ON THE COMMUTATIVITY OF s-UNITAL RINGS AND PERIODIC RINGS (*)

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Sommario. - *In questo lavoro vengono provati due teoremi relativi alla commutatività di anelli s-unitali e di anelli periodici.*

Summary. - *In this paper two theorems have been proved for the commutativity of s-unital rings and periodic rings respectively.*

Let R be a ring, Z its centre and $x, y \in R$. For the following properties (1) and (2), n is a fixed positive integer and for the properties (3), (4) and (5) n is a positive integer depending on pair x, y. For the last property ($5'$) the positive integer n' depends on pair y, x.

1) $C(n) : n[x, y] = 0$ implies $[x, y] = 0$.
2) $P_{11}^{*}(Z) : x^n(xy)^n - (yx)^n x^n \in Z$.
3) $P'(n) : [x^n, y^n] = 0$.
4) $C'(n) : n[x, y] = 0$ implies $[x, y] = 0$.
5) $C'_{n+1}(xy) : [(xy)^{n+1} - x^{n+1}y^{n+1}, xy] = 0$.
5') $C'_{n'+1}(yx) : [(yx)^{n'+1} - y^{n'+1}x^{n'+1}, yx] = 0$.

Infact (5) and ($5'$) represent the same condition provided $n = n'$. They are different in case otherwise. In theorem 1, it is shown that if R is an s-unital ring satisfying conditions $P'(n)$, $C'(n)$ and [$C'_{n+1}(xy)$ and $C'_{n'+1}(yx)$], then R is commutative. In theorem 2, it has been proved that a periodic ring, in which nilpotents of R forms a commutative set and the ring satisfying conditions $C((n+1)n)$

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and $P_{11}^*(Z)$ is commutative. In the end counter examples are given which show that the hypotheses of our theorems are not altogether superfluous. It is to be noticed that $C(1)$, $C'(1)$, $P'(1)$ are vacuously true namely empty conditions.

1. Introduction.

Throughout this paper R represents an associative ring (may be without unity 1), Z the centre of R, $C(R)$ the commutator ideal of R, N the set of nilpotent elements of R and $[x,y]=xy-yx$ for all $x,y\in R$.

A ring R is called a left (resp. right) s-unital if $x\in Rx$ (resp. $x\in xR$) for each $x\in R$. Further R is called s-unital if it is both left as well as right s-unital, that is, $x\in xR\cap Rx$, for each $x\in R$.

If R is s-unital (resp. left or right s-unital) then for any finite subset F of R there exists an element $e\in R$ such that $ex=xe=x$ (resp. $ex=x$ and $xe=x$) for all $x\in F$. Such an element e is called the pseudo (resp. pseudo left or pseudo right) identity of F in R.

A ring R is called periodic if for every x in R, there exist distinct positive integers $m=m(x)$, $n=n(x)$ such that $x^m=x^n$. By a theorem of Chacron (cf. [7, Theorem 1]) R is periodic if and only if for each $x\in R$, there exists a positive integer $k=k(x)$ and a polynomial $f(\lambda)$, $f_x(\lambda)$ with integer coefficients such that $x^k = x^{k+1}f(x)$.

In the present paper we use the following notations for the different properties. Among the following first 8 properties we take n to be a fixed positive integer.

1) $P(n):[x^n,y^n]=0$, for all $x,y\in R$.
2) $C(n):n[x,y]=0$ implies $[x,y]=0$, for all $x,y\in R$.
3) $P_{n+1}(xy):(xy)^{n+1}-x^{n+1}y^{n+1}=0$, for all $x,y\in R$.
4) $P_{n+1}(Z):(xy)^{n+1}-x^{n+1}y^{n+1}\in Z$, for all $x,y\in R$.
5) $P_{11}(Z):(xy)^n-(yx)^n\in Z$, for all $x,y\in R$.
6) $C_{11}(Z):[x,(xy)^n-(yx)^n]=0$, for all $x,y\in R$.
7) $P_{11}^*(Z):x^n(xy)^n-(yx)^nx^n\in Z$, for all $x,y\in R$.
8) $C_{11}^*(Z):[x,x^n(xy)^n-(yx)^nx^n]=0$, for all $x,y\in R$.

For the properties (9) to (11) mentioned below the positive integer $n = n(x, y)$ depends on pair x, y. Whereas for property $(11')$ the positive integer $n' = n'(y, x)$ depends on pair y, x.

9) $P'(n) : [x^n, y^n] = 0$, for all $x, y \in R$.

10) $C'(n) : n[x, y] = 0$ implies $[x, y] = 0$, for all $x, y \in R$.

11) $C'_{n+1}(xy) : [(xy)^{n+1} - x^{n+1}y^{n+1}, xy] = 0$, for all $y, x \in R$.

11') $C'_{n'+1}(yx) : [(yx)^{n'+1} - y^{n'+1}x^{n'+1}, yx] = 0$, for all $y, x \in R$.

Obviously $P'(n) = P'(n')$ and $C'(n) = C'(n')$ when $n = n'$.

Let $\mathcal{P}$ be a ring property. If $\mathcal{P}$ is inherited by every finitely generated subring and every natural homomorphic image modulo the annihilator of a central element, then $\mathcal{P}$ is called an H-property.

EXAMPLE. $C(n)$ is an H-property.

If $\mathcal{P}$ is a ring property such that the ring R has the property $\mathcal{P}$ if and only if all its finitely generated sub-rings have property $\mathcal{P}$ then $\mathcal{P}$ is called an F-property.

EXAMPLE. Commutativity is an F-property.

Abu-Khuzam et al [1, Theorem 2] proved that an s-unital ring satisfying conditions $C((n+1)n)$ and $P_{n+1}(xy)$ is commutative. Later on, Abu-Khuzam and others [4, Theorem 1] proved that an n-torsion free ring with unity satisfying $P(n)$ and $P_{n+1}(Z)$ is commutative. By weakening the hypotheses of the foregoing theorems, we prove a general result on commutativity of s-unital rings with properties $P'(n)$, $C'(n)$ and $[C'_{n+1}(xy)$ and $C'_{n'+1}(yx)]$ which is given as our theorem 1.

In their paper [4, Theorem 2] Abu-Khuzam and others, showed that n-torsion free periodic ring (not necessarily with unity) with conditions, N commutative and $P_{11}(Z)$; is commutative. We generalize this result by weakening the condition $P_{11}(Z)$ by $P^*_{11}(Z)$. In fact $P_{11}(Z)$ is wekened as $C_{11}(Z)$ and $C_{11}(Z)$ is weakened as $C^*_{11}(Z)$ and is obtained by replacing first coordinate x of the commutator by x^{n+1}. However $P^*_{11}(Z)$ is the condition between $C_{11}(Z)$ and $C^*_{11}(Z)$. Hence it is the generalisation of condition $P_{11}(Z)$. We also weaken the condition of the ring R to be n-torsion free by the condition that commutators are $(n+1)n$-torsion free.

2. Preparatory Results.

To make the ground work for theorems 1 and 2, we use the following known results.

PROPOSITION 1 [10, Proposition 1]. Let $\mathcal{P}$ be an H-property and $\mathcal{P}'$ be an F-property. If every ring with 1 having the property $\mathcal{P}$ has the property $\mathcal{P}'$, then every s-unital ring having $\mathcal{P}$ has $\mathcal{P}'$.

LEMMA 2 [9, Theorem]. *Let R be a ring in which, given $a, b \in R$, there exist integers $m = m(a,b) \geq 1$, $n = n(a,b) \geq 1$ such that $a^m b^n = b^n a^m$. Then the commutator ideal of R is nil.*

LEMMA 3 [2]. *If $[x,y]$ commutes with x, then $[x^k, y] = kx^{k-1}[x,y]$ for all positive integers k.*

LEMMA 4 [12, Lemma]. *Suppose that R is a ring with identity 1. If $x^m[x,y] = 0$ and $(x+1)^m[x,y] = 0$ for some x, y in R and some integer $m > 0$, then $[x,y] = 0$. A similar statement holds if we assume $[x,y]x^m = 0$ and $[x,y](x+1)^m = 0$ instead.*

LEMMA 5 [8, Lemma 4]. *Let R be a ring with identity satisfying the properties $P'(n)$ and $C'(n)$. Then*

i) $a \in N$, $x \in R$ imply $[a, x^n] = 0$,

ii) $a \in N$, $b \in N$ imply $[a,b] = 0$.

Part (ii) is special case of part (i).

LEMMA 6 [5]. *Let R be a periodic ring such that N is commutative. Then the commutator ideal of R is nil, and N forms an ideal of R.*

LEMMA 7 [8, Theorem 1]. *If R is an s-unital ring satisfying the identities $P(n)$, $C(n)$ and $C^*_{11}(Z)$, then R is commutative.*

3. Theorems.

Now we come to our own theorems introduced in section 1.

THEOREM 1. *If R is an s-unital ring, satisfying the identities $P'(n)$, $C'(n)$ and $[C'_{n+1}(xy)$ and $C'_{n'+1}(yx)]$, where n and n' are positive integers depending on pair x, y and pair y, x respectively, then R is commutative.*

Proof. According to proposition 1, we may assume that R has unity 1. Since R satisfies the hypothesis $P'(n)$ viz $[x^n, y^n] = 0$, which by lemma 2 yields that the commutator ideal is nil. This implies that the set of nilpotent elements N forms an ideal.

$$\text{This implies that } N^2 \subseteq Z \tag{1}$$

Let $a \in N$ and $b \in R$. Put $x = (a+1)$ and $y = b$ in the hypothesis $C'_{n+1}(xy)$ i.e. $[(xy)^{n+1} - x^{n+1}y^{n+1}, xy] = 0$, to obtain $[(ab+b)^{n+1} - (a+1)^{n+1}b^{n+1}](ab+b) - (ab+b)$

$$[(ab+b)^{n+1} - (a+1)^{n+1}b^{n+1}] = 0\ . \tag{2}$$

Using same substitutions for x and y in the identity $C'_{n'+1}(yx)$, we get $[(ba+b)^{n'+1} - b^{n'+1}(a+1)^{n'+1}](ba+b) - (ba+b)$

$$[(ba+b)^{n'+1} - b^{n'+1}(a+1)^{n'+1}] = 0 \tag{2'}$$

The conditions (11) and (11′) are same provided $n = n'$. Therefore we can write (2′) as follows.

$$[(ba+b)^{n+1} - b^{n+1}(a+1)^{n+1}](ba+b) - (ba+b)$$

$$[(ba+b)^{n+1} - b^{n+1}(a+1)^{n+1}] = 0\ . \tag{3}$$

On subtracting (3) from (2), we get

$$(ab+b)(a+1)^{n+1}b^{n+1} - (a+1)^{n+1}b^{n+1}(ab+b) +$$

$$b^{n+1}(a+1)^{n+1}(ba+b) - (ba+b)b^{n+1}(a+1)^{n+1} = 0\ .$$

Using binomial expansion for terms having powers $(n+1)$, condition $N^2 \subseteq Z$ and lemma (5), we obtain after simplification that

$$n[ba\ b^{n+1} - ab^{n+2} + b^{n+1}ab - b^{n+2}a] = 0\ .$$

After rearranging the terms, we get

$$n[b^{n+1}, [a, b]] = 0 \ .$$

Since R satisfies the property $C'(n)$, we get

$$[b^{n+1}, [a, b]] = 0 \ .$$

By using the identity $[x \cdot y, z] = x[y, z] + [x, z]y$, we obtain

$$b^n[b, [a, b]] + [b^n, [a, b]]b = 0 \ .$$

Or $b^n[b,\ [a, b]) = 0$ (Since $[b^n, [a, b]] = 0$ by lemma 5(i)).

Replacing b by $b + 1$ and using lemma 4, we get

$$[b, [a, b]] = 0 \quad (a \in N,\ b \in R) \ . \tag{4}$$

Using lemma 5(i), eq. (4) and lemma (3), we get

$$0 = [a, b^n] = nb^{n-1}[a, b] \ .$$

By the property $C'(n)$ and lemma (4) we get

$$[a, b] = 0 \quad (a \in N,\ b \in R) \ .$$

Thus the nilpotents of R are central and since $C(R)$ is nil

$$[x, [x, y]] = 0 \quad \text{for all } x, y \in R \ . \tag{5}$$

Using eq.(5) and lemma (3), we have

$$0 = [x^n, y^n] = n\ x^{n-1}[x, y^n] \text{ for all } x, y \text{ in } R \ .$$

By lemma (4) and property $C'(n)$ this yields

$$[x, y^n] = 0 \text{ for all } x, y \text{ in } R \ .$$

Similarly $0 = [x, y^n] = ny^{n-1}[x, y]$ yields $[x, y] = 0$, for all x, y in R. Thus R is commutative.

THEOREM 2. *Let n be a fixed positive integer and R be a periodic ring (not necessarily with identity). If R satisfies the identities*

$C((n+1)n)$, $P^*_{11}(Z)$ and if N is commutative, then R is commutative.

Proof. We consider the proof in two parts.

Part I: When R has an identity 1. By lemma (6), N is an ideal of R, also since N is commutative, $N^2 \subseteq Z$. Note that $a \in N$ gives that a is quasi regular namely a has quasi inverse so $(1+a)$ has inverse in R. Now for $a \in N$, $b \in R$ we choose $x = (1+a)$ and $y = b(1+a)^{-1}$ in the hypothesis $x^n(xy)^n - (yx)^n x^n \in Z$, to obtain,

$$(1+a)^{n+1} b^n (1+a)^{-1} - b^n (1+a)^n \in Z \ . \tag{6}$$

This gives, in particular,

$$\{(1+a)^{n+1} b^n (1+a)^{-1} - b^n (1+a)^n\}(1+a) =$$

$$(1+a)\{(1+a)^{n+1} b^n (1+a)^{-1} - b^n (1+a)^n\} \ .$$

Using binomial expansion and condition $N^2 \subseteq Z$, we get

$$(n+1)(ab^n - b^n a) = (1+a)\{(1+a)^{n+1} b^n (1+a)^{-1} - b^n (1+a)^n\} \tag{7}$$

Since N is commutative ideal.

So $(1+a)(ab^n - b^n a) = ab^n - b^n a$ therefore (7) yields

$$(n+1)(1+a)(ab^n - b^n a) = (1+a)\{(1+a)^{n+1} b^n (1+a)^{-1} - b^n (1+a)^n\} \ .$$

Further since $a \in N$, $(1+a)$ is a unit in R and thus

$$(n+1)(ab^n - b^n a) = \{(1+a)^{n+1} b^n (1+a)^{-1} - b^n (1+a)^n\} \in Z \ ,$$

by (6). Thus $(n+1)[a, b^n] \in Z$.

Since in R every commutator is $(n+1)$ n-torsion free so

$$[a, b^n] \in Z \ . \tag{8}$$

Now suppose $x_1, x_2 \ldots x_k \in R$. Since $R/C(R)$ is commutative $(x_1, \ldots, x_k)^n - x_1^n \ldots x_k^n \in C(R) \subseteq N$ by lemma (6). But N is commutative, therefore

$$[a, (x_1 \ldots x_k)^n] = [a, x_1^n \ldots x_k^n] \text{ for } a \in N \tag{9}$$

Combining (8) and (9), we conclude that

$$[a, x_1^n \dots x_k^n] \in Z \text{ for } a \in N,\ x_1 \dots x_k \in R \text{ and } k \geq 1 \tag{10}$$

Let S be the sub ring of R generated by the n^{th}-powers of the elements of R. Then by (10),

$$[a, x] \in Z(S) \text{ for all } a \in N(S),\ x \in S \tag{11}$$

(where $Z(S)$ and $N(S)$ have their usual meanings).

Combining the facts that S is periodic, $N(S)$ is commutative and the condition (11), Abu-Khuzam's theorem [3] shows that S is commutative and hence

$$[x^n, y^n] = 0 \text{ for all } x, y \in R\ . \tag{12}$$

Since every commutator in R is $(n+1)$ n-torsion free and R satisfies the properties $P_{11}^*(Z)$ and (12), lemma (7) yields that R is commutative. (Lemma (7), which is for an s-unital ring is true for ring with unity also, because ring with unity is an s-unital too.).

Part II: When R does not have identity 1.

First we establish two claims,

1) Idempotents of R are central.
2) Homomorphic image of nilpotent elements of the ring R is the set of nilpotent elements of the homomorphic image S of R.

Claim 1. Let $e_0 \in R$ be an idempotent and $r \in R$.
Put $x = e_0$, $y = e_0 + e_0 r - e_0 r e_0$ in $P_{11}^*(Z)$, to get

$$e_0^n(e_0(e_0 + e_0 r - e_0 r e_0))^n - ((e_0 + e_0 r - e_0 r e_0)e_0)^n e_0^n \in Z$$

and hence $e_0^n(e_0 + e_0 r - e_0 r e_0) - e_0 \cdot e_0^n \in Z$

Or $e_0 r - e_0 r e_0 \in Z$.

Therefore in particular $e_0(e_0 r - e_0 r e_0) = (e_0 r - e_0 r e_0)e_0 = 0$. Or $e_0 r - e_0 r e_0 = 0$.

i.e. $e_0 r = e_0 r e_0$. Similarly $r e_0 = e_0 r e_0$ and so $e_0 r = r e_0$ and the claim follows.

Claim 2. If $\sigma : R \to S$ is a homomorphism of R on to S then the nilpotents of S coincide with $\sigma(N)$, where N is the set of nilpotents of R.

This claim is essentially proved in [11].

Now we come to main proof. A ring R is isomorphic to a subdirect sum of sub-directly irreducible rings $R_i (i \in \Gamma)$. Suppose that $\sigma_i : R \to R_i$ is the natural homomorphism of R on to R_i, let $x_i \in R_i$ and $\sigma_i(x) = x_i$, $x \in R$. Since R is periodic, $x^s = x^r$ for some integers $s > r > 0$, and hence

$$e_0 = x^{(s-r)r} \text{ is an idempotent.} \tag{13}$$

By claim 1, $\sigma_i(e_0)$ is central idempotent of R_i. Since R_i is subdirectly irreducible, so $\sigma_i(e_0) = 0$ or $\sigma_i(e_0) = 1_i$ provided $1_i \in R_i$.

Now there arise two cases:

Case 1. When R_i does not have an identity then $\sigma_i(e_0) = 0$ i.e. $x_i^{(s-r)r} = 0$. Thus R_i is nil and hence by claim (2), $R_i = \sigma_i(N)$. By hypothesis N is commutative, therefore R_i is commutative.

Case 2. When R_i has an identity 1_i.

Let $\sigma_i(e_0') = 1_i$, $e_0' \in R$.

Since R is periodic, we choose $s > r > 0$ such that $e_0'^s = e_0'^r$. Let $e_0 = e_0'^{(s-r)r}$, then e_0 is also an idempotent and moreover, $\sigma_i(e_0) = 1_i^{(s-r)r} = 1_i$. By claim 1, e_0 is a central idempotent element of R. Thus e_0R is a ring with identity e_0. Clearly e_0R inherits all the hypotheses of the ground ring R including the property $C((n+1)n)$, but R_i may not have $C((n+1)n)$. However by part I it follows that e_0R is commutative.

i.e. $[e_0x, e_0y] = 0$ for all $x, y \in R$, which implies $[\sigma_i(x), \sigma_i(y)] = 0$ (since $\sigma_i(e_0) = 1_i$ and thus

$R_i = \sigma_i(R)$ is commutative. Hence the ground ring R is also commutative, which proves the theorem.

4. Counter Examples.

In this section we provide some counter examples showing that all the hypotheses of theorems 1 and 2 are individually essential.

Example 1. The following example is to show that the condition $C'(n)$ is indispensable in theorem 1.

Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in GF(2) \right\} .$$

For all pairs of elements $x, y \in R$, we can find $n = n(x, y) = n'(y, x) = n' \geq 1$ such that R satisfies all the hypotheses of theorem 1 except the condition $C'(n)$, and R is not commutative.

For example, choose

$$x = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and $n = n' = 2$. Then the foregoing fact can easily be verified.

For other choices of elements x, y and $n = n(x, y) = n'(y, x) = n' \geq 1$, the similar verifications can be made.

EXAMPLE 2. Following example shows that the condition $[C'_{n+1}(xy)$ and $C'_{n'+1}(yx)]$ cannot be omitted in Theorem 1.
Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c \in GF(5) \right\} .$$

For all pairs of elements $x, y \in R$, we can find $n = n(x, y) = n'(y, x) = n' \geq 1$ such that R satisfies all the hypotheses of theorem 1 except the condition $[C'_{n+1}(xy)$ and $C'_{n'+1}(yx)]$ and R is not commutative. For example choose

$$x = \begin{pmatrix} 2 & 4 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \; y = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } n = n' = 4$$

then the foregoing fact can easily be verified. Similar verifications are also true for other choices of elements x, y and $n = n(x, y) = n'(y, x) = n' \geq 1$.

EXAMPLE 3. Let R be as in example 1 but with entries in $GF(3)$, and let

$$x = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \ y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

and for this pair of x, y choose $n = n' = 2$. This shows that the condition $P'(n)$ is indispensable in theorem 1, and similarly for other choices of elements of x, y and $n = n(x, y) = n'(y, x) = n' \geq 1$ we can easily verify the indispensability of $P'(n)$.

REMARK. The rings in the above examples are with unity hence s-unital too.

EXAMPLE 4. Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in GF(3) \right\}$, and let $n = 4$. Then R satisfies all the hypotheses of theorem 2 except the hypothesis; "N is commutative". However the ring R is not commutative. This shows that the said hypothesis is essential in Theroem 2.

EXAMPLE 5. Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c \in GF(5) \right\},$$

and let $n = 2$. Then R satisfies all the hypotheses of theorem 2 except the condition $P_{11}^*(Z)$ and R is not commutative. This shows that condition $P_{11}^*(Z)$ can not be dropped in Theorem 2.

EXAMPLE 6. Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c \in GF(3) \right\},$$

and let $n = 5$. Then R satisfies all the conditions except $C((n+1)n)$ yet R is non-commutative. Thus we cannot drop this condition.

Infact with $n = 5$, R satisfies all the hypotheses but the commutators are n-torsion free yet R is not-commutative.

With $n = 6$, R satisfies all the hypotheses but the commutators are $(n+1)$-torsion free, yet R is not commutative. This shows that the condition $C((n+1)n)$ cannot be substituted by either $C(n)$ or $C(n+1)$ in theorem 2. This shows that the condition $C((n+1)n)$ is essential in theorem 2.

References

[1] Abu-Khuzam H., Bell H.E. and Yaqub A., *Commutativity theorems for s-unital rings satisfying polynomial identities*, Math. J. Okayama Univ. **22** (1980), 111-114.

[2] Abu-Khuzam H. and Yaqub A., *n-torsion free rings wih commuting powers*, Math. Japan **25** (1980), 37-42.

[3] Abu-Khuzam H., *A commutativity theorem for periodic rings*, Math. Japo. **32** (1987), 1-3.

[4] Abu-Khuzam H., Bell H.E. and Yaqub A., *Commutativity of rings satisfying certain polynimial identities*, Bull. Austral. Math. Soc. Vol. **44** (1991), 63-69.

[5] Bell H.E., *Some commutativity results for periodic rings*, Acta Math. Acad. Sci. Hungar **28** (1976), 279-283.

[6] Bell H.E., *On rings with commuting powers*, Math. Japo. **24** (1979), 473-478.

[7] Bell H.E., *On commutativity of periodic rings and near-rings*, Acta Math. Acad. Sci. Hungar. **36** (1980), 293-302.

[8] Giri R.D. and Tiwari Shraddha, *Some commutativity theorems for s-unital rings*, Far East Jour. Math. Sci. **1**(2) (1993), 169-178.

[9] Herstein I.N., *A commutativity theorem*, J. Algebra **38** (1976), 112-118.

[10] Hirano Y., Kobayashi Y. and Tominaga H., *Some polynomial identities and commutativity of s-unital rings*, Math. J. Oka. Univ. **24** (1982) 7-13.

[11] Ikehata S. and Tominaga H., *A commutativity theorem*, Math. Japon. **24** (1979), 29-30.

[12] Nicholson W.K. and Yaqub A., *A commutativity theorem*, Algebra Universalis **10** (1980), 260-263.