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Weighted Strichartz Estimate for the Wave Equation and Low Regularity Solutions

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Summary. - *In this work we study weighted Sobolev spaces in* $\mathbf{R}^n$ *generated by the Lie algebra of vector fields*

$$(1+|x|^2)^{1/2}\partial_{x_j},\ j=1,...,n.$$

Interpolation properties and Sobolev embeddings are obtained on the basis of a suitable localization in $\mathbf{R}^n$. *As an application we derive weighted* L^q *estimates for the solution of the homogeneous wave equation. For the inhomogeneous wave equation we generalize the weighted Strichartz estimate established in [5] and establish global existence result for the supercritical semilinear wave equation with non compact small initial data in these weighted Sobolev spaces.*

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1. Introduction

In this work we study the decay properties of the wave equation

$$\Box u \equiv \partial_t^2 u - \Delta u = 0, \tag{1}$$
$$u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x). \tag{2}$$

Among the most important a priori estimates for this classical equation we mention the standard energy estimate, the estimate of von Wahl [18], and the Strichartz type estimates [15].

The energy estimate gives a control of derivatives of L^2-norms of the solution:

$$\|\nabla_{t,x} u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq C(\|\nabla_x u_0\|_{L^2(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}).$$

The estimate of von Wahl controls the L^∞ norm of the solution:

$$(1+t+|x|)^{\frac{n-1}{2}} |u(t,x)| \leq C(\|u_0\|_{W^{[n/2]+1,1}} + \|u_1\|_{W^{[n/2],1}}).$$

Strichartz estimates give an estimate of the $L^q(\mathbb{R}^{n+1}_+)$ norm of the solution in terms of the L^p norm of the data, for suitable values of p, q.

Our goal is to obtain unified decay estimates of the solution in terms of the norm of the data in suitable weighted Sobolev spaces. These spaces are natural extensions of the weighted Sobolev spaces studied by Y. Choquet-Bruhat and D. Christodoulou [2]. They defined, for any integer $s \geq 0$ and real δ,

$$\|u\|_{H^{s,\delta}} = \sum_{|\alpha| \leq s} \|\langle x \rangle^{\delta + |\alpha|} D_x^\alpha u\|_{L^2(\mathbb{R}^n)}$$

(where $\langle x \rangle = (1+|x|^2)^{1/2}$). Here we extend their definition to the L^p case and more generally to any real order s. This is essential to handle initial data of minimal regularity for Problem (1), (2). To this end, we consider a dyadic partition of unity in $\mathbf{R^n}$, i.e., a sequence of functions $\phi_j \in C_c^\infty(\mathbb{R}^n)$ such that $\phi_j \geq 0$, $\sum \phi_j = 1$, and

$$\operatorname{supp} \phi_0 \subseteq \{|x| \leq 2\}, \qquad \operatorname{supp} \phi_j \subseteq \{2^{j-1} \leq |x| \leq 2^{j+1}\} \quad j \geq 1.$$

Moreover, we define the pseudodifferential operators Λ_j^s as

$$\Lambda_j^s \text{ has symbol } \langle 2^j\xi\rangle^s = (1+2^{2j}|\xi|^2)^{s/2}. \tag{3}$$

Then the norm of the space $H_p^{s,\delta}$ is defined as follows:

$$\|u\|^p_{H_p^{s,\delta}} = \sum_{j\geq 0} \|\Lambda_j^s(\langle x\rangle^\delta \phi_j u)\|^p_{L^p} \sim \sum_{j\geq 0} 2^{j\delta p}\|\Lambda_j^s(\phi_j u)\|^p_{L^p}.$$

Notice that the dyadic decomposition used is in the x-variables and not in the dual ξ-variables as usual. When $p=2$, we write simply $H^{s,\delta}$ instead of $H_p^{s,\delta}$.

We develop a fairly complete theory of the spaces $H_p^{s,\delta}$, with special attention to interpolation, duality and embedding properties. A brief account of these results is given in the next section. Then we prove the following estimates:

THEOREM 1.1. *Let $n\geq 2$. For $d\in[0,(n-1)/2]$, the solution $u(t,x)$ of* (1),(2) *satisfies for $t\geq 0$ the estimate*

$$(1+t+|x|)^{(n-1)/2}(1+|t-|x||)^d|u(t,x)| \leq C(\|u_0\|_{H^{s_0,\delta_0}} + \|u_1\|_{H^{s_1,\delta_1}}) \tag{4}$$

provided

$$s_0 > \frac{n}{2}, \quad \delta_0 > -\frac{1}{2}+d, \quad s_1 > \frac{n}{2}-1, \quad \delta_1 > \frac{1}{2}+d,$$

with a constant $C=C(d,\delta_0,\delta_1,s_0,s_1,n)>0$ independent of t,x,u_0,u_1.

THEOREM 1.2. *Let $n\geq 3$. For any real $a<-1/2$, $b\in]-1/2,0]$ the solution $u(t,x)$ of* (1),(2) *satisfies the estimate*

$$\begin{aligned} &\|(1+t+|x|)^a(1+|t-|x||)^b u\|_{L^2(\mathbb{R}^{n+1}_+)} \\ &\leq C(\|u_0\|_{H^{-b,b}} + \|u_1\|_{H^{-b-1,b+1}}) \end{aligned} \tag{5}$$

with a constant $C=C(a,b,n)>0$ independent of u_0,u_1.

The estimate is also true for $n=2$, provided $b<0$ strictly.

Theorems 1.1, 1.2 are proved by a combination of techniques, using two different representations of the solution to the wave operator, namely the Fourier representation and the fundamental solution expression.

Moreover, interpolating between Theorems 1.1 and 1.2 we prove the following

THEOREM 1.3. *Let $n \geq 3$, $q \in [2, \infty]$. For any*

$$\rho < \frac{n-1}{2} - \frac{n}{q}, \quad 0 \leq \sigma \leq \frac{n-1}{2} - \frac{n-1}{q}$$

the solution $u(t,x)$ of (1),(2) *satisfies the estimate*

$$\|(1+t+|x|)^{\rho}(1+|t-|x||)^{\sigma} u\|_{L^q(\mathbb{R}^{n+1}_+)} \leq C(\|u_0\|_{H^{s_0,\delta_0}} + \|u_1\|_{H^{s_1,\delta_1}}) \tag{6}$$

provided

$$s_0 > \frac{n}{2} - \frac{n}{q}, \quad \delta_0 > \frac{1}{q} - \frac{1}{2} + \sigma, \quad s_1 > \frac{n}{2} - \frac{n}{q} - 1, \quad \delta_1 > \frac{1}{q} + \frac{1}{2} + \sigma,$$

with a constant $C = C(\sigma, \rho, \delta_0, \delta_1, s_0, s_1, n) > 0$ independent of u_0, u_1.

Moreover, (6) *is also true for any $\rho < (n-1)/2 - n/q$, $-1/q < \sigma \leq 0$ provided $s_0 > n/2 - n/q - \sigma$, $\delta_0 > 1/q - 1/2 + \sigma$, $s_1 > n/2 - n/q - \sigma - 1$, $\delta_1 > 1/q + 1/2 + \sigma$.*

The above estimates hold also for $n = 2$, provided $\sigma < (n-1)/2 - (n-1)/q$ strictly.

Notice in particular that choosing $\rho = \sigma$ we obtain the estimate

$$\|(1+t+|x|)^{\sigma}(1+|t-|x||)^{\sigma} u\|_{L^q(\mathbb{R}^{n+1}_+)} \leq C(\|u_0\|_{H^{s_0,\delta_0}} + \|u_1\|_{H^{s_1,\delta_1}}) \tag{7}$$

which is valid for:

$$2 + \frac{2}{n-1} \leq q \leq \infty, \quad 0 \leq \sigma < \frac{n-1}{2} - \frac{n}{q}$$

and

$$s_0 > \frac{n}{2} - \frac{n}{q}, \quad \delta_0 > \frac{1}{q} - \frac{1}{2} + \sigma, \quad s_1 > \frac{n}{2} - \frac{n}{q} - 1, \quad \delta_1 > \frac{1}{q} + \frac{1}{2} + \sigma.$$

Finally, the above estimates are applied to the initial value problem with small data for the semilinear wave equations of the form

$$\Box u = F(u) \qquad \text{in } \mathbb{R}^{n+1}_+, \tag{8}$$
$$u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x) \qquad \text{for } x \in \mathbb{R}^n, \tag{9}$$

where $n \geq 2$. We shall assume that $F \in C^1(\mathbb{R})$ satisifes

$$F(0) = 0, \qquad |F'(u)| \leq C|u|^{\lambda-1}, \tag{10}$$

where $C > 0$ and $\lambda > 1$. Typical examples are $F = |u|^\lambda$ and $F = |u|^{\lambda-1}u$.

Equation (8) has a long history. In 1979 Fritz John [8] proved that (8), (9) has global solution for $n = 3$, provided the initial data are smooth and small enough, and $\lambda > 1 + \sqrt{2}$; he also proved that for λ below this value in general solutions blow up in a finite time even with small data. This agreed with Walter Strauss' conjecture [14] that for $n \geq 2$ and λ greater than the positive root $\lambda_0(n)$ of the equation

$$\lambda\Big(\frac{n-1}{2}\lambda - \frac{n+1}{2}\Big) = 1 \tag{11}$$

Problem (8), (9) has a global solution. The conjecture was proved true for $n = 2$ by Robert Glassey [6], who also proved the blow up below $\lambda_0(2)$ [7]. The critical case $\lambda = \lambda_0$ was considered by Jack Schaeffer [12] who proved blow up for $n = 2, 3$. Sideris [13] completely solved the subcritical case, showing that one has always blow up in general for $\lambda < \lambda_0(n)$, $n \geq 2$. On the other hand, the supercritical case has been treated by many authors (see, e.g., [3], [16], [1], [19], [9], [10] and the references cited therein). The gobal existence result is established in [5] for any $\lambda > \lambda_0(n)$ (see also [4] and [17]).

Our aim is to extend the result of [5], in two directions: on one hand, we relax the regularity assumptions on the initial data; on the other hand, we remove the assumption that the initial data are compactly supported. This result is obtained combining estimate (7) with a suitable extension of the weighted Strichartz type estimate established in [5] and [17]. The extension is contained in the following Lemma:

LEMMA 1.4. *Assume that*

$$\frac{n-1}{2(n+1)} \le \frac{1}{q} \le \frac{1}{2}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \tag{12}$$

$$a < \frac{n-1}{2} - \frac{n}{q}, \qquad b > \frac{1}{q}, \qquad \delta > 0. \tag{13}$$

Then for any $F \in L^p(\mathbb{R}^{n+1}_+)$ *we have*

$$\begin{aligned} &\|(1+|t-|x||)^a(1+t+|x|)^a S(F)\|_{L^q} \qquad (14)\\ &\le C(\delta,a,b,p,n)\|(1+|t-|x||)^{b+\delta}(1+t+|x|)^b F\|_{L^p}. \end{aligned}$$

Combining estimates (14) and (7) we obtain

THEOREM 1.5. *Assume* $n \ge 2$, $F(u) \in C^1(\mathbb{R})$ *satisfies* (10) *with*

$$\lambda_0(n) < \lambda \le \frac{n+3}{n-1} \tag{15}$$

and that the initial data (9) *satisfy* $u_0 \in H^{s_0,\delta_0}$, $u_1 \in H^{s_1,\delta_1}$ *with*

$$s_0 > \frac{\lambda-1}{\lambda+1}\cdot\frac{n}{2}, \quad \delta_0 > \frac{1}{\lambda} - \frac{1}{2}, \quad s_1 > \frac{\lambda-1}{\lambda+1}\cdot\frac{n}{2} - 1, \quad \delta_1 > \frac{1}{\lambda} + \frac{1}{2}. \tag{16}$$

Then there exists $\epsilon > 0$ *such that, for all data with* $\|u_0\|_{H^{s_0,\delta_0}} + \|u_1\|_{H^{s_1,\delta_1}} < \epsilon$, *Problem* (8), (9) *has a unique weak global solution*

$$u(t,x) \in L^{\lambda+1}(\mathbb{R}^{n+1}_+). \tag{17}$$

Actually, we have $(1+|t-|x||)^a(1+t+|x|)^a u \in L^{\lambda+1}(\mathbb{R}^{n+1}_+)$ *for any* $a < (n-1)/2 - n/(\lambda+1)$.

By weak solution we mean as usual a solution of the integral equation corresponding to (8), (9). For instance, in $n = 4$ space dimensions, and for λ close to the critical value $\lambda_0(4) = 2$, Theorem 1.5 implies global existence for any small initial data $u_0 \in H^1$, $u_1 \in L^2$ such that $\langle x\rangle\nabla u_0$ and $\langle x\rangle u_1$ are in L^2; actually the regularity can be even lower, indeed (16) give for $\lambda = 2$

$$s_0 > \frac{2}{3}, \quad s_1 > -\frac{1}{3}.$$

The complete results and the proofs will appear in [11].

2. The spaces $H_p^{s,\delta}$

We list in the following statements several properties of the spaces $H_p^{s,\delta}$ and $H^{s,\delta}$; for more complete results and proofs see [11].

LEMMA 2.1. *Let* $p, p_0, p_1 \in]1, \infty[,\ a, s, s_0, s_1, \delta, \delta_0, \delta_1 \in \mathbb{R}$.

1. *The following duality relation holds:*

$$(H_p^{s,\delta})' = H_q^{-s,-\delta}, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{18}$$

Moreover, the complex interpolation property holds:

$$(H_{p_0}^{s_0,\delta_0}, H_{p_1}^{s_1,\delta_1})_\theta = H_p^{s,\delta}, \tag{19}$$

where

$$\begin{aligned} &0 < \theta < 1, \\ &\delta = (1-\theta)\delta_0 + \theta\delta_1, \\ &s = (1-\theta)s_0 + \theta s_1, \\ &\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \end{aligned}$$

2. *The following Sobolev type embeddings hold: for any* $1 < p < \infty,\ \delta \in \mathbb{R},\ s > n/p$,

$$\langle x\rangle^{\delta+n/p}|u(x)| \leq C\|u\|_{H_p^{s,\delta}} \tag{20}$$

with $C = C(p, s, \delta, n)$ *independent of* $u \in H_p^{s,\delta}$*; and for any* $1 < p \leq q < \infty,\ \delta \in \mathbb{R},\ s \geq n/p - n/q$,

$$\|\langle x\rangle^{\delta+n/p-n/q}u\|_{L^q} \leq C\|u\|_{H_p^{s,\delta}} \tag{21}$$

with $C = C(p, q, s, \delta, n)$ *independent of* $u \in H_p^{s,\delta}$*. Moreover, if* $s_0 \geq s_1$ *and* $\delta_0 \geq \delta_1$,

$$H_p^{s_0,\delta_0} \subseteq H_p^{s_1,\delta_1}$$

3. *Multiplication by a function* $\psi \in C_c^\infty(\mathbb{R}^n)$ *is a bounded operator on* $H_p^{s,\delta}$. *More generally, let* $\psi \in C^\infty(\mathbb{R}^n)$ *be a smooth function such that*

$$|D^\alpha \psi| \leq C_\alpha \quad \text{for } |\alpha| \leq N.$$

Then multiplication by ψ *is a bounded operator on* $H_p^{s,\delta}$ *provided* $|s| \leq N$:

$$\|\psi u\|_{H_p^{s,\delta}} \leq C\|u\|_{H_p^{s,\delta}} \tag{22}$$

with C *depending only on* s, δ, p *and on* C_α *for* $|\alpha| \leq N$.

4. *The multiplication operator by* $\langle x\rangle^a$ *is an isometry of* $H_p^{s,\delta}$ *onto* $H_p^{s,\delta-a}$; *moreover, for any multiindex* α,

$$x^\alpha : H_p^{s,\delta} \to H_p^{s,\delta-|\alpha|}, \qquad D^\alpha : H_p^{s,\delta} \to H_p^{s-|\alpha|,\delta+|\alpha|}, \tag{23}$$

are bounded operators. Thus in particular

$$\langle x\rangle^{|\alpha|} D^\alpha, x^\alpha D^\alpha : H_p^{s,\delta} \to H_p^{s-|\alpha|,\delta} \tag{24}$$

are bounded.

LEMMA 2.2. *Let* $s, \delta \in \mathbb{R}$, $R \geq 1$. *If* $u \in H^{s,\delta}$ *vanishes on the ball* $B(0, R)$, *then for all* $a \geq 0$ *we have*

$$\||x|^{-a} u\|_{H^{s,\delta}} \leq CR^{-a}\|u\|_{H^{s,\delta}} \tag{25}$$

and

$$R^a\|u\|_{H^{s,\delta}} \leq C\|u\|_{H^{s,\delta+a}} \tag{26}$$

with $C = C(s, \delta, a)$ *independent of* R *and* u.

Of special interest are the spaces $H^{s,-s}$, whose norm on power 2 is equivalent to

$$\|u\|_{H^{s,-s}}^2 \sim \sum_{j\geq 0} 2^{-2js}\|\Lambda_j^s(\phi_j u)\|_{L^2}^2.$$

Notice in particular that, as it follows from the next Lemma, the space $H^{s,-s}$ coincides with the homogeneous Sobolev space $\dot{H}^s$ provided s is in the range $0 \leq s < n/2$.

LEMMA 2.3. *The spaces $H^{s,-s}$ have the following properties.*

1. *For any $s \geq 0$, we have the equivalence on $H^{s,-s}$*

$$\|u\|_{H^{s,-s}} \sim \|\langle x\rangle^{-s}u\|_{L^2} + \||\xi|^s\widehat{u}\|_{L^2}. \tag{27}$$

If in addition $0 \leq s < n/2$, we have the equivalence

$$\|u\|_{H^{s,-s}} \sim \||\xi|^s\widehat{u}\|_{L^2}. \tag{28}$$

2. *For any $\lambda > 0$, $0 \leq s < n/2$, we have*

$$C^{-1}\|u\|_{H^{s,-s}} \leq \lambda^{n/2-s}\|S_\lambda u\|_{H^{s,-s}} \leq C\|u\|_{H^{s,-s}} \tag{29}$$

with $C = C(s,n)$ independent of λ and $u \in H^{s,-s}$.

3. *For any $s \geq 0$ we have*

$$\|u\|_{H^{-s,s}} \leq C\|\langle x\rangle^s u\|_{L^2} \tag{30}$$

with $C = C(s,n)$ independent of $u \in H^{-s,s}$.

4. *For any $s > -n/2$ we have*

$$\||\xi|^s\widehat{u}\|_{L^2} \leq C\|u\|_{H^{s,-s}} \tag{31}$$

with $C = C(s,n)$ independent of $u \in H^{s,-s}$.

The following property can be regarded as an extension of the classical Hardy inequality:

THEOREM 2.4. *Let $s \in [0,1/2[$, $\lambda \geq 0$. Then*

$$\left\|\frac{u}{||x|-\lambda|^s}\right\|_{L^2} \leq C\|u\|_{H^{s,-s}} \tag{32}$$

with $C = C(s,n)$ independent of $u \in H^{s,-s}$, λ.

References

[1] Yvonne Choquet-Bruhat, *Global existence for solutions of* $\Box u = a|u|^p$., J. Differential Equations **82** (1989), 98–108.

[2] Yvonne Choquet-Bruhat and Demetrios Christodoulou, *Elliptic systems in* $H_{s,\delta}$ *spaces on manifolds which are Euclidean at infinity*, Acta Math. **146** (1981), no. 1-2, 129–150.

[3] Demetrios Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. **39** (1986), 267–282.

[4] Vladimir Georgiev, *Existence of global solutions to supercritical semilinear wave equations*, Serdica Math. J. **22** (1996), no. 2, 125–164.

[5] Vladimir Georgiev, Hans Lindblad, and Christopher D. Sogge, *Weighted Strichartz estimates and global existence for semilinear wave equations*, Amer. J. Math. **119** (1997), no. 6, 1291–1319.

[6] Robert Glassey, *Existence in the large for* $u_{tt} - \Delta u = f(u)$ *in two space dimensions*, Math. Z. **178** (1981), 233–260.

[7] Robert Glassey, *Finite time blow up for solutions of nonlinear wave equations*, Math. Z. **177** (1981), 323–341.

[8] Fritz John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. **28** (1979), no. 1-3, 235–268.

[9] Hideo Kubo, *On the critical decay and power for semilinear wave equations in odd space dimensions*, Discrete and Continuos Dynamical Systems **2** (1996), 173–190.

[10] Hideo Kubo and Kôji Kubota, *Asymptotic behaviors of radially symmetric solutions of* $\Box u = |u|^p$ *for super critical values* p *in even space dimensions*, Japanese J. Math. **24** (1998), 191–256.

[11] Vladimir Georgiev Piero D'Ancona and Hideo Kubo, *Weighted decay estimates for the wave equation*, to appear.

[12] Jack Schaeffer, *The equation* $u_{tt} - \delta u = |u|^p$ *for the critical value of* p, Proc. Roy. Soc. Edinburgh Sect. A **101** (1985), no. 1-2, 31–44.

[13] Thomas Sideris, *Nonexistence of global solutions to semilinear wave equations in high dimensions*, J. Differential Equations **52** (1984), 378–406.

[14] Walter A. Strauss, *Nonlinear scattering theory at low energy*, J. Funct. Anal. **41** (1981), 110–133.

[15] Robert S. Strichartz, *A priori estimates for the wave equation and some applications*, J. Funct. Anal. **5** (1970), 218–235.

[16] Li Ta-tsien and Chen Yun-mei, *Initial value problems for nonlinear wave euations*, Comm. Partial Differential Equations **13** (1988), 383–422.

[17] DANIEL TATARU, *Strichartz estimates for the wave equation in the hyperbolic space and global existence for the semi inear wave equation*, Transactions AMS.

[18] WOLF VON WAHL, l^p *decay rates for homogeneous wave equations*, Math. Z. **120** (1971), 93–106.

[19] YI ZHOU, *Cauchy problem for semilinear wave equations in four space dimensions with small initial data*, J. Differential Equations **8** (1995), 135–144.

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