

## On Global Solutions to a Semilinear Elliptic Boundary Problem in an Unbounded Domain

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SUMMARY. - *We consider solutions to the elliptic linear equation*

$$Lu := \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) = 0 \quad (1)$$

*of second order in an unbounded domain*

$$\{x = (x', x_n) : |x'| < Ax_n^\sigma + B, \quad 0 < x_n < \infty\}, \quad 0 \leq \sigma \leq 1,$$

*in  $\mathbf{R}^n$ . We study the asymptotic behavior as  $x_n \rightarrow \infty$  of the solutions of (1) satisfying the nonlinear boundary condition*

$$\frac{\partial u}{\partial N} - b(x)|u(x)|^{p-1}u(x) = 0 \quad (2)$$

*on the lateral surface*

$$S = \{x \in \partial Q, \quad 0 < x_n < \infty\},$$

*where  $p > 0$ ,  $b(x) \geq b_0 > 0$ .*

*We show that a global solution of the problem can exist not for all values of parameters  $p, \sigma$  and indicate these values.*

*The boundary problem in the cylinder was studied by us in [1], [2]. The obtained results generalize some results of B. Hu in [4].*

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## 1. Introduction

We consider the solutions to the elliptic linear equation

$$Lu := \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) = 0 \quad (3)$$

of second order in an unbounded domain

$$Q = \{x = (x', x_n) : |x'| < Ax_n^\sigma + B, \ 0 < x_n < \infty\}, \ 0 \leq \sigma \leq 1,$$

in  $\mathbf{R}^n$ . We study the asymptotic behavior as  $x_n \rightarrow \infty$  of the solutions of (3) satisfying the nonlinear boundary condition

$$\frac{\partial u}{\partial N} - b(x)|u(x)|^{p-1}u(x) = 0 \quad (4)$$

on the lateral surface

$$S = \{|x'| = Ax_n^\sigma + B, \ 0 < x_n < \infty\},$$

where  $p > 0$ ,  $b(x) \geq b_0 > 0$  and

$$\frac{\partial u}{\partial N} = \sum_{i=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \cos \theta_i,$$

$\theta_i$  is the angle between the axis  $x_i$  and the outer normal vector.

We suppose that

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq c_0 |\xi|^2, \ c_0 > 0, \ x \in Q,$$

and that  $|a_{ij}(x)| \leq C$  for  $i, j = 1, \dots, n$  and for all  $x \in Q$ . We don't assume that  $a_{ij}$  are continuous.

Let us denote  $\Omega_T$  and  $\Sigma_T$  the sections of the domain  $Q$  and the boundary  $S$  by the plane  $x_n = T$ , and  $Q_T$  and  $S_T$  the parts of  $Q$  and  $S$  between the planes  $x_n = 1$  and  $x_n = T$ .

We consider weak solutions  $u$  satisfying (3) and (4). It means that  $u \in H_{loc}^1(Q) \cap L_{loc}^{p+1}(S)$  and

$$\int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx + \int_S b(x)|u(x)|^{p-1}u(x)\varphi(x)dS = 0 \quad (5)$$

for all functions  $\varphi(x) \in H^1(Q)$ , equal to 0 as  $x_n = 0$  and in a neighborhood of  $x_n = \infty$ .

We are looking for the conditions of global existence of solutions to the problem (3)-(4).

## 2. Auxiliary results

LEMMA 2.1 (HARNACK'S INEQUALITY). *Suppose that  $u > 0$  is a weak solution to the equation  $Lu = 0$  in a domain  $\Omega \subset \mathbf{R}^n$  and  $\Omega' \subset\subset \Omega$ . There exists a positive constant  $C$  such that*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u.$$

*If  $\Omega = R \cdot \Omega_1, \Omega' = R \cdot \Omega'_1$ , then the constant  $C$  is independent of  $R$  for  $R \geq 1$ .*

*Proof.* See [6, Theorem 8.1, p. 237]. □

LEMMA 2.2 (HARNACK'S INEQUALITY IN A CLOSED DOMAIN). *Let  $u$  be a weak solution to the equation  $Lu = 0$  and  $u > 0$  in a domain  $\Omega \subset \mathbf{R}^n$  with smooth (of class  $C^1$ ) boundary,  $\partial u / \partial N = 0$  on a neighborhood of  $\partial\Omega \cap \overline{\Omega}'$ , in  $\partial\Omega$  where  $\Omega'$  is a bounded subdomain in  $\mathbf{R}^n$  with boundary of class  $C^1$ ,  $\overline{\Omega}' \subset\subset \overline{\Omega}$ . Then there exists a positive constant  $C$  such that*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u. \tag{6}$$

*If  $\Omega = R \cdot \Omega_1, \Omega' = R \cdot \Omega'_1$ , then the constant  $C$  is independent of  $R$  for  $R \geq 1$ .*

*Proof.* Let us fix a point  $y_0 \in \Gamma$ . There exists a neighborhood  $\omega$  of this point and a system of local coordinates  $(y_1, \dots, y_n)$  such that

1)  $\omega \cap \Omega' \subset \{y_1 > 0\}; \quad \omega \cap \Gamma \subset \{y_1 = 0\};$

2) operator  $L$  has the form

$$Lv = \sum_{i,j=1}^n \frac{\partial}{\partial y_i} (b_{ij}(y) \frac{\partial v}{\partial y_j}).$$

We can suppose that this neighborhood is of the form  $\omega = \{y : |y - y_0| < r\}$ . The operator  $L_1(y, D)$  coincides with  $L$  in  $\omega^+ = \omega \cap \Omega'$  and with  $L(-y_1, y', -D_1, D')$  in  $\omega \setminus \Omega'$ . Let  $u_1(y) = u(y)$  in  $\omega \cap \Omega'$  and  $u_1(y) = u(-y_1, y')$  in  $\omega \setminus \Omega'$ . Then

$$L_1(y, D)u_1(y) = 0$$

in  $\omega \setminus \{y_1 = 0\}$  and  $\partial u(y)/\partial y_1 = 0$  for  $y_1 = 0^+$ . Let  $v(y)$  be the solution of the equation  $L_1v(y) = 0$  in  $\omega$ , coinciding with  $u_1$  on the boundary of  $\omega$ . Obviously,  $v(y) = u_1(y)$  in  $\omega$ .

Let  $\omega_1 = \{y; |y - y_0| < r/2\}$ . By Lemma 2.1, we have

$$\sup_{\omega_1} v(y) \leq C \inf_{\omega_1} v(y),$$

and therefore,

$$\sup_{\omega_1^+} u(y) \leq C \inf_{\omega_1^+} u(y).$$

Choosing a finite covering of the boundary  $\Gamma$  by the sets of the form as  $\omega_1$  and applying once again Lemma 2.1, we obtain the inequality (6). The proof is complete.  $\square$

LEMMA 2.3 (HARDY'S INEQUALITY). *Let  $n \geq 3$ . Let  $u \in C(\overline{Q}) \cap C^1(Q)$  and  $u = 1$  if  $x_n = 1$ ,  $\sigma(n-1) > 1$ , and  $u = 0$  if  $x_n = 1$ ,  $\sigma(n-1) < 1$ . There exists a constant  $C$  such that*

$$\int_Q \frac{u^2}{|x|^2} dx \leq C \int_Q |\nabla u|^2 dx.$$

*Proof.* Put  $u = 1$  if  $x_n \leq 1$ ,  $\sigma(n-1) > 1$  and  $u = 0$  if  $x_n \leq 1$ ,  $\sigma(n-1) < 1$ . Let  $y' = x'$ ,  $y_n = x_n^\sigma$ . The domain  $Q$  in  $y$ -coordinates has the form

$$|y'| \leq Ay_n + B.$$

Using the Hardy inequality

$$\int_Q u^2 |y|^{-1-1/\sigma} dy \leq c_1 \int_Q |\nabla_y u|^2 |y|^{1-1/\sigma} dy,$$

(see, for example [3]), we obtain in  $x$ -coordinates the inequality

$$\int_Q u^2 x_n^{-2} dy \leq c_2 \int_Q (x_n^{2(\sigma-1)} |\nabla_{x'} u|^2 + u_{x_n}^2) dx,$$

what implies the inequality

$$\int_Q \frac{u^2}{|x|^2} dx \leq c_3 \int_Q |\nabla u|^2 dx.$$

□

LEMMA 2.4. Let  $\sigma(n-1) > 1, \sigma \leq 1$ ,

$$Q = \{x = (x', x_n) : |x'| < Ax_n^\sigma + B, 1 < x_n < \infty\}, A > 0, B \geq 0.$$

There exists a weak solution to the following problem:

$$LE = 0 \text{ in } Q, \quad \frac{\partial E}{\partial N} = 0 \text{ on } S, \quad E = 1 \text{ for } x_n = 1,$$

from the class  $H_{loc}^1(Q)$ , such that  $E(x) \geq cx_n^{\sigma(2-n)}$ ,  $c > 0$ . Moreover,

$$\lim_{x_n \rightarrow \infty} E(x) = 0, \quad \left| \int_{\Omega_1} \sum_{j=1}^n a_{nj}(x) \frac{\partial E}{\partial x_j} dx' \right| = c_0 \neq 0.$$

*Proof.* Consider the problem of minimization of the functional

$$\int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} dx$$

in the class of functions  $u$  from  $H^1(Q)$ , equal 1 for  $x_n = 1$ . By the Hardy inequality of Lemma 2.3 we have

$$\int_Q \frac{u^2}{|x|^2} dx \leq C \int_Q |\nabla u|^2 dx, \quad (7)$$

so it is easy to prove the existence of the minimizing function  $E$  and its uniqueness.

This function is positive and satisfies the equality

$$\int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx = 0 \quad (8)$$

for all functions  $\varphi$  from  $H^1(Q)$ , equal 0 in a neighborhood of infinity and for  $x_n = 1$ .

The function  $E$  cannot be equal to a constant in  $Q$ , since for  $\sigma \geq 1/(n-1)$

$$\int_Q \frac{1}{|x|^2} dx = \int_1^\infty x_n^{\sigma(n-1)-2} dx_n = \infty$$

and it contradicts to (7).

The same arguments show that  $E(x)$  cannot be estimated from below with a positive constant. The maximum principle implies then that

$$\lim_{x_n \rightarrow \infty} E(x) = 0.$$

Note that the function  $E$  is the limit in  $H^1$  of the sequence of the functions  $u_k$  from  $C^1(\bar{Q})$ , equal 0 for  $x_n > k$  and 1 for  $x_n = 1$ . Therefore,

$$\begin{aligned} \int_{\Omega_1} \sum_{j=1}^n a_{nj}(x) \frac{\partial E}{\partial x_j} dx' &= \int_{\Omega_1} u_k(x) \sum_{j=1}^n a_{nj}(x) \frac{\partial E}{\partial x_n} dx' \\ &= - \int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial u_k}{\partial x_i} dx. \end{aligned}$$

The limit of the last integral as  $k \rightarrow \infty$  is equal to

$$- \int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial E}{\partial x_i} dx$$

and therefore, is not equal to 0. Moreover, using [6, Theorem 7.1], we can show that  $u_k \rightarrow E$  uniformly on each compact subset of  $Q$ . Thus,

$$\left| \int_{\Omega_1} \sum_{j=1}^n a_{nj}(x) \frac{\partial E}{\partial x_j} dx' \right| = c_0 \neq 0.$$

Let us fix a point  $x_0 \in Q$  and set  $x_{0n} = T$ . Let  $E(x_0) = m$ . By Lemma 2.2,

$$c_1 m \leq E(x) \leq c_2 m,$$

if  $x \in Q$ ,  $T - R \leq x_n \leq T + R$  and  $c_1 > 0$ , if  $R = \min(T^\sigma, T/2)$ . Using the homothety  $x = Ry$ , we obtain the equation  $L_R E = 0$  in

a bounded domain of diameter  $\leq 2$ , and the coefficients of the operators  $L_R$  are uniformly bounded and these operators are uniformly elliptic.

By Lemma 2.2, the constants  $c_1, c_2$  do not depend on  $T$ . Therefore,

$$\int_{T-R \leq x_n \leq T+R} E(x)^2 dx \leq C_1 m^2 T^{\sigma n}.$$

Let  $h \in C^1(\mathbf{R})$ ,  $h(x_n) = 1$  if  $T - R/2 \leq x_n \leq T + R/2$ ,  $h(x_n) = 0$  if  $x_n \leq T - R$  or  $x_n \geq T + R$ ,  $|h'(x_n)| \leq C_2/R$ . Put in (8)  $\varphi(x) = h(x_n)E(x)$ . We obtain

$$\begin{aligned} & \int_{T-R \leq x_n \leq T+R} h \sum_{i,j=1}^n a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial E}{\partial x_i} dx \\ & + \int_{T-R/2 \geq x_n \geq T-R} \sum_{j=1}^n a_{nj}(x) \frac{\partial E}{\partial x_j} h'(x_n) E(x) dx \\ & + \int_{T+R/2 \leq x_n \leq T+R} \sum_{j=1}^n a_{nj}(x) \frac{\partial E}{\partial x_j} h'(x_n) E(x) dx = 0, \end{aligned}$$

and therefore,

$$\int_{T-R/2 \leq x_n \leq T+R/2} |\nabla E|^2 dx \leq C_3 m^2 T^{\sigma(n-2)}.$$

There exists a constant  $\theta \in ]T - R/2, T + R/2[$  such that

$$\int_{x_n=\theta} |\nabla E|^2 dx' \leq C_3 m^2 T^{\sigma(n-3)}.$$

Put in (5) the function  $\varphi(x) = h_\varepsilon(x)$ , where  $h_\varepsilon \in C^1(\mathbf{R})$ ,  $h_\varepsilon(x_n) = 1$  if  $1 + \varepsilon \leq x_n \leq \theta - \varepsilon$ ,  $h_\varepsilon(x_n) = 0$  if  $x_n \leq 1$  or  $x_n \geq \theta$ , and pass to the limit for  $\varepsilon \rightarrow 0$ . We obtain

$$\left| \int_{x_n=\theta} \sum_{j=1}^n a_{nj}(x) \frac{\partial E}{\partial x_j} dx' \right| = c_0$$

for all  $\theta > 1$ . We have

$$c_0 = \left| \int_{x_n=T} \sum_{j=1}^n a_{nj}(x) \frac{\partial E}{\partial x_j} dx' \right|$$

$$\leq C_4 \left( \int_{x_n=\theta} |\nabla E|^2 dx' \right)^{1/2} T^{\sigma(n-1)/2} \leq C_5 m T^{\sigma(n-2)},$$

so that

$$m \geq C_6 T^{\sigma(2-n)},$$

i.e.

$$E(x) \geq C_6 x_n^{\sigma(2-n)},$$

q.e.d. □

LEMMA 2.5. *Let  $0 \leq \sigma(n-1) < 1$ ,  $n > 2$ ,*

$$Q = \{x = (x', x_n) : |x'| < Ax_n^\sigma + B\}, \quad 1 < x_n < \infty, \quad A > 0, \quad B \geq 0.$$

*There exists a weak solution  $E$  of the problem:*

$$LE = 0 \text{ in } Q, \quad \frac{\partial E}{\partial N} = 0 \text{ on } S, \quad E = 0 \text{ at } x_n = 1,$$

*from the class  $H_{loc}^1(Q)$ , satisfying the estimate  $E(x) \geq c > 0$  for  $x_n \geq 2$ .*

*Proof.* Let  $u_{T,\lambda}$  be the solution to the equation  $Lu = 0$  in  $Q_T$ , satisfying the boundary condition  $\frac{\partial E}{\partial N} = 0$  on  $S$  and such that

$$u_{T,\lambda}(x', 1) = 0, \quad u_{T,\lambda}(x', T) = \lambda. \quad (9)$$

We can find this solution, minimizing the functional

$$F_T(u) = \int_{Q_T} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u(x)}{\partial x_j} \frac{\partial u(x)}{\partial x_i} dx$$

in the class of positive functions  $u$  from  $C^\infty(Q_T)$ , satisfying (9). The minimizing function  $u_{T,\lambda}$  is positive,  $0 < u_{T,\lambda}(x) < \lambda$  in  $Q_T$ . Choosing the value of  $\lambda$  we can obtain  $u_{T,\lambda}(y_0) = 1$ , where  $y_0$  is a fixed point from  $Q$ , whose  $n$ -th coordinate is equal to 2.

Moreover,

$$\int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u_{T,\lambda}}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx = 0 \quad (10)$$



for all functions  $\varphi(x) \in H^1(Q)$ , equal 0 for  $x_n \leq 1$  and for  $x_n = T$ . The function  $u_{T,\lambda}(x)$  is continuous in  $\overline{Q_T}$ , see [6].

Let  $K$  be a compact subset in  $\overline{Q}$ , and let  $T_0$  be such that  $K \subset \overline{Q_{T_0}}$ . By Lemma 2.2, we have  $|u_{T,\lambda}(x)| \leq C(T_0)$  in  $Q_{T_0}$  for  $T > T_0 + 1$ .

Let  $h(x_n)$  be such a piece-wise linear function that  $h(x_n) = 1$  for  $1 < x_n < T_0 - 1$ ,  $h(x_n) = x_n$  for  $0 < x_n < 1$ ,  $h(x_n) = 0$  for  $x_n > T_0$ .

Put in (10) the function  $\varphi(x) = h(x_n)u_{T,\lambda}(x)$ , where  $T > T_0$ . We obtain

$$\begin{aligned} J(T, T_0) &\equiv \int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u_{T,\lambda}}{\partial x_j} \frac{\partial u_{T,\lambda}}{\partial x_i} h(x_n) dx \\ &= - \int_Q \sum_{j=1}^n a_{nj}(x) \frac{\partial u_{T,\lambda}}{\partial x_j} h'(x_n) u_{T,\lambda}(x) dx \\ &\leq C_1 J(T, T_0)^{1/2} \left( \int_{Q_{T_0}} u_{T,\lambda}(x)^2 dx \right)^{1/2} \\ &\leq C_1 J(T, T_0)^{1/2} \left( \int_{Q_{T_0}} dx \right)^{1/2} \leq C(T_0) J(T, T_0)^{1/2}. \end{aligned}$$

Thus  $J(T, T_0) \leq C_1(T_0)$  and  $C_1(T_0)$  does not depend on  $\lambda$ .

Therefore, the set of bounded functions  $u_{T,\lambda}$  on  $K$  is weakly compact, and there exists a subsequence  $\{u_{T_k, \lambda_k}\}$ , which converges weakly in  $H^1(K)$  to a function  $E$ . Considering a sequence of compact sets  $K_m$ , converging to  $Q$  and using diagonalization, we can find a sequence  $\{u_k\}$ , converging weakly in the space  $H_{loc}^1(Q)$  to the function  $E$  from this space. Moreover, using [6, Theorem 7.1], we can show that  $u_k \rightarrow E$  uniformly on each compact subset of  $Q$ .

We have also

$$\int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u_k}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx = 0$$

for all functions  $\varphi(x) \in H^1(Q)$ , equal to 0 for  $x_n = 0$  and for  $x_n = T_k$ . Passing to the limit we obtain that

$$\int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx = 0$$

for all functions  $\varphi(x) \in H^1(Q)$ , equal to 0 for  $x_n = 0$  and in a neighborhood of infinity, i.e.  $E$  is a weak solution of the Neumann problem. Moreover,  $E \geq 0$  in  $Q$  and  $E(y_0) = 1$ , so that  $E \not\equiv 0$ .

Let us show that the function  $E$  cannot be bounded. Indeed, let  $E(x) \leq C$  in  $Q$ . Let  $h(x_n)$  be a smooth function such that  $h(x_n) = 1$  if  $1 < x_n < T$ ,  $h(x_n) = 0$  for  $x_n > 2T$ . One can suppose that  $|h'(x_n)| \leq C_1 T^{-1}$ ,  $|h''(x_n)| \leq C_2 T^{-2}$ . We have

$$\begin{aligned} \int_Q \sum_{i,j=1}^n h(x_n) a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial E}{\partial x_i} dx &= \int_Q \sum_{j=1}^n h'(x_n) E(x) a_{nj}(x) \frac{\partial E}{\partial x_j} dx \\ &\leq C_3 \left( \int_Q \sum_{i,j=1}^n h(x_n) a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial E}{\partial x_i} dx \right)^{1/2} T^{-1/2+\sigma(n-1)/2}. \end{aligned}$$

Therefore,

$$\int_Q \sum_{i,j=1}^n h(x_n) a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial E}{\partial x_i} dx \leq C_4 T^{-1+\sigma(n-1)},$$

and we obtain a contradiction when  $T \rightarrow \infty$ .

Thus there exists a subsequence  $y_k$ , tending to  $\infty$  and such that  $E(y_k) \geq c_1$ , where  $c_1 = \min_{x_n=2} E(x) > 0$ . By the maximum principle and Lemma 2.2, it follows that  $E(x) \geq c_0$  in  $Q$  for  $x_n \geq 2$ , q.e.d.  $\square$

### 3. Existence of positive solutions

Consider the solution  $v(x)$  of equation (3), with boundary condition (4).

**THEOREM 3.1.** *Let*

$$Q = \{x = (x', x_n) : |x'| < Ax_n^\sigma + B, \ 1 < x_n < \infty\}, \ \frac{1}{n-1} < \sigma \leq 1.$$

*Suppose that  $v(x)$  satisfies (3), (4) and  $v(x) \geq 0$  in  $Q$ . If*

$$1 < p \leq 1 + \frac{2-\sigma}{\sigma(n-2)},$$

*then  $v(x) \equiv 0$ .*

*Proof.* The maximum principle implies that  $v(x) \geq cE(x)$  for  $x_n \geq 1$ , where  $E$  is the function, constructed in Lemma 4,  $c > 0$ , or  $v(x) \equiv 0$ . Indeed, put  $w(x) = v(x) - cE(x)$ , where  $c$  is so small that  $w(x) \geq 0$ , when  $x_n = 1$ . Since  $Lw = 0$  in  $Q$ ,  $\frac{\partial w}{\partial N} \geq 0$  on  $S$ , the function  $w$  cannot have a minimum point  $x_1$  in  $Q_T$  if  $x_{1n} > 1$ . On the other hand, by the maximum principle,

$$\lim_{N \rightarrow \infty} \inf_{x_n > N} w(x) \geq 0.$$

Therefore,

$$w(x) \geq 0 \text{ for } x \in Q_T,$$

i.e.

$$v(x) \geq cE(x) \geq c_1 x_n^{\sigma(2-n)}.$$

Put in (5)

$$\varphi(x) = h(x_n)/v(x),$$

where  $h(x_n) = 1$  for  $1 < x_n < T$ ,  $h(x_n) = 0$  for  $2T < x_n$  and for  $x_n < 1/2$ ,  $h$  is a smooth function for  $x_n > 1$ ,

$$|h'(x_n)| \leq c_1 T^{-1}, \quad |h''(x_n)| \leq c_1 T^{-2}.$$

We see that

$$\begin{aligned} 0 = J(T) + \int_{Q_T} h'(x_n) \frac{1}{v} \sum_{j=1}^n a_{nj}(x) \frac{\partial v(x)}{\partial x_j} dx \\ + \int_{\Omega_1} \frac{1}{v} \sum_{j=1}^n a_{nj}(x) \frac{\partial v(x', 1)}{\partial x_j} dx', \end{aligned} \quad (11)$$

where

$$\begin{aligned} J(T) = \int_{Q_T} h(x_n) \frac{1}{v^2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial v(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_i} dx \\ + \int_{S_T} b(x) h(x_n) |v(x)|^{p-1} dS. \end{aligned}$$

Using the inequality  $|h'(x_n)|^2 \leq c_1 h(x_n) T^{-2}$  for  $x_n > 1$ , we see that

$$\left| \int_{Q_T} h'(x_n) \frac{1}{v} \sum_{j=1}^n a_{nj}(x) \frac{\partial v(x)}{\partial x_j} dx \right|^2 \leq C_1 T^{\sigma(n-1)-1} J(T).$$

Therefore,

$$\begin{aligned} J(T) &\leq - \int_{\Omega} \sum_{j=1}^n a_{nj}(x) \frac{\partial v(x', 1)}{\partial x_j} dx' + C_2 T^{\sigma(n-1)/2-1/2} \sqrt{J(T)} \\ &= C_3 + C_4 T^{\sigma(n-1)/2-1/2} \sqrt{J(T)} \end{aligned} \quad (12)$$

and

$$J(T) \leq C_5 T^{\sigma(n-1)-1} \quad \text{for } T \geq 1. \quad (13)$$

However,

$$J(T) \geq \int_{S_T} b(x) h(x_n) |v(x)|^{p-1} dS \geq C_6 T^{-(p-1)\sigma(n-2)+1+\sigma(n-2)},$$

$C_6 > 0$ , and this inequality leads to a contradiction for large  $T$  if

$$1 < p < 1 + \frac{2 - \sigma}{\sigma(n-2)}.$$

Therefore, in this case  $v(x) \equiv 0$ .

If

$$p = 1 + \frac{2 - \sigma}{\sigma(n-2)},$$

then (13) implies that the integral

$$\int x_n^{1-\sigma(n-1)} |\nabla v|^2 v^{-2} dx$$

is converging and therefore,

$$T^{1-\sigma(n-1)} \int_{Q_T \cap \text{supp } h'} |\nabla v|^2 v^{-2} dx \rightarrow 0$$

for  $T \rightarrow \infty$ .

On the other hand, from (3) and (11) we have

$$0 < C_6 \leq T^{(p-1)\sigma(n-2)-1-\sigma(n-2)} J(T) = T^{1-\sigma(n-1)} J(T)$$

$$\leq C_7 T^{1-\sigma(n-1)} + C_8 \left( T^{1-\sigma(n-1)} \int_{Q_T \cap \text{supp } h'} \frac{1}{v^2} \left( \sum_{j=1}^n a_{nj}(x) \frac{\partial v(x)}{\partial x_j} \right)^2 dx \right)^{1/2}.$$

Since the right hand side of the last inequality tends to 0 as  $T \rightarrow \infty$  if  $\sigma(n-1) > 1$ , we obtain a contradiction, i.e. in this case also  $v(x) \equiv 0$ .

The proof is complete.  $\square$

**THEOREM 3.2.** *Let*

$$Q = \{x = (x', x_n) : |x'| < Ax_n^\sigma + B, 1 < x_n < \infty\}, 0 \leq \sigma(n-1) < 1.$$

*Suppose that  $v(x)$  satisfies (3), (4) and  $v(x) \geq 0$  in  $Q$ . If  $p > 1$ , then  $v(x) \equiv 0$ .*

*Proof.* Let us show that  $v(x) > c_1 > 0$  for  $x_n > 1$  or  $v(x) \equiv 0$ . Indeed, let  $c > 0$  be so small that  $v(x) > cE(x)$  at  $x_n = 1$ , where  $E$  is the function found in Lemma 2.5. Put

$$z(x) = \max(cE(x) - v(x), 0).$$

Then  $z(x) = 0, \nabla z(x) = 0$  at  $x_n = 1, z(x) \partial z(x) / \partial N \leq 0$  on  $S$ . Let  $h$  be a smooth function for  $x_n > 1, h(x_n) = 1$  for  $x_n < T, h(x_n) = 0$  for  $x_n > 2T$ . Therefore,

$$\begin{aligned} & - \int_Q h(x_n) \sum_{j=1}^n a_{ij}(x) \frac{\partial z(x)}{\partial x_j} \frac{\partial z(x)}{\partial x_i} dx + \int_S h(x_n) z(x) \frac{\partial z(x)}{\partial N} dS \\ & - \int_{Q_T} h'(x_n) z(x) \sum_{j=1}^n a_{nj}(x) \frac{\partial z(x)}{\partial x_j} dx \geq 0. \end{aligned}$$

Therefore,

$$\int_Q h(x_n) \sum_{j=1}^n a_{ij}(x) \frac{\partial z(x)}{\partial x_j} \frac{\partial z(x)}{\partial x_i} dx \leq C \int_{Q \cap \text{supp } h'} h'(x_n)^2 z(x)^2 dx$$

$$\leq C_1 T^{-2} \int_{Q \cap \text{supph}'} E(x)^2 dx \leq C_2 \int_{Q \cap \text{supph}'} x_n^{-2} E(x)^2 dx.$$

Since by Lemma 3, the integral  $\int_Q x_n^{-2} E(x)^2 dx$  is converging, the integral  $\int_{Q \cap \text{supph}'} x_n^{-2} E(x)^2 dx$  tends to 0 as  $x_n \rightarrow \infty$ . In particular, it means that  $v(x) > cE(x) > c_1 > 0$  for  $x_n \geq 1$ .

Let  $h(x_n) = 1$  for  $x_n < T - 1$ ,  $h(x_n) = 0$  for  $T < x_n$ ,  $h$  be a smooth function for  $x_n > 1$ ,

$$|h'(x_n)| \leq c_1, \quad |h''(x_n)| \leq c_1.$$

Put in the definition of weak solutions the function  $\varphi(x) = h(x_n)/v(x)$ . We have

$$0 = J(T) - \int_{Q_T} h'(x_n) \frac{1}{v} \sum_{j=1}^n a_{nj} \frac{\partial v(x)}{\partial x_j} dx + \int_{\Omega_1} \frac{1}{v} \sum_{j=1}^n a_{nj} \frac{\partial v(x', 1)}{\partial x_j} dx',$$

where

$$\begin{aligned} J(T) &= \int_{Q_T} h(x_n) \frac{1}{v^2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial v(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_i} dx \\ &\quad - \int_{S_T} b(x) h(x_n) |v(x)|^{p-1} dS. \end{aligned}$$

Using the inequality  $|h'(x_n)|^2 \leq 2c_1 h(x_n)$  for  $x_n > 1$ , we see that

$$\left| \int_{Q_T} h'(x_n) \frac{1}{v} \sum_{j=1}^n a_{nj}(x) \frac{\partial v(x)}{\partial x_j} dx \right|^2 \leq C_1 J(T) T^{\sigma(n-1)/2}.$$

Therefore,

$$\begin{aligned} J(T) &\leq \left| \int_{\Omega} \sum_{j=1}^n a_{nj}(x) \frac{\partial v(x', 1)}{\partial x_j} dx' \right| + C_2 \sqrt{J(T)} T^{\sigma(n-1)/2} \\ &= C_3 + C_2 \sqrt{J(T)} T^{\sigma(n-1)/2}. \end{aligned}$$

and  $J(T) \leq C_4 T^{\sigma(n-1)}$ . However,

$$J(T) \geq \int_{S_T} b(x) h(x_n) |v(x)|^{p-1} dS \geq C_5 T^{1+\sigma(n-2)}, \quad C_5 > 0,$$

and the inequality  $C_5 T^{1+\sigma(n-2)} \leq C_4 T^{\sigma(n-1)}$ , following from the above inequality, is impossible for large  $T$ . Therefore,  $v(x) \equiv 0$ .  $\square$

THEOREM 3.3. *Let*

$$Q = \{x = (x', x_n) : |x'| < Ax_n^\sigma + B, 1 < x_n < \infty\}, 0 \leq \sigma(n-1) \leq 1.$$

*Suppose that  $v(x)$  satisfies (3) and  $v(x) \geq 0$  in  $Q$ . Suppose that*

$$\sum_{j=1}^n a_{nj}(x) \frac{\partial v(x)}{\partial x_j} \leq 0 \quad \text{as } x_n = 1$$

*and  $\partial v(x)/\partial N \geq 0$  on  $S$ ,  $p > 0$ . Then  $v(x) \equiv 0$ .*

*Proof.* Let  $\varepsilon$  be a small positive number. Put in (5)  $\varphi(x) = h(x_n)/(v(x) + \varepsilon)$ , where  $h(x_n) = 1$  for  $1 < x_n < T$ ,  $h(x_n) = 0$  for  $2T < x_n$ ,  $h$  is a smooth function for  $x_n > 1$ . We have

$$\begin{aligned} J(T) &\equiv \int_{Q_T} h(x_n) \frac{1}{(v + \varepsilon)^2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial v(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_i} dx \\ &\quad + \int_{S_T} b(x) h(x_n) \frac{v(x)^p}{(v + \varepsilon)^2} dS \\ &\leq \int_{Q_T} h'(x_n) \frac{1}{v + \varepsilon} \sum_{j=1}^n a_{nj}(x) \frac{\partial v(x)}{\partial x_j} dx, \end{aligned}$$

i.e.

$$J(T) \leq C \int_{Q_T} T^{-2} dx \leq C_1 T^{-2+1+\sigma(n-1)}.$$

We see tending  $T \rightarrow \infty$  that  $v(x) \equiv 0$ .

The proof is complete.  $\square$

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