Rend. Istit. Mat. Univ. Trieste
Vol. XXIX, 207–220 (1997)

White Noise Perturbation of the Equations of Linear Parabolic Viscoelasticity

PH. CLÉMENT, G. DA PRATO and J. PRÜSS (*)

SUMMARY. - *Evolutionary integral equations as appearing in the theory of linear parabolic viscoelasticity are studied in the presence of white noise. It is shown that the stochastic convolution leads to regular solutions, and that under suitable assumptions the samples are Hölder-continuous. These results are put in a wider perspective by consideration of equations with fractional derivatives which are also studied in this paper. This way, known results are recovered and put into broader perspective.*

1. Statement of the problem

Let H be a separable Hilbert space, A a closed linear densely defined operator in H, and $b \in L_{1,loc}(\mathbb{R}_+)$ a scalar kernel. In this paper we consider the integro-differential equations

$$\dot{u}(t) + \int_0^t b(t-\tau)Au(\tau)d\tau = f(t),\ t \geq 0, \quad u(0) = u_0, \qquad (1)$$

on the halfline, and

$$\dot{v}(t) + \int_{-\infty}^t b(t-\tau)Av(\tau)d\tau = g(t),\ t \in \mathbb{R}, \qquad (2)$$

(*) Authors' addresses: Ph. Clément: Technical University Delft (The Netherlands). G. Da Prato: Scuola Normale Superiore di Pisa (Italy). J. Prüss: Martin-Luther-Universität Halle-Wittenberg (Germany).
Key words: Linear viscoelasticity, parabolicity, stochastic convolution, fractional derivatives, Volterra equations.

on the whole real line. Here initial value $u_0 \in H$ and the forcing functions f and g, loosely speaking, are of the form

$$f(t) = h(t) + \sqrt{Q}\dot{W}(t),$$

with deterministic part $h \in L_{1,loc}(\mathbb{R}_+; H)$, covariance Q, and a cylindrical H-valued Wiener process W with corresponding *white noise* $\dot{W}$.

In the recent papers [1], [2], the first two authors studied integrated versions of (1) and (2) under assumptions arising from the application of these equations in the theory of heat conduction in materials with memory. It is the purpose of this note to extend this study to (1) and (2) under hypotheses which are typical in the theory of linear viscoelastic material behaviour.

In such applications, the Hilbert space H will be a space of square integrable functions on a bounded domain $\mathcal{O} \subset \mathbb{R}^n$, and the operator $-A$ an elliptic differential operator like the Laplacian, the elasticity operator, or the Stokes operator, together with appropriate boundary conditions; see e.g. the monograph [6], Section 5. To formulate this abstractly, we impose the following assumption on A.

Hypothesis (A) *A is an unbounded, selfadjoint, positive definite operator in H with compact resolvent. Consequently, the eigenvalues μ_k of A form a nondecreasing sequence with $\lim_{k\to\infty} \mu_k = \infty$, the corresponding eigenvectors e_k form an orthonormal basis of H.*

We are particularly interested in the case $H = L_2(\mathcal{O})$, where $\mathcal{O}$ denotes a bounded open domain in $\mathbb{R}^N$, and A is the L_2-realization of an elliptic boundary value problem of the second order, e.g. $A = -\Delta$ with Dirichlet boundary conditions. In such a situation we require that the following hypothesis about the eigenfunctions $e_k \in L_2(\mathcal{O})$ of A for the eigenvalues μ_k is met; see Da Prato and Zabczyk [4], Section 5.5.

Hypothesis (E) *There is a constant $M > 0$ such that*

$$|e_k(\xi)| \leq M \quad \textit{and} \quad |\nabla e_k(\xi)| \leq M\mu_k^{1/2}, \quad k \in \mathbb{N},\ \xi \in \mathcal{O}.$$

The kernel $b(t)$ should be thought of as a stress relaxation kernel, which means that b is at least nonnegative, nonincreasing, and of positive type. Nevertheless, our assumptions on b are more stringent.

Hypothesis (b) *$b \in L_1(\mathbb{R}_+)$ is 3-monotone, i.e. b and $-\dot{b}$ are nonnegative, nonincreasing, convex; in addition,*

$$\lim_{t\to 0} \frac{\frac{1}{t}\int_0^t sb(s)ds}{\int_0^t -s\dot{b}(s)ds} < \infty. \tag{3}$$

For the following discussion we use as a general reference the third authors monograph [6].

In case **(A)** and **(b)** are valid, problems (1) and (2) are well-posed. There exists the resolvent family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(H)$ which is strongly continuous, uniformly bounded by 1, with $\lim_{t\to\infty} |S(t)|_{\mathcal{B}(H)} = 0$, and uniformly integrable on $\mathbb{R}_+$; the latter means $S \in L_1(\mathbb{R}_+; \mathcal{B}(H))$. Observe that **(A)** and **(b)** imply that the problems under consideration are *parabolic*; define

$$\theta_b := \sup\{|\arg \widehat{b}(\lambda)| : \ \mathrm{Re}\lambda > 0\}; \tag{4}$$

then parabolicity means $\theta_b < \pi/2$. For 3-monotone kernels, condition (3) is in fact equivalent to parabolicity. Typical examples of kernels subject to **(b)** are the functions

$$g_{\beta,\eta}(t) := t^{\beta-1}e^{-\eta t}/\Gamma(\beta), \quad t > 0,$$

where $\beta \in (0,1)$ and $\eta \geq 0$. In this case we have $\theta_{g_{\beta,\eta}} = \beta\pi/2$.

The unique mild solution of (1) is given by the variation of parameters formula

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds, \quad t \geq 0, \tag{5}$$

whenever $u_0 \in H$ and $f \in L_{1,loc}(\mathbb{R}_+; H)$. Similarly, since S is uniformly integrable, we have the representation

$$v(t) = \int_{-\infty}^t S(t-s)g(s)ds, \quad t \in \mathbb{R}, \tag{6}$$

for the unique mild solution of (2), for each $g \in L_1(\mathbb{R}; H)+L_\infty(\mathbb{R}; H)$. The relation between these solutions is

$$\lim_{t\to\infty} |u(t)-v(t)|_H = 0, \quad \text{whenever} \quad \lim_{t\to\infty} |f(t)-g(t)|_H = 0.$$

The resolvent family $S(t)$ can be written explicitly by means of the spectral decomposition of A as

$$S(t)x = \sum_{k=0}^{\infty} s_k(t)(x|e_k)e_k, \quad t \geq 0, \tag{7}$$

where the scalar functions $s_k(t)$ are the solutions of the scalar problems

$$\dot{s}_k(t) + \mu_k \int_0^t b(t-\tau)s_k(\tau) = 0, \quad t \geq 0, \quad s_k(0) = 1. \tag{8}$$

Next we consider the assumptions on the white noise and the covariance; see Da Prato and Zabczyk [4].

Hypothesis (W): *Q is selfadjoint positive semi-definite; there exists a sequence $\gamma_k \geq 0$ such that*

$$Qe_k = \gamma_k e_k, \quad k \in \mathbb{N}_0.$$

$W(t)$ is of the form

$$(W(t)|x) = \sum_{k=0}^{\infty} w_k(t)(x|e_k), \quad t \in \mathbb{R},\ x \in H,$$

where $w_k(t)$ are mutually independent real Wiener processes on the $(\Omega, \mathcal{F}, \mathbb{P})$.

Our plan for this paper is as follows. In Section 2 we state the main results about white noise perturbations of equations in linear viscoelasticity, i.e. equations (1) and (2), assuming the hyptheses **(A)**, **(E)**, **(b)**, and **(W)** explained above. These results are proved in Section 3 by means of the methods introduced in the monograph by Da Prato and Zabczyk [4], adapted to evolutionary integral equations in Clément and Da Prato [1], [2]. The required estimates are already available and are taken from Monniaux and Prüss [5]. Section 4 is devoted to a study of the equation

$$u + g_\alpha \star Au = g_\beta \star \dot{W}$$

on the halfline, where $g_\gamma(t) = t^{\gamma-1}/\Gamma(\gamma)$, $t > 0$ for $\gamma > 0$ denotes the Riemann-Liouville kernel of fractional integration. The results for this problem are compared with those above and with results available in the literature.

2. Main results

Concentrating on the stochastic case we let $h(t) = 0$, i.e. $f(t) = \dot{W}(t)$. This means that we have to investigate the stochastic convolutions

$$W_S^+(t) = \int_0^t S(t-\tau)\sqrt{Q}dW(\tau), \quad t \geq 0, \tag{9}$$

on the halfline, and

$$W_S(t) = \int_{-\infty}^t S(t-\tau)\sqrt{Q}dW(\tau), \quad t \in \mathbb{R}, \tag{10}$$

on the real line, where by means of a second independent Wiener process $W_1(t)$, $W(t)$ is extended to $\mathbb{R}$ by the definition

$$W(t) = \begin{cases} W(t) & t \geq 0 \\ W_1(-t) & t \leq 0. \end{cases}$$

In virtue of the spectral decompositions of A and Q we may rewrite

$$W_S^+(t) = \sum_{k=1}^{\infty} \sqrt{\gamma_k} \int_0^t s_k(t-\tau) e_k dw_k(\tau), \quad t \geq 0, \tag{11}$$

and

$$W_S(t) = \sum_{k=1}^{\infty} \sqrt{\gamma_k} \int_{-\infty}^t s_k(t-\tau) e_k dw_k(\tau), \quad t \geq 0. \tag{12}$$

Our main result on (1) reads as follows.

THEOREM 2.1. *Assume that Hypotheses* **(A), (b), (W)** *are valid, and suppose*

$$\mathrm{Tr}[\mathrm{QA}^{-1/\rho}] = \sum_{\mathrm{k}=1}^{\infty} \gamma_{\mathrm{k}}/\mu_{\mathrm{k}}^{1/\rho} < \infty, \tag{13}$$

where

$$\rho := 1 + \frac{2}{\pi} \sup\{|\arg \widehat{b}(\lambda)| : \ \mathrm{Re}\,\lambda > 0\} \tag{14}$$

Then the series (11) and (12) converges in $L_2(\Omega; H)$*, uniformly in* t *on bounded subsets of* $\mathbb{R}_+$ *resp.* $\mathbb{R}$. $W_S^+(t)$ *is a Gaussian random variable with mean zero and covariance operator* Q_t *given by*

$$Q_t = \int_0^t S(\tau) Q S^*(\tau), \quad t \geq 0,$$

and we have $\mathrm{Tr}[\mathrm{Q_t}] \leq \mathrm{Tr}[\mathrm{QA}^{-1/\rho}]$. $W_S(t)$ *is a stationary Gaussian process with mean zero and covariance*

$$Q_\infty = \lim_{t\to\infty} Q_t = \int_0^\infty S(t) Q S^*(t) dt.$$

If in addition, there is $\theta \in (0,1)$ *such that*

$$\mathrm{Tr}[\mathrm{QA}^{(\theta-1)/\rho}] = \sum_{\mathrm{k}=1}^{\infty} \gamma_{\mathrm{k}}/\mu_{\mathrm{k}}^{(1-\theta)/\rho} < \infty, \tag{15}$$

then for each $\alpha \in (0, \theta/2)$*, the trajectories of* $W_S^+(t)$ *and* $W_S(t)$ *are almost surely* α*-Hölder-continuous.*

In case $H = L_2(\mathcal{O})$ *and Hypothesis* **(E)** *as well as*

$$\mathrm{Tr}[\mathrm{QA}^{(\theta-1/\rho)}] = \sum_{\mathrm{k}=1}^{\infty} \gamma_{\mathrm{k}}/\mu_{\mathrm{k}}^{(1-\rho\theta)/\rho} < \infty, \tag{16}$$

are met, the trajectories of $W_S^+(t,\xi)$ *and* $W_S(t,\xi)$ *are almost surely Hölder-continuous in* ξ*, for each exponent* $\alpha \in (0,\theta)$.

Observe that in case $H = L_2(0,1)$, i.e. $N = 1$, and $A = -d^2/d\xi^2$ with Dirichlet boundary conditions we have $\mu_k = \pi^2 k^2$. Therefore in case $Q = I$ (15) is satisfied whenever $0 < \theta < 1 - \rho/2$, and (16) holds provided $0 < \theta < (1-\rho/2)/\rho$. Note that by Hypothesis **(b)** we have $\rho \in [1,2)$.

3. Proofs of the main results

To a large extent the proofs of our main results follow the arguments for the Cauchy problem presented in Da Prato and Zabczyk [4], and those of Clément and Da Prato [1], [2] for a different class of evolutionary integral equations. We first collect some properties of the functions $s_\mu(t)$ introduced above.

LEMMA 3.1. *Suppose the kernel $b(t)$ is subject to Hypothesis* **(b)**, *and let $\rho \in (1,2)$ be defined by (14). Then*

(i) $|s_\mu(t)| \leq 1$ *for all* $t, \mu > 0$;

(ii) $|\dot{s}_\mu|_1 \leq C$ *for all* $\mu > 0$;

(iii) $|t\dot{s}_\mu|_1 \leq C\mu^{-1/\rho}$ *for all* $\mu > 0$;

(iv) $|s_\mu|_1 \leq C\mu^{-1/\rho}$ *for all* $\mu > 0$,

where $C > 0$ denotes a constant which is independent of $\mu > 0$, and $|\cdot|_1$ denotes the norm in $L_1(\mathbb{R}_+)$.

Proof. Assertion (i) follows from the proof of Corollary 1.2 in Prüss [6], while (ii) and (iii) are contained in Proposition 6 of Monniaux and Prüss [5]. (Observe the relation $\dot{s}_\mu(t) = -\mu r_\mu(t)$, to connect the notations.) To prove (iv), observe that

$$s_\mu(t) = s_\mu(R) - \int_t^R \dot{s}_\mu(\tau)d\tau$$

and (ii) imply that the limit of $s_\mu(R)$ for $R \to \infty$ exists and satisfies

$$\lim_{R\to\infty} s_\mu(R) = \lim_{\lambda\to 0+} \lambda\widehat{s}_\mu(\lambda) = \lim_{\lambda\to 0+} \frac{\lambda}{\lambda + \mu\widehat{b}(\lambda)} = 0.$$

Therefore

$$s_\mu(t) = -\int_t^\infty \dot{s}_\mu(\tau)d\tau$$

yields

$$|s_\mu|_1 \leq \int_0^\infty \int_t^\infty |\dot{s}_\mu(\tau)|d\tau dt = \int_0^\infty \tau|\dot{s}_\mu(\tau)|d\tau = |t\dot{s}_\mu|_1 \leq C\mu^{-1/\rho}$$

by assertion (iii). □

Observe that for the case $b = g_{\beta,\eta}$ we have $\rho = 1 + \beta$. It has been shown in Monniaux and Prüss [5] that the exponent $1/\rho$ in (iii) and (iv) of Lemma 3.1 is optimal.

Now, let the hypotheses of Theorem 2.1 be fulfilled. Then by (i) and (iv) of Lemma 3.1, e.g. $u(t) = W_S^+(t)$ satisfies

$$\mathbb{E}|u(t)|_H^2 = \sum_{k=1}^{\infty} \gamma_k \int_0^t s_{\mu_k}^2(\tau)d\tau \le \sum_{k=1}^{\infty} \gamma_k |s_{\mu_k}|_1 \le C \sum_{k=1}^{\infty} \gamma_k \mu_k^{-1/\rho} < \infty.$$

Therefore we may argue as in the proof of Theorem 2.2 of Clément and Da Prato [1] to obtain the first statements of Theorem 2.1.

Concerning Hölder-continuity, we derive two estimates which are similar to those in Hypothesis 2 of Clément and Da Prato [1].

LEMMA 3.2. *Suppose that the kernel $b(t)$ is subject to Hypothesis* **(b)**. *Then for each $\theta \in (0,1)$ there is a constant $C_\theta > 0$ such that*

$$\int_\tau^t s_\mu^2(r)dr \le C_\theta \mu^{(\theta-1)/\rho}|t-\tau|^\theta, \quad 0 < \tau < t, \tag{17}$$

and

$$\int_{-\infty}^\tau [s_\mu(\tau - r) - s_\mu(t-r)]^2 dr \le C_\theta \mu^{(\theta-1)/\rho}|t-\tau|^\theta, \quad \tau < t. \tag{18}$$

Proof. From Lemma 3.1 (i), we obtain

$$\int_\tau^t s_\mu^2(r)dr \le \int_\tau^t |s_\mu(\tau)|d\tau \le |s_\mu|_1,$$

as well as

$$\int_\tau^t s_\mu^2(r)dr \le |t-\tau|,$$

hence by interpolation, employing Lemma 1 (iv)

$$\int_\tau^t s_\mu^2(r)dr \le |t-\tau|^\theta |s_\mu|_1^{1-\theta} \le C^{1-\theta}|t-\tau|^\theta \mu^{(\theta-1)/\rho},$$

which proves (17).

To prove (18) we employ once more Lemma 3.1, this time (i), (ii) and (iii).

$$\begin{aligned}
&\int_{-\infty}^{\tau}[s_\mu(\tau-r)-s_\mu(t-r)]^2dr\\
&\quad\le 2\int_{-\infty}^{\tau}|s_\mu(\tau-r)-s_\mu(t-r)|dr\\
&\quad\le 2\int_{-\infty}^{\tau}\int_{\tau}^{t}|\dot{s}_\mu(\sigma-r)|d\sigma dr = 2\int_{\tau}^{t}\int_{-\infty}^{\tau}|\dot{s}_\mu(\sigma-r)|drd\sigma\\
&\quad\le 2\int_{\tau}^{t}(\sigma-\tau)^{\theta-1}\\
&\qquad\quad [\int_{-\infty}^{\tau}(\sigma-r)|\dot{s}_\mu(\sigma-r)|dr]^{1-\theta}[\int_{-\infty}^{\tau}|\dot{s}_\mu(\sigma-r)|dr]^{\theta}dr\\
&\quad\le 2C\mu^{(\theta-1)/\rho}\int_{\tau}^{t}(\sigma-\tau)^{\theta-1}d\sigma = 2[C/\theta]|t-\tau|^{\theta}\mu^{(\theta-1)/\rho}. \qquad \square
\end{aligned}$$

Since

$$\mathbb{E}|u(t)-u(s)|_H^2 = \sum_{k=1}^{\infty}\gamma_k[\int_0^s|s_k(t-r)-s_k(\tau-r)|^2dr+\int_\tau^t|s_k(r)|^2dr],$$

with Lemma 3.2 we may conclude Hölder-continuity of $u(t)=W_S^+(t)$ or $u(t)=W_S(t)$ as in the proofs given in Clément and Da Prato [1] or [2]. Similarly, in case **(E)** holds, we obtain Hölder-continuity in space from the identity

$$\mathbb{E}|u(t,\xi)-u(t,\eta)|^2 = \sum_{k=1}^{\infty}\gamma_k\int_0^t|s_k(r)|^2dr|e_k(\xi)-e_k(\eta)|^2.$$

4. Fractional derivatives and white noise

In the remaining part of this paper we take up a different viewpoint to equations with noise which will set the results of this paper and those obtained in [1], [2] in another perspective. We consider the problems

$$u + g_\alpha \star Au = g_\beta \star \dot{W} \tag{19}$$

in the Hilbert space H, where the operator A is subject to Hypothesis **(A)** and also to **(E)** if appropriate, the Wiener process W is subject to **(W)** and g_γ denotes the fractional integration kernel

$$g_\gamma(t) = t^{\gamma-1}/\Gamma(\gamma)\ , \quad t > 0,$$

where $\gamma > 0$. In case $\alpha = \beta$, applying the fractional derivative D^α to (19) we obtain

$$D^\alpha u + Au = \dot{W} \tag{20}$$

which interpolates between the first order equation $\dot{u} + Au = \dot{W}$ and the second order problem $\ddot{u} + Au = \dot{W}$. If we set $\beta = 1$, $a = g_\alpha$ for $\alpha \in (0,1]$ equation (19) is a special case of the problem studied in [1].

For $\alpha \in (0,2)$, $\beta > 0$, define the scalar fundamental solution of (19) by

$$\hat{r}_\mu(\lambda) = \frac{\widehat{g}_\beta(\lambda)}{1 + \mu\widehat{g}_\alpha(\lambda)} = \frac{1}{\lambda^\beta} \cdot \frac{\lambda^\alpha}{\lambda^\alpha + \mu}, \qquad Re\lambda > 0,\ \mu > 0. \tag{21}$$

Then with $r_k = r_{\mu_k}$, the solution of (19) can be written as

$$u(t) = \sum_{k=1}^\infty \sqrt{\gamma_k} \int_0^t r_k(t-\tau) dw_k(\tau) e_k\ , \quad t > 0, \tag{22}$$

and therefore as above

$$\mathbb{E}|u(t)|_H^2 = \sum_{k=1}^\infty \gamma_k \int_0^t |r_k(\tau)|^2 d\tau \tag{23}$$

as well as

$$\mathbb{E}|u(t)-u(s)|_H^2 = \sum_{k=1}^\infty \gamma_k [\int_0^t |r_k(t-s+\tau)-r_k(\tau)|^2 d\tau + \int_s^t |r_k(t-\tau)|^2 d\tau] \tag{24}$$

and in case $H = L_2(\mathcal{O})$ and **(E)** is valid

$$\mathbb{E}|u(t,\xi) - u(t,\eta)|^2 = \sum_{k=1}^\infty \gamma_k [\int_0^t r_k^2(\tau) d\tau] |e_k(\xi) - e_k(\eta)|^2. \tag{25}$$

Identities (23) and (24) show that the solution $u(t)$ of (19) exists and is continuous in $L_2(\Omega;H)$ iff

$$\sigma_1 := \sum_{k=1}^{\infty} \gamma_k |r_k|_2^2 < \infty. \tag{26}$$

Next observe that with the convention $r_\mu(t) = 0$ for $t < 0$ we have

$$\begin{aligned}\int_0^s |r_k(t-s+\tau) - r_k(\tau)|^2 d\tau + \int_s^t |r_k(t-\tau)|^2 \\ \leq \int_{-\infty}^{\infty} |r_k(t-s+\tau) - r_k(\tau)|^2 d\tau \\ = |r_k(t-s+\cdot) - r_k(\cdot)|_2^2 \leq |r_k|^2_{B^\theta_{2,\infty}(\mathbb{R})} |t-s|^{2\theta},\end{aligned}$$

where $B^\theta_{2,\infty}(\mathbb{R})$ denotes a Besov space. Now we have the embedding

$$H_2^\theta(\mathbb{R}) = B^\theta_{2,2}(\mathbb{R}) \hookrightarrow B^\theta_{2,\infty}(\mathbb{R}),$$

and so the condition

$$\sigma_2 := \sum_{k=1}^{\infty} \gamma_k |r_k|^2_{\theta,2} < \infty \tag{27}$$

implies Hölder-continuity of $u(t)$ in time t of the order θ. Finally, from **(E)** we obtain by interpolation

$$|e_k(\xi) - e_k(\eta)| \leq C|\xi - \eta|^\theta \mu_k^{\theta/2},$$

hence

$$\sigma_3 := \sum_{k=1}^{\infty} \gamma_k \mu_k^\theta |r_k|_2^2 < \infty \tag{28}$$

yields Hölder-continuity of $u(t,\xi)$ in space ξ of order θ. Therefore the goal is to estimate the H_2^θ-norms of the fundamental solution $r_\mu(t)$ of the scalar problems

$$r_\mu + \mu g_\alpha \star r_\mu = g_\beta. \tag{29}$$

This will be done in the following Lemma.

LEMMA 4.1. *Suppose $\alpha \in (0,2)$, $\beta > 0$, $\theta \in [0,1]$, and let $r_\mu(t)$ denote the solution of (29). Then*

$$|r_\mu|_{\theta,2}^2 \le C_{\alpha,\beta,\theta}\mu^{\frac{1-2(\beta-\theta)}{\alpha}}, \quad \mu > 0,$$

whenever $1/2+\theta < \beta < 1/2+\alpha$, where $|\cdot|_{\theta,2}$ denotes the norm in $H_2^\theta(\mathbb{R})$.

Proof. We first consider the case $\theta = 0$. Then by the Paley-Wiener theorem, $r_\mu \in L_2(\mathbb{R}_+)$ iff $\widehat{r}_\mu \in \mathcal{H}_2(\mathbb{C}_+)$, the Hardy space of exponent 2 and $|r_\mu|_2 = (1/\sqrt{2\pi})|\hat{r}_\mu|_{\mathcal{H}_2}$. Now we may compute

$$|\widehat{r}_\mu|_{\mathcal{H}_2}^2 = 2\int_0^\infty [\frac{\rho^\alpha}{\rho^\beta(\rho^\alpha+\mu)}]^2 d\rho = 2\mu^{\frac{1-2\beta}{\alpha}} \int_0^\infty [\frac{s^{\alpha-\beta}}{1+s^\alpha}]^2 ds,$$

and the last integral is finite iff $1/2 < \beta < 1/2+\alpha$. In case $\theta \ne 0$, observe that $|r_\mu|_2 + |D^\theta r_\mu|_2$ defines an equivalent norm on H_2^θ, hence replacing β by $\beta - \theta$ the result follows. □

Now we are in position to state our result on (19).

THEOREM 4.2. *Let $\alpha \in (0,2)$, $\beta > 0$, $\theta \in [0,1]$ such that $1/2+\theta < \beta < 1/2+\alpha$. Assume that* **(A)** *and* **(W)** *are satisfied. Then*

(i) *If $Tr(QA^{(1-2\beta)/\alpha}) = \sum_{k=1}^\infty \gamma_k \mu_k^{(1-2\beta)/\alpha} < \infty$ then the solution u of (19) exists and belongs to $C_b(\mathbb{R}_+; L_2(\Omega; H))$.*

(ii) *If $Tr(QA^{(1-2\beta+2\theta)/\alpha}) = \sum_{k=1}^\infty \gamma_k \mu_k^{(1-2\beta+2\theta)/\alpha} < \infty$ then $u \in C_b^\theta(\mathbb{R}_+; L_2(\Omega; H))$.*

(iii) *If $H = L_2(\mathcal{O})$,* **(E)** *holds, and*

$$Tr(QA^{(1-2\beta+\alpha\theta)/\alpha}) = \sum_{k=1}^\infty \gamma_k \mu_k^{(1-2\beta+\alpha\theta)\alpha} < \infty,$$

then $u \in C_b(\mathbb{R}_+; C^\theta(\mathcal{O}; L_2(\Omega)))$.

Proof. Use Lemma 4.1 to estimate the quantities σ_i, $i = 1,2,3$. □

REMARK. From the proof of the Lemma it is apparent that the upper bound $1/2+\alpha$ for β is not needed if we only want local results, i.e. on a finite time interval $[0,T]$.

It is instructive to discuss the following one-dimensional example.

Example. Let $H = L_2(0,\pi)$, $A = A_0^m$, where $A_0 = -(d/dx)^2$ say with Dirichlet boundary conditions. Then $\mu_k = k^{2m}$ hence with pure white noise $\gamma_k = 1$ we obtain

$$\sum_{k=1}^{\infty} \gamma_k \mu_k^{\frac{1-2\beta}{\alpha}} < \infty \quad \Leftrightarrow \quad \beta > \frac{1}{2} + \frac{\alpha}{4m}.$$

$$\sum_{k=1}^{\infty} \gamma_k \mu_k^{\frac{1-2\beta+2\theta}{\alpha}} < \infty \quad \Leftrightarrow \quad \beta > \theta + \frac{1}{2} + \frac{\alpha}{4m}.$$

$$\sum_{k=1}^{\infty} \gamma_k \mu_k^{\frac{1-2\beta+\alpha\theta}{\alpha}} < \infty \quad \Leftrightarrow \quad \beta > \frac{\alpha}{2}\theta + \frac{1}{2} + \frac{\alpha}{4m}.$$

Observe that the condition on existence $\beta > 1/2 + \alpha/4m$ already implies Hölder-regularity in time and in space. Spatial regularity is better than that in time, by the factor $2/\alpha$. Note also that $\beta = 1$ works for all $\alpha \in (0,2)$, $m \geq 1$.

We conclude with a brief discusson of the case $\alpha = 2$. Then

$$\widehat{r}_\mu(\lambda) = \lambda^{2-\beta}/(\lambda^2 + \mu),$$

hence there are poles $\pm i\sqrt{\mu}$ on the imaginary axis, and so Lemma 4.1 is not valid in this case. Therefore we proceed differently. For $\beta > 1/2$ we may employ the complex inversion formula for the Laplace transform to the result

$$r_\mu(t) = \frac{1}{2\pi i} \int_{\gamma-\infty}^{\gamma+\infty} \frac{\lambda^{2-\beta}}{\lambda^2+\mu} e^{\lambda t} d\lambda, \quad t > 0,$$

where $\gamma > 0$. Contracting the contour to the negative real half-axis we obtain

$$r_\mu(t) = \mu^{(1-\beta)/2}\{\sin(\sqrt{\mu}t + (2-\beta)\pi/2) + \\ -\pi^{-1}\sin((2-\beta)\pi) \int_0^\infty e^{-\sqrt{\mu}tr} \frac{r^{2-\beta}dr}{1+r^2}\}, \quad t > 0.$$

This formula is valid for $1/2 < \beta < 3$ and it shows

$$\int_0^T r_\mu^2(t)dt \sim c\mu^{1-\beta} \quad \text{as} \quad \mu \to \infty,$$

for any fixed $T > 0$. Therefore the condition for local existence in the case $\alpha = 2$ becomes

$$\sum_{k=1}^{\infty} \gamma_k \mu_k^{1-\beta} < \infty.$$

Note that this is not the limiting case of Theorem 4.2 (i) as $\alpha \to 2$. This is an extension of the result of Da Prato and Zabczyk [4] on $\ddot{u} + Au = \dot{W}$.

REFERENCES

[1] CLÉMENT PH. and DA PRATO G., *Some results on stochastic convolutions arising in Volterra equations perturbed by noise*, Rend. Mat. Acc. Lincei **7** (1996), 147–153.

[2] CLÉMENT PH. and DA PRATO G., *White noise perturbations of the heat equation in materials with memory*, Dynam. Systems Appl. **6** (1997), 441–460.

[3] CLÉMENT PH. and PRÜSS J., *Completely positive measures and Feller semigroups*, Math. Ann. **287** (1990), 73–105.

[4] DA PRATO G. and ZABCZYK J., *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and its Applications, University Press, Cambridge, 1992.

[5] MONNIAUX S. and PRÜSS J., *A theorem of the Dore-Venni type for noncommuting operators*, Trans. Amer. Math. Soc. **349** (1997), 4787–4814.

[6] PRÜSS J., *Evolutionary Integral Equations and Applications*, Birkhäuser Verlag, Basel, 1993.

Received June 8, 1998.