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Some additive decompositions of semisimple matrices

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Abstract. *Under suitable hypotheses on the ground field and on the semisimple matrix M, we discuss some additive decompositions of M and of its image through a convergent power series.*

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1. Introduction

In this paper we carry on the study of some additive decompositions of matrices, started in [2] and in [1] with applications to the image of a matrix through a power series (see for instance also [3]). We work in a fixed algebraic closure $\overline{\mathbb{K}}$ of a fixed field $\mathbb{K}$ of characteristic 0.

In §2 we define a projection, $\tau_{\mathbb{K}} : \overline{\mathbb{K}} \to \overline{\mathbb{K}}$, whose image is $\mathbb{K}$ and which allows to decompose $\overline{\mathbb{K}}$ as direct sum $\mathbb{K} \oplus Ker(\tau_{\mathbb{K}})$ (see Remark 2.1). As a consequence, we get the $\mathbb{K}$-*trace decomposition* of a semisimple matrix $M \in M_n(\mathbb{K})$, i.e. we write (in a unique way) $M = T + F$, where $T, F \in M_n(\mathbb{K})$ are mutually commuting, T diagonalizable over $\mathbb{K}$ and F semisimple, with all eigenvalues in $Ker(\tau_{\mathbb{K}})$ (see Proposition 2.10). Finally, we obtain the *fine* $\mathbb{K}$-*trace decomposition* of any semisimple matrix $M \in M_n(\mathbb{K})$ (see Proposition 2.11 and Remark 2.13), which decomposes each summand of the $\mathbb{K}$-trace decomposition in simpler summands.

In §3, starting from the fine $\mathbb{K}$-trace decomposition of a semisimple matrix M, we get a formula for the image $f(M)$ through a power series under the further conditions that $\mathbb{K}$ is a *valued field* and that M is $\mathbb{K}$-*quadratic*, i.e. its eigenvalues have degree at most 2 over $\mathbb{K}$ (see Proposition 3.5 and in particular Examples 3.6).

In §4, we *normalize* the fine $\mathbb{K}$-trace decomposition of a semisimple $\mathbb{K}$-quadratic matrix M, when the field $\mathbb{K}$ is *ordered quadratically closed* and we write its image through a power series as above (see Proposition 4.10). When $\mathbb{K}$ is *real closed* too, this formula holds for every semisimple matrix in the domain of convergence of the series and it can be viewed as a generalization of the classical Rodrigues' formula for the exponential of a real skew symmetric matrix (see Examples 4.11).

2. Fine $\mathbb{K}$-trace decomposition

In this paper $\mathbb{K}$ is a fixed field of characteristic 0 and $\overline{\mathbb{K}}$ one of its algebraic closures.

REMARK 2.1 ($\mathbb{K}$-decomposition): Let $\mathbb{L} \subseteq \overline{\mathbb{K}}$ be any finite extension of $\mathbb{K}$ and $\lambda \in \mathbb{L}$ be of degree d with minimal polynomial over $\mathbb{K}$:

$$m_\lambda(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0\,.$$

The multiplication by λ is a $\mathbb{K}$-linear mapping from $\mathbb{L}$ into $\mathbb{L}$, whose characteristic polynomial is $m_\lambda(X)^{[\mathbb{L}:\mathbb{K}(\lambda)]}$ and whose trace is:

$$tr_{\mathbb{L}/\mathbb{K}}(\lambda) = -[\mathbb{L} : \mathbb{K}(\lambda)]a_{d-1} = -\frac{[\mathbb{L} : \mathbb{K}]}{[\mathbb{K}(\lambda) : \mathbb{K}]}a_{d-1} = -\frac{[\mathbb{L} : \mathbb{K}]}{d}a_{d-1}$$

(see for instance [6, Ch. VI, Proposition 5.6]).

Hence the expression: $\dfrac{tr_{\mathbb{L}/\mathbb{K}}(\lambda)}{[\mathbb{L} : \mathbb{K}]} = -\dfrac{a_{d-1}}{d}$ depends only on λ and $\mathbb{K}$ and not on the finite extension $\mathbb{L} \subset \overline{\mathbb{K}}$ of $\mathbb{K}$, containing λ.

Therefore, for each $\lambda \in \overline{\mathbb{K}}$, we call $\tau_{\mathbb{K}}(\lambda) := -\dfrac{a_{d-1}}{d}$ the *normalized* $\mathbb{K}$-*trace of* λ.

It is easy to check that $\tau_{\mathbb{K}}$ is a $\mathbb{K}$-linear mapping from $\overline{\mathbb{K}}$ onto $\mathbb{K} \subseteq \overline{\mathbb{K}}$ with $\tau_{\mathbb{K}}^2 = \tau_{\mathbb{K}}$ (i.e. $\tau_{\mathbb{K}}$ is a *projection* as a $\mathbb{K}$-linear endomorphism of $\overline{\mathbb{K}}$) and therefore we get a canonical decomposition as $\mathbb{K}$-vector spaces: $\overline{\mathbb{K}} = \mathbb{K} \oplus Ker(\tau_{\mathbb{K}})$, (see for instance [4, p. 211]), i.e. every element $\lambda \in \overline{\mathbb{K}}$ has a unique decomposition $\lambda = \tau_{\mathbb{K}}(\lambda) + \varphi_{\mathbb{K}}(\lambda)$ with $\tau_{\mathbb{K}}(\lambda) \in \mathbb{K}$ and $\varphi_{\mathbb{K}}(\lambda) \in Ker(\tau_{\mathbb{K}})$.

We call this decomposition, $\tau_{\mathbb{K}}(\lambda)$ and $\varphi_{\mathbb{K}}(\lambda)$ respectively, the $\mathbb{K}$-*decomposition*, the $\mathbb{K}$-*part* and the $\mathbb{K}$-*trace-free part* of λ. We will write $\tau(\lambda)$ and $\varphi(\lambda)$ in absence of ambiguity about the field $\mathbb{K}$.

REMARK 2.2 ($\mathbb{K}$-trace-free polynomial): Recall that two elements $\lambda, \lambda' \in \overline{\mathbb{K}}$ are said to be *conjugate over* $\mathbb{K}$, if they have the same minimal polynomial over $\mathbb{K}$ or, equivalently, if they are in the same orbit under $Aut(\overline{\mathbb{K}}/\mathbb{K})$: the group of all automorphisms of the field $\overline{\mathbb{K}}$ fixing each element of $\mathbb{K}$. Hence $\lambda, \lambda' \in \overline{\mathbb{K}}$ are conjugate over $\mathbb{K}$ if and only if $\tau(\lambda) = \tau(\lambda')$ and $\varphi(\lambda)$, $\varphi(\lambda')$ are conjugate over $\mathbb{K}$.

For every $\lambda \in \overline{\mathbb{K}}$ we denote $\nu_{\mathbb{K}}(\lambda) = \nu(\lambda)$ the *normalized norm of* λ *over* $\mathbb{K}$ as $\nu(\lambda) = (-1)^d a_0 = \lambda_1\lambda_2 \ldots \lambda_d$, where $d = deg_{\mathbb{K}}(\lambda)$, a_0 is the constant term of the minimal polynomial of λ over $\mathbb{K}$ and $\{\lambda = \lambda_1, \lambda_2, \ldots, \lambda_d\}$ is the conjugacy class of λ over $\mathbb{K}$.

If $m_\lambda(X)$ is the minimal polynomial of λ over $\mathbb{K}$, then we call $\mathbb{K}$-*trace-free polynomial of* λ to be the polynomial $\widetilde{m}_\lambda(X) = m_\lambda(X+\tau(\lambda)) - m_\lambda(\tau(\lambda))$.

Note that $m_\lambda(X+\tau(\lambda))$ is the minimal polynomial of $\varphi(\lambda)$ over $\mathbb{K}$ and so $m_\lambda(\tau(\lambda)) = (-1)^d\nu(\varphi(\lambda))$, thus $\widetilde{m}_\lambda(X) = m_\lambda(X+\tau(\lambda)) + (-1)^{d+1}\nu(\varphi(\lambda))$. Moreover $\widetilde{m}_\lambda(X)$ is monic of degree d, its coefficient of the term of degree $d-1$ and its constant term are both zero. In particular, if $d=2$, then $\widetilde{m}_\lambda(X) = X^2$ for every λ.

Remark 2.3 ($\mathbb{K}$-linear involution): The mapping $\lambda \mapsto \overline{\lambda} = \overline{\tau(\lambda)+\varphi(\lambda)} = \tau(\lambda) - \varphi(\lambda)$ is called the $\mathbb{K}$-*linear involution of* $\overline{\mathbb{K}}$. The $\mathbb{K}$-linear involution is an automorphism of $\overline{\mathbb{K}}$ as $\mathbb{K}$-vector space, but in general not as field; $\mathbb{K}$ is the set of fixed points of the $\mathbb{K}$-linear involution.

Lemma 2.4. *a) If $\lambda = \tau(\lambda)+\varphi(\lambda) \in \overline{\mathbb{K}}$ has degree 2 over $\mathbb{K}$, then $\varphi(\lambda)^2 \in \mathbb{K}$ and the unique conjugate of λ, distinct from λ, is $\overline{\lambda}$.*

b) If $\mathbb{L} \subseteq \overline{\mathbb{K}}$ is an extension of $\mathbb{K}$ of degree 2, then the $\mathbb{K}$-linear involution restricted to $\mathbb{L}$ is an element of $Aut(\mathbb{L}/\mathbb{K})$.

c) If $\beta, \beta' \in Ker(\tau)$ have degree 2 over $\mathbb{K}$, then $\beta\beta' \in \mathbb{K}$ if and only if β and β' are linearly dependent over $\mathbb{K}$, otherwise $\beta\beta' \in Ker(\tau)$ and its degree over $\mathbb{K}$ is 2.

d) If $\lambda \in \overline{\mathbb{K}}$ has degree 2 over $\mathbb{K}$, then $\lambda \in Ker(\tau)$ if and only if $\lambda^2 \in \mathbb{K}$.

Proof. a) Since $\varphi(\lambda)$ has degree 2 over $\mathbb{K}$ and its normalized $\mathbb{K}$-trace is zero, its minimal polynomial has the form $X^2 - \varphi(\lambda)^2$, hence $\varphi(\lambda)^2 \in \mathbb{K}$ and $\varphi(\lambda)$ and $-\varphi(\lambda)$ are conjugate over $\mathbb{K}$ and also λ and $\overline{\lambda}$ are conjugate over $\mathbb{K}$ (remember Remark 2.2).

b) Choose an element $\lambda \in \mathbb{L}$ of degree 2 over $\mathbb{K}$, having normalized $\mathbb{K}$-trace equal to zero. Hence the elements of $\mathbb{L}$ are of the form $k_1 + k_2\lambda$ with $k_1, k_2 \in \mathbb{K}$. We conclude by standard computations, because $\lambda^2 \in \mathbb{K}$ by (a).

c) If β and β' are as in (c), then, from (a), β^2, β'^2 are both in $\mathbb{K}$. Hence $\beta\beta'$ is root of $X^2 - \beta^2\beta'^2 \in \mathbb{K}[X]$ and so the degree of $\beta\beta'$ is at most 2. The degree of $\beta\beta'$ is 1 if and only if $\beta\beta' = t \in \mathbb{K}$, i. e. if and only if $\beta = \dfrac{t}{\beta'} = \dfrac{t}{\beta'^2}\beta'$ and so if and only if β, β' are linearly dependent over $\mathbb{K}$, because $\dfrac{t}{\beta'^2} \in \mathbb{K}$. Otherwise the degree of $\beta\beta'$ is 2; thus $X^2 - \beta^2\beta'^2$ is its minimal polynomial and so: $\tau(\beta\beta') = 0$.

Finally, one implication of (d) follows directly from (a), since $\tau(\lambda) = 0$. For the other implication it suffices to note that, if λ has degree 2 over $\mathbb{K}$ and $\lambda^2 \in \mathbb{K}$, then the minimal polynomial of λ over $\mathbb{K}$ is $X^2 - \lambda^2$ and so: $\tau(\lambda) = 0$. □

REMARK 2.5 (minimal polinomial): From now on M is a fixed *semisimple* matrix in the set $M_n(\mathbb{K})$ of the square matrices of order $n \geq 2$ with entries in $\mathbb{K}$ and with minimal polynomial $m(X) = (X-\gamma_1)\cdots(X-\gamma_s)\, m_1(X)\cdots m_t(X)$, where $\gamma_1, \cdots, \gamma_s$ are mutually distinct elements of $\mathbb{K}$ and $m_1(X), \cdots, m_t(X)$ are mutually distinct irreducible monic polynomials in $\mathbb{K}[X]$ of degrees $d_1, \cdots, d_t \geq 2$ respectively. We denote by $\lambda_{h1}, \cdots, \lambda_{hd_h}$ the d_h distinct roots of the factor $m_h(X)$ and by $\mathbb{F} \subseteq \overline{\mathbb{K}}$ the *splitting field* of $m(X)$. By conciseness we put $\lambda_h = \lambda_{h1}$ for every $h = 1, \cdots, t$.

DEFINITION 2.6. *With the notations of Remark 2.5 we say that the semisimple matrix $M \in M_n(\mathbb{K})$ is* $\mathbb{K}$-quadratic *if every factor $m_h(X)$ has degree 2 (or if $m(X) = (X-\gamma_1)\cdots(X-\gamma_s)$).*

REMARK 2.7 (Frobenius decomposition): The semisimple matrix M has a unique decomposition:

$$M = \sum_{i=1}^{s} \gamma_i \mathcal{A}_i + \sum_{h=1}^{t} \sum_{j=1}^{d_h} \lambda_{hj} C_{hj}\,,$$

where $\{\gamma_1\} \cup \cdots \{\gamma_s\} \cup_{h=1}^{t} \{\lambda_{h1}, \cdots, \lambda_{hd_h}\}$ is the set of all distinct eigenvalues of M (arranged in conjugacy classes) and the matrices $\mathcal{A}_i$'s, C_{hj}'s are *idempotent* and satisfying the conditions: $\mathcal{A}_i \mathcal{A}_{i'} = 0$ (if $i \neq i'$), $\mathcal{A}_i C_{hj} = C_{hj} \mathcal{A}_i = 0$ (for every i, j, h), $C_{hj} C_{h'j'} = 0$ (if $(h,j) \neq (h',j')$), $\sum_{i=1}^{s} \mathcal{A}_i + \sum_{h=1}^{t} \sum_{j=1}^{d_h} C_{hj} = I_n$ (the identity matrix of order n).

The above decomposition is called *Frobenius decomposition of* M and the matrices $\mathcal{A}_i$'s and C_{hj}'s, called *Frobenius covariants* of M, are uniquely determined by M (and by the previous conditions) and are polynomial expressions of M of degree strictly less than $deg(m(X))$; finally the matrices $\mathcal{A}_i$'s have coefficients in $\mathbb{K}$ and the matrices C_{hj}'s in $\mathbb{F}$ (see [1, § 1]).

DEFINITION 2.8. *A* $\mathbb{K}$-trace decomposition *of M is an additive decomposition: $M = T + F$ where $T, F \in M_n(\mathbb{K})$ are mutually commuting, T is diagonalizable over $\mathbb{K}$ and F is semisimple with eigenvalues in $Ker(\tau_{\mathbb{K}})$. We say that T and F respectively are a* $\mathbb{K}$-part *and a* $\mathbb{K}$-trace-free part of M.

REMARK 2.9: If A is a matrix in $M_n(\mathbb{K})$ such that all eigenvalues of A are in $Ker(\tau)$, then its usual *trace* is zero.

Indeed if λ is an eigenvalue of A of degree d over $\mathbb{K}$, then every conjugate of λ over $\mathbb{K}$ is again an eigenvalue of A, moreover the sum of the eigenvalues of the whole conjugacy class of λ over $\mathbb{K}$ is $d \cdot \tau(\lambda) = 0$ and so the trace of A is zero.

Therefore, if the matrix $M \in M_n(\mathbb{K})$ has a $\mathbb{K}$-trace decomposition, then the trace of its $\mathbb{K}$-trace-free part is zero.

Proposition 2.10. *The semisimple matrix $M \in M_n(\mathbb{K})$ has a unique $\mathbb{K}$-trace decomposition: $M = T(M) + F(M)$, where, with the notations of Remark 2.5 and Remark 2.7,*

$$T(M) = \sum_{i=1}^{s} \gamma_i \mathcal{A}_i + \sum_{h=1}^{t} \tau(\lambda_h) \sum_{j=1}^{d_h} C_{hj} \quad \text{and} \quad F(M) = \sum_{h=1}^{t} \sum_{j=1}^{d_h} \varphi(\lambda_{hj}) C_{hj}\,.$$

In particular $T(M)$, $F(M)$ are polynomial expressions of M.

Proof. By Remark 2.2, $\tau(\lambda_{hj}) = \tau(\lambda_h)$ for every j and, for every h, the set $\{\varphi(\lambda_{h1}), \cdots, \varphi(\lambda_{hd_h})\}$ is a conjugacy class over $\mathbb{K}$.

Now let $M = \sum_{i=1}^{s} \gamma_i \mathcal{A}_i + \sum_{h=1}^{t} \sum_{j=1}^{d_h} \lambda_{hj} C_{hj}$ be the Frobenius decomposition of M. By decomposing each λ_{hj} as $\tau(\lambda_h) + \varphi(\lambda_{hj})$, we get: $M = T(M) + F(M)$, with $T(M)$ and $F(M)$ as in the statement and therefore polynomial expressions of M. Arguing as in the proof of [1, Theorem 1.6], $T(M)$ and $F(M)$ are matrices with coefficients in $\mathbb{K}$, i.e. in the fixed field of $Aut(\mathbb{F}/\mathbb{K})$. Standard computations show that $T(M)$ and $F(M)$ are respectively diagonalizable over $\mathbb{K}$ and over $\mathbb{F}$ with eigenvalues $\{\gamma_i, \tau(\lambda_h)\}$ and $\{\varphi(\lambda_{hj})\}$ (see also Remark 2.7 and [1, Proposition 1.9]). This allows to conclude about the existence of a $\mathbb{K}$-trace decomposition in terms of polynomial expressions of M.

Now let $M = T' + F'$ be any other $\mathbb{K}$-trace decomposition of M. This implies $T(M) - T' = F' - F(M)$. Now T', F' commute with M and so with $T(M), F(M)$; moreover the four matrices are semisimple, hence, by simultaneous diagonalizability, every eigenvalue σ' of F' can be written as $\sigma' = \delta - \delta' + \sigma$ with δ, δ', σ eigenvalues of $T(M)$, T', $F(M)$ respectively. From the uniqueness of the $\mathbb{K}$-decomposition $\delta - \delta' = 0$ and $\sigma = \sigma'$. Therefore $T' = T(M)$ and $F' = F(M)$. □

Proposition 2.11. *Let $M \in M_n(\mathbb{K})$ be semisimple with eigenvalues: $\gamma_1, \cdots, \gamma_s$ distinct and belonging to $\mathbb{K}$ and the remaining (not in $\mathbb{K}$) $\{\lambda_h = \lambda_{h1}, \cdots, \lambda_{hd_h}\}$, $h = 1, \cdots, t$, arranged in distinct conjugacy classes. We have the decomposition*

$$M = \sum_{i=1}^{s} \gamma_i \mathcal{A}_i + \sum_{h=1}^{t} \frac{(-1)^{d_h+1}\,\tau(\lambda_h)}{\nu(\varphi(\lambda_h))} \widetilde{m}_h(\mathcal{B}_h) + \sum_{h=1}^{t} \mathcal{B}_h \tag{1}$$

with $\mathcal{B}_h = \mathcal{B}_h(M) = \sum_{j=1}^{d_h} \varphi(\lambda_{hj})\, C_{hj}$ $(h = 1, \cdots, t)$ matrix in $M_n(\mathbb{K})$ and $\widetilde{m}_h(X)$ $(h = 1, \cdots, t)$ the $\mathbb{K}$-trace free polynomial of λ_h.

Proof. By Proposition 2.10, we have:

$$M = \sum_{i=1}^{s} \gamma_i \mathcal{A}_i + \sum_{h=1}^{t} \tau(\lambda_h) \sum_{j=1}^{d_h} C_{hj} + \sum_{h=1}^{t} \sum_{j=1}^{d_h} \varphi(\lambda_{hj}) C_{hj} =$$
$$= \sum_{i=1}^{s} \gamma_i \mathcal{A}_i + \sum_{h=1}^{t} \tau(\lambda_h) \sum_{j=1}^{d_h} C_{hj} + \sum_{h=1}^{t} \mathcal{B}_h$$

As in [1, Proposition 1.5], it is easy to check that $\sigma(\mathcal{B}_h) = \mathcal{B}_h$ for every $\sigma \in Aut(\mathbb{F}/\mathbb{K})$, hence $\mathcal{B}_h \in M_n(\mathbb{K})$. Thus it suffices to prove that, for every $h = 1, \cdots, t$: $\widetilde{m}_h(\mathcal{B}_h) = (-1)^{d_h+1}\nu(\varphi(\lambda_h)) \sum_{j=1}^{d_h} C_{hj}$.

Since $\widetilde{m}_h(X)$ has constant term zero, from the properties of the matrices C_{hj}'s, we obtain $\widetilde{m}_h(\mathcal{B}_h) = \sum_{j=1}^{d_h} \widetilde{m}_h(\varphi(\lambda_{hj}))C_{hj}$ and so, by Remark 2.2, we can conclude that $\widetilde{m}_h(\mathcal{B}_h) = (-1)^{d_h+1}\nu(\varphi(\lambda_h)) \sum_{j=1}^{d_h} C_{hj}$. □

Remark 2.12: The matrices $\mathcal{A}_i$'s and $\mathcal{B}_h$'s in Proposition 2.11 are polynomial functions of M satisfying the following properties:

1) $\mathcal{A}_i \mathcal{A}_{i'} = \delta_{ii'} A_i$, for every i, i';

2) $\mathcal{A}_i \mathcal{B}_h = \mathcal{B}_h \mathcal{A}_i = 0$, for every i, h;

3) $\mathcal{B}_h \mathcal{B}_{h'} = 0$, provided $h \neq h'$;

4) $\mathcal{B}_h \, \widetilde{m}_h(\mathcal{B}_h) = (-1)^{d_h+1}\nu(\varphi(\lambda_h))\, \mathcal{B}_h$, for every h.

Some of the previous properties have been already noted in Remark 2.7 and the others are easy to get by standard computations.

Remark 2.13 (fine $\mathbb{K}$-trace decomposition): It is easy to note that in Proposition 2.11, formula (1), every $\mathcal{B}_h$ is $\mathbb{K}$-trace free, while the $\mathbb{K}$-part and the $\mathbb{K}$-trace free part of M are respectively:

$$T(M) = \sum_{i=1}^{s} \gamma_i \mathcal{A}_i + \sum_{h=1}^{t} \frac{(-1)^{d_h+1}\, \tau(\lambda_h)}{\nu(\varphi(\lambda_h))} \widetilde{m}_h(\mathcal{B}_h)$$

with eigenvalues γ_i and $\tau(\lambda_h)$;

$$F(M) = \sum_{h=1}^{t} \mathcal{B}_h$$

with eigenvalues $\varphi(\lambda_{hj})$ and possibly 0.

Therefore we call the decomposition (1) in Proposition 2.11, the *fine* $\mathbb{K}$-*trace decomposition* of the semisimple matrix $M \in M_n(\mathbb{K})$.

Remark 2.14: By Lemma 2.4-(a) and Remark 2.2, if the matrix $M \in M_n(\mathbb{K})$ is semisimple and $\mathbb{K}$-quadratic, then the fine $\mathbb{K}$-trace decomposition of M becomes:

$$M = \sum_{i=1}^{s} \gamma_i \mathcal{A}_i + \sum_{h=1}^{t} \frac{\tau(\lambda_h)}{\varphi(\lambda_h)^2} \mathcal{B}_h^2 + \sum_{h=1}^{t} \mathcal{B}_h \,, \tag{2}$$

while the property (4) in Remark 2.12 becomes

4') $\mathcal{B}_h^3 = \varphi(\lambda_h)^2 \, \mathcal{B}_h$, for every h.

Moreover, from the properties of the Frobenius covariants of M and from Lemma 2.4-(a), we get:

$$I_n = \sum_{i=1}^{s} \mathcal{A}_i + \sum_{h=1}^{t} \frac{\mathcal{B}_h^2}{\varphi(\lambda_h)^2} \,.$$

3. A formula for power series of matrices over a valued field.

Remark 3.1: In this section we assume that $\mathbb{K}$ (of characteristic 0) is endowed with an *absolute value* $|\cdot|$. We call such a pair $(\mathbb{K}, |\cdot|)$ a *valued field*. We refer for instance to [5, Ch. 9], to [10, Ch. III], to [6, Ch. XII] and to [7, Ch. 23] for more information.

Let $(\mathbb{K}, |\cdot|)$ be a valued field. The absolute value over $\mathbb{K}$ extends in a unique way to its *completion* $\mathbb{K}^c$; this one extends in a unique way to an absolute value over a fixed algebraic closure $\overline{\mathbb{K}^c}$ of $\mathbb{K}^c$ and finally the last one extends in a unique way to the completion $(\overline{\mathbb{K}^c})^c$ (see for instance [7] Theorem 2 p. 48, Ostrowski's Theorem p. 55 and Theorem 4' p. 60). We denote all extensions always by the same symbol $|\cdot|$.

Note that the field $\{\alpha \in \overline{\mathbb{K}^c} \;/\; \alpha \text{ is algebraic over } \mathbb{K}\}$ is the unique algebraic closure of $\mathbb{K}$ contained in $\overline{\mathbb{K}^c}$ and therefore it can be identified with $\overline{\mathbb{K}}$.

By restriction, we get an absolute value over $\overline{\mathbb{K}}$, extending the absolute value of $\mathbb{K}$.

Lemma 3.2. *Let $(\mathbb{K}, |\cdot|)$ be a valued field.*

a) If $\mathbb{K}$ is algebraically closed, then its completion $\mathbb{K}^c$ (with respect to $|\cdot|$) is algebraically closed too.

b) $(\overline{\mathbb{K}^c})^c$ is algebraically closed and complete with respect to the unique extension to it of $|\cdot|$.

Proof. The proof of (a) follows easily from Ostrowski's Theorem in archimedean case (see [7, p. 55]), while for the non-archimedean case we refer to [7, Appendix 24.15, p. 316]. Part (b) follows directly from (a). □

DEFINITION 3.3. *Let $f(X) = \sum_{m=0}^{+\infty} a_m X^m$ be a series with coefficients in the valued field $(\mathbb{K}, |\cdot|)$ and $R_f \in \mathbb{R} \cup \{+\infty\}$ be the* radius of convergence *of the associated real series $\sum_{m=0}^{+\infty} |a_m| X^m$.*

We denote by $\Lambda_{f,\mathbb{K}}$ the subset of the matrices $A \in M_n(\mathbb{K})$ such that all of their eigenvalues $\lambda = \tau(\lambda) + \varphi(\lambda)$ (with their $\mathbb{K}$-decompositions) satisfy:

i) $|\tau(\lambda)| + |\varphi(\lambda)| < R_f$, if the absolute value of $\mathbb{K}$ is archimedean,

ii) $max(|\tau(\lambda)|, |\varphi(\lambda)|) < R_f$, if the absolute value of $\mathbb{K}$ is non-archimedean.

For every eigenvalue λ of a matrix $A \in \Lambda_{f,\mathbb{K}}$ with $\mathbb{K}$-decomposition $\lambda = \tau(\lambda) + \varphi(\lambda)$, denoted by $\lfloor x \rfloor$ the integer part *of the real number x, we introduce the formal series:*

$$\mathcal{T}f(\lambda) = \sum_{m=0}^{+\infty} a_m \sum_{h=0}^{\lfloor m/2 \rfloor} \binom{m}{2h} \tau(\lambda)^{m-2h} \varphi(\lambda)^{2h},$$

$$\mathcal{F}f(\lambda) = \sum_{m=1}^{+\infty} a_m \sum_{h=1}^{\lfloor (m+1)/2 \rfloor} \binom{m}{2h-1} \tau(\lambda)^{m-2h+1} \varphi(\lambda)^{2h-1}.$$

REMARK 3.4: a) If $A \in \Lambda_{f,\mathbb{K}}$, then $f(A) \in M_n(\mathbb{K}^c)$ (see [1, Remark-Definition 3.1(c)]).

b) If λ is an eigenvalue of a matrix in $\Lambda_{f,\mathbb{K}}$, then $f(\lambda), \mathcal{T}f(\lambda), \mathcal{F}f(\lambda)$ converge in $(\overline{\mathbb{K}})^c \subseteq (\overline{\mathbb{K}^c})^c$ and $f(\lambda) = \mathcal{T}f(\lambda) + \mathcal{F}f(\lambda)$.

The previous assertions follow from the definitions, by standard computations.

c) Assume that λ is an eigenvalue of degree 2 over $\mathbb{K}$ of a matrix in $\Lambda_{f,\mathbb{K}}$. Then: $\mathcal{T}f(\lambda) \in \mathbb{K}^c$, $\mathcal{F}f(\lambda), f(\lambda) \in \mathbb{K}^c(\lambda)$. Moreover, if $\lambda \notin \mathbb{K}^c$, then $f(\lambda) = \mathcal{T}f(\lambda) + \mathcal{F}f(\lambda)$ is the $\mathbb{K}^c$-decomposition of $f(\lambda)$.

Indeed if $\lambda \notin \mathbb{K}^c$ has degree 2 over $\mathbb{K}$ and $\lambda = \tau(\lambda) + \varphi(\lambda)$ is its $\mathbb{K}$-decomposition, then $\varphi(\lambda) \notin \mathbb{K}^c$ and $\varphi(\lambda)^2 \in \mathbb{K}$. Hence, looking at the partial sums and at their limits, we get that $\mathcal{T}f(\lambda) \in \mathbb{K}^c$, while $\mathcal{F}f(\lambda)$ is a multiple of $\varphi(\lambda)$ with coefficient in $\mathbb{K}^c$ and so it belongs to $Ker(\tau_{\mathbb{K}^c})$. We can conclude by uniqueness of $\mathbb{K}^c$-decomposition.

PROPOSITION 3.5. *Let $f(X) = \sum_{m=0}^{+\infty} a_m X^m$ be a series with coefficients in the valued field $(\mathbb{K}, |\cdot|)$ and $M \in \Lambda_{f,\mathbb{K}}$ be a semisimple $\mathbb{K}$-quadratic matrix with* *fine $\mathbb{K}$-trace decomposition:*

$$M = \sum_{i=1}^{s} \gamma_i \mathcal{A}_i + \sum_{h=1}^{t} \frac{\tau_{\mathbb{K}}(\lambda_h)}{\varphi_{\mathbb{K}}(\lambda_h)^2} \mathcal{B}_h^2 + \sum_{h=1}^{t} \mathcal{B}_h$$

as in Remark 2.14. Then

a) $f(M) = \sum_{i=1}^{s} f(\gamma_i)\mathcal{A}_i + \sum_{h=1}^{t} \left[\frac{\mathcal{T}f(\lambda_h)}{\varphi_{\mathbb{K}}(\lambda_h)^2} \mathcal{B}_h^2 + \frac{\mathcal{F}f(\lambda_h)}{\varphi_{\mathbb{K}}(\lambda_h)} \mathcal{B}_h \right]$,

with $f(\gamma_i)$, $\frac{\mathcal{T}f(\lambda_h)}{\varphi_{\mathbb{K}}(\lambda_h)^2}, \frac{\mathcal{F}f(\lambda_h)}{\varphi_{\mathbb{K}}(\lambda_h)} \in \mathbb{K}^c$ *for every* i, h*;*

b) furthermore, if no eigenvalue λ_h *of degree* 2 *over* $\mathbb{K}$ *is in* $\mathbb{K}^c$*,*

$$f(M) = \sum_{i=1}^{s} f(\gamma_i)\mathcal{A}_i + \sum_{h=1}^{t} \left[\frac{\tau_{\mathbb{K}^c}(f(\lambda_h))}{\varphi_{\mathbb{K}}(\lambda_h)^2} \mathcal{B}_h^2 + \frac{\varphi_{\mathbb{K}^c}(f(\lambda_h))}{\varphi_{\mathbb{K}}(\lambda_h)} \mathcal{B}_h \right];$$

c) in general $f(M)$ *is semisimple,* $\mathbb{K}^c$*-quadratic and its* $\mathbb{K}^c$*-trace decomposition is*

$$f(M) = T(f(M)) + F(f(M)),$$

$$\textit{where} \qquad T(f(M)) = \sum_{i=1}^{s} f(\gamma_i)\mathcal{A}_i + \sum_{h=1}^{t} \mathcal{T}f(\lambda_h) \frac{\mathcal{B}_h^2}{\varphi_{\mathbb{K}}(\lambda_h)^2},$$

whose (possibly repeated) eigenvalues are $f(\gamma_i)$ *and* $\mathcal{T}f(\lambda_h)$ *for every* i, h *and*

$$F(f(M)) = \sum_{h=1}^{t} \frac{\mathcal{F}f(\lambda_h)}{\varphi_{\mathbb{K}}(\lambda_h)} \mathcal{B}_h,$$

whose (possibly repeated) eigenvalues are $\pm\mathcal{F}f(\lambda_h)$ *for every* h *and possibly* 0*.*

Proof. Parts (b) and (c) follow from part (a) via Remark 3.4 and ordinary computations.

For (a), we denote $\alpha_j = \tau_{\mathbb{K}}(\lambda_j)$, $n_j = -\varphi_{\mathbb{K}}(\lambda_j)^2$, so that we can write the $\mathbb{K}$-decomposition of λ_j as $\lambda_j = \alpha_j + \sqrt{-n_j}$. From $\mathcal{B}_j^3 = -n_j\mathcal{B}_j$ we get:

$$\mathcal{B}_j^{2k} = (-n_j)^{k-1}\mathcal{B}_j^2 \text{ and } \mathcal{B}_j^{2k-1} = (-n_j)^{k-1}\mathcal{B}_j \text{ for every } k \geq 1.$$

Therefore, for every $m \geq 1$, standard computations allow to get:

$$\left[\frac{\tau_{\mathbb{K}}(\lambda_j)}{\varphi_{\mathbb{K}}(\lambda_j)^2}\mathcal{B}_j^2+\mathcal{B}_j\right]^m = \left[\sum_{h=0}^{\lfloor m/2\rfloor}\frac{\binom{m}{2h}\alpha_j^{m-2h}(\sqrt{-n_j})^{2h}}{(-n_j)}\right]\mathcal{B}_j^2 + \left[\sum_{h=1}^{\lfloor (m+1)/2\rfloor}\frac{\binom{m}{2h-1}\alpha_j^{m-2h+1}(\sqrt{-n_j})^{2h-1}}{\sqrt{-n_j}}\right]\mathcal{B}_j.$$

We have:

$$f(M)=a_oI_n+\sum_{m=1}^{+\infty}a_m\left[\sum_{i=1}^{s}\gamma_i\mathcal{A}_i+\sum_{j=1}^{t}(\frac{\alpha_j}{(-n_j)}\mathcal{B}_j^2+\mathcal{B}_j)\right]^m,$$

thus, remembering the properties of the various matrices on the right:

$$\begin{aligned} f(M) &= a_oI_n+\sum_{m=1}^{+\infty}a_m\sum_{i=1}^{s}\gamma_i^m\mathcal{A}_i+\sum_{m=1}^{+\infty}a_m\sum_{j=1}^{t}\left[\frac{\alpha_j}{(-n_j)}\mathcal{B}_j^2+\mathcal{B}_j\right]^m \\ &= a_0\left[I_n-\sum_{i=1}^{s}\mathcal{A}_i+\sum_{j=1}^{t}\frac{\mathcal{B}_j^2}{n_j}\right]+\sum_{i=1}^{s}f(\gamma_i)\mathcal{A}_i \\ &\quad+\sum_{j=1}^{t}\left[\frac{\sum_{m=0}^{+\infty}a_m\sum_{h=0}^{\lfloor m/2\rfloor}\binom{m}{2h}\alpha_j^{m-2h}\sqrt{-n_j}^{2h}}{(-n_j)}\right]\mathcal{B}_j^2 \\ &\quad+\sum_{j=1}^{t}\left[\frac{\sum_{m=1}^{+\infty}a_m\sum_{h=1}^{\lfloor (m+1)/2\rfloor}\binom{m}{2h-1}\alpha_j^{m-2h+1}\sqrt{-n_j}^{2h-1}}{\sqrt{-n_j}}\right]\mathcal{B}_j. \end{aligned}$$

Now, remembering Remark 2.14 and the definitions of the various matrices, we get the expression of $f(M)$ in the statement. Note that the expressions of $\dfrac{\mathcal{T}f(\lambda_j)}{\varphi_{\mathbb{K}}(\lambda_j)^2}$ and of $\dfrac{\mathcal{F}f(\lambda_j)}{\varphi_{\mathbb{K}}(\lambda_j)}$ are invariant under exchanging λ_j with its conjugate $\overline{\lambda}_j$. □

EXAMPLES 3.6: Assume that $(\mathbb{K}, |\cdot|)$ is a valued field. Then the restriction to fundamental field $\mathbb{Q}$ of $|\cdot|$ is equivalent either to the usual euclidean absolute value (archimedean case), or to the trivial absolute value, or to a p-adic absolute value for some prime number p (see for instance [7, Ch. 23, Theorem 1, p. 44]). Hence, if the absolute value is non-archimedean, we say that the valued field $(\mathbb{K}, |\cdot|)$ has *trivial fundamental restriction* or *p-adic fundamental restriction* respectively. In all cases we can define as power series, as in ordinary real case, the exponential function, the sine, the cosine, the hyperbolic sine and the hyperbolic cosine. These series have the same radius of convergence: $R = +\infty$ if the absolute value is archimedean, $R = 1$ if the absolute value has trivial fundamental restriction, and $R = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$ if the absolute value has p-adic fundamental restriction (see for instance [9, pp. 70–72]).

We put $\Lambda_{\mathbb{K}} = \Lambda_{exp,\mathbb{K}}$ (remember Definition 3.3). If $M \in \Lambda_{\mathbb{K}}$ and $\lambda = \tau_{\mathbb{K}}(\lambda)+\varphi_{\mathbb{K}}(\lambda)$ is an eigenvalue of M with its $\mathbb{K}$-decomposition, then, for $f(\lambda) = \exp(\lambda)$, we have:

$$\mathcal{T}f(\lambda) = \exp(\tau_{\mathbb{K}}(\lambda))\cosh(\varphi_{\mathbb{K}}(\lambda)) \text{ and } \mathcal{F}f(\lambda) = \exp(\tau_{\mathbb{K}}(\lambda))\sinh(\varphi_{\mathbb{K}}(\lambda))\,.$$

Now if $M \in \Lambda_{\mathbb{K}}$ is a semisimple $\mathbb{K}$-quadratic matrix, then from Proposition 3.5 we get:

$$\exp(M) = \sum_{i=1}^{s} \exp(\gamma_i)\mathcal{A}_i + \sum_{j=1}^{t} \frac{\exp(\tau_{\mathbb{K}}(\lambda_j))\cosh(\varphi_{\mathbb{K}}(\lambda_j))}{\varphi_{\mathbb{K}}(\lambda_j)^2}\mathcal{B}_j^2 + \sum_{j=1}^{t} \frac{\exp(\tau_{\mathbb{K}}(\lambda_j))\sinh(\varphi_{\mathbb{K}}(\lambda_j))}{\varphi_{\mathbb{K}}(\lambda_j)}\mathcal{B}_j.$$

Analogously we can get the formulas for other power series; for instance if $M \in \Lambda_{\mathbb{K}}$ is semisimple and $\mathbb{K}$-quadratic, then

$$\cos(M) = \sum_{i=1}^{s} \cos(\gamma_i)\mathcal{A}_i + \sum_{j=1}^{t} \frac{\cos(\tau_{\mathbb{K}}(\lambda_j))\cos(\varphi_{\mathbb{K}}(\lambda_j))}{\varphi_{\mathbb{K}}(\lambda_j)^2}\mathcal{B}_j^2 - \sum_{j=1}^{t} \frac{\sin(\tau_{\mathbb{K}}(\lambda_j))\sin(\varphi_{\mathbb{K}}(\lambda_j))}{\varphi_{\mathbb{K}}(\lambda_j)}\mathcal{B}_j.$$

4. Matrices over an ordered quadratically closed field.

REMARK 4.1: In this section we assume that $\mathbb{K}$ is an *ordered quadratically closed field*, i.e. $\mathbb{K}$ is an ordered field such that all of its positive elements have square root in $\mathbb{K}$ (for this notion we follow [6, Ch. XI, p. 462] rather than other definitions of quadratically closed field in literature).

For every $a \in \mathbb{K}$, $a > 0$, we denote by $\sqrt{a}$ its unique positive square root in $\mathbb{K}$. Moreover we fix a square root of -1 in $\overline{\mathbb{K}} \setminus \mathbb{K}$, denoted by $\sqrt{-1}$.

REMARK 4.2: It is known that an ordered quadratically closed field has characteristic 0 and it has a unique order as field (see for instance [6, Ch. XI, p. 462]).

Definition 4.3. *The field* $\mathbb{K}$ *is said to be a* real closed field*, if it can be endowed with a structure of ordered field such that its positive elements have a square root in* $\mathbb{K}$ *and every polynomial of odd degree of* $\mathbb{K}[X]$ *has a root in* $\mathbb{K}$*.*

Remark 4.4: It follows directly from the definitions that every real closed field is an ordered quadratically closed field. It is known that, for every ordered field $\mathbb{K}$, there exists an algebraic extension, contained in $\overline{\mathbb{K}}$, which is real closed and whose order extends the order of $\mathbb{K}$ and, moreover, that any two such extensions are $\mathbb{K}$-isomorphic (see for instance [5, Theorem 11.4] or [8, Theorem 15.9]). We call any such extension $\mathbb{L}$ a *real closure of the ordered field* $\mathbb{K}$ *in* $\overline{\mathbb{K}}$. Note that $\overline{\mathbb{K}}$ is the algebraic closure of $\mathbb{L}$ too.

For more information, further characterizations and properties of real closed fields we refer for instance to [6, Ch. XI.2] and to [8, Ch. 15]. In particular it is known that $\mathbb{K}$ is a real closed field if and only if $\sqrt{-1} \notin \mathbb{K}$ and $\mathbb{K}(\sqrt{-1})$ is algebraically closed (see for instance [8, characterization (1), p. 221]).

In Proposition 4.12 we point out other simple characterizations of real closed fields.

Proposition 4.5. *Assume that* $\mathbb{K}$ *is an ordered quadratically closed field and choose one of its real closures,* $\mathbb{L}$ *in* $\overline{\mathbb{K}}$*.*

a) For every $z \in \overline{\mathbb{K}}$ *there exist* $x, y \in \mathbb{L}$ *such that* $z = x + \sqrt{-1}\,y$*; such elements are uniquely determined by* $\mathbb{L}$ *and by* $\sqrt{-1}$*,.*

We denote $x = \mathbf{Re}(z)$ *and* $y = \mathbf{Im}(z)$*: the* real *and the* imaginary *part of* z*.*

b) For every $z \in \overline{\mathbb{K}}$ *of degree* 2 *over* $\mathbb{K}$*,* $\mathbf{Re}(z)$ *and* $\mathbf{Im}(z)$ *are both elements of* $\mathbb{K}$ *and moreover* $\tau_{\mathbb{K}}(z) = \mathbf{Re}(z)$ *and* $\varphi_{\mathbb{K}}(z) = \sqrt{-1}\,\mathbf{Im}(z)$*; hence, in this case,* $\mathbf{Re}(z)$ *and* $\mathbf{Im}(z)$ *are independent of* $\mathbb{L}$*.*

Proof. Part (a) follows from Remark 4.4 since $\overline{\mathbb{K}} = \mathbb{L}(\sqrt{-1})$.

Let $z \in \overline{\mathbb{K}}$ as in (b) and write $z = \tau_{\mathbb{K}}(z) + \varphi_{\mathbb{K}}(z)$, where the minimal polynomial of $\varphi_{\mathbb{K}}(z)$ is $g(X) = X^2 - \varphi_{\mathbb{K}}(z)^2$ (remember Lemma 2.4-(d)); thus $-\varphi_{\mathbb{K}}(z)^2 > 0$, being $g(X)$ irreducible; so $\pm\sqrt{-\varphi_{\mathbb{K}}(z)^2}$ are both elements of the ordered quadratically closed field $\mathbb{K}$. Now $\varphi_{\mathbb{K}}(z) = \sqrt{-1}\,[\pm\sqrt{-\varphi_{\mathbb{K}}(z)^2}]$ and we conclude (b) by uniqueness of the decomposition in (a). □

Lemma 4.6. *Let* $(\mathbb{K}, |.|)$ *be a valued field and* $\lambda \in \overline{\mathbb{K}} \subseteq \overline{\mathbb{K}^c}$*.*
Then $(\mathbb{K}(\lambda))^c = \mathbb{K}^c(\lambda)$*.*

Proof. The element λ is algebraic over $\mathbb{K}^c$ too. Hence, by [6, Ch. XII, Proposition 2.5], we get that $\mathbb{K}^c(\lambda)$ is complete. Since it contains $\mathbb{K}(\lambda)$, it contains also its completion and this gives the first inclusion.

Now let $\vartheta \in \mathbb{K}^c(\lambda)$ and denote by l the degree of λ over $\mathbb{K}^c$. Therefore $\vartheta = \sum_{i=0}^{l-1} h_i \lambda^i$ with $h_0, \cdots, h_{l-1} \in \mathbb{K}^c$. Since $\mathbb{K}$ is dense in $\mathbb{K}^c$, there exist some sequences in $\mathbb{K}$, $\{k_m^{(i)}\}_{m\in\mathbb{N}}$, $0 \le i \le l-1$, such that each $k_m^{(i)}$ converges to h_i. Since the topology over $\mathbb{K}^c(\lambda)$ is the product topology (see the proof of [6, Ch. XII, Proposition 2.2]), we have that: $k_m^{(0)} + k_m^{(1)}\lambda + \cdots + k_m^{(l-1)}\lambda^{l-1}$ is a sequence in $\mathbb{K}(\lambda)$, which converges to ϑ. Thus $\vartheta \in (\mathbb{K}(\lambda))^c$. □

Proposition 4.7. *Let* $(\mathbb{K}, |.|)$ *be a real closed valued field and denote by* $\mathbb{K}^c$ *its completion.*

If $\sqrt{-1} \in \mathbb{K}^c$*, then* $\mathbb{K}^c$ *is algebraically closed.*

If $\sqrt{-1} \notin \mathbb{K}^c$*, then* $\mathbb{K}^c$ *is real closed.*

Proof. By Lemma 4.6, we have: $(\overline{\mathbb{K}})^c = (\mathbb{K}(\sqrt{-1}))^c = \mathbb{K}^c(\sqrt{-1})$. Since the completion of an algebraically closed field is algebraically closed too (remember Lemma 3.2-(a)), $\mathbb{K}^c(\sqrt{-1})$ is algebraically closed. Hence, if $\sqrt{-1} \in \mathbb{K}^c$, then $\mathbb{K}^c$ is algebraically closed. Otherwise $\mathbb{K}^c$ is real closed by the characterization recalled in Remark 4.4. □

Corollary 4.8. *If* $(\mathbb{K}, |.|)$ *is a real closed valued field with* p*-adic fundamental restriction for some prime* p*, then its completion* $\mathbb{K}^c$ *is algebraically closed.*

Proof. Let us consider the sequence $\{x_n\}_{n\ge1}$, $x_n = \sqrt{p^n - 1}$. Since $\mathbb{K}$ is real closed (hence ordered and quadratically closed), $x_n \in \mathbb{K}$. Now $x_n^2 + 1 = p^n \to 0$ in $(\mathbb{K}, |.|)$.

If there exists a subsequence $\{x_{n_k}\}$ such that $(x_{n_k} + \sqrt{-1}) \to 0$, then $\pm\sqrt{-1} \in \mathbb{K}^c$.

Otherwise there exists a real number $\delta > 0$ such that $|x_n + \sqrt{-1}| \ge \delta > 0$ for every n. In this case $|x_n - \sqrt{-1}| = \dfrac{|x_n^2+1|}{|x_n + \sqrt{-1}|} \le \dfrac{|x_n^2+1|}{\delta} \to 0$. So $x_n \to \sqrt{-1}$ and again $\sqrt{-1} \in \mathbb{K}^c$. Hence, by the previous Proposition, $\mathbb{K}^c$ is algebraically closed. □

Remark 4.9 (normalized fine $\mathbb{K}$-trace decomposition): Let $\mathbb{K}$ be an ordered quadratically closed field and $M \in M_n(\mathbb{K})$ be a semisimple $\mathbb{K}$-quadratic matrix. As remarked in Proposition 4.5 (after choosing $\sqrt{-1} \in \overline{\mathbb{K}}$), for every eigenvalue λ of M, the decomposition $\lambda = \mathbf{Re}(\lambda) + \sqrt{-1}\,\mathbf{Im}(\lambda)$ is independent of the choice of a real closure of $\mathbb{K}$ into $\overline{\mathbb{K}}$, being $\mathbf{Re}(\lambda) = \tau_{\mathbb{K}}(\lambda)$ and $\sqrt{-1}\,\mathbf{Im}(\lambda) = \varphi_{\mathbb{K}}(\lambda)$. Remembering Remark 2.12, we choose $\lambda_1, \ldots, \lambda_t$, the eigenvalues of M not in $\mathbb{K}$, so that $\mathbf{Im}(\lambda_j) > 0$ for every j, and we define the matrices:

$$\mathbf{A}_i = \mathcal{A}_i \text{ for every } i = 1, \cdots, s,,$$

$$\text{and} \qquad \mathbf{B}_j = \frac{\mathcal{B}_j}{\mathbf{Im}(\lambda_j)} \text{ for every } j = 1, \cdots, t\,.$$

They are in $M_n(\mathbb{K}) \setminus \{0\}$ and are polynomial expressions of M. Moreover

$$\begin{aligned}
&\mathbf{A}_i\mathbf{A}_j = \delta_{ij}\mathbf{A}_i \text{ for every } i, j\,;\\
&\mathbf{A}_i\mathbf{B}_j = \mathbf{B}_j\mathbf{A}_i = 0 \text{ for every } i, j\,;\\
&\mathbf{B}_i\mathbf{B}_j = 0 \text{ for every } i \neq j \text{ and}\\
&\mathbf{B}_j^3 = -\mathbf{B}_j \text{ for every } j\,.
\end{aligned}$$

Then by Remark 2.14 we get

$$M = \sum_{i=1}^{s} \gamma_i\,\mathbf{A}_i - \sum_{j=1}^{t} \mathbf{Re}(\lambda_j)\,\mathbf{B}_j^2 + \sum_{j=1}^{t} \mathbf{Im}(\lambda_j)\,\mathbf{B}_j \quad (\text{with } \mathbf{Im}(\lambda_j) > 0). \tag{3}$$

We call the above decomposition *normalized fine $\mathbb{K}$-trace decomposition of M.*

PROPOSITION 4.10. *Assume that $(\mathbb{K}, |\cdot|)$ is an ordered quadratically closed valued field, that $f(X)$ is a power series with coefficients in $\mathbb{K}$ and that $M \in \Lambda_{f,\mathbb{K}}$ is a semisimple $\mathbb{K}$-quadratic matrix with normalized fine $\mathbb{K}$-trace decomposition:*

$$M = \sum_{i=1}^{s} \gamma_i\,\mathbf{A}_i - \sum_{j=1}^{t} \mathbf{Re}(\lambda_j)\,\mathbf{B}_j^2 + \sum_{j=1}^{t} \mathbf{Im}(\lambda_j)\,\mathbf{B}_j, \quad \textit{with } \mathbf{Im}(\lambda_j) > 0.$$

Then

a) $f(M) = \sum_{i=1}^{s} f(\gamma_i)\mathbf{A}_i - \sum_{j=1}^{t} \mathcal{T}f(\lambda_j)\,\mathbf{B}_j^2 + \sum_{j=1}^{t} \mathcal{G}f(\lambda_j)\mathbf{B}_j$,

where $f(\gamma_i)$, $\mathcal{T}f(\lambda_j)$ and $\mathcal{G}f(\lambda_j) := \dfrac{\mathcal{F}f(\lambda_j)}{\sqrt{-1}}$ belong to $\mathbb{K}^c$ for every i, j;

b) if $\mathbb{K}^c$ is ordered quadratically closed too , then

$$f(M) = \sum_{i=1}^{s} f(\gamma_i)\mathbf{A}_i - \sum_{j=1}^{t} \mathbf{Re}(f(\lambda_j))\,\mathbf{B}_j^2 + \sum_{j=1}^{t} \mathbf{Im}(f(\lambda_j))\mathbf{B}_j,$$

where $f(\gamma_i)$, $\mathbf{Re}(f(\lambda_j))$, $\mathbf{Im}(f(\lambda_j)) \in \mathbb{K}^c$ for every i, j.

Proof. Part (a) follows from Proposition 3.5-(a). Indeed it suffices to remark that

$$\mathcal{G}f(\lambda_j)\mathbf{B}_j = \frac{\mathcal{F}f(\lambda_j)}{\sqrt{-1}}\,\frac{\mathcal{B}_j}{\mathbf{Im}(\lambda_j)} = \frac{\mathcal{F}f(\lambda_j)}{\varphi_{\mathbb{K}}(\lambda_j)}\,\mathcal{B}_j\,.$$

From the expression of $\mathcal{F}f(\lambda_j)$ it follows that $\mathcal{G}f(\lambda_j) \in \mathbb{K}^c$.

Part (b) follows from part (a), from Remark 3.4-(c) and from Proposition 4.5-(b), since, for every j, $f(\lambda_j)$ has degree at most 2 over $\mathbb{K}^c$ and $\lambda_j \notin \mathbb{K}^c$ (because $\varphi_{\mathbb{K}}(\lambda_j)^2 < 0$ and $\mathbb{K}^c$ is an ordered field). □

EXAMPLES 4.11: Assume that $(\mathbb{K}, |\cdot|)$ is an ordered quadratically closed valued field. If $M \in \Lambda_{\mathbb{K}}$ is semisimple and $\mathbb{K}$-quadratic, then, by Proposition 4.10 and remarking that $\mathcal{T}\exp(\lambda_j) = \exp(\mathbf{Re}(\lambda_j))\cos(\mathbf{Im}(\lambda_j))$ and $\mathcal{G}\exp(\lambda_j) = \exp(\mathbf{Re}(\lambda_j))\sin(\mathbf{Im}(\lambda_j))$, we have:

$$\exp(M) = \sum_{i=1}^{s} \exp(\gamma_i)\mathbf{A}_i - \sum_{j=1}^{t} \exp(\mathbf{Re}(\lambda_j))\cos(\mathbf{Im}(\lambda_j))\mathbf{B}_j^2 + \sum_{j=1}^{t} \exp(\mathbf{Re}(\lambda_j))\sin(\mathbf{Im}(\lambda_j))\mathbf{B}_j,$$

and, analogously,

$$\cos(M) = \sum_{i=1}^{s} \cos(\gamma_i)\mathbf{A}_i - \sum_{j=1}^{t} \cos(\mathbf{Re}(\lambda_j))\cosh(\mathbf{Im}(\lambda_j))\mathbf{B}_j^2 - \sum_{j=1}^{t} \sin(\mathbf{Re}(\lambda_j))\sinh(\mathbf{Im}(\lambda_j))\mathbf{B}_j,$$

where the λ_j's are the eigenvalues of M not in $\mathbb{K}$, having positive imaginary part.

The previous formulas point out the analogous formulas in Examples 3.6. Moreover the expression of $\exp(M)$ extends the classical Rodrigues' formula for the exponential of a real skew symmetric matrix (see for instance [3, Theorem 2.2]).

PROPOSITION 4.12. *If char$(\mathbb{K}) = 0$, the following assertions are equivalent:*

a) $\mathbb{K}$ *is real closed;*

b) $\mathbb{K}$ *is not algebraically closed and the irreducible polynomials of* $\mathbb{K}[X]$ *have degree at most* 2*;*

c) the $\mathbb{K}$*-linear involution of* $\overline{\mathbb{K}}$ *is an element of Aut*$(\overline{\mathbb{K}}/\mathbb{K})$ *different from the identity;*

d) $Ker(\tau_{\mathbb{K}})$ *is the* $\mathbb{K}$*-vector space generated by* $\sqrt{-1}$*;*

e) $\mathbb{K}$ *is not algebraically closed and every semisimple matrix with entries in* $\mathbb{K}$ *is* $\mathbb{K}$*-quadratic.*

Proof. The implication (a) $\Rightarrow$ (b) follows from Remark 4.4. For the converse it suffices to prove that $\overline{\mathbb{K}} = \mathbb{K}(\sqrt{-1})$, since $\mathbb{K}$ is not algebraically closed. Let $t \in \overline{\mathbb{K}} \setminus \mathbb{K}$, thus it has degree 2. We decompose $t = \alpha + \beta$ as sum of an element $\alpha \in \mathbb{K}$ and of an element $\beta \in Ker(\tau_{\mathbb{K}}) \setminus \{0\}$. By Lemma 2.4-(a) the conjugate of t is $\bar{t} = \alpha - \beta$ and $\beta^2 \in \mathbb{K}$. Now we consider the polynomial of $\mathbb{K}[X]$: $q(X) = X^4 - \beta^2 = (X - \sqrt{\beta})(X + \sqrt{\beta})(X - \sqrt{-\beta})(X + \sqrt{-\beta})$ with its factorization in $\overline{\mathbb{K}}[X]$ (note that its roots are not in $\mathbb{K}$). Since $q(X)$ has degree 4, it is reducible over $\mathbb{K}$, so it is product of two irreducible polynomials of $\mathbb{K}[X]$. Since $\beta \notin \mathbb{K}$, one of the two factors must have the form $(X - \sqrt{\beta})(X \pm \sqrt{-\beta})$ and therefore $\sqrt{-\beta^2} \in \mathbb{K} \setminus \{0\}$. Hence $\beta = \pm\sqrt{-\beta^2}\sqrt{-1} \in \mathbb{K}(\sqrt{-1})$. This implies that $t = \alpha + \beta \in \mathbb{K}(\sqrt{-1})$, therefore $\overline{\mathbb{K}} \setminus \mathbb{K} \subseteq \mathbb{K}(\sqrt{-1})$ and so $\overline{\mathbb{K}} = \mathbb{K}(\sqrt{-1})$.

By Lemma 2.4-(b), (a) implies (c). Now assume (c), so that $\mathbb{K}$ is not algebraically closed. Let $\lambda = \alpha + \beta \in \overline{\mathbb{K}}$ with its $\mathbb{K}$-decomposition. In particular $\mathbb{K}(\lambda) = \mathbb{K}(\beta)$. From (c) we have: $\overline{\beta^2} = \overline{\beta}^2 = (-\beta)^2 = \beta^2$. Hence, by Remark 2.3, we get that $\beta^2 \in \mathbb{K}$ and so both β and λ have degree at most 2 over $\mathbb{K}$. This gives (b).

Next we prove the equivalence between (a) and (d). Assume first (d). By Remark 2.1, $\overline{\mathbb{K}} = \mathbb{K} \oplus Ker(\tau_{\mathbb{K}}) = \mathbb{K} \oplus Span(\sqrt{-1})$, thus $\sqrt{-1} \notin \mathbb{K}$ and $\mathbb{K}(\sqrt{-1}) = \overline{\mathbb{K}}$ is algebraically closed. For the converse, every element in $\overline{\mathbb{K}} \setminus \mathbb{K}$ is algebraic of order 2 over $\mathbb{K}$. Since $\mathbb{K}$ is ordered quadratically closed too, by Lemma 2.4-(d), $\beta \in Ker(\tau_{\mathbb{K}})$ if and only if $\beta = \pm\sqrt{t}$ with $t \in \mathbb{K}$ and $t \leq 0$, i.e. if and only if $\beta = \pm\sqrt{-t}\,\sqrt{-1}$ with $\sqrt{-t} \in \mathbb{K}$ and this allows to conclude.

Now (b) implies (e) by obvious reasons. For the converse, we note that every monic irreducible polynomial $q(X) \in \mathbb{K}[X]$ with $deg(q(X)) \geq 2$ is the minimal polynomial of its *companion matrix*, which is therefore semisimple and so $\mathbb{K}$-quadratic, by (e). Since $q(X)$ is irreducible, we get that $deg(q(X)) = 2$. □

REMARK 4.13: Assume that $(\mathbb{K}, |\cdot|)$ is a real closed valued field and that $f(X)$ is a power series with coefficients in $\mathbb{K}$, then (see Proposition 4.12-(e)) the formula of Proposition 4.10-(a) (and possibly of Proposition 4.10-(b)) holds for every semisimple matrix $M \in \Lambda_{f,\mathbb{K}}$. In particular, the same formulas of Examples 4.11 hold for every semisimple matrix $M \in \Lambda_{\mathbb{K}}$.

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