t-1}] = \frac{\alpha_{x,t}}{\theta_{x,t}} = \frac{\alpha_{x,t}}{\beta_{x,t}} n_{x,t-1} q_{x,t}^*$$

while, given the best estimate mortality rate  $q_{x,t}^*$ , the expected number of deaths was previously

$$\mathbb{E}[D_{x,t}|q_{x,t}^*; n_{x,t-1}] = n_{x,t-1} q_{x,t}^*. \tag{3.2.8}$$

Hence  $E[D_{x,t}|n_{x,t-1}] \geq E[D_{x,t}|q_{x,t}^*; n_{x,t-1}]$ , depending on the value taken the ratio  $\frac{\alpha_{x,t}}{\beta_{x,t}}$ . As we will see later, this ratio represents the expected number of systematic deviation in mortality.

The trend in mortality implies some correlation in time among the mortality rates. Assuming that  $Z_{x,t}$ 's are correlated in time and the mortality experience of the portfolio is informative to capture the mortality trend, an inferential procedure can be used to capture possible correlations among the mortality rates and update the parameters of the probability distribution of  $Z_{x,t}$  to the experience.

At time 0, when no experience is available, we adopt (3.2.4) with a value for the parameters which is the same for all times  $t$ ,  $t = 1, 2, \dots$  and ages  $x$ ,  $x = x_0 + t$ . Denoting these initial value of parameters  $\bar{\alpha}$  and  $\bar{\beta}$ ,

$$Z_{x,t} \sim \text{Gamma}(\bar{\alpha}, \bar{\beta}) \quad (3.2.9)$$

The values  $\bar{\alpha}$  and  $\bar{\beta}$  could be chosen, for example, such that  $\mathbb{E}[Q_{x,t}] = q_{x,t}^*$ .

From (3.2.9), it follows

$$[D_{x_0,1}|n_{x_0,0}] \sim \text{NBin}\left(\bar{\alpha}, \frac{\theta_{x_0,1}}{\theta_{x_0,1} + 1}\right), \quad (3.2.10)$$

where  $\theta_{x_0,1} = \frac{\bar{\beta}}{n_{x_0,0}q_{x_0,1}^*}$ .

At time 1, the number of deaths observed in year (0, 1) is available. Let  $d_{x_0,1}$  be such number. Then  $n_{x_0+1,1} = n_{x_0,0} - d_{x_0,1}$ . The posterior probability distribution of  $Q_{x_0,1}$  conditional on the observation  $D_{x_0,1} = d_{x_0,1}$  is calculated as

$$h(q_{x_0,1}|d_{x_0,1}) \propto h(q_{x_0,1}) \mathcal{L}(q_{x_0,1}|d_{x_0,1}) \quad (3.2.11)$$

where  $h(q_{x_0,1})$  is the prior pdf of  $Q_{x,t}$  and  $\mathcal{L}(q_{x_0,1}|d_{x_0,1})$  is the likelihood of the

observation. The calculations yields the posterior distribution of  $Q_{x_0,1}$  as

$$\begin{aligned}
h(q_{x_0,1}|d_{x_0,1}) &= \frac{h(q_{x_0,1}) f(d_{x_0,1}|q_{x_0,1}, n_{x_0,0})}{\int_0^1 f(d_{x_0,1}|q_{x_0,1}, n_{x_0,0}) h(q_{x_0,1}) dq_{x_0,1}} \\
&= \frac{\left(\frac{\bar{\beta}}{q_{x_0,1}^*}\right)^{\bar{\alpha}} (q_{x_0,1})^{\bar{\alpha}-1} e^{-\left(\frac{\bar{\beta}}{q_{x_0,1}^*}\right) q_{x_0,1}} \frac{(n_{x_0,0} q_{x_0,1})^{d_{x_0,1}} e^{-n_{x_0,0} q_{x_0,1}}}{d_{x_0,1}!}}{\frac{\Gamma(\bar{\alpha}+d_{x_0,1})}{\Gamma(\bar{\alpha})d_{x_0,1}!} \left(\frac{\theta_{x_0,1}}{1+\theta_{x_0,1}}\right)^{\bar{\alpha}} \left(1 - \frac{\theta_{x_0,1}}{1+\theta_{x_0,1}}\right)^{d_{x_0,1}}} \\
&= \frac{\left(\frac{\bar{\beta}}{q_{x_0,1}^*}\right)^{\bar{\alpha}} (n_{x_0,0})^{d_{x_0,1}} (q_{x_0,1})^{\bar{\alpha}+d_{x_0,1}-1} e^{-\left(\frac{\bar{\beta}}{q_{x_0,1}^*}+n_{x_0,0}\right) q_{x_0,1}}}{\Gamma(\bar{\alpha} + d_{x_0,1}) \left(\frac{\bar{\beta}}{\bar{\beta}+n_{x_0,0} q_{x_0,1}^*}\right)^{\bar{\alpha}} \left(\frac{n_{x_0,0} q_{x_0,1}^*}{\bar{\beta}+n_{x_0,0} q_{x_0,1}^*}\right)^{d_{x_0,1}}} \\
&= \frac{\left(\frac{\bar{\beta}}{q_{x_0,1}^*} + n_{x_0,0}\right)^{\bar{\alpha}+d_{x_0,1}}}{\Gamma(\bar{\alpha} + d_{x_0,1})} (q_{x_0,1})^{\bar{\alpha}+d_{x_0,1}-1} e^{-\left(\frac{\bar{\beta}}{q_{x_0,1}^*}+n_{x_0,0}\right) q_{x_0,1}} \quad (3.2.12)
\end{aligned}$$

which means

$$[Q_{x_0,1}|d_{x_0,1}] \sim \text{Gamma}\left(\bar{\alpha} + d_{x_0,1}, \frac{\bar{\beta}}{q_{x_0,1}^*} + n_{x_0,0}\right) \quad (3.2.13)$$

Then posterior distribution of mortality adjustments  $Z_{x,t}$  results from (3.2.13) as

$$[Z_{x,t}|d_{x_0,1}] \sim \text{Gamma}\left(\bar{\alpha} + d_{x_0,1}, \bar{\beta} + n_{x_0,0} q_{x_0,1}^*\right) \quad (3.2.14)$$

where  $\bar{\alpha} + d_{x_0,1} = \alpha_{x_0+1,2}$  and  $\bar{\beta} + n_{x_0,0} q_{x_0,1}^* = \beta_{x_0+1,2}$  are the updated parameters to the experience in the first period.

Comparing (3.2.9) to (3.2.14) indicates that the first parameter of  $Z_{x,t}$  is increased by the observed number of deaths,  $d_{x_0,1}$ , while the second parameter is increased by the expected number of deaths for the first year,  $n_{x_0,0} q_{x_0,1}^*$ . Besides, while the prior expected value of  $Z_{x,t}$  was

$$\mathbb{E}[Z_{x,t}] = \frac{\bar{\alpha}}{\bar{\beta}},$$

the posterior expected value at time 1 is found as

$$\mathbb{E}[Z_{x,t}|d_{x_0,1}] = \frac{\bar{\alpha} + d_{x_0,1}}{\bar{\beta} + n_{x_0,0} q_{x_0,1}^*}.$$

So,  $\mathbb{E}[Z_{x,t}|d_{x_0,1}] \geq \mathbb{E}[Z_{x,t}]$  depends on the comparison of the actual,  $d_{x_0,1}$ , and expected values  $n_{x_0,0} q_{x_0,1}^*$  of deaths in the first period.

The posterior distribution (3.2.14) at time 1 is the starting point for the calculations for next period, which means the prior probability distribution of  $Q_{x_0+1,2}$  for the second period (1,2) results from (3.2.14) as

$$[Q_{x_0+1,2}|d_{x_0,1}] \sim \text{Gamma} \left( \bar{\alpha} + d_{x_0,1}, \frac{\bar{\beta} + n_{x_0,0} q_{x_0,1}^*}{q_{x_0+1,2}^*} \right) \quad (3.2.15)$$

The unconditional distribution of the number of deaths for the second periods is then becomes

$$[D_{x_0+1,2}|n_{x_0,0}, d_{x_0,1}] \sim \text{NBin} \left( \bar{\alpha} + d_{x_0,1}, \frac{\theta_{x_0+1,2}}{\theta_{x_0+1,2} + 1} \right), \quad (3.2.16)$$

where  $\theta_{x_0+1,2} = \frac{\beta_{x_0+1,2}}{n_{x_0+1,1} q_{x_0+1,2}^*}$ .

At time 2, the number of deaths observed in year (1,2) gets available. Let  $d_{x_0+1,2}$  be such number. The posterior probability distribution of  $Q_{x_0+1,2}$  conditional on the observation  $D_{x_0+1,2} = d_{x_0+1,2}$  is calculated as

$$h(q_{x_0+1,2}|d_{x_0,1}, d_{x_0+1,2}) \propto h(q_{x_0+1,2}|d_{x_0,1}) \mathcal{L}(q_{x_0+1,2}|d_{x_0,1}, d_{x_0+1,2}) \quad (3.2.17)$$

where  $\mathcal{L}(q_{x_0,1}|d_{x_0,1}, d_{x_0+1,2})$  is the likelihood of the observation and  $h(q_{x_0+1,2}|d_{x_0,1})$  is the prior pdf of  $Q_{x,t}$  defined by (3.2.15). The similar calculations which is done in the first period yields the posterior distribution of  $Q_{x_0+1,2}$  as

$$\begin{aligned} h(q_{x_0+1,2}|d_{x_0,1}, d_{x_0+1,2}) &= \frac{h(q_{x_0,1}|d_{x_0,1}) f(d_{x_0,1}|q_{x_0+1,2}, n_{x_0+1,1}, d_{x_0,1})}{\int_0^1 f(d_{x_0,1}|q_{x_0+1,2}, n_{x_0+1,1}, d_{x_0,1}) h(q_{x_0,1}|d_{x_0,1}) dq_{x_0,1}} \\ &= \vdots \\ &= \frac{\left( \frac{\beta_{x_0+1,2}}{q_{x_0+1,2}^*} + n_{x_0+1,1} \right)^{\alpha_{x_0+1,2} + d_{x_0+1,2}}}{\Gamma(\alpha_{x_0+1,2} + d_{x_0+1,2})} \\ &\times (q_{x_0+1,2})^{\alpha_{x_0+1,2} + d_{x_0+1,2} - 1} e^{-\left( \frac{\beta_{x_0+1,2}}{q_{x_0+1,2}^*} + n_{x_0+1,1} \right) q_{x_0+1,2}} \end{aligned} \quad (3.2.18)$$

which means

$$[Q_{x_0+1,2}|d_{x_0,1}, d_{x_0+1,2}] \sim \text{Gamma} \left( \alpha_{x_0+1,2} + d_{x_0+1,2}, \frac{\beta_{x_0+1,2}}{q_{x_0+1,2}^*} + n_{x_0+1,1} \right) \quad (3.2.19)$$

Then posterior distribution of mortality adjustments  $Z_{x,t}$  is resulted from (3.2.19) as

$$[Z_{x,t}|d_{x_0,1}, d_{x_0+1,2}] \sim \text{Gamma} \left( \alpha_{x_0+1,2} + d_{x_0+1,2}, \beta_{x_0+1,2} + n_{x_0+1,1} q_{x_0+1,2}^* \right) \quad (3.2.20)$$

where  $\alpha_{x_0+1,2} + d_{x_0+1,2} = \alpha_{x_0+2,3}$  and  $\beta_{x_0+1,2} + n_{x_0+1,1} q_{x_0+1,2}^* = \beta_{x_0+2,3}$  are the updated parameters to the experience in the second period.

Similarly, to the case at time 1, the posterior distribution (3.2.20) at time 2 is the starting point for the calculations for next period, which means the prior probability distribution of  $Q_{x_0+2,3}$  for the period (2,3) results from (3.2.20) as

$$[Q_{x_0+2,3}|d_{x_0,1}, d_{x_0+1,2}] \sim \text{Gamma} \left( \alpha_{x_0+2,3}, \frac{\beta_{x_0+1,2} + n_{x_0+1,1} q_{x_0+1,2}^*}{q_{x_0+2,3}^*} \right) \quad (3.2.21)$$

It yields the unconditional distribution of the number of deaths as

$$[D_{x_0+2,3}|n_{x_0,0}, d_{x_0,1}, d_{x_0+1,2}] \sim \text{NBin} \left( \alpha_{x_0+2,3}, \frac{\theta_{x_0+2,3}}{\theta_{x_0+2,3} + 1} \right), \quad (3.2.22)$$

where  $\theta_{x_0+2,3} = \frac{\beta_{x_0+2,3}}{n_{x_0+2,2} q_{x_0+2,3}^*}$ .

Following the same steps, at time  $t - 1$  we have the observation of the annual number of deaths,  $D_{x_0+h-1,h} = d_{x_0+h-1,h}$  and the number of survivors  $n_{x_0+h,h} = n_{x_0+h-1,h-1} - d_{x_0+h-1,h}$  where  $h = 1, 2, \dots$ . The general structure of the unconditional distribution of the number of deaths can be written as

$$\begin{aligned} & [D_{x_0+t-1,t}|n_{x_0,0}, d_{x_0,1}, d_{x_0+1,2}, \dots, d_{x_0+t-2,t-1}] \\ & \sim \text{NBin} \left( \alpha_{x_0+t-1,t}, \frac{\theta_{x_0+t-1,t}}{\theta_{x_0+t-1,t} + 1} \right) \end{aligned} \quad (3.2.23)$$

where  $\alpha_{x_0+t-1,t} = \bar{\alpha} + \sum_{h=1}^{t-1} d_{x_0+h,h+1}$  and  $\theta_{x_0+t-1,t} = \frac{\bar{\beta} + \sum_{h=1}^{t-1} n_{x_0+h-1,h-1} q_{x_0+h-1,h}^*}{n_{x_0+t-1,t-1} q_{x_0+t-1,t}^*}$ .

The expected number of deaths in each year  $t - 1$ ,  $t = 1, 2, \dots$ , is given by

$$\begin{aligned} & \mathbb{E}[D_{x_0+t-1} | n_{x_0,0}, d_{x_0,1}, d_{x_0+1,1}, \dots, d_{x_0+t-2,t-1}] \\ &= \frac{\bar{\alpha} + \sum_{h=1}^{t-1} d_{x_0+h-1,h}}{\bar{\beta} + \sum_{h=1}^{t-1} n_{x_0+h-1,h-1} q_{x_0+h-1,h}^*} n_{x_0+t-1,t-1} q_{x_0+t-1,t}^* \end{aligned} \quad (3.2.24)$$

The unconditional expected number of deaths in a year is given by the expected value of the best-estimate mortality rate, i.e.  $n_{x_0+t-1,t-1} q_{x_0+t-1,t}^*$ , adjusted by a coefficient, which is simply the expected value of random mortality adjustment  $Z_{x,t}$ . As it can be seen, this expected value depends on the relation between the observed number of deaths (seen in nominator) and those expected at the beginning of the year (seen in denominator). If the experience is consistent with what is expected, such value will remain stable in time; conversely, if experience is worse than the expected value, that is the number of deaths is lower than expected, then that value will decrease in time.

### Multiple cohorts

Recalling from the beginning of the Sect. 3.2, when more than one cohort is included in the portfolio, the number of deaths in year  $(t - 1, t)$  is denoted as  $D_t$  with  $D_t = \sum_{x=x_0}^{\min\{x_0+t-1,\omega\}} D_{x,t}$  where  $D_{x,t}$  representing the random number of deaths for those aged  $x$  at time  $t - 1$ . Similarly, the random number of survivors at time  $t$  is denoted as  $N_t$  with  $N_t = \sum_{x=x_0}^{\min\{x_0+t-1,\omega\}} N_{x,t}$  where  $N_{x,t}$  represents the random number survivors at time  $t$  for those aged  $x$  at time  $t$ .

For any age  $x$  and time  $t$ , we assume (3.2.2). The assumption (3.2.2) requires that, conditional on  $q_{x,t}$ , the lifetimes of the individuals belonging to one cohort are independent and identically distributed. We further assume that, conditional on the life table  $q_{x,t}$ , at any time  $t$  the number of deaths  $D_{x,t}$  are independent in respect of age, which means that the individual lifetimes are independent also among cohorts. So we find

$$[D_t | \{q_{x,t}\}; \{n_{x,t-1}\}] \sim \text{Poi} \left( \sum_{x=x_0}^{\min\{x_0+t-1,\omega\}} n_{x,t-1} q_{x,t} \right) \quad (3.2.25)$$

Uncertainty in aggregate mortality can be introduced as follows. For age  $x$  and time  $t$ , we let

$$Q_{x,t} = q_{x,t}^* Z_t \quad (3.2.26)$$

where  $Z_t$  is a (positive) random coefficient, expressing a systematic deviation in mortality. We point out that, the systematic deviation is now assumed to be time-specific, but age-independent, which means the source of uncertainty is assumed to be common to all the cohorts.

Conditional on  $Z_t = z$ , the assumption (3.2.25) can be extended as follows

$$[D_t | \{z q_{x,t}^*\}; \{n_{x,t-1}\}] \sim \text{Poi} \left( \sum_{x=x_0}^{\min\{x_0+t-1, \omega\}} n_{x,t-1} z q_{x,t}^* \right)$$

Futhermore, assuming that the random coefficient  $Z_t$  is Gamma distributed, i.e.

$$Z_t \sim \text{Gamma}(\alpha_t, \beta_t)$$

the unconditional distribution of deaths becomes having a Negative Binomial distribution, i.e.

$$[D_t | \{n_{x,t-1}\}] \sim \text{NBin} \left( \alpha_t, \frac{\theta_t}{\theta_t + 1} \right) \quad (3.2.27)$$

where  $\theta_t = \frac{\beta_t}{\sum_{x=x_0}^{\min\{x_0+t-1, \omega\}} n_{x,t-1} q_{x,t}^*}$ .

*Proof.*

$$\begin{aligned}
f(d_t|\{n_{x,t-1}\}) &= \int_0^\infty f(d_t|z q_{x,t}^*; n_{x,t-1}) f(z q_{x,t}^*) dz \\
&= \int_0^\infty \frac{\left(z \sum_{x=x_0}^{\min\{x_0+t-1, \omega\}} n_{x,t-1} q_{x,t}^*\right)^{d_t} e^{-z \sum_{x=x_0}^{\min\{x_0+t-1, \omega\}} n_{x,t-1} q_{x,t}^*}}{d_t!} \\
&\quad \times \frac{(\beta_t)^{\alpha_t}}{\Gamma(\alpha_t)} z^{\alpha_t-1} e^{-\beta_t z} dz \\
&= \frac{\left(\sum_{x=x_0}^{\min\{x_0+t-1, \omega\}} n_{x,t-1} q_{x,t}^*\right)^{d_t} (\beta_t)^{\alpha_t} \left(\beta_t + \sum_{x=x_0}^{\min\{x_0+t-1, \omega\}} n_{x,t-1} q_{x,t}^*\right)^{\alpha_t-d_t}}{d_t! \Gamma(\alpha_t)} \\
&\quad \times \int_0^\infty y^{\alpha_t+d_t-1} e^{-y} dy \\
&= \frac{\Gamma(\alpha_t + d_t)}{\Gamma(\alpha_t) d_t!} \left(\frac{\beta_t}{\beta_t + \sum_{x=x_0}^{\min\{x_0+t-1, \omega\}} n_{x,t-1} q_{x,t}^*}\right)^{\alpha_t} \\
&\quad \times \left(\frac{\sum_{x=x_0}^{\min\{x_0+t-1, \omega\}} n_{x,t-1} q_{x,t}^*}{\beta_t + \sum_{x=x_0}^{\min\{x_0+t-1, \omega\}} n_{x,t-1} q_{x,t}^*}\right)^{d_t}
\end{aligned}$$

which proves that

$$[D_t|\{n_{x,t-1}\}] \sim \text{NBin}\left(\alpha_t, \frac{\theta_t}{1 + \theta_t}\right)$$

where  $\theta_t = \frac{\beta_t}{\sum_{x=x_0}^{\min\{x_0+t-1, \omega\}} n_{x,t-1} q_{x,t}^*}$ . □

Let's assume that the  $Z_t$ 's are correlated in time. So, similar to the one-cohort case, an inferential procedure can be defined.

At time 0, when no experience is available, it is assumed that

$$Z_t \sim \text{Gamma}(\bar{\alpha}, \bar{\beta})$$

for all future times  $t$ . So we have unconditional distribution of the number of deaths in first period (proved above) as

$$[D_1|\{n_{x,0}\}] \sim \text{NBin}\left(\bar{\alpha}, \frac{\theta_1}{1 + \theta_1}\right)$$

where  $\theta_1 = \frac{\bar{\beta}}{\sum_{x=x_0}^{\min\{x_0, \omega\}} n_{x,0} q_{x,1}^*} = \frac{\bar{\beta}}{n_{x_0,0} q_{x_0,1}^*}$ . So

$$f(d_1|\{n_{x,0}\}) = \frac{\Gamma(\bar{\alpha} + d_1)}{\Gamma(\bar{\alpha}) d_1!} \left( \frac{\bar{\beta}}{\bar{\beta} + n_{x_0,0} q_{x_0,1}^*} \right)^{\bar{\alpha}} \left( \frac{n_{x_0,0} q_{x_0,1}^*}{\bar{\beta} + n_{x_0,0} q_{x_0,1}^*} \right)^{d_1}$$

At time 1, the observation in year (0, 1) is available and let  $D_1 = d_1$  denote the number. Then the posterior distribution of  $Z_t$ , conditional on the new information is found by

$$g(z|d_1) \propto g(z) L(z|d_1)$$

where  $g(z) = \frac{(\bar{\beta})^{\bar{\alpha}}}{\Gamma(\bar{\alpha})} z^{\bar{\alpha}-1} e^{-\bar{\beta}z}$  and  $L(z|d_1) = \frac{(z n_{x_0,0} q_{x_0,1}^*)^{d_1} e^{-z n_{x_0,0} q_{x_0,1}^*}}{d_1!}$

Then doing some calculations yields the updated distribution of the mortality adjustment for the next periods  $t$ , i.e,

$$\begin{aligned} g(z|d_1) &= \frac{\frac{(\bar{\beta})^{\bar{\alpha}}}{\Gamma(\bar{\alpha})} z^{\bar{\alpha}-1} e^{-\bar{\beta}z} \frac{(z n_{x_0,0} q_{x_0,1}^*)^{d_1} e^{-z n_{x_0,0} q_{x_0,1}^*}}{d_1!}}{\frac{\Gamma(\bar{\alpha}+d_1)}{\Gamma(\bar{\alpha}) d_1!} \left( \frac{\bar{\beta}}{\bar{\beta}+n_{x_0,0} q_{x_0,1}^*} \right)^{\bar{\alpha}} \left( \frac{n_{x_0,0} q_{x_0,1}^*}{\bar{\beta}+n_{x_0,0} q_{x_0,1}^*} \right)^{d_1}} \\ &= \frac{(\bar{\beta} + n_{x_0,0} q_{x_0,1}^*)^{\bar{\alpha}+d_1}}{\Gamma(\bar{\alpha} + d_1)} z^{\bar{\alpha}+d_1-1} e^{-(\bar{\beta}+n_{x_0,0} q_{x_0,1}^*)z} \end{aligned}$$

indicating

$$[Z_t|d_1] \sim \text{Gamma}(\bar{\alpha} + d_1, \bar{\beta} + n_{x_0,0} q_{x_0,1}^*)$$

At time 1, a new cohort enters to the portfolio. The information available in second period is  $\{n_{x,1}\} = \{n_{x_0,1}, n_{x_0+1,1}\}$  and  $\{q_{x,2}^*\} = \{q_{x_0,2}^*, q_{x_0+1,2}^*\}$ . The unconditional distribution of the number of deaths in the second period is then

$$[D_2|\{n_{x_0,1}, n_{x_0+1,1}\}, d_1] \sim \text{NBin}\left(\bar{\alpha} + d_1, \frac{\theta_2}{1 + \theta_2}\right)$$

where  $\theta_2 = \frac{\bar{\beta} + n_{x_0,0} q_{x_0,1}^*}{n_{x_0,1} q_{x_0,2}^* + n_{x_0+1,1} q_{x_0+1,2}^*}$ .

At time 2, the number of deaths observed in year (1, 2) is available,  $D_2 = d_2$ . Then the posterior distribution of  $Z_t$  conditioned on the information  $D_t = d_t$  is

$$g(z|d_1, d_2) \sim g(z|d_1) L(z|d_1, d_2)$$

where

$$g(z|d_1) = \frac{(\bar{\beta} + n_{x_0,0} q_{x_0,1}^*)^{\bar{\alpha}+d_1}}{\Gamma(\bar{\alpha} + d_1)} z^{\bar{\alpha}+d_1-1} e^{-(\bar{\beta}+n_{x_0,0} q_{x_0,1}^*)z}$$

$$L(z|d_1, d_2) = \frac{(z(n_{x_0,1} q_{x_0,2}^* + n_{x_0+1,1} q_{x_0+1,2}^*))^{d_2} e^{-z(n_{x_0,1} q_{x_0,2}^* + n_{x_0+1,1} q_{x_0+1,2}^*)}}{d_2!}$$

Then the posterior distribution of  $Z_t$  for the next periods is calculated as

$$g(z|d_1, d_2) = \frac{(\bar{\beta} + \sum_{h=1}^2 \sum_{x=x_0}^{x_0+1} n_{x,h-1} q_{x,h}^*)^{\bar{\alpha}+d_1+d_2}}{\Gamma(\bar{\alpha} + d_1 + d_2)} \times z^{\bar{\alpha}+d_1+d_2-1} e^{-z(\bar{\beta} + \sum_{h=1}^2 \sum_{x=x_0}^{x_0+1} n_{x,h-1} q_{x,h}^*)}$$

indicating

$$[Z_t|d_1, d_2] \sim \text{Gamma} \left( \bar{\alpha} + d_1 + d_2, \bar{\beta} + \sum_{h=1}^2 \sum_{x=x_0}^{x_0+1} n_{x,h-1} q_{x,h}^* \right)$$

After a new cohort enters the portfolio at time  $t = 2$ , the information available at the end of the second period (begining of the third period) is  $\{n_{x,2}\} = \{n_{x_0,2}, n_{x_0+1,2}, n_{x_0+2,2}\}$  and  $\{q_{x,3}^*\} = \{q_{x_0,3}^*, q_{x_0+1,3}^*, q_{x_0+2,3}^*\}$ . Then the unconditional distribution of the number of deaths in the third period is

$$[D_3|\{n_{x_0,2}, n_{x_0+1,2}, n_{x_0+2,2}\}, d_1, d_2] \sim \text{NBin} \left( \bar{\alpha} + d_1 + d_2, \frac{\theta_3}{1 + \theta_3} \right)$$

where  $\theta_3 = \frac{\bar{\beta} + \sum_{h=1}^2 \sum_{x=x_0}^{x_0+1} n_{x,h-1} q_{x,h}^*}{\sum_{x=x_0}^{x_0+1} n_{x,t-1} q_{x,t}^*}$ .

Following the same steps, generalization of the multi-cohort model can be given as follows:

At time  $t - 1$ , after observing

- the annual number of new entrants  $n_{x_0,0}, n_{x_0,1}, \dots, n_{x_0,t-1}$ ,
- the annual number of deaths  $D_1 = d_1, D_2 = d_2, \dots, D_{t-1} = d_{t-1}$ ,
- the number of survivors  $n_{x,h}$  in each cohort time  $h$ ,  $h = 0, 1, \dots, t - 1$ ,

it is obtained that

$$[D_t | \{n_{x,0}, n_{x,1}, \dots, n_{x,t-1}\}; d_1, d_2, \dots, d_{t-1}] \sim \text{NBin} \left( \alpha_t, \frac{\theta_t}{1 + \theta_t} \right)$$

where  $\alpha_t = \bar{\alpha} + \sum_{h=1}^{t-1} d_h$  and  $\theta_t = \frac{\bar{\beta} + \sum_{h=1}^{t-1} \sum_{x=x_0}^{\min\{x_0+h-1, \omega\}} n_{x,h-1} q_{x,h}^*}{\sum_{x=x_0}^{\min\{x_0+t-1, \omega\}} n_{x,t-1} q_{x,t}^*}$ . As in Sect. ??, we have that the prior expected value of  $Z_t$  is

$$\mathbb{E}[Z_t] = \frac{\bar{\alpha}}{\bar{\beta}}$$

while the posterior expected value at time  $s$ ,  $s = 1, 2, \dots, t$ , is

$$\mathbb{E}[Z_t | d_1, d_2, \dots, d_{s-1}] = \frac{\bar{\alpha} + \sum_{h=1}^{s-1} d_h}{\bar{\beta} + \sum_{h=1}^{s-1} \sum_{x=x_0}^{\min\{x_0+h-1, \omega\}} n_{x,h-1} q_{x,h}^*}$$

As it was mentioned in Sect. 3.2.1, it might be the case  $E[Z_t] \gtrless E[Z_t | d_1, d_2, \dots, d_{s-1}]$  depending on the comparison between the experienced number of deaths and the relevant expected value.

### 3.2.2 Life expectancy

In this section will be defined the life expectancy of an individual age  $x$  at time  $t$  under the dynamic model. As we recall from Chp. 2, the appropriate use of projected life tables leads to the life expectancy of a person age  $x$  at time  $t$ , of course conditioned on the projected life table used, i.e.  $\{q_{x,t}\}$ , defined as

$$\dot{e}_{x,t} | \{q_{x,t}\} = \sum_{h=0}^{\omega-x} \left( h + \frac{1}{2} \right) {}_h|1q_{x,t} \quad (3.2.28)$$

(as in Eqn. (2.3.4)). Using the deferred probabilities derived from the diagonal of projected life table for the person age  $x$  at time  $t$ , i.e. the relevant cohort life table, we get the life expectancy as

$$\dot{e}_{x,t} | \{q_{x,t}\} = \frac{1}{2} + \sum_{h=1}^{\omega-x} h (1 - q_{x,t}) (1 - q_{x+1,t+1}) \dots (1 - q_{x+h-1,t+h-1}) q_{x+h,t+h} \quad (3.2.29)$$

As can be seen in the Eqn. (3.2.29), the life expectancy of an individual age  $x$  at time  $t$  is a function of (or conditioned on) the random variables

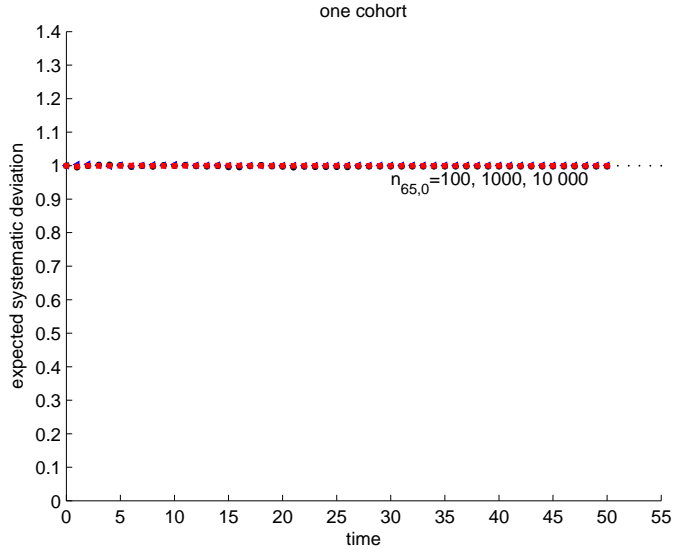
$$(1 - q_{x,t}), (1 - q_{x+1,t+1}) \cdots (1 - q_{x+h-1,t+h-1}), q_{x+h,t+h}. \quad (3.2.30)$$

The calculation of the unconditional life expectancy requires the joint distribution function of the random variables (3.2.30). Noting that trends imply some correlation in time among mortality rates, they cannot be assumed independent, and so their joint distribution cannot be written as a product of marginal distributions of the random variables (3.2.30). Considering that the marginal distributions can be obtained under the Gamma distribution assumption of the mortality rates, in Eqn. (3.2.5), the joint distribution can be derived by use of an appropriate copula function, representing the correlation between the mortality rates. The calculation of the joint distribution via a copula function hasn't been taken to the scope of this work, however it can be considered as a further research.

### 3.2.3 Numerical results

In this section, we will provide some numerical findings on the expected mortality adjustment, i.e.  $\mathbb{E}[Z_t | \{d_1, d_2, \dots, d_s\}]$ . At time 0, we set  $\bar{\alpha} = \bar{\beta}$ , namely setting expected future mortality trend at time 0 as the best estimate mortality,  $\mathbb{E}[Q_{x,t}] = q_{x,h}^*$ . Assuming that  $\bar{\beta} = 100$  (based on the expert judgement), the expected mortality adjustment over time in the portfolios having one cohort and multiple cohorts (assuming a constant number of new policies in each year) is analyzed. Figs. 3.36 – 3.37 show results under the assumption of that the annual number of deaths follows the best estimate scenario, i.e.  $d_{x,s} = n_{x,s-1} q_{x,s}^*$ . The expected systematic deviation keeps close to 1, with some fluctuations at the beginning of the portfolio, in particular when the portfolio size is small. The fluctuations offset when multiple cohorts are addressed. Under the assumption that the number of deaths is 25% lower than expected under the best estimate scenario, i.e.  $d_{x,s} = 0.75 n_{x,s-1} q_{x,s}^*$ , the Figs. 3.38 – 3.39 show that the expected systematic deviation

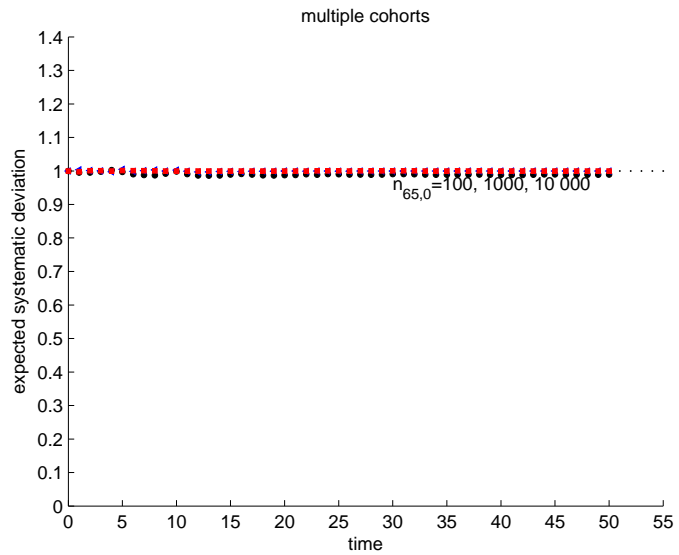
**Figure 3.36:** Ex-pected systematic deviation in mortality:  
 $\mathbb{E}[Z_t | \{d_1, d_2, \dots, d_s\}_{1 \leq s \leq t}]$   
 $\bar{\alpha} = \bar{\beta}, \bar{\beta} = 100;$   
 $d_{x,s} = n_{x,s-1} q_{x,s}^*$   
 One cohort



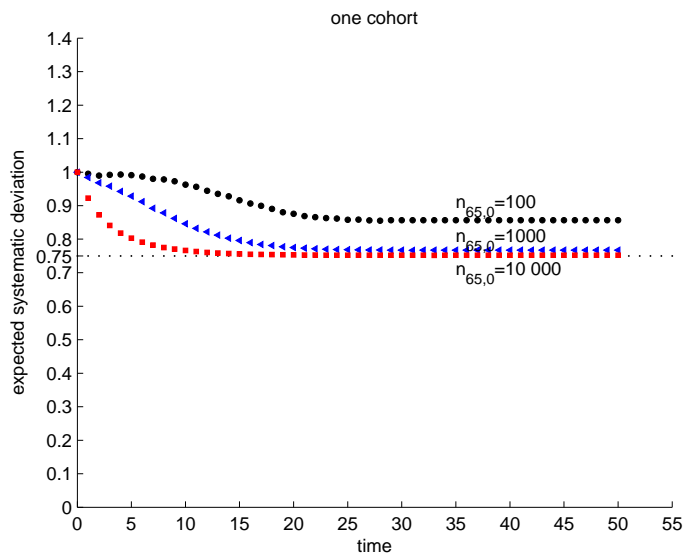
goes to 0.75, more slowly in small portfolios than in large portfolios. The results in Figs. 3.40- 3.41 assuming 25% higher number of deaths than expected under the best estimate scenario, i.e.  $d_{x,s} = 1.25 n_{x,s-1} q_{x,s}^*$  show that the expected systematic deviation goes to 1.25, again more slowly in small portfolios than in large portfolios. The convergence, both in cases 25% lower and higher number of death than expected under the best estimate scenario, is more rapid when multiple cohorts are addressed in the portfolio.

In Figs. 3.42 – 3.47, the role of the initial value of the coefficient of variation of the mortality rate is investigated, namely alternative values for  $\bar{\beta}$  are tested. The previous assumption  $\bar{\beta} = 100$  implies  $\mathbb{CV}[Q_{x,t}] = 0.10$ . The values  $\bar{\beta} = 25$  and  $\bar{\beta} = 400$  provide  $\mathbb{CV}[Q_{x,t}] = 0.20$  and  $\mathbb{CV}[Q_{x,t}] = 0.05$ , respectively. The results show that if the observed number of deaths is close to the expected under the best estimate scenario, in Figs. 3.42- 3.43, then the volatility of the mortality rate assumed at time  $t = 0$  does not affect the output and the expected systematic deviation keeps close to 1. However, if the observed number of deaths is different than the expected under the best estimate scenario, then the expected systematic deviation is affected by the volatility of the mortality rate assumed at time 0, the expected systematic deviation converges to the value of reduction observed in the

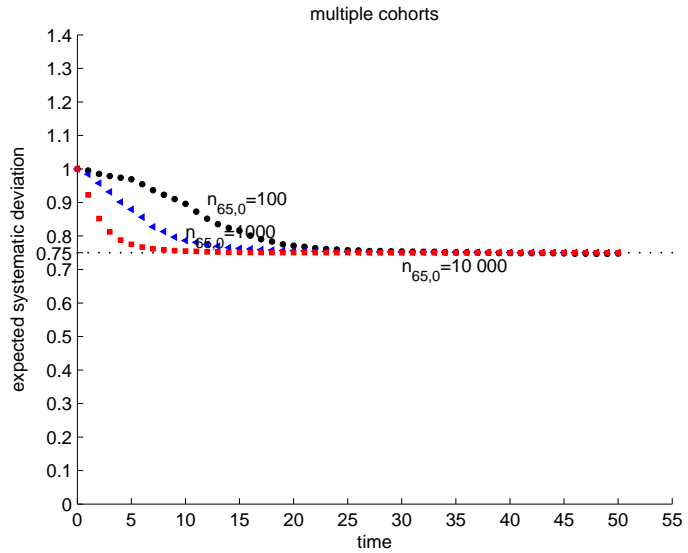
**Figure 3.37:** Expected systematic deviation in mortality:  
 $\mathbb{E}[Z_t | \{d_1, d_2, \dots, d_s\}_{1 \leq s \leq t}]$   
 $\bar{\alpha} = \bar{\beta}, \bar{\beta} = 100;$   
 $d_{x,s} = n_{x,s-1} q_{x,s}^*$   
 Multiple cohorts



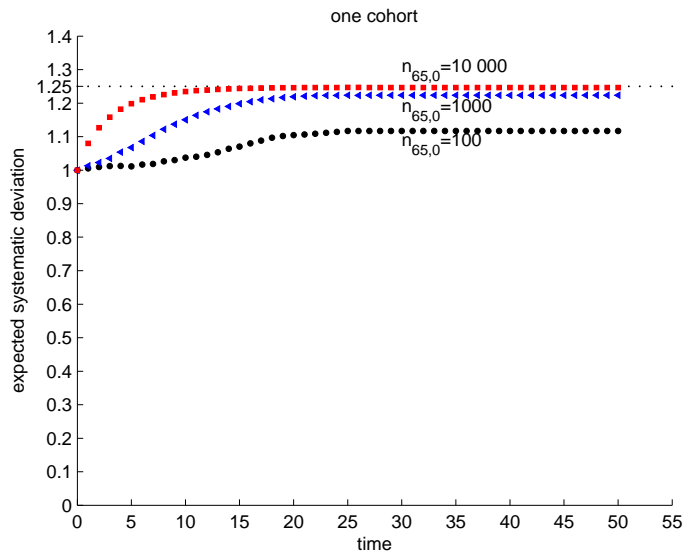
**Figure 3.38:** Expected systematic deviation in mortality:  
 $\mathbb{E}[Z_t | \{d_1, d_2, \dots, d_s\}_{1 \leq s \leq t}]$   
 $\bar{\alpha} = \bar{\beta}, \bar{\beta} = 100;$   
 $d_{x,s} = 0.75 n_{x,s-1} q_{x,s}^*$   
 One cohort



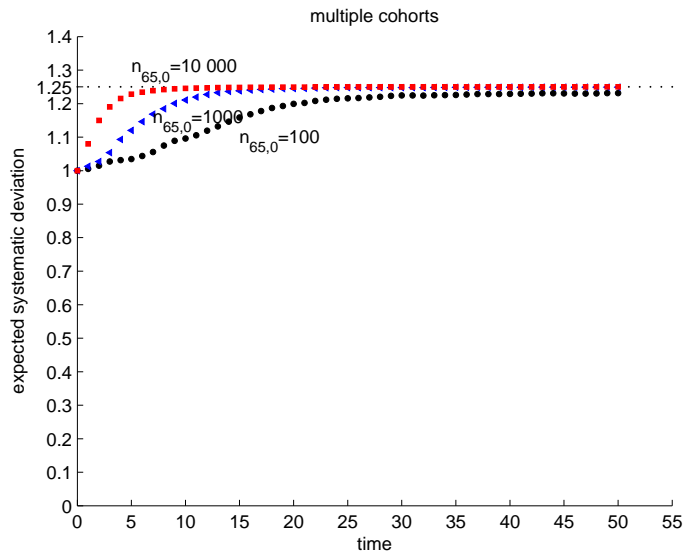
**Figure 3.39:** Ex-pected systematic deviation in mortality:  
 $\mathbb{E}[Z_t | \{d_1, d_2, \dots, d_s\}_{1 \leq s \leq t}]$   
 $\bar{\alpha} = \bar{\beta}, \bar{\beta} = 100;$   
 $d_{x,s} = 0.75 n_{x,s-1} q_{x,s}^*.$   
 Multiple cohorts



**Figure 3.40:** Ex-pected systematic deviation in mortality:  
 $\mathbb{E}[Z_t | \{d_1, d_2, \dots, d_s\}_{1 \leq s \leq t}]$   
 $\bar{\alpha} = \bar{\beta}, \bar{\beta} = 100;$   
 $d_{x,s} = 1.25 n_{x,s-1} q_{x,s}^*.$   
 One cohort

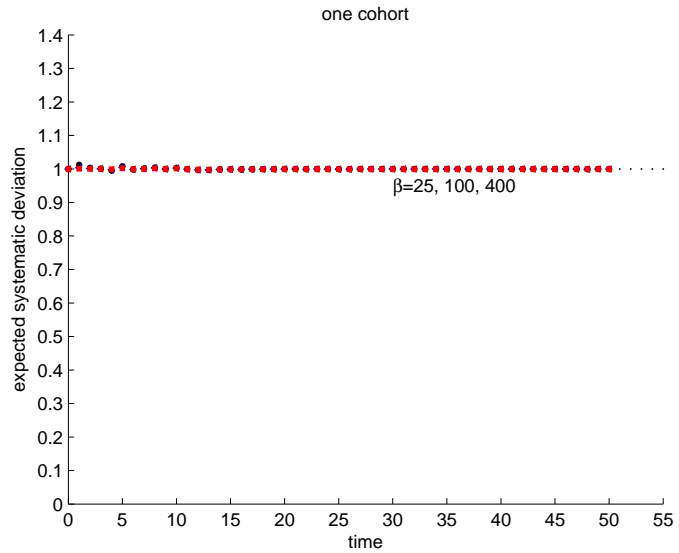


**Figure 3.41:** Ex-pected systematic deviation in mortality:  
 $\mathbb{E}[Z_t | \{d_1, d_2, \dots, d_s\}_{1 \leq s \leq t}]$   
 $\bar{\alpha} = \bar{\beta}, \bar{\beta} = 100;$   
 $d_{x,s} = 1.25 n_{x,s-1} q_{x,s}^*.$   
 Multiple cohorts

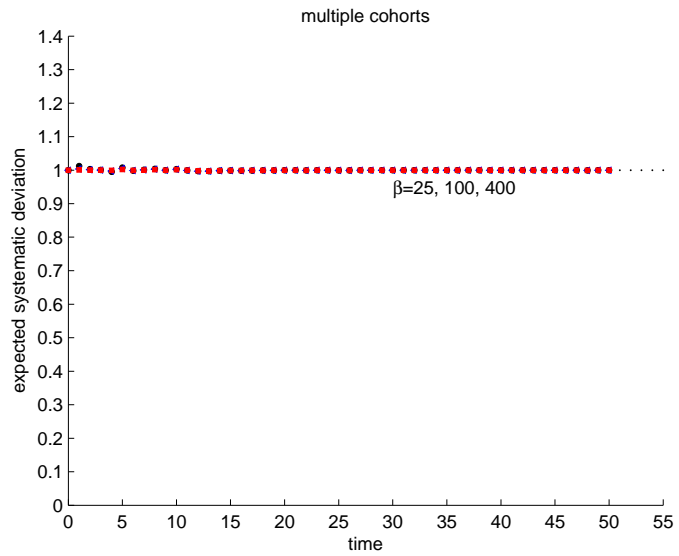


number of deaths respect to the expected under the best estimate scenario. The speed of convergence depends on the volatility level of the mortality rate assumed at time 0, i.e. the lower is the assumed volatility, the lower is the speed.

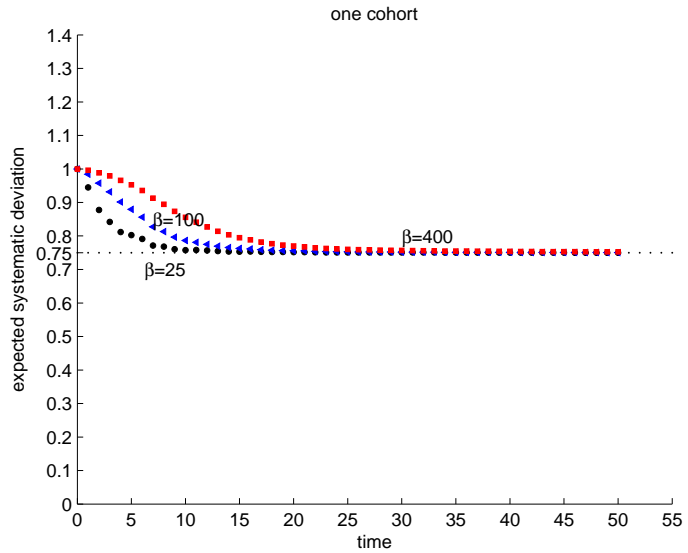
**Figure 3.42:** Ex-pected systematic deviation in mortality:  
 $\mathbb{E}[Z_t|\{d_1, d_2, \dots, d_s\}_{1 \leq s \leq t}]$   
 $\bar{\alpha} = \bar{\beta}, n_{65,s} = 1000;$   
 $d_{x,s} = n_{x,s-1} q_{x,s}^*$ . One cohort



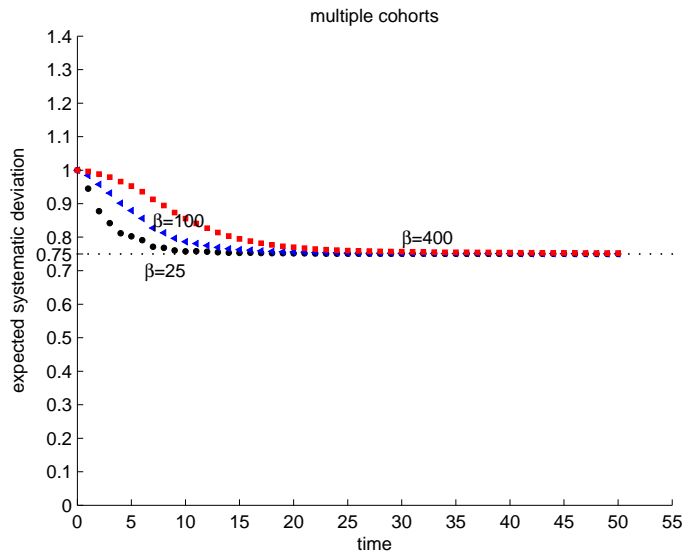
**Figure 3.43:** Ex-pected systematic deviation in mortality:  
 $\mathbb{E}[Z_t|\{d_1, d_2, \dots, d_s\}_{1 \leq s \leq t}]$   
 $\bar{\alpha} = \bar{\beta}, n_{65,s} = 1000;$   
 $d_{x,s} = n_{x,s-1} q_{x,s}^*$ . Multiple cohorts



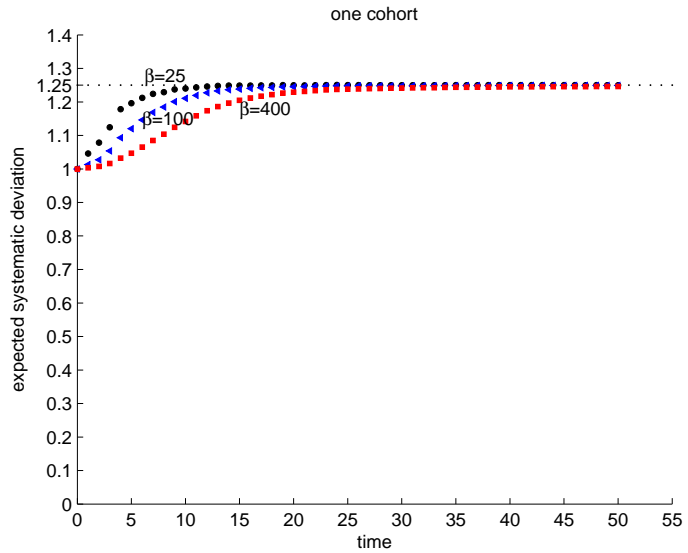
**Figure 3.44:** Ex-pected systematic deviation in mortality:  
 $\mathbb{E}[Z_t|\{d_1, d_2, \dots, d_s\}_{1 \leq s \leq t}]$   
 $\bar{\alpha} = \bar{\beta}, n_{65,s} = 1000;$   
 $d_{x,s} = 0.75 n_{x,s-1} q_{x,s}^*.$   
 One cohort



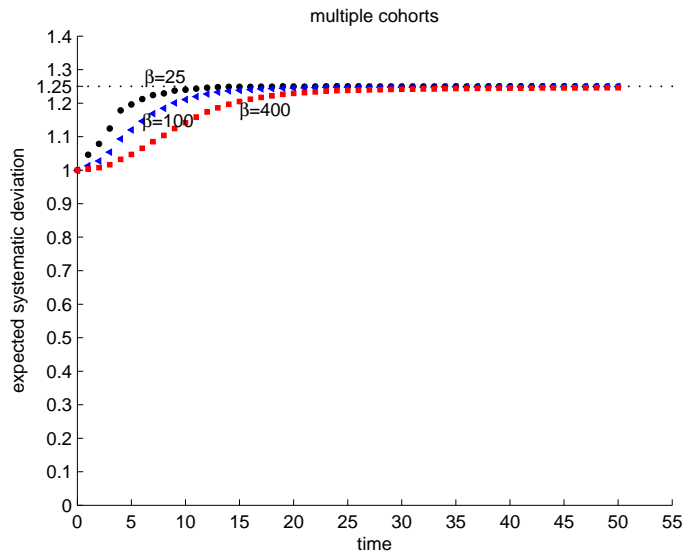
**Figure 3.45:** Ex-pected systematic deviation in mortality:  
 $\mathbb{E}[Z_t|\{d_1, d_2, \dots, d_s\}_{1 \leq s \leq t}]$   
 $\bar{\alpha} = \bar{\beta}, n_{65,s} = 1000;$   
 $d_{x,s} = 0.75 n_{x,s-1} q_{x,s}^*.$   
 Multiple cohorts



**Figure 3.46:** Ex-pected systematic deviation in mortality:  
 $\mathbb{E}[Z_t|\{d_1, d_2, \dots, d_s\}_{1 \leq s \leq t}]$   
 $\bar{\alpha} = \bar{\beta}, n_{65,s} = 1000;$   
 $d_{x,s} = 1.25 n_{x,s-1} q_{x,s}^*.$   
 One cohort



**Figure 3.47:** Ex-pected systematic deviation in mortality:  
 $\mathbb{E}[Z_t|\{d_1, d_2, \dots, d_s\}_{1 \leq s \leq t}]$   
 $\bar{\alpha} = \bar{\beta}, n_{65,s} = 1000;$   
 $d_{x,s} = 1.25 n_{x,s-1} q_{x,s}^*.$   
 Multiple cohorts



## CHAPTER 4

# CONCLUSION

In this thesis two stochastic mortality models in the literature have been investigated. Defining the stochastic nature of mortality trends via random variable(s), they use Bayesian inferential procedure to update the distribution defined on the variable(s). One of the models, the static model, has been extended to the continuous case with the allowance of the multiple cohorts. While defining the model, two approximation methods has adopted for the total number of deaths in the portfolio. The analysis showed that the model resulted from these two approximation methods provides the same level of projections regarding to the individuals lifetimes and their riskiness.

The numerical evidences show that the static model better performs in the portfolio with multiple cohorts. Although some fluctuations are observed at the beginning of the portfolio duration when the experienced mortality is deviated from the expected one, it resulted in increasing life expectancies, Lexis points and stable, some cases decreasing, random fluctuations in expected lifetimes. Besides it proved its capability of capturing the uncertainty risk in mortality trends and showed a decreasing behaviour over time. Considering that the model performs in the portfolios with multiple cohorts, the market data could be used in the inference procedure, instead of data of only one insurer. Some technical details still need further investigation; for example a proper estimation of the parameters defining the link between generations born in consecutive years.

Based on the analysis performed and considering its practicability, the dynamic model can suggest a stochastic approach to the aggregate mortality. However, the model is sensitive to the assumptions, i.e. the initial value of the param-

eters, in particular in regard of the volatility initially assigned to the mortality rate. Although the Bayesian inference procedure naturally leads to a reduction of volatility in time, a good judgment might be required in setting the initial parameters in order to avoid underestimating the volatility in short periods. The calculation of the life expectancy under dynamic model requires a joint distribution of survival and mortality rates. A further research can concern the derivation of the joint distributions of the random mortality rates under dynamic model by use of an appropriate copula function.

In the thesis, the static model and dynamic model's has been analysed independently and proved their capability in modelling uncertainty risk. A further research can concern a joint investigation, such as implementation on life annuities or solvency requirements etc.

# APPENDIX A

## KOLMOGOROV-TYPE APPROXIMATION

Kolmogorov approximation is based on the idea of taking multiples of differences of probabilities aiming to match the moments of the true and approximating distribution.

Let  $S$  be the sum of the random variables  $X_1, X_2, \dots, X_m$  with  $X_i \sim \text{Bin}(n_i, p_i)$ . Let  $p(i)$  denote the true  $\mathbb{P}(S = i)$ , and let  $p_k(i)$  denote the approximation to this probability based on differences up to order  $k$ . Then, the approximation probability  $p_0(i)$  to  $\mathbb{P}(S = i)$  is taken directly from the approximating distribution. The  $k$ -th backward difference at  $i$ , denoted  $\nabla^{(k)}p_0(i)$ , is

$$\nabla^{(k)}p_0(i) = \nabla^{(k-1)}p_0(i) - \nabla^{(k-1)}p_0(i-1) \quad (\text{A.0.1})$$

where  $p_0(i) = 0$  for  $i < 0$  and  $\nabla^{(0)}p_0(i) = p_0(i)$

The approximating distribution is improved by adding to it a linear combination of its backward differences, up to order  $k$ :

$$p_k(i) = p_0(i) + \sum_{j=1}^k a_j \nabla^{(j)}p_0(i) \quad (\text{A.0.2})$$

where the coefficients  $a_j$  have the values which matches the first  $k$  moments of the true and approximated distributions. The  $l$ -th moment of the  $k$ -th order approximation, from the formula  $\sum_{i=1}^{\infty} (i - \theta)^l p_k(i)$ , about some value  $\theta$ , is calculated as

$$\mu_{lk} = \mu_{l0} + \sum_{j=1}^k a_j b_{lj} \quad (\text{A.0.3})$$

where

$$b_{lj} = \sum_{i=0}^{\infty} (i - \theta)^l \nabla^{(j)} p_0(i) \quad (\text{A.0.4})$$

$\mu_{l0}$  denotes the  $l$ -th moment of the 'original' approximating distribution  $p_0(i)$ . Some algebra yields that

$$b_{lj} = \sum_{i=j}^l c_{lji} \mu_{l-i,0} \quad (\text{A.0.5})$$

where  $c_{lji} = \binom{l}{i} (-1)^j j! \mathcal{S}_i^{(j)}$ , and  $\mathcal{S}_i^{(j)}$  is the Stirling number of the second kind (see Abramowitz and Stegun, 1970).

As a conclusion, the  $l$ -th moment of the  $k$ -th order approximation can be written in terms of the moments of the original approximation distribution  $p_0(i)$  as

$$\mu_{lk} = \mu_{l0} + \sum_{j=1}^k a_j \sum_{i=j}^l c_{lji} \mu_{l-i,0} \quad (\text{A.0.6})$$

The  $\theta$  value about which the moments are calculated can be considered an input to the calculations. It could be, for example, 0 or could be assumed as the true mean (i.e.  $\nu$ ). The true mean for  $\theta$  is recommended by the authors in order to keep each moment as small as possible in the computations, and so as minimizing possible numerical instability in implementing the algorithm which is in the following.

1. Calculate the probabilities of the initial (original) approximating distribution  $p_0(i)$  for all  $i$ .
2. For  $k = 1, 2, \dots$ :
  - (a) Find the  $k$ -th moment of  $\nu_k$  of the true distribution about the true mean  $\nu$ .
  - (b) Find the  $k$ -th differences  $\nabla^{(k)} p_0(i)$  for all  $i$  using (A.0.1).
  - (c) From (A.0.6), calculate  $\mu_{k,k-1}$ , which is the  $k$ -th moment of the current approximating distribution (having probabilities  $p_{k-1}(i)$ ) about  $\nu$ .

- (d) Calculate  $a_k = (-1)^k(\nu_k - \mu_{k,k-1})/k!$ .
  - (e) Then improve the approximation to the true distribution as calculating  $p_k(i) = p_{k-1}(i) + a_k \nabla^{(k)} p_0(i)$  for all  $i$ .
3. Continue until  $|p_k(i) - p_{k-1}(i)|$  is sufficiently small for all  $i$ , which indicates that further differences will not improve the approximation, or until the desired number of moments has been matched.

The difficulty in Kolmogorov-type approximation, and implementing the algorithm defined above, lies in the calculation of the moments of the true distribution. The following section describes the computations to calculate the moments of the sum of the independently distributed random variables (the true distribution) using the moment and cumulant generating functions.

## Moments of the sum of the random variables

Let  $S$  be the sum of the independently distributed random variables  $X_1, X_2, \dots, X_m$  and let  $M_{X_i}(t)$  be the moment generating function (mgf) of the  $i$ -th random variable in the summation. The mgf of  $S$ , defined as  $M_S(t)$ , is written in terms of  $M_{X_i}(t)$ s as

$$M_S(t) = \prod_{i=1}^m M_{X_i}(t) \quad (\text{A.0.7})$$

As well known, the moments of a random variable,  $\nu_k$ , is calculated from the mgf as  $\nu_k = M_S^{(k)}(0)$  where  $M_S^{(k)}(0)$  is the  $k$ -th derivative of mgf at 0. However, taking the derivatives of the product defined above is not an easy task. Instead, the cumulant generating function (cgf), which is defined as the logarithm of the mgf, can be used. Let  $K_S(t)$  and  $K_{X_i}(t)$  be the cgf for the corresponding random variables. Then we have

$$K_S(t) = \sum_{i=1}^m K_{X_i}(t) \quad (\text{A.0.8})$$

The summation (A.0.8) will be a key point in linking the moments of  $X_i$ 's to the moments of  $S$ . The derivatives, so as the moments, are calculated up to order 6.

The relation between the mgf and cgf, i.e.

$$\begin{aligned} K_S(t) &= \ln M_S(t) \\ M_S(t) &= e^{K_S(t)} \end{aligned} \tag{A.0.9}$$

yields the following equations for the derivatives of the mgf of  $S$ :

$$\begin{aligned} M_S^{(1)}(t) &= K_S^{(1)}(t) M_S(t) \\ M_S^{(2)}(t) &= K_S^{(2)}(t) M_S(t) + K_S^{(1)}(t) M_S^{(1)}(t) \\ M_S^{(3)}(t) &= K_S^{(3)}(t) M_S(t) + 2 K_S^{(2)}(t) M_S^{(1)}(t) + K_S^{(1)}(t) M_S^{(2)}(t) \\ M_S^{(4)}(t) &= K_S^{(4)}(t) M_S(t) + 3 K_S^{(3)}(t) M_S^{(1)}(t) + 3 K_S^{(2)}(t) M_S^{(2)}(t) \\ &\quad + K_S^{(1)}(t) M_S^{(3)}(t) \\ M_S^{(5)}(t) &= K_S^{(5)}(t) M_S(t) + 4 K_S^{(4)}(t) M_S^{(1)}(t) + 6 K_S^{(3)}(t) M_S^{(2)}(t) \\ &\quad + 4 K_S^{(2)}(t) M_S^{(3)}(t) + K_S^{(1)}(t) M_S^{(4)}(t) \\ M_S^{(6)}(t) &= K_S^{(6)}(t) M_S(t) + 5 K_S^{(5)}(t) M_S^{(1)}(t) + 10 K_S^{(4)}(t) M_S^{(2)}(t) \\ &\quad + 10 K_S^{(3)}(t) M_S^{(3)}(t) + 5 K_S^{(2)}(t) M_S^{(4)}(t) + K_S^{(1)}(t) M_S^{(5)}(t) \end{aligned} \tag{A.0.10}$$

As seen from the equations in (A.0.10), the  $k$ -th derivative of mgf,  $M_S^{(k)}(t)$ , is a function of the  $k$ -th derivative of cgf,  $K_S^{(k)}(t)$ , and the derivatives of mgf at smaller orders, i.e.  $M_S(t), M_S^{(1)}(t), M_S^{(2)}(t), \dots, M_S^{(k-1)}(t)$ . What is needed to calculate  $M_S^{(k)}(t)$  is the term  $K_S^{(k)}(t)$  because the other terms have already been calculated. This takes us to the equation (A.0.8), which yields  $K_S^{(k)}(t) = \sum_{i=1}^m K_{X_i}^{(k)}(t)$ . So the next step will be calculation of derivatives  $K_{X_i}^{(k)}(t)$ ,  $i = 1, 2, \dots, m$ .

Starting from the relation between the mgf and cgf, i.e.  $K_{X_i}(t) = \ln M_{X_i}(t)$ ,

the derivatives of the cgf of  $X_i$  is calculated as follows:

$$\begin{aligned}
K_{X_i}^{(1)}(t) &= M_{X_i}^{(1)}(t)/M_S(t) \\
K_{X_i}^{(2)}(t) &= M_{X_i}^{(2)}(t)/M_S(t) - \left(K_{X_i}^{(1)}(t)\right)^2 \\
K_{X_i}^{(3)}(t) &= M_{X_i}^{(3)}(t)/M_S(t) - 3 K_{X_i}^{(2)}(t) K_{X_i}^{(1)}(t) - \left(K_{X_i}^{(1)}(t)\right)^3 \\
K_{X_i}^{(4)}(t) &= M_{X_i}^{(4)}(t)/M_S(t) - 4 K_{X_i}^{(3)}(t) K_{X_i}^{(1)}(t) - 6 K_{X_i}^{(2)}(t) \left(K_{X_i}^{(1)}(t)\right)^2 \\
&\quad - \left(K_{X_i}^{(2)}(t)\right)^2 - \left(K_{X_i}^{(1)}(t)\right)^4 \\
K_{X_i}^{(5)}(t) &= M_{X_i}^{(5)}(t)/M_S(t) - 5 K_{X_i}^{(4)}(t) K_{X_i}^{(1)}(t) - 10 K_{X_i}^{(3)}(t) \left(K_{X_i}^{(1)}(t)\right)^2 \\
&\quad - 10 K_{X_i}^{(2)}(t) \left(K_{X_i}^{(1)}(t)\right)^3 - 10 K_{X_i}^{(3)}(t) K_{X_i}^{(2)}(t) \\
&\quad - 15 \left(K_{X_i}^{(2)}(t)\right)^2 K_{X_i}^{(1)}(t) - \left(K_{X_i}^{(1)}(t)\right)^5 \\
K_{X_i}^{(6)}(t) &= M_{X_i}^{(6)}(t)/M_S(t) - 6 K_{X_i}^{(5)}(t) K_{X_i}^{(1)}(t) - 15 K_{X_i}^{(4)}(t) K_{X_i}^{(2)}(t) \\
&\quad - 15 K_{X_i}^{(4)}(t) \left(K_{X_i}^{(1)}(t)\right)^2 - 20 K_{X_i}^{(3)}(t) \left(K_{X_i}^{(1)}(t)\right)^3 \\
&\quad - 15 K_{X_i}^{(2)}(t) \left(K_{X_i}^{(1)}(t)\right)^4 - 45 \left(K_{X_i}^{(2)}(t)\right)^2 \left(K_{X_i}^{(1)}(t)\right)^2 \\
&\quad - 60 K_{X_i}^{(3)}(t) K_{X_i}^{(2)}(t) K_{X_i}^{(1)}(t) - 10 \left(K_{X_i}^{(3)}(t)\right)^2 \\
&\quad - 15 \left(K_{X_i}^{(2)}(t)\right)^3 - \left(K_{X_i}^{(1)}(t)\right)^6
\end{aligned} \tag{A.0.11}$$

As seen from the equations in (A.0.11), the  $k$ -th derivative of cgf,  $K_{X_i}^{(k)}(t)$ , is a function of the  $k$ -th derivative of mgf,  $M_{X_i}^{(k)}(t)$ , and the derivatives of cgf at smaller orders, i.e.  $K_{X_i}^{(1)}(t), K_{X_i}^{(2)}(t), \dots, K_{X_i}^{(k-1)}(t)$ . What is needed to calculate  $K_{X_i}^{(k)}(t)$  is the term  $M_{X_i}^{(k)}(t)$  because the other terms have already been calculated.

Under the binomial assumption the random variables  $X_i$  with index  $n_i$  and the probability  $p_i$ , the mgf  $M_{X_i}(t)$  is

$$M_{X_i}(t) = \left(1 - p_i - p_i e^t\right)^{n_i} \tag{A.0.12}$$

and the derivatives  $M_{X_i}^{(k)}(t)$  are calculated as follows:

$$\begin{aligned}
M_{X_i}^{(1)}(t) &= n_i p_i e^t (1 - p_i - p_i e^t)^{n_i-1} \\
M_{X_i}^{(2)}(t) &= M_{X_i}^{(1)}(t) + \frac{n_i!}{(n_i-2)!} p_i^2 e^{2t} (1 - p_i - p_i e^t)^{n_i-2} \\
M_{X_i}^{(3)}(t) &= 3 M_{X_i}^{(2)}(t) - 2 M_{X_i}^{(1)}(t) + \frac{n_i!}{(n_i-3)!} p_i^3 e^{3t} (1 - p_i - p_i e^t)^{n_i-3} \\
M_{X_i}^{(4)}(t) &= 6 M_{X_i}^{(3)}(t) - 11 M_{X_i}^{(2)}(t) + 6 M_{X_i}^{(1)}(t) \\
&\quad + \frac{n_i!}{(n_i-4)!} p_i^4 e^{4t} (1 - p_i - p_i e^t)^{n_i-4} \\
M_{X_i}^{(5)}(t) &= 10 M_{X_i}^{(4)}(t) - 35 M_{X_i}^{(3)}(t) + 50 M_{X_i}^{(2)}(t) - 24 M_{X_i}^{(1)}(t) \\
&\quad + \frac{n_i!}{(n_i-5)!} p_i^5 e^{5t} (1 - p_i - p_i e^t)^{n_i-5} \\
M_{X_i}^{(6)}(t) &= 15 M_{X_i}^{(5)}(t) - 85 M_{X_i}^{(4)}(t) + 225 M_{X_i}^{(3)}(t) - 274 M_{X_i}^{(2)}(t) \\
&\quad + 120 M_{X_i}^{(1)}(t) + \frac{n_i!}{(n_i-6)!} p_i^6 e^{6t} (1 - p_i - p_i e^t)^{n_i-6}
\end{aligned} \tag{A.0.13}$$

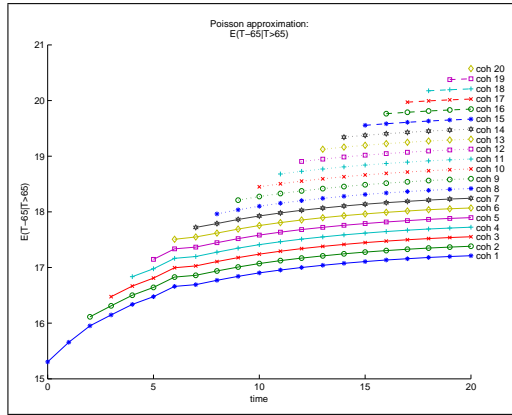
As a conclusion, the overall process to calculate the moments of a random variable  $S$ , which is defined as the sum of independently distributed random variables  $X_1, X_2, \dots, X_m$  with  $X_i \sim \text{Bin}(n_i, p_i)$ , consists in following steps: for each  $k = 1, 2, \dots$

1. Calculate of  $M_{X_i}^{(k)}(t)$  (defined in equations (A.0.13))
2. Calculate of  $K_{X_i}^{(k)}(t)$  using  $M_{X_i}^{(k)}(t)$  (defined in equations (A.0.11))
3. Calculate of  $K_S^{(k)}(t)$  as  $\sum_{i=1}^m K_{X_i}^{(k)}(t)$
4. Calculate of  $M_S^{(k)}(t)$  using  $K_S^{(k)}(t)$  (defined in equations (A.0.10))

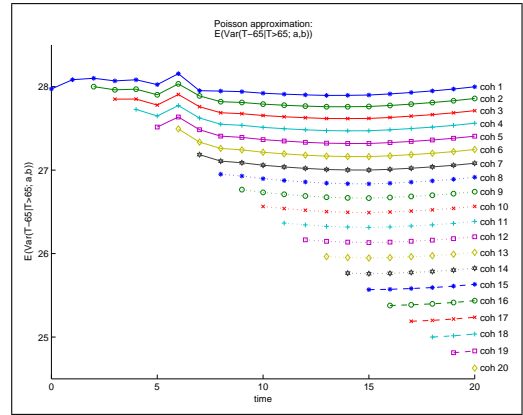
# APPENDIX B

## STATIC MODEL FIGURES

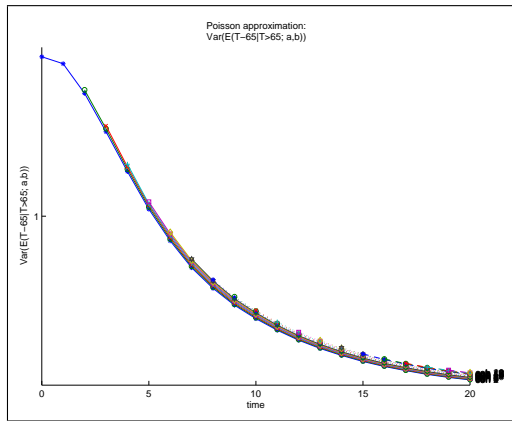
Static model has been analysed under different parameter values, i.e. different cohort sizes,  $n_{x_0,j-1} = 100$  and  $1000$ , and assuming the mortality observed in the portfolio equal to the expected mortality, 25% lower and higher than expected mortality, namely  $d_{x,j} = n_{x,j-1}q_{x,j}$ ,  $d_{x,j} = 0.75 n_{x,j-1}q_{x,j}$ ,  $d_{x,j} = 1.25 n_{x,j-1}q_{x,j}$ , respectively. The analysis provided a number of graphical results, which is not feasible to include all in the text. Hence, some of them, being not included in the text, has been provided in this Appendix.



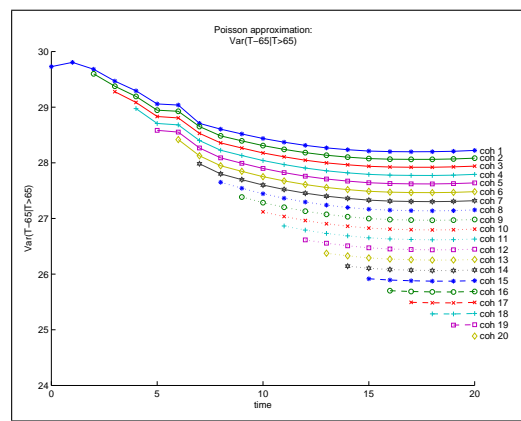
(a) Expected lifetime



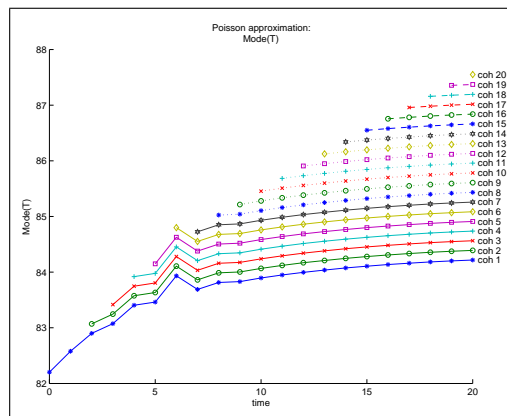
(b) Random fluctuations



(c) Uncertainty risk



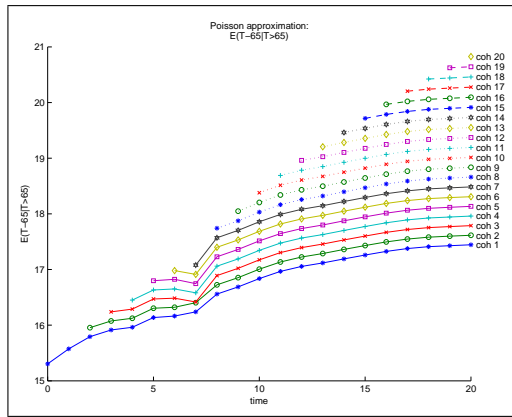
(d) Variance of the lifetime



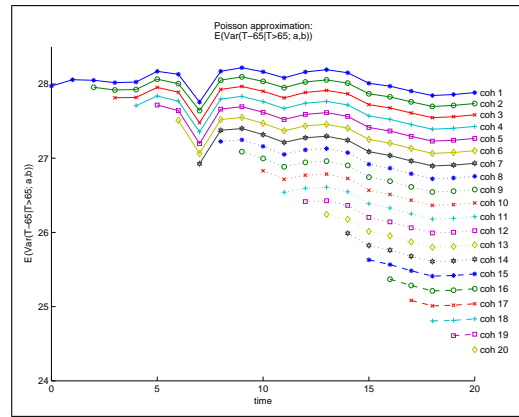
(e) Lexis point

**Figure B.1:** The quantities of the random lifetime under Poisson approximation:

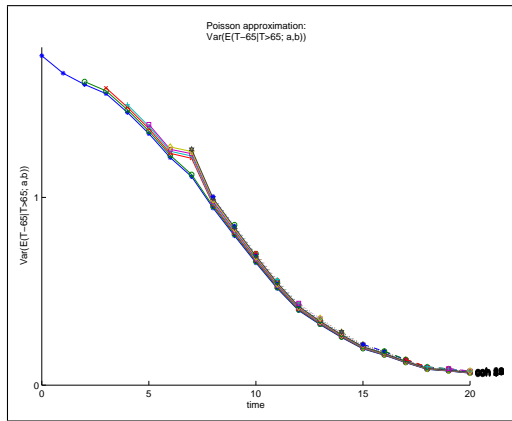
$$n_{x_0, j-1} = 100, d_{x, j} = n_{x, j-1} q_{x, j}, \text{ multiple cohorts}$$



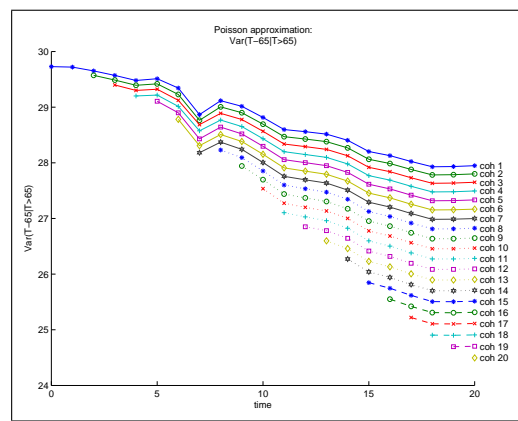
(a) Expected lifetime



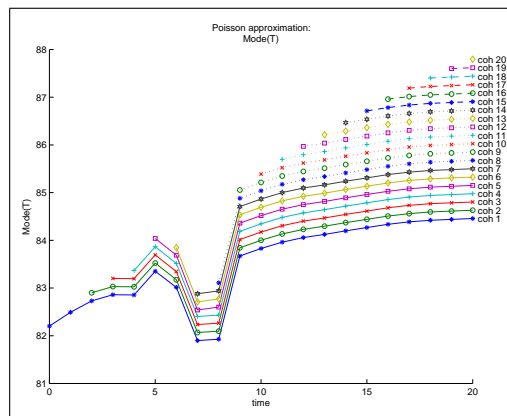
(b) Random fluctuations



(c) Uncertainty risk

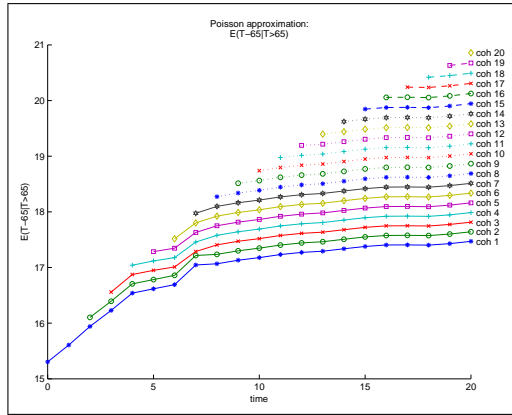


(d) Variance of the lifetime

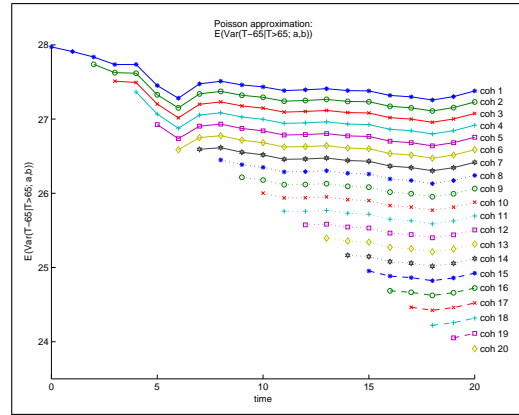


(e) Lexis point

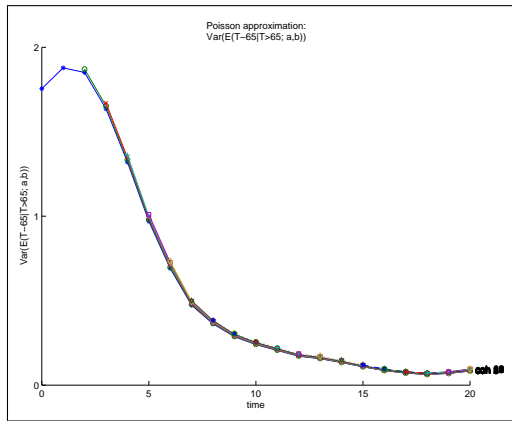
**Figure B.2:** The quantities of the random lifetime under Poisson approximation:  
 $n_{x_0, j-1} = 100$ ,  $d_{x, j} = 0.75 n_{x, j-1} q_{x, j}$ , multiple cohorts



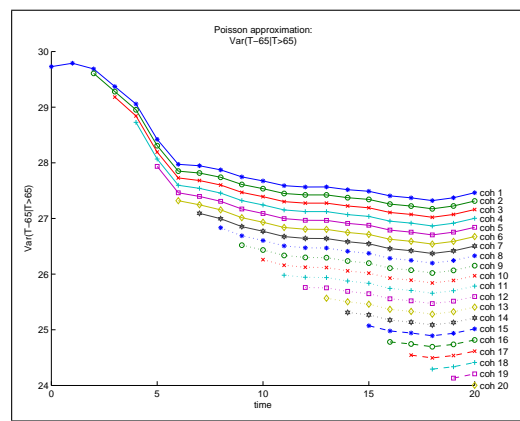
(a) Expected lifetime



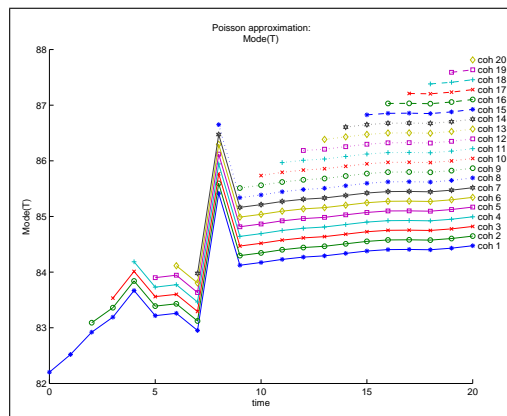
(b) Random fluctuations



(c) Uncertainty risk

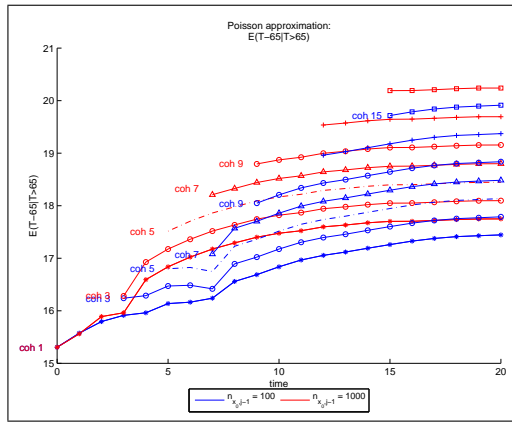


(d) Variance of the lifetime

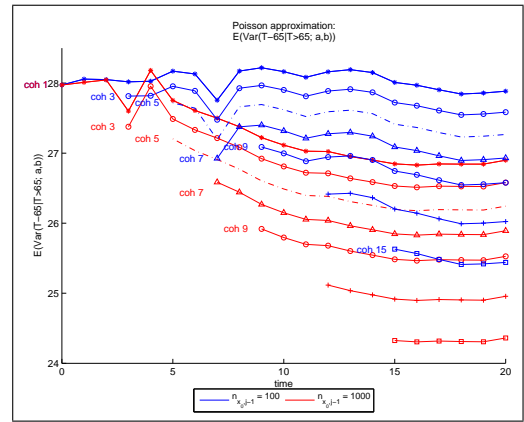


(e) Lexis point

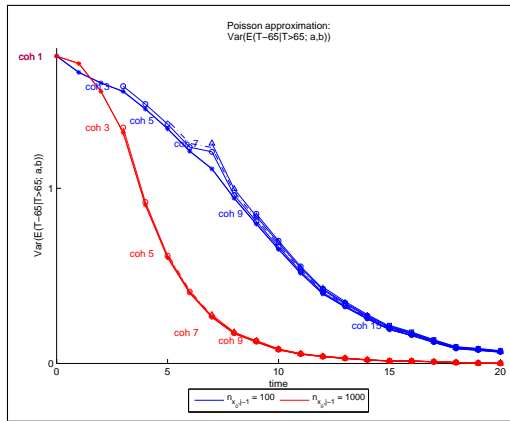
**Figure B.3:** The quantities of the random lifetime under Poisson approximation:  
 $n_{x_0,j-1} = 100$ ,  $d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ , multiple cohorts



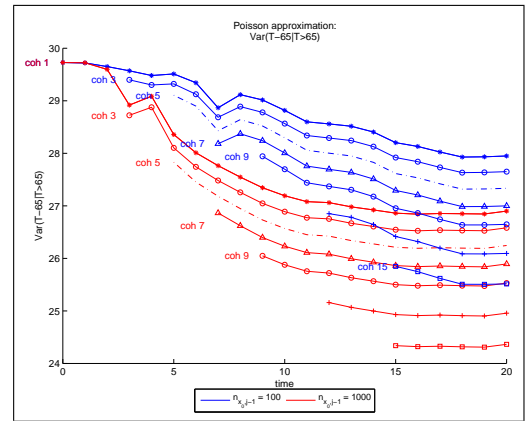
(a) Expected lifetime



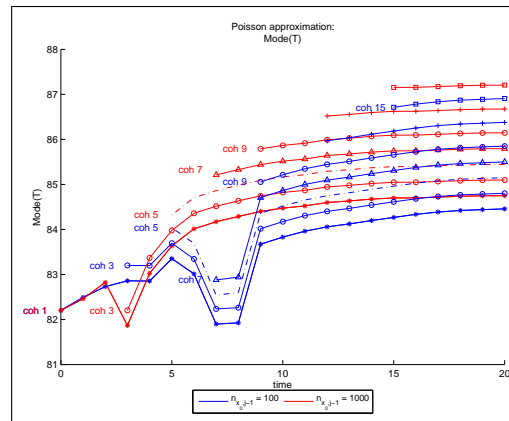
(b) Random fluctuations



(c) Uncertainty risk

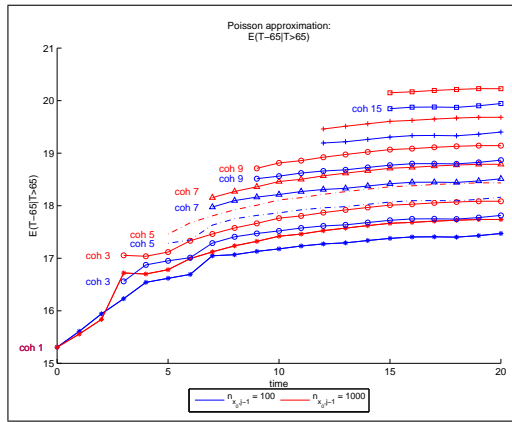


(d) Variance of the lifetime

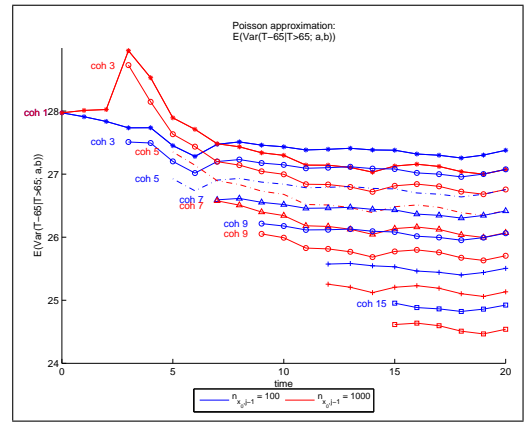


(e) Lexis point

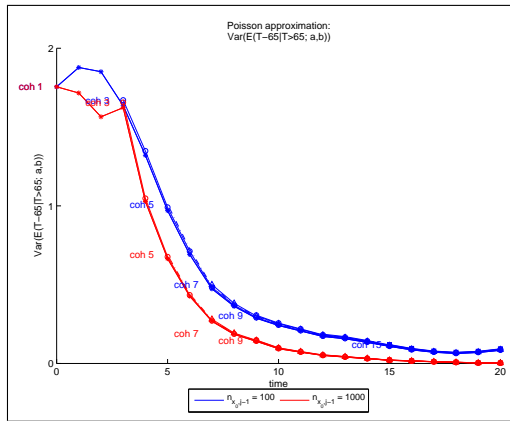
**Figure B.4:** The quantities of the random lifetime under Poisson approximation different cohort sizes:  $n_{x_0, j-1} = 100, 1000$ ,  $d_{x, j} = 0.75 n_{x, j-1} q_{x, j}$ , multiple cohorts



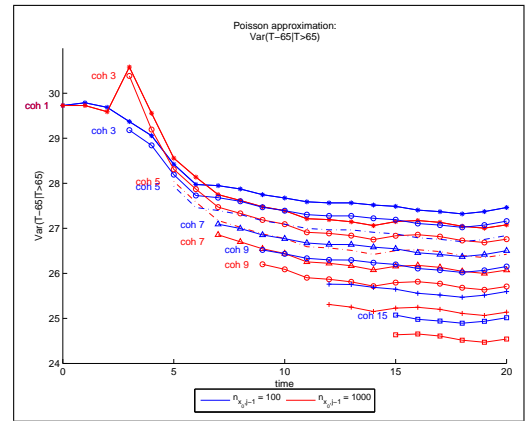
(a) Expected lifetime



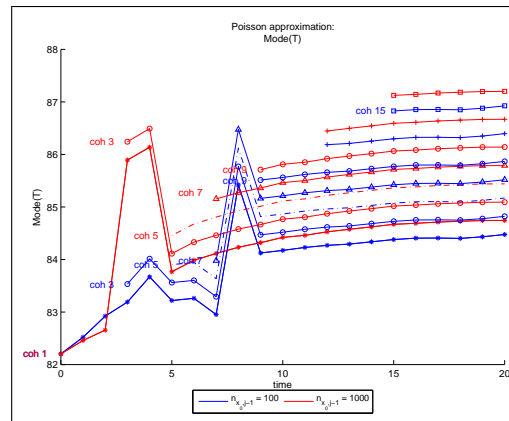
(b) Random fluctuation



(c) Uncertainty risk

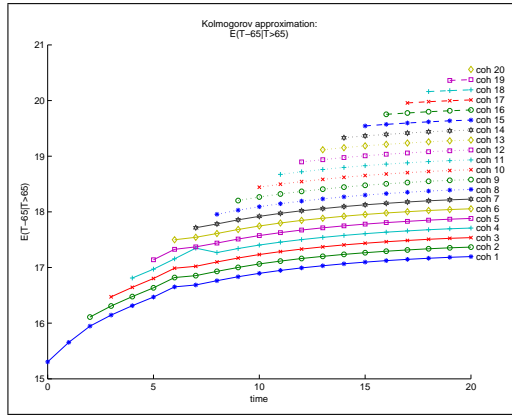


(d) Variance of the lifetime

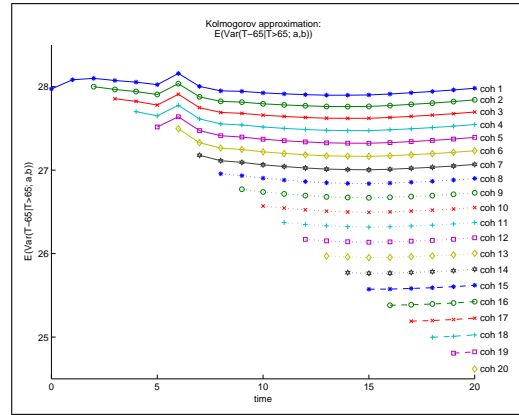


(e) Lexis point

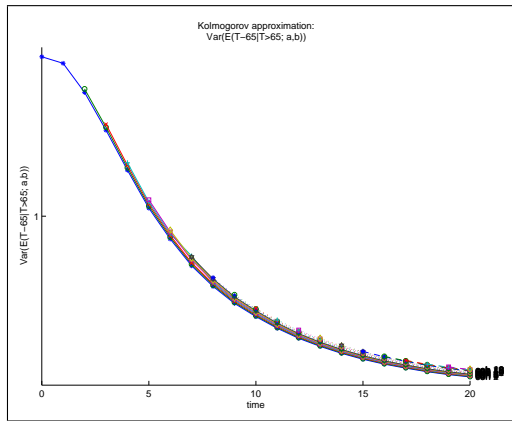
**Figure B.5:** The quantities of the random lifetime under Poisson approximation different cohort sizes:  $n_{x_{0,j-1}} = 100, 1000$ ,  $d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ , multiple cohorts



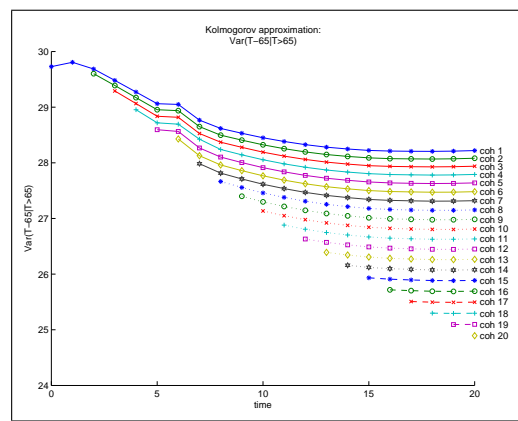
(a) Expected lifetime



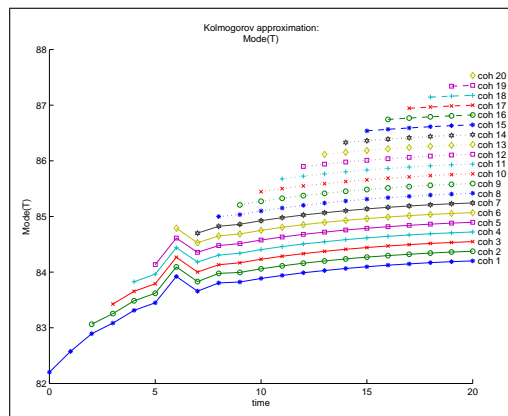
(b) Random fluctuations



(c) Uncertainty risk

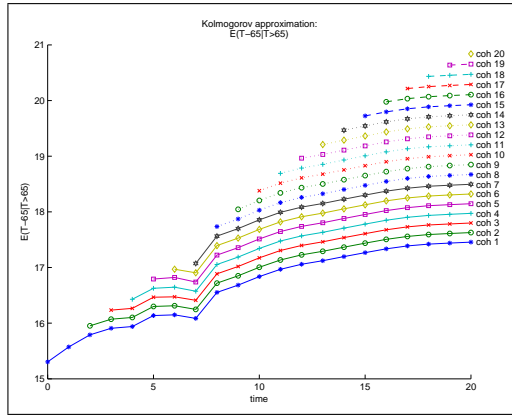


(d) Variance of the lifetime

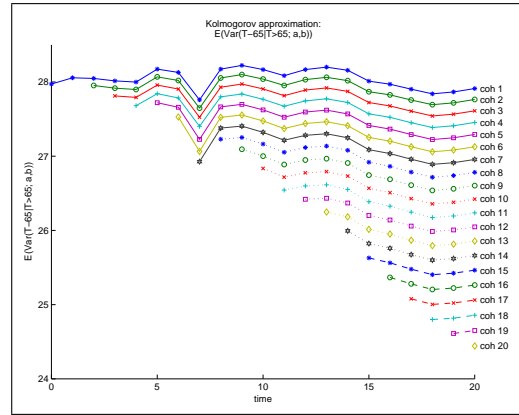


(e) Lexis point

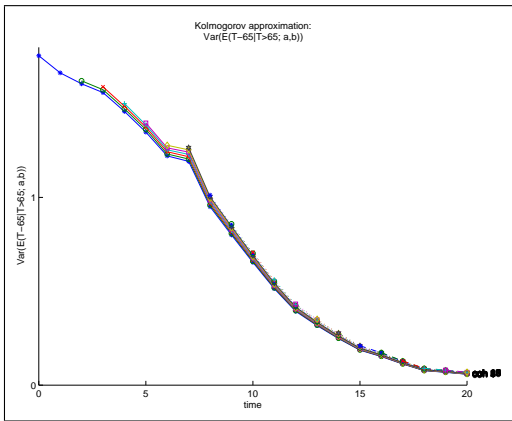
**Figure B.6:** The quantities of the random lifetime under Kolmogorov-type approximation:  $n_{x_0,j-1} = 100$ ,  $d_{x,j} = n_{x,j-1}q_{x,j}$



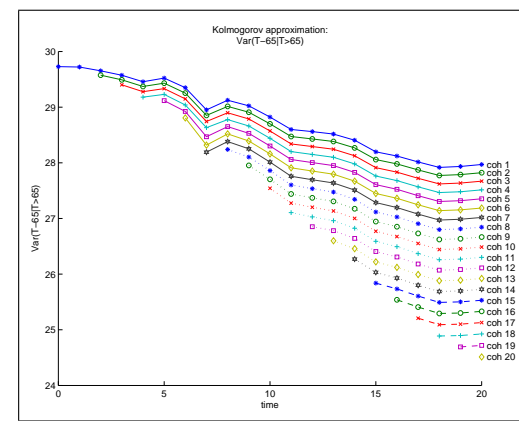
(a) Expected lifetime in cohorts



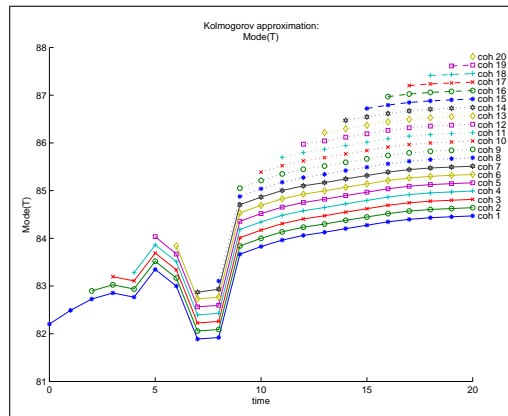
(b) Random fluctuation risk



(c) Uncertainty risk

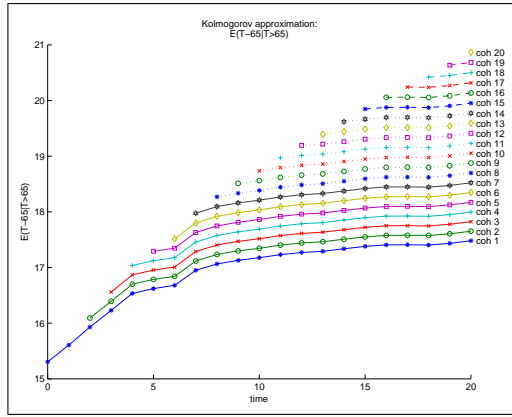


(d) Variance of the lifetime

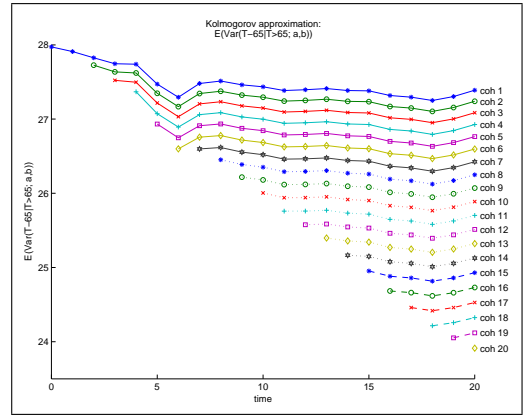


(e) Lexis point

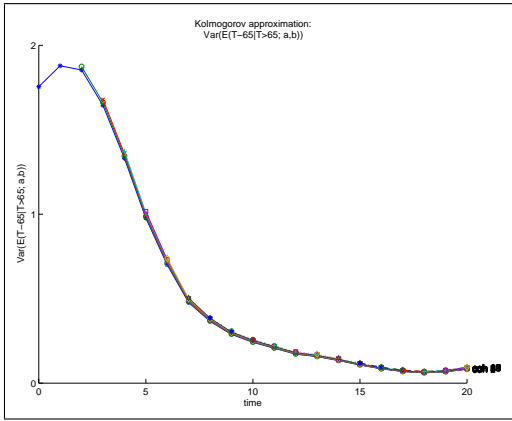
**Figure B.7:** The quantities of the random lifetime under Kolmogorov-type approximation:  $n_{x_0,j-1} = 100$ ,  $d_{x,j} = 0.75 n_{x,j-1} q_{x,j}$



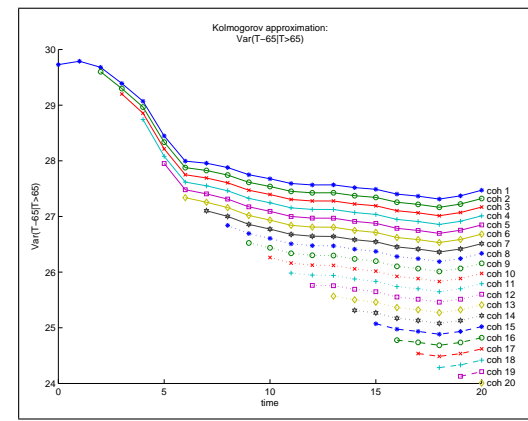
(a) Expected lifetime



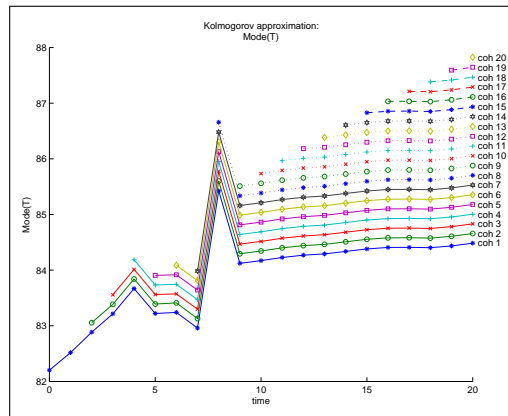
(b) Random fluctuation risk



(c) Uncertainty risk

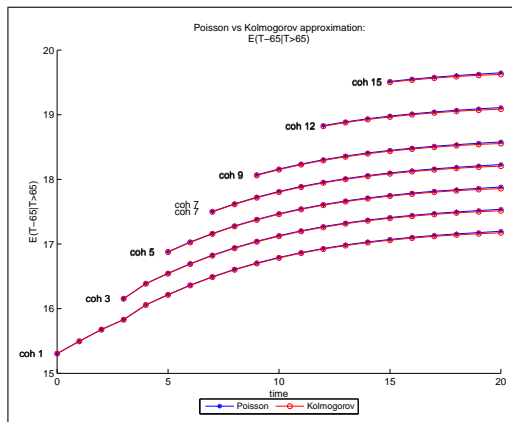


(d) Variance of the lifetime

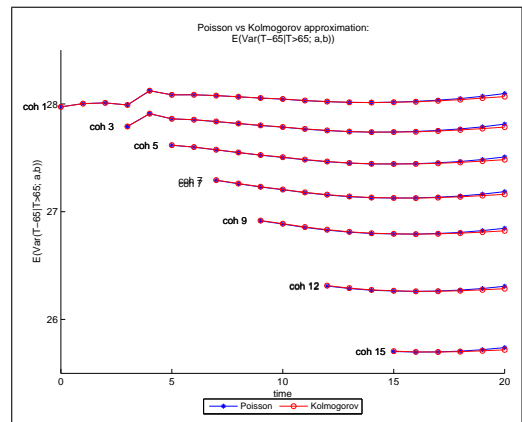


(e) Lexis point

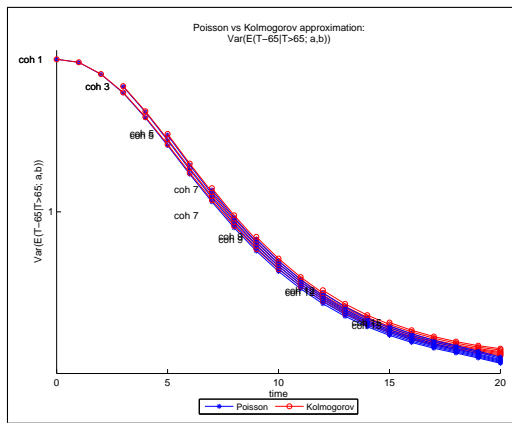
**Figure B.8:** The quantities of the random lifetime under Kolmogorov-type approximation:  $n_{x_0,j-1} = 100$ ,  $d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$



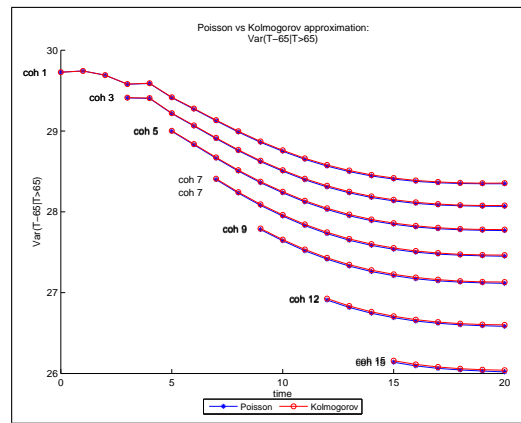
(a) Expected lifetime



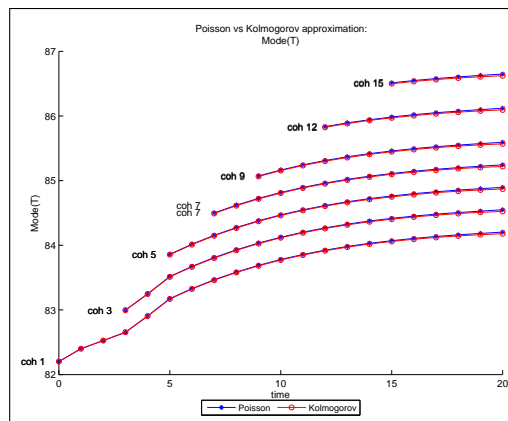
(b) Random fluctuations



(c) Uncertainty risk

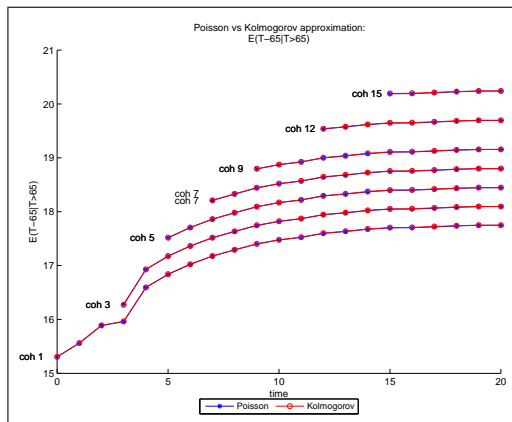


(d) Variance of the lifetime

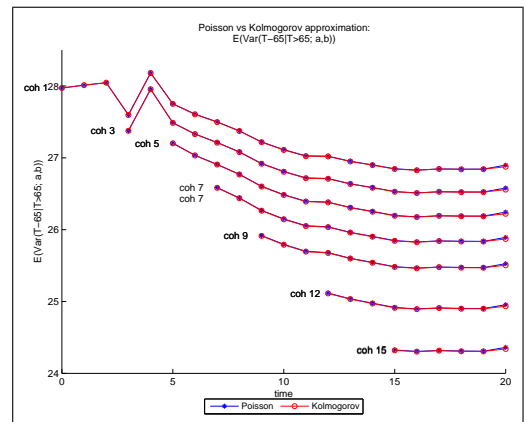


(e) Lexis point

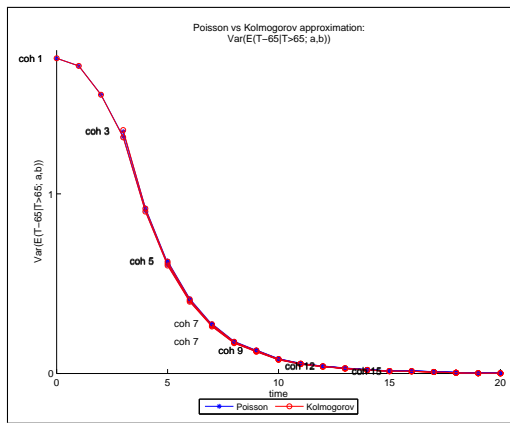
**Figure B.9:** The quantities of the random lifetime under Poisson and Kolmogorov-type approximations:  $n_{x_0, j-1} = 1000$ ,  $d_{x, j} = n_{x, j-1}q_{x, j}$



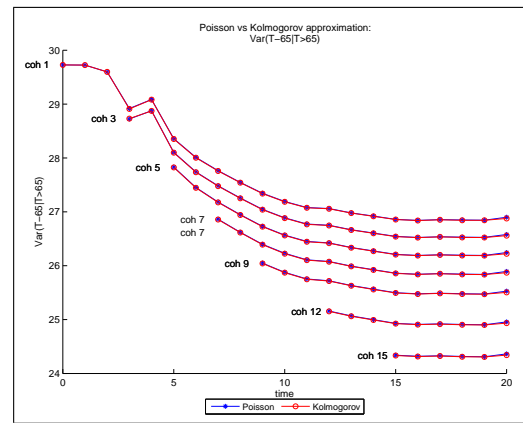
(a) Expected lifetime



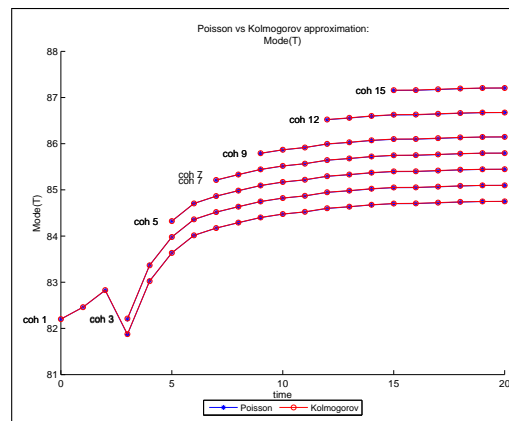
(b) Random fluctuations



(c) Uncertainty risk

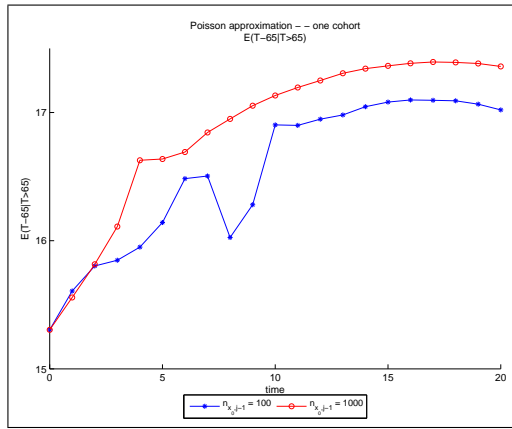


(d) Variance of the lifetime

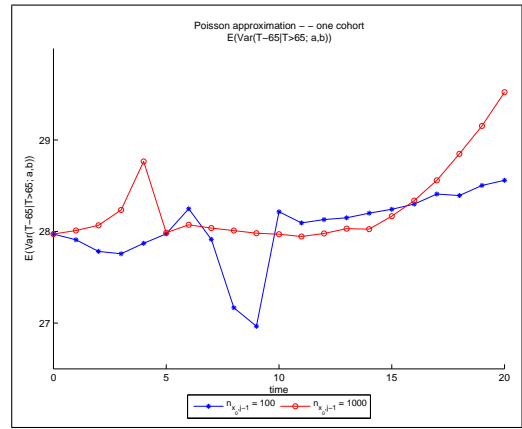


(e) Lexis point

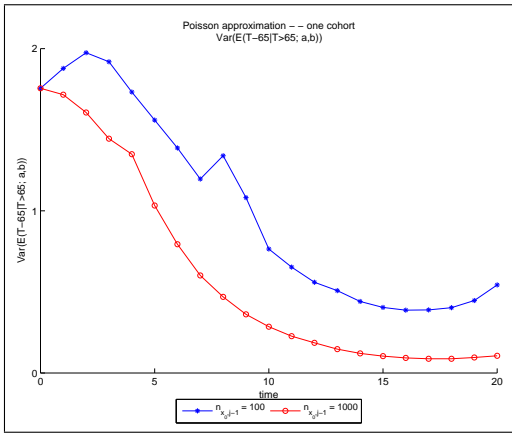
**Figure B.10:** The quantities of the random lifetime under Poisson and Kolmogorov-type approximations:  $n_{x_0, j-1} = 1000$ ,  $d_{x, j} = 0.75 n_{x, j-1} q_{x, j}$



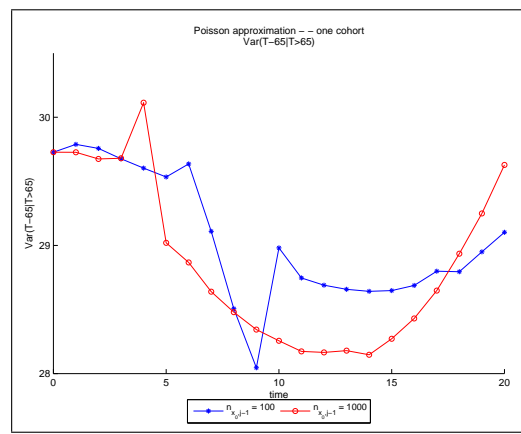
(a) Expected lifetime



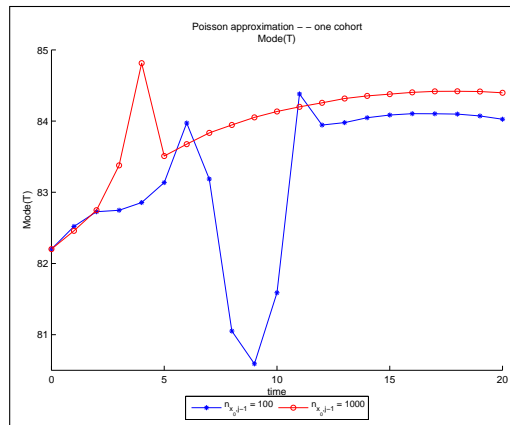
(b) Random fluctuations



(c) Uncertainty risk



(d) Variance of the lifetime



(e) Lexis point

**Figure B.11:** The quantities of the random lifetime under Poisson approximation:  $n_{x_0,j-1} = 100, 1000$ ,  $d_{x,j} = 1.25 n_{x,j-1} q_{x,j}$ , one cohort

## REFERENCES

- [1] BUTLER, K., AND STEPHENS, M. The distribution of a sum of binomial random variables. Tech. rep., Dep. of Statistics, Stanford University, 1993.
- [2] CMI. An interim basis for adjusting the "92" series mortality projections for cohort effects. Working Paper 1, The Faculty of Actuaries and Institute of Actuaries, 2002.
- [3] CMI. Stochastic projection methodologies: Further progress and p-spline model features, eexample results and implications. Working Paper 20, The Faculty of Actuaries and Institute of Actuaries, 2006.
- [4] LEE, R. D., AND CARTER, L. R. Modeling and forecasting u.s. mortality. *Journal of the American Statistical Association* 87, 419 (1992), 659–671.
- [5] OLIVIERI, A. Uncertainty in mortality projections: An actuarial perspective. *Insurance Mathematics and Economics* 29, 2 (2001), 231–245.
- [6] OLIVIERI, A., AND PITACCO, E. Inference about mortality improvements in life annuity portfolios. In: Transaction of the 27th International Congress of Actuaries, Cancun (Mexico) (2002), 2002.
- [7] OLIVIERI, A., AND PITACCO, E. Solvency requirements for pension annuities. *Journal of Pension Economics and Finance* 2, 2 (2003), 127–157.
- [8] OLIVIERI, A., AND PITACCO, E. Stochastic mortality: The impact on target capital. *Astin Bulletin* 39, 2 (2009), 541–563.
- [9] OLIVIERI, A., AND PITACCO, E. *Introduction to Insurance Mathematics*. Springer, 2010.

- [10] OLIVIERI, A., AND PITACCO, E. Life tables in actuarial models: from the deterministic setting to a bayesian approach. *AStA Advances in Statistical Analysis* 96, 2 (2011), 127–153.