

Complex Weyl symbols of metaplectic operators: An elementary approach

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ABSTRACT. *We give explicit formulas for the Berezin symbols and the complex Weyl symbols of the metaplectic representation operators by using the holomorphic representations of the Jacobi group. Then we recover some known formulas for the symbols of the metaplectic operators in the classical Weyl calculus, in particular for the classical Weyl symbol of the exponential of an operator whose Weyl symbol is a quadratic form.*

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1. Introduction

The metaplectic representation (also called oscillator representation or Weil representation) is a projective unitary representation of the symplectic group $Sp(n, \mathbb{R})$ which was first investigated by I. E. Segal, D. Shale and A. Weil, see for instance [21] and its references. The metaplectic representation plays an important role in very different areas of mathematics such as number theory (automorphic forms) and mathematical physics (quantum mechanics).

For the computations, it is sometimes convenient to realize $Sp(n, \mathbb{R})$ as a subgroup S of $SU(n, n)$, see [21, p. 175]. The group S acts naturally on the $(2n+1)$ -dimensional (real) Heisenberg group H_n and then on the generic representations (the non-degenerated unitary irreducible representations) of H_n on the Fock space \mathcal{F} . Let $k \cdot \rho$ denote the action of $k \in S$ on the generic representation ρ of H_n . Then $k \cdot \rho$ and ρ are unitarily equivalent representations and there exists a unitary operator $\sigma(k)$ on \mathcal{F} (defined up to a unit scalar) such that

$$(k \cdot \rho)(h)\sigma(k) = \sigma(k)\rho(h) \tag{1}$$

for each $h \in H_n$. The map σ is thus the metaplectic representation of S .

One can find in the literature various methods to construct the metaplectic representation of S , that is, to obtain explicit formulas for $\sigma(k)$, $k \in S$, see for

instance [21] and [28]. However, as mentioned in [30, p. 533], the most direct method is to use the holomorphic representations of the (multi-dimensional) Jacobi group $G := H_n \rtimes S$. Here we apply this method in full details in order to get explicit expressions for the kernels of the metaplectic operators $\sigma(k)$ and then for the Berezin symbols of $\sigma(k)$. Note that this method also works for the harmonic representation of $SU(p, q)$, see [15].

The complex Weyl calculus W_0 is the correspondence between operators on \mathcal{F} and functions on \mathbb{C}^n obtained by translating the usual Weyl correspondence (see [21, 26]) by means of the Bargmann transform. It is known that W_0 is the unitary component in the polar decomposition of the Berezin correspondence on \mathcal{F} , see [11, 29].

In the present paper, we give explicit formulas for the complex Weyl symbols of the metaplectic operators. More precisely, we compute $W_0(\sigma(k))$ for $k \in S$ and $W_0(d\sigma(X))$ for X in the Lie algebra \mathfrak{s} of S . Our method is quite elementary, let us describe it briefly. For any operator A on \mathcal{F} , $W_0(A)$ can be expressed by an integral formula involving the kernel of A . This allows us to reduce the computation of $W_0(d\sigma(X))$ to that of some Gaussian integral.

As an immediate consequence of these results, we recover some known formulas for $W_1(\sigma'(g))$, $g \in Sp(n, \mathbb{R})$ and $W_1(d\sigma'(X))$, $X \in sp(n, \mathbb{R})$ where σ' denotes the metaplectic representation of $Sp(n, \mathbb{R})$ and W_1 the inverse map of the classical Weyl correspondence, [17, 21]. In particular, since $W_1(d\sigma'(X))$ is a quadratic form on \mathbb{R}^{2n} for each $X \in sp(n, \mathbb{R})$, we recover a formula of [27] for the Weyl symbol of the exponential of an operator having a quadratic form as Weyl symbol. A similar formula for the star exponential of a quadratic form for the Moyal star product can be found in [6]. Note that these formulas were then established in [27] and [6] by solving some differential systems, see also [16]. Note also that the study of the Weyl symbol of the function of an operator whose Weyl symbol is quadratic is still a subject of active research, see for instance [19] and references therein.

The plan of this paper is as follows. In Section 2, we review some generalities about the non-degenerated unitary irreducible representations of H_n on the Fock space and about the Berezin correspondence. In Section 3, we introduce the complex Weyl correspondence and we emphasize its connection with the Berezin correspondence and also with the classical Weyl correspondence. In Section 4, we consider the metaplectic representation σ and we give a functional equation satisfied by the kernel of $\sigma(k)$ for $k \in S$. Section 5 is devoted to a short presentation of the (multi-dimensional) Jacobi group $G = H_n \rtimes S$ and its holomorphic representations [9]. We deduce from the formula for the Berezin symbol of a holomorphic representation operator $\pi(h, k)$ given in [12] a functional equation for the kernel of $\pi(id, k)$ which is used in Section 6 to find formula for the kernel of $\sigma(k)$. From this, we derive in Section 7 a formula for $W_0(\sigma(k))$ ($k \in S$). Finally, we compute $W_0(d\sigma(X))$ ($X \in \mathfrak{s}$) in Section 8 and

we relate the results to those of [27] and [6].

2. Heisenberg group: Berezin quantization

In this section, we first review some general facts on the the Bargmann-Fock model for the unitary irreducible (non-degenerated) representations of the Heisenberg group, see [21]. We follow the presentation of [13], see also [14].

For each $z, w \in \mathbb{C}^n$, we denote $zw := \sum_{k=1}^n z_k w_k$. For each $z, z', w, w' \in \mathbb{C}^n$, let

$$\omega((z, w), (z', w')) = \frac{i}{2}(zw' - z'w).$$

Then the $(2n + 1)$ -dimensional real Heisenberg group is

$$H_n := \{((z, \bar{z}), c) : z \in \mathbb{C}^n, c \in \mathbb{R}\}$$

endowed with the multiplication law

$$((z, \bar{z}), c) \cdot ((z', \bar{z}'), c') = ((z + z', \bar{z} + \bar{z}'), c + c' + \frac{1}{2}\omega((z, \bar{z}), (z', \bar{z}'))).$$

Let $\lambda > 0$. By the Stone-von Neumann theorem, there exists a unique (up to unitary equivalence) unitary irreducible representation ρ_λ of H_n whose restriction to the center of H_n is the character $(0, c) \rightarrow e^{i\lambda c}$ [32]. The Bargmann-Fock realization of ρ_λ is defined as follows [3].

Let \mathcal{F}_λ be the Hilbert space of all holomorphic functions f on \mathbb{C}^n such that

$$\|f\|_{\mathcal{F}_\lambda}^2 := \int_{\mathbb{C}^n} |f(z)|^2 e^{-\lambda|z|^2/2} d\mu_\lambda(z) < +\infty$$

where $d\mu_\lambda(z) := (2\pi)^{-n} \lambda^n dm(z)$. Here $z = x + iy$ with x and y in \mathbb{R}^n and $dm(z) := dx dy$ is the standard Lebesgue measure on \mathbb{C}^n .

Then

$$(\rho_\lambda(h)f)(z) = \exp(i\lambda c_0 + \frac{\lambda}{2}\bar{z}_0 z - \frac{\lambda}{4}|z_0|^2) f(z - z_0)$$

for each $h = ((z_0, \bar{z}_0), c_0) \in H_n$ and $z \in \mathbb{C}^n$.

For each $z \in \mathbb{C}^n$, consider the *coherent state* $e_z(w) = \exp(\lambda\bar{z}w/2)$. Then we have the reproducing property $f(z) = \langle f, e_z \rangle_{\mathcal{F}_\lambda}$ for each $f \in \mathcal{F}_\lambda$.

We can introduce the Berezin calculus on \mathcal{F}_λ [7, 8, 11]. The Berezin (co-variant) symbol of an operator A on \mathcal{F}_λ is the function $S_\lambda(A)$ defined on \mathbb{C}^n by

$$S_\lambda(A)(z) := \frac{\langle A e_z, e_z \rangle_{\mathcal{F}_\lambda}}{\langle e_z, e_z \rangle_{\mathcal{F}_\lambda}}$$

and the double Berezin symbol s_λ is defined by

$$s_\lambda(A)(z, w) := \frac{\langle A e_w, e_z \rangle_{\mathcal{F}_\lambda}}{\langle e_w, e_z \rangle_{\mathcal{F}_\lambda}}$$

for each $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$ such that $\langle e_w, e_z \rangle_{\mathcal{F}_\lambda} \neq 0$.

Note that $s_\lambda(A)(z, w)$ is holomorphic in the variable z and anti-holomorphic in the variable w , then $s_\lambda(A)$ is determined by its restriction to the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$, that is, by $S_\lambda(A)$. Moreover, the operator A can be recovered from $s_\lambda(A)$ as follows. We have

$$\begin{aligned} Af(z) &= \langle Af, e_z \rangle_{\mathcal{F}_\lambda} = \langle f, A^* e_z \rangle_{\mathcal{F}_\lambda} \\ &= \int_{\mathbb{C}^n} f(w) \overline{A^* e_z(w)} e^{-\lambda|w|^2/2} d\mu_\lambda(w) \\ &= \int_{\mathbb{C}^n} f(w) \overline{\langle A^* e_z, e_w \rangle_{\mathcal{F}_\lambda}} e^{-\lambda|w|^2/2} d\mu_\lambda(w) \\ &= \int_{\mathbb{C}^n} f(w) s_\lambda(A)(z, w) \langle e_w, e_z \rangle_{\mathcal{F}_\lambda} e^{-\lambda|w|^2/2} d\mu_\lambda(w). \end{aligned}$$

In particular, we see that the map $A \rightarrow S_\lambda(A)$ is injective and that the kernel of A is the function

$$k_A(z, w) = \langle Ae_w, e_z \rangle_{\mathcal{F}_\lambda} = s_\lambda(A)(z, w) \langle e_w, e_z \rangle_{\mathcal{F}_\lambda}. \quad (2)$$

The map S_λ is then a bounded operator from the space $\mathcal{L}_2(\mathcal{F}_\lambda)$ of all Hilbert-Schmidt operators on \mathcal{F}_λ (endowed with the Hilbert-Schmidt norm) to $L^2(\mathbb{C}^n, \mu_\lambda)$ which is one-to-one and has dense range [33]. Let us introduce the Berezin transform which will be needed later. Let S_λ^* be the adjoint operator of S_λ . Then the Berezin transform is the operator B_λ on $L^2(\mathbb{C}^n, \mu_\lambda)$ defined by $B_\lambda := S_\lambda S_\lambda^*$. We have the integral formula

$$(B_\lambda f)(z) = \int_{\mathbb{C}^n} f(w) e^{-\lambda|z-w|^2/2} d\mu_\lambda(w),$$

see [7, 8, 33]. Note also that we have $B_\lambda = \exp(\Delta/2\lambda)$ where

$$\Delta = 4 \sum_{k=1}^n \partial^2 / \partial z_k \partial \bar{z}_k,$$

see [29, 33].

3. Complex Weyl correspondence for Heisenberg group

The complex Weyl correspondence can be constructed from a *Stratonovich-Weyl quantizer* see [13, 23, 31] and [1, Example 2.2 and Example 4.2].

Let R_0 be the parity operator on \mathcal{F}_λ defined by

$$(R_0 f)(z) = 2^n f(-z).$$

Then we define

$$\Omega_0(z) := \rho_\lambda((z, \bar{z}), 0) R_0 \rho_\lambda((z, \bar{z}), 0)^{-1}$$

for each $z \in \mathbb{C}^n$. By an easy computation we get

$$(\Omega_0(z)f)(w) = 2^n \exp(\lambda(w\bar{z} - |z|^2)) f(2z - w) \quad (3)$$

for each $z, w \in \mathbb{C}^n$ and $f \in \mathcal{F}_\lambda$. The map Ω_0 is called a *Stratonovich-Weyl quantizer*. For each trace-class operator A on \mathcal{F}_λ , we define

$$W_0(A)(z) := \text{Tr}(A\Omega_0(z))$$

for each $z \in \mathbb{C}^n$.

Recall that we denote by k_A the kernel of the trace-class (or more generally Hilbert-Schmidt) operator A on \mathcal{F}_λ , see Section 2. We have the following result, see [1, 13, 14].

PROPOSITION 3.1. *For each trace-class operator A on \mathcal{F}_λ and each $z \in \mathbb{C}^n$, we have*

$$W_0(A)(z) = 2^n \int_{\mathbb{C}^n} k_A(w, 2z - w) \exp(\lambda(-z\bar{z} + z\bar{w} - \frac{1}{2}w\bar{w})) d\mu_\lambda(w). \quad (4)$$

and, equivalently, on a more symmetric form

$$W_0(A)(z) = 2^n \int_{\mathbb{C}^n} k_A(z+w, z-w) \exp(\frac{\lambda}{2}(-z\bar{z} - w\bar{w} + z\bar{w} - \bar{z}w)) d\mu_\lambda(w). \quad (5)$$

These integral formulas allow us to extend W_0 to operators on \mathcal{F}_λ which are not necessarily trace-class, for instance Hilbert-Schmidt operators. It is known that $W_0 : \mathcal{L}_2(\mathcal{F}_\lambda) \rightarrow L^2(\mathbb{C}^n, \mu_\lambda)$ is the unitary part in the polar decomposition of S_λ , that is we have $S_\lambda = B_\lambda^{1/2} W_0$, see [29, Theorem 6] and [11, 14].

Now, with the aim of linking W_0 to the classical Weyl correspondence, we consider another realization of the unitary irreducible representation of H_n with central character $((0, 0), c) \rightarrow e^{i\lambda c}$, namely the Schrödinger representation ρ'_λ defined on $L^2(\mathbb{R}^n)$ by

$$(\rho'_\lambda((a + ib, a - ib), c)\phi)(x) = \exp(i\lambda(c - bx + \frac{1}{2}ab)) \phi(x - a)$$

for each $a, b, x \in \mathbb{R}^n$.

An (unitary) intertwining operator between ρ_λ and ρ'_λ is the Bargmann transform $\mathcal{B} : L^2(\mathbb{R}^n) \rightarrow \mathcal{F}_\lambda$ defined by

$$(\mathcal{B}f)(z) = \left(\frac{\lambda}{\pi}\right)^{n/4} \int_{\mathbb{R}^n} \exp\left(-\frac{\lambda}{4}z^2 + \lambda z x - \frac{\lambda}{2}x^2\right) \phi(x) dx,$$

see [11, 21].

Starting from the parity operator R_1 on $L^2(\mathbb{R}^n)$ defined by

$$(R_1\phi)(x) = 2^n\phi(-x),$$

we can define the Stratonovich-Weyl quantizer Ω_1 on \mathbb{R}^{2n} by

$$\Omega_1(a, b) := \rho'_\lambda((a + ib, a - ib), 0) R_1 \rho'_\lambda((a + ib, a - ib), 0)^{-1}$$

or, equivalently, by

$$(\Omega_1(a, b)\phi)(x) = 2^n \exp(2i\lambda b(a - x)) \phi(2a - x) \quad (6)$$

for each $\phi \in L^2(\mathbb{R}^n)$. Then, for each trace-class operator A on $L^2(\mathbb{R}^n)$, we define the function $W_1(A)$ on \mathbb{R}^{2n} by

$$W_1(A)(x, y) := \text{Tr}(A\Omega_1(x, y))$$

for each $x, y \in \mathbb{R}^n$.

On the other hand, recall that the classical Weyl correspondence on \mathbb{R}^{2n} is defined as follows [21, 26]. For each function f in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$, we define the operator $\mathcal{W}(f)$ acting on the Hilbert space $L^2(\mathbb{R}^n)$ by

$$(\mathcal{W}(f)\phi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{iyt} f\left(x + \frac{1}{2}y, t\right) \phi(x + y) dy dt. \quad (7)$$

Consider the Fourier transform $\mathcal{F}_2 f$ of $f \in \mathcal{S}(\mathbb{R}^{2n})$ with respect to the second variable

$$(\mathcal{F}_2 f)(x, y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-iyt} f(x, t) dt.$$

Then we can write

$$(\mathcal{W}(f)\phi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} (\mathcal{F}_2 f)\left(\frac{1}{2}(x + y), x - y\right) \phi(y) dy$$

and we can see that the kernel of $\mathcal{W}(f)$ is

$$k_{\mathcal{W}(f)}(x, y) = (2\pi)^{-n/2} (\mathcal{F}_2 f)\left(\frac{1}{2}(x + y), x - y\right).$$

LEMMA 3.2. For each $f \in \mathcal{S}(\mathbb{R}^{2n})$ such that $\mathcal{W}(f)$ is trace-class we have

$$\mathrm{Tr}(\Omega_1(a, b)\mathcal{W}(f)) = f(a, \lambda b)$$

for each $a, b \in \mathbb{R}^n$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^{2n})$ and $a, b \in \mathbb{R}^n$. Then we have

$$\begin{aligned} (\Omega_1(a, b)\mathcal{W}(f)\phi)(x) &= 2^n \exp(2i\lambda b(a-x)) (\mathcal{W}(f)\phi)(2a-x) \\ &= 2^n \exp(2i\lambda b(a-x)) \int_{\mathbb{R}^n} k_{\mathcal{W}(f)}(2a-x, y)\phi(y) dy. \end{aligned}$$

Thus the kernel of $\Omega_1(a, b)\mathcal{W}(f)$ is

$$k'(x, y) := 2^n \exp(2i\lambda b(a-x)) k_{\mathcal{W}(f)}(2a-x, y)$$

and by Mercer's theorem we have

$$\begin{aligned} \mathrm{Tr}(\Omega_1(a, b)\mathcal{W}(f)) &= \int_{\mathbb{R}^n} k'(x, x) dx \\ &= 2^n \int_{\mathbb{R}^n} \exp(2i\lambda b(a-x)) k_{\mathcal{W}(f)}(2a-x, x) dx \\ &= (2\pi)^{-n/2} 2^n \int_{\mathbb{R}^n} \exp(2i\lambda b(a-x)) (\mathcal{F}_2 f)(a, 2a-2x) dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(i\lambda bx) (\mathcal{F}_2 f)(a, x) dx \\ &= f(a, \lambda b), \end{aligned}$$

by the Fourier inversion theorem. □

As an immediate consequence of this lemma, we get the following proposition.

PROPOSITION 3.3. Let $f \in \mathcal{S}(\mathbb{R}^{2n})$ such that $\mathcal{W}(f)$ is trace-class. Then, for each $x, y \in \mathbb{R}^n$, we have

$$W_1(\mathcal{W}(f))(x, y) = f(x, \lambda y).$$

If, in particular, we take $\lambda = 1$, we see that W_1 and \mathcal{W} are inverse to each other.

We can now specify the connection between W_0 and W_1 , hence between W_0 and \mathcal{W} .

PROPOSITION 3.4. For each trace-class operator A on $L^2(\mathbb{R}^n)$ and each $a, b \in \mathbb{R}^n$, we have

$$W_1(A)(a, b) = W_0(\mathcal{B}A\mathcal{B}^{-1})(a + ib).$$

Proof. We can easily verify that $\mathcal{B}R_1 = R_0\mathcal{B}$. Let A be a trace-class operator on $L^2(\mathbb{R}^n)$ and let $a, b \in \mathbb{R}^n$. Recall that \mathcal{B} intertwines ρ_λ and ρ'_λ . Then we have

$$\begin{aligned}
W_1(A)(a, b) &= \text{Tr}(A\Omega_1(a, b)) \\
&= \text{Tr}(A\rho'_\lambda((a + ib, a - ib), 0)R_1\rho'_\lambda((a + ib, a - ib), 0)^{-1}) \\
&= \text{Tr}(A\rho'_\lambda((a + ib, a - ib), 0)\mathcal{B}^{-1}R_0\mathcal{B}\rho'_\lambda((a + ib, a - ib), 0)^{-1}) \\
&= \text{Tr}(A\mathcal{B}^{-1}\rho_\lambda((a + ib, a - ib), 0)R_0\rho_\lambda((a + ib, a - ib), 0)^{-1}\mathcal{B}) \\
&= \text{Tr}(\mathcal{B}A\mathcal{B}^{-1}\Omega_0(a + ib)) \\
&= W_0(\mathcal{B}A\mathcal{B}^{-1})(a + ib).
\end{aligned}$$

□

Of course, Proposition 3.4 can be extended to operators which are not necessarily of trace-class.

4. The metaplectic representation

Here we consider the group $S := Sp(n, \mathbb{C}) \cap SU(n, n)$ which is isomorphic to $Sp(n, \mathbb{R})$ via the map $M \rightarrow UMU^{-1}$ where $U := \begin{pmatrix} I_n & iI_n \\ & -iI_n \end{pmatrix}$, see [21, p. 175]. Then S consists of all matrices

$$k = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}, \quad P, Q \in M_n(\mathbb{C}), \quad PP^* - QQ^* = I_n, \quad PQ^t = QP^t.$$

Note that we also have

$$P^*P - Q^t\bar{Q} = I_n, \quad P^*Q = Q^t\bar{P}.$$

This implies, in particular, that $P^{-1}Q$ and $\bar{Q}P^{-1}$ are symmetric.

The group S acts on \mathbb{C}^n by

$$k(z, \bar{z}) = (Pz + Q\bar{z}, \bar{Q}z + \bar{P}\bar{z})$$

where $k = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}$. Then ω is invariant for the action of S , that is, we have

$$\omega(k(z, \bar{z}), k(z', \bar{z}')) = \omega((z, \bar{z}), (z', \bar{z}'))$$

for each $z, z' \in \mathbb{C}^n$. We will denote $kz = Pz + Q\bar{z}$.

Thus S also acts on H_n by $k \cdot ((z, \bar{z}), c) = (k(z, \bar{z}), c)$.

Let $\lambda > 0$ and $\rho := \rho_\lambda$. For each $k \in S$, we define ρ_k by $\rho_k(h) := \rho(k \cdot h)$ for each $h \in H_n$. Since $k \in S$ acts on H_n as a group isomorphism, we see that ρ_k is also a generic representation of H_n . Moreover, for each $h = ((0, 0), c)$

in the center of H_n , we have $\rho_k(h) = \rho(h)$. Hence, by the Stone-von Neumann theorem, ρ_k and ρ are unitarily equivalent, that is, there exists a unitary operator A_k of \mathcal{F}_λ (defined up to a unit complex number) such that

$$\rho_k(h) = A_k \rho(h) A_k^{-1} \quad (8)$$

for each $h \in H_n$.

Now we express Eq. 8 in terms of kernels.

PROPOSITION 4.1. *Let $k \in S$. Then the operator A_k on \mathcal{F}_λ satisfies Eq. 8 if and only if its kernel $b_k(z, w) := k_{A_k}(z, w)$ satisfies the relation*

$$\begin{aligned} \exp\left(-\frac{\lambda}{4}|kz_0|^2 + \frac{\lambda}{2}\overline{kz_0}z\right) b_k(z - kz_0, w) \\ = \exp\left(-\frac{\lambda}{4}|z_0|^2 - \frac{\lambda}{2}\overline{w}z_0\right) b_k(z, w + z_0) \end{aligned} \quad (9)$$

for each z_0, z and w in \mathbb{C}^n .

Proof. Let $k \in S$ and $h = ((z_0, \overline{z_0}), 0) \in H_n$. We have

$$(A_k f)(z) = \int_{\mathbb{C}^n} b_k(z, w) f(w) e^{-\lambda|w|^2/2} d\mu_\lambda(w).$$

Then, on the one hand, we get

$$\begin{aligned} \rho_\lambda(k \cdot h)(A_k f)(z) &= \exp\left(-\frac{\lambda}{4}|kz_0|^2 + \frac{\lambda}{2}\overline{kz_0}z\right) \\ &\times \int_{\mathbb{C}^n} b_k(z - kz_0, w) f(w) e^{-\lambda|w|^2/2} d\mu_\lambda(w). \end{aligned}$$

On the other hand, we have

$$(A_k \rho_\lambda(h) f)(z) = \int_{\mathbb{C}^n} b_k(z, w) \exp\left(-\frac{\lambda}{4}|z_0|^2 + \frac{\lambda}{2}\overline{z_0}w\right) f(w - z_0) e^{-\lambda|w|^2/2} d\mu_\lambda(w).$$

By performing the change of variables $w \rightarrow w + z_0$ in this integral and then writing that the kernels of $\rho_\lambda(k \cdot h)A_k$ and $A_k \rho_\lambda(h)$ are the same, we obtained the desired equation. \square

5. Holomorphic representations of the Jacobi group

In this section, we essentially follow [12] in which we constructed the holomorphic representations of the Jacobi group by applying the general method of [30, Chapter XII] (see also [9, 10]).

The (multi-dimensional) Jacobi group is the semi-direct product $G := H_n \rtimes S$ with respect to the action of S on H_n introduced in Section 4. The elements

of G can be written as $((z, \bar{z}), c, k)$ where $z \in \mathbb{C}^n$, $c \in \mathbb{R}$ and $k \in S$. The multiplication of G is given by

$$((z, \bar{z}), c, k) \cdot ((z', \bar{z}'), c', k') = ((z, \bar{z}) + k(z', \bar{z}'), c + c' + \frac{1}{2}\omega((z, \bar{z}), k(z', \bar{z}')), kk').$$

The complexification G^c of G is then the semi-direct product $G^c = H_n^c \rtimes Sp(n, \mathbb{C})$ whose elements can be written as $((z, w), c, k)$ where $z, w \in \mathbb{C}^n$, $c \in \mathbb{C}$, $k \in Sp(n, \mathbb{C})$ and the multiplication of G^c is obtained by replacing \bar{z} and \bar{z}' by w and w' in the preceding formula.

Let \mathfrak{g} and \mathfrak{g}^c the Lie algebras of G and G^c . For each $X = ((z, w), c, \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix})$ in \mathfrak{g}^c we define

$$X^* = ((-\bar{w}, -\bar{z}), -\bar{c}, \begin{pmatrix} \bar{A}^t & -\bar{C} \\ -\bar{B} & -\bar{A} \end{pmatrix}).$$

We denote by $g \rightarrow g^*$ the involutive anti-automorphism of G^c which is obtained by exponentiating $X \rightarrow X^*$ to G^c .

Let K be the subgroup of G consisting of all elements $((0, 0), c, \begin{pmatrix} P & 0 \\ 0 & \bar{P} \end{pmatrix})$ where $c \in \mathbb{R}$ and $P \in U(n)$. Let P^+ and P^- be the subgroups of G defined by

$$P^+ = \left\{ \left((y, 0), 0, \begin{pmatrix} I_n & Y \\ 0 & I_n \end{pmatrix} \right) : y \in \mathbb{C}^n, Y \in M_n(\mathbb{C}), Y^t = Y \right\}$$

and

$$P^- = \left\{ \left((0, v), 0, \begin{pmatrix} I_n & 0 \\ V & I_n \end{pmatrix} \right) : v \in \mathbb{C}^n, V \in M_n(\mathbb{C}), V^t = V \right\}$$

and let \mathfrak{p}^+ and \mathfrak{p}^- be the Lie algebras of P^+ and P^- . For convenience, we denote by $a(y, Y)$ the element $((y, 0), 0, \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix})$ of \mathfrak{p}^+ .

We can verify that each element $g = ((z_0, w_0), c_0, \begin{pmatrix} A & B \\ C & D \end{pmatrix}) \in G^c$ has a $P^+K^cP^-$ -decomposition if and only if $\text{Det}(D) \neq 0$ and, in this case, we have

$$g = \left((y, 0), 0, \begin{pmatrix} I_n & Y \\ 0 & I_n \end{pmatrix} \right) \cdot \left((0, 0), c, \begin{pmatrix} P & 0 \\ 0 & (P^t)^{-1} \end{pmatrix} \right) \cdot \left((0, v), 0, \begin{pmatrix} I_n & 0 \\ V & I_n \end{pmatrix} \right)$$

where $y = z_0 - BD^{-1}w_0$, $Y = BD^{-1}$, $v = D^{-1}w_0$, $V = D^{-1}C$, $P = A - BD^{-1}C = (D^t)^{-1}$ and $c = c_0 - \frac{i}{4}(z_0 - BD^{-1}w_0)w_0$. We then denote by $\zeta : P^+K^cP^- \rightarrow P^+$ and $\kappa : P^+K^cP^- \rightarrow K^c$ the projections onto P^+ - and K^c -components.

Consider the action (defined almost everywhere) of G^c on \mathfrak{p}^+ defined as follows. For $Z \in \mathfrak{p}^+$ and $g \in G^c$ with $g \exp Z \in P^+K^cP^-$, we define the element $g \cdot Z$ of \mathfrak{p}^+ by $g \cdot Z := \log \zeta(g \exp Z)$. We can verify that the action of $g = ((z_0, w_0), c_0, \begin{pmatrix} A & B \\ C & D \end{pmatrix}) \in G^c$ on $a(y, Y) \in \mathfrak{p}^+$ is given by $g \cdot a(y, Y) = a(y', Y')$ where $Y' = (AY + B)(CY + D)^{-1}$ and

$$y' = z_0 + Ay - (AY + B)(CY + D)^{-1}(w_0 + Cy).$$

Consequently, we have

$$\mathcal{D} := G \cdot 0 = \{a(y, Y) \in \mathfrak{p}^+ : I_n - Y\bar{Y} > 0\} \cong \mathbb{C}^n \times \mathcal{B}.$$

where $\mathcal{B} := \{Y \in M_n(\mathbb{C}) : Y^t = Y, I_n - Y\bar{Y} > 0\}$.

Let χ be a unitary character of K whose extension to K^c is also denoted by χ . Following [30], we can introduce the functions $K_\chi(Z, W) := \chi(\kappa(\exp W^* \exp Z))^{-1}$ for $Z, W \in \mathcal{D}$ and $J_\chi(g, Z) := \chi(\kappa(g \exp Z))$ for $g \in G$ and $Z \in \mathcal{D}$. We consider the Hilbert space \mathcal{H}_χ of all holomorphic functions f on \mathcal{D} such that

$$\|f\|_\chi^2 := \int_{\mathcal{D}} |f(Z)|^2 K_\chi(Z, Z)^{-1} c_\chi d\mu(Z) < +\infty.$$

Here the G -invariant measure μ on \mathcal{D} is defined by

$$d\mu(Z) = \text{Det}(1 - Y\bar{Y})^{-(n+2)} d\mu_L(y, Y),$$

where $d\mu_L$ is the Lebesgue measure on $\mathcal{D} \cong \mathbb{C}^n \times \mathcal{B}$, see [30, p. 538], and the constant c_χ is defined by

$$c_\chi^{-1} = \int_{\mathcal{D}} K_\chi(Z, Z)^{-1} d\mu(Z).$$

Let us fix χ as follows. Let $\lambda > 0$ and $m \in \mathbb{Z}$. Then, for each $k = ((0, 0), c, \begin{pmatrix} P & 0 \\ 0 & \bar{P} \end{pmatrix}) \in K$, we set $\chi(k) = e^{i\lambda c} (\text{Det } P)^m$.

PROPOSITION 5.1. [12, 30]

1. Let $Z = a(y, Y) \in \mathcal{D}$ and $W = a(v, V) \in \mathcal{D}$. We have

$$\begin{aligned} K_\chi(Z, W) &= \text{Det}(I_n - Y\bar{V})^m \\ &\times \exp\left(\frac{\lambda}{4} (2y(I_n - \bar{V}Y)^{-1}\bar{v} + y(I_n - \bar{V}Y)^{-1}\bar{V}y + \bar{v}Y(I_n - \bar{V}Y)^{-1}\bar{v})\right). \end{aligned}$$

2. $\mathcal{H}_\chi \neq (0)$ if and only if $m + n + 1/2 < 0$. In this case, \mathcal{H}_χ contains the polynomials.

3. For each $g = ((z_0, \bar{z}_0), c_0, \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}) \in G$ and each $Z = a(y, Y) \in \mathcal{D}$, we have

$$\begin{aligned} J(g, Z) &= e^{i\lambda c_0} \text{Det}(\bar{Q}Y + \bar{P})^{-m} \\ &\times \exp\left(\frac{\lambda}{4} (z_0\bar{z}_0 + 2\bar{z}_0Py + yP^t\bar{Q}y)\right) \\ &\times \exp\left(-\frac{\lambda}{4} (\bar{z}_0 + \bar{Q}y)(PY + Q)(\bar{Q}Y + \bar{P})^{-1}(\bar{z}_0 + \bar{Q}y)\right). \end{aligned}$$

We assume now that $m + n + 1/2 < 0$. Then the formula

$$\pi_\chi(g)f(Z) = J_\chi(g^{-1}, Z)^{-1} f(g^{-1} \cdot Z)$$

defines a unitary representation of G on \mathcal{H}_χ , [30, p. 540].

Note that \mathcal{H}_χ is a reproducing kernel Hilbert space. Indeed, if we set $e_Z(W) := K_\chi(W, Z)$ then we have the reproducing property $f(Z) = \langle f, e_Z \rangle_\chi$ for each $f \in \mathcal{H}_\chi$ and each $Z \in \mathcal{D}$ [30, p. 540]. Here $\langle \cdot, \cdot \rangle_\chi$ is the inner product on \mathcal{H}_χ .

By using the coherent states $(e_Z)_{Z \in \mathcal{D}}$, we can define the Berezin symbol $S_\chi(A)(Z)$ and the double Berezin symbol $s_\chi(A)(Z, W)$ of an operator A on \mathcal{H}_χ as this was done for operators on \mathcal{F}_λ in Section 2. We can also verify that the kernel $k_A(Z, W)$ of A satisfying

$$f(Z) = \int_{\mathcal{D}} k_A(Z, W) f(W) K_\chi(W, W)^{-1} c_\chi d\mu(W)$$

for each $f \in \mathcal{H}_\chi$ is given by

$$k_A(Z, W) = \langle A e_W, e_Z \rangle_\chi = s_\chi(A)(Z, W) \langle e_W, e_Z \rangle_\chi.$$

6. Kernels of metaplectic operators

By Proposition 5.1, we have

$$\begin{aligned} (\pi_\chi((z_0, \bar{z}_0), 0, I_{2n})f)(Z) \\ = \exp\left(\frac{\lambda}{4}(-|z_0|^2 + 2\bar{z}_0 y + \bar{z}_0 Y \bar{z}_0)\right) f(a(y - z_0 + Y \bar{z}_0, Y)) \end{aligned}$$

for $Z = a(y, Y) \in \mathcal{D}$, $z_0 \in \mathbb{C}^n$ and $f \in \mathcal{H}_\chi$.

Since this formula for $\pi_\chi((z_0, \bar{z}_0), 0, I_{2n})$ is close to that of $\rho_\lambda((z_0, \bar{z}_0), 0)$, see Section 2, a natural idea is then to use the commutation relation

$$\pi_\chi(k(z_0, \bar{z}_0), 0, I_{2n})\pi_\chi((0, 0), 0, k) = \pi_\chi((0, 0), 0, k)\pi_\chi((z_0, \bar{z}_0), 0, I_{2n})$$

for $k \in S$ in order to find a solution $b_k(z, w)$ of equation (9).

PROPOSITION 6.1. *For $k \in S$, let $B_k(Z, W)$ be the kernel of $\pi_\chi((0, 0), 0, k)$. Then*

$$b_k(z, w) := B_k(a(0, z), a(0, w))$$

is a solution of equation (9).

Proof. Let $k \in S$ and $Z = a(y, Y) \in \mathcal{D}$. For each $f \in \mathcal{H}_\chi$, we have

$$(\pi_\chi((0, 0), 0, k)f)(Z) = \int_{\mathcal{D}} B_k(Z, W) f(W) K_\chi(W, W)^{-1} c_\chi d\mu(W).$$

Then, on the one hand, we have

$$\begin{aligned} & (\pi_\chi(k(z_0, \bar{z}_0), 0, I_{2n})\pi_\chi((0, 0), 0, k)f)(Z) \\ &= \exp\left(\frac{\lambda}{4}(-|kz_0|^2 + 2(\overline{kz_0})y + (\overline{kz_0})Y(\overline{kz_0}))\right) \\ & \quad \times \int_{\mathcal{D}} B_k(a(y - kz_0 + Y(\overline{kz_0}), Y), W)f(W)K_\chi(W, W)^{-1}c_\chi d\mu(W). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & (\pi_\chi((0, 0), 0, k)\pi_\chi((z_0, \bar{z}_0), 0, I_{2n})f)(Z) \\ &= \int_{\mathcal{D}} B_k(Z, W)(\pi_\chi((z_0, \bar{z}_0), 0, I_{2n})f)(W)K_\chi(W, W)^{-1}c_\chi d\mu(W) \\ &= \int_{\mathcal{D}} B_k(Z, W) \exp\left(\frac{\lambda}{4}(-|z_0|^2 + 2\bar{z}_0v + \bar{z}_0V(\bar{z}_0))\right) \\ & \quad \times f(a(v - z_0 + V\bar{z}_0, V))K_\chi(W, W)^{-1}c_\chi d\mu(W) \end{aligned}$$

with the notation $W = a(v, V)$. We perform in this integral the change of variables $v \rightarrow v + z_0 - V\bar{z}_0$ and we find

$$\begin{aligned} & (\pi_\chi((0, 0), 0, k)\pi_\chi((z_0, \bar{z}_0), 0, I_{2n})f)(Z) \\ &= \int_{\mathcal{D}} B_k(Z, a(v + z_0 - V\bar{z}_0, V)) \exp\left(\frac{\lambda}{4}(|z_0|^2 + 2\bar{z}_0v - \bar{z}_0V\bar{z}_0)\right) f(W) \\ & \quad \times K_\chi(a(v + z_0 - V\bar{z}_0, V), a(v + z_0 - V\bar{z}_0, V))^{-1}c_\chi d\mu(W). \end{aligned}$$

Then, by writing that $\pi_\chi(k(z_0, \bar{z}_0), 0, I_{2n})\pi_\chi((0, 0), 0, k)$ and

$$\pi_\chi((0, 0), 0, k)\pi_\chi((z_0, \bar{z}_0), 0, I_{2n})$$

have the same kernel, we obtain

$$\begin{aligned} & \exp\left(\frac{\lambda}{4}(-|kz_0|^2 + 2(\overline{kz_0})y + (\overline{kz_0})Y(\overline{kz_0}))\right) \\ & \quad \times B_k(a(y - kz_0 + Y(\overline{kz_0}), Y), W)K_\chi(W, W)^{-1} \\ &= \exp\left(\frac{\lambda}{4}(|z_0|^2 + 2\bar{z}_0v - \bar{z}_0V\bar{z}_0)\right) B_k(a(y, Y), a(v + z_0 - V\bar{z}_0, V)) \\ & \quad \times K_\chi(a(v + z_0 - V\bar{z}_0, V), a(v + z_0 - V\bar{z}_0, V))^{-1}. \end{aligned}$$

Note that $K_\chi(a(v, 0), a(v, 0)) = \exp\left(\frac{\lambda}{2}|v|^2\right)$. Then, taking $Y = V = 0$ in the above equality we get

$$\begin{aligned} & \exp\left(\frac{\lambda}{4}(-|kz_0|^2 + 2(\overline{kz_0})y - 2|v|^2)\right) B_k(a(y - kz_0, 0), a(v, 0)) \\ &= \exp\left(\frac{\lambda}{4}(|z_0|^2 + 2\bar{z}_0v - 2|v + z_0|^2)\right) B_k(a(y, 0), a(v + z_0, 0)), \end{aligned}$$

hence the desired result. \square

The next step is then to compute $B_k(a(y, 0), a(v, 0))$.

PROPOSITION 6.2. *Let $Z = a(y, 0)$ and $W = a(v, 0)$. Then we have*

$$B_k(Z, W) = (\text{Det } P)^m \exp\left(\frac{\lambda}{4} (y(\bar{Q}P^{-1}y) + 2y(P^t)^{-1}\bar{v} - \bar{v}(P^{-1}Q\bar{v}))\right). \quad (10)$$

Proof. Let $Z = a(y, 0)$, $W = a(v, 0)$ and $g = ((0, 0), 0, k)$ where $k \in S$. Then we have

$$\begin{aligned} B_k(Z, W) &= K_{\pi_\chi(g)}(Z, W) = \langle \pi_\chi(g)e_W, e_Z \rangle_\chi \\ &= (\pi_\chi(g)e_W)(Z) = J_\chi(g^{-1}, Z)^{-1}e_W(g^{-1} \cdot Z) \\ &= J_\chi(g^{-1}, Z)^{-1} \langle e_W, e_{g^{-1} \cdot Z} \rangle_\chi = J_\chi(g^{-1}, Z)^{-1}K_\chi(g^{-1} \cdot Z, W). \end{aligned}$$

Note that $k^{-1} = \begin{pmatrix} P^* & -Q^t \\ -Q^* & P^t \end{pmatrix}$. Then we get

$$\begin{aligned} g^{-1} \cdot Z &= a(P^*y - Q^t(P^t)^{-1}Q^*y, -Q^t(P^t)^{-1}) \\ &= a(P^{-1}y, -Q^t(P^t)^{-1}) \\ &= a(P^{-1}y, -P^{-1}Q) \end{aligned}$$

since

$$P^* - Q^t(P^t)^{-1}Q^* = P^* - P^{-1}QQ^* = P^{-1}(PP^* - QQ^*) = P^{-1}.$$

Thus, by Proposition 5.1, we have

$$K_\chi(g^{-1} \cdot Z, W) = \exp\left(\frac{\lambda}{4} (2y(P^t)^{-1}\bar{v} - \bar{v}P^{-1}Q\bar{v})\right).$$

Moreover, by Proposition 5.1 again, we find

$$J_\chi(g^{-1}, Z)^{-1} = (\text{Det } P)^m \exp\left(\frac{\lambda}{4} (y(P^*)^t Q^*y - y\bar{Q}Q^t(P^t)^{-1}Q^*y)\right).$$

Since we have

$$\begin{aligned} (P^*)^t - \bar{Q}Q^t(P^t)^{-1} &= ((P^*)^t P^t - \bar{Q}Q^t)(P^t)^{-1} \\ &= (PP^* - QQ^*)^t (P^t)^{-1} = (P^t)^{-1}, \end{aligned}$$

the result follows. □

For each $k \in S$, we denote by A_k the operator on \mathcal{F}_λ with kernel

$$b_k(z, w) = B_k(a(z, 0), a(w, 0)).$$

By Schur's lemma, there exists, for each $k, k' \in S$ a scalar $\alpha(k, k')$ such that $A_{kk'} = \alpha(k, k')A_k A_{k'}$.

LEMMA 6.3. Let $k = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}$, $k' = \begin{pmatrix} P' & Q' \\ \bar{Q}' & \bar{P}' \end{pmatrix}$ and $k'' = \begin{pmatrix} P'' & Q'' \\ \bar{Q}'' & \bar{P}'' \end{pmatrix}$ in S . Then we have

$$(\text{Det } P'')^m = \alpha(k, k')(\text{Det } P)^m(\text{Det } P')^m(\text{Det}(P^{-1}P''P'^{-1}))^{-1/2}.$$

Here, if $z \in \mathbb{C}$, we define $z^{1/2}$ as the principal determination of the square-root (with branch cut along the negative real axis).

Proof. Since the kernel of $A_k A_{k'}$ is the convolution of the kernels of A_k and $A_{k'}$, we have

$$b_{kk'}(z, w) = \alpha(k, k') \int_{\mathbb{C}^n} b_k(z, u) b_{k'}(u, w) e^{-\lambda|u|^2/2} d\mu_\lambda(u).$$

Taking $z = w = 0$, we get

$$\begin{aligned} (\text{Det } P'')^m &= \alpha(k, k')(\text{Det } P)^m(\text{Det } P')^m \\ &\times \int_{\mathbb{C}^n} \exp\left(\frac{\lambda}{4}(-\bar{u}P^{-1}Q\bar{u} + u\bar{Q}'P'^{-1}u)\right) e^{-\lambda|u|^2/2} d\mu_\lambda(u). \end{aligned}$$

Recall that $P^{-1}Q$ and $\bar{Q}'P'^{-1}$ are symmetric. Then, the integral in the preceding equality can be evaluated by using [21, Theorem 3, p. 258] and its value is

$$\begin{aligned} \text{Det}^{-1/2}(I_n + P^{-1}Q\bar{Q}'P'^{-1}) &= \text{Det}^{-1/2}(P^{-1}(PP' + Q\bar{Q}')P'^{-1}) \\ &= \text{Det}^{-1/2}(P^{-1}P''P'^{-1}). \end{aligned}$$

The result follows. \square

We are now in position to recover [21, Theorem 4.37] (this result is due to V. Bargmann and C. Itzykson). For $k \in S$, we denote by $\sigma(k)$ the operator A_k with kernel $b_k(z, w)$ corresponding to $m = -1/2$. Then σ is called the metaplectic representation of S . Note that the value $m = -1/2$ does not correspond to a holomorphic representation of G , see 2 of Proposition 5.1.

PROPOSITION 6.4. 1. For each $k, k' \in S$, we have $\sigma(kk') = \pm\sigma(k)\sigma(k')$.

2. For each $k \in S$, $\sigma(k)$ is unitary.

Proof. 1. This is an immediate consequence of Lemma 6.3. 2. From the formula for b_k , see equation (10), we deduce that we have $b_{k^{-1}}(z, w) = \bar{b}_k(z, w)$ for each $z, w \in \mathbb{C}^n$, hence $A_{k^{-1}} = A_k^*$. This implies that $A_k A_k^* = \pm Id$. Since $A_k A_k^*$ is positive, we have $A_k A_k^* = Id$. By the same argument, we also obtain $A_k^* A_k = Id$. \square

We can also give a formula for the differential of σ .

PROPOSITION 6.5. *Let $X = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \in \mathfrak{s}$. Then we have*

$$(d\sigma(X)f)(z) = \left(-\frac{1}{2} \operatorname{Tr}(A) + \frac{\lambda}{4} z(\bar{B}z)\right) f(z) - \sum_{j=1}^n (Az)_j \frac{\partial f}{\partial z_j} - \frac{1}{\lambda} \sum_{j,k} b_{jk} \frac{\partial^2 f}{\partial z_j \partial z_k}$$

where $B = (b_{jk})$.

Proof. By differentiating the following formula for the kernel b_k of $\sigma(k)$

$$b_k(z, w) = (\operatorname{Det} P)^{-1/2} \exp\left(\frac{\lambda}{4} (z(\bar{Q}P^{-1}z) + 2(P^{-1}z)\bar{w} - \bar{w}(P^{-1}Q\bar{w}))\right),$$

we obtain a formula for the kernel $b_X(z, w)$ of $d\sigma(X)$:

$$b_X(z, w) = \left(-\frac{1}{2} \operatorname{Tr}(A) + \frac{\lambda}{4} z(\bar{B}z) - \frac{\lambda}{2} (Az)\bar{w} - \frac{\lambda}{4} \bar{w}(B\bar{w})\right) \exp\left(\frac{\lambda}{2} z\bar{w}\right).$$

Remark that by differentiating the reproducing property

$$f(z) = \langle f, e_z \rangle_{\mathcal{F}_\lambda} = \int_{\mathbb{C}^n} e^{\lambda z \bar{w}/2} f(w) e^{-\lambda |w|^2/2} d\mu_\lambda(w)$$

under the integral sign, we get

$$\frac{\partial f}{\partial z_j} = \frac{\lambda}{2} \int_{\mathbb{C}^n} \bar{w}_j e^{\lambda z \bar{w}/2} f(w) e^{-\lambda |w|^2/2} d\mu_\lambda(w)$$

for each $j = 1, 2, \dots, n$ and, by differentiating again, we also obtain

$$\frac{\partial^2 f}{\partial z_j \partial z_k} = \left(\frac{\lambda}{2}\right)^2 \int_{\mathbb{C}^n} \bar{w}_j \bar{w}_k e^{\lambda z \bar{w}/2} f(w) e^{-\lambda |w|^2/2} d\mu_\lambda(w)$$

for each $j, k = 1, 2, \dots, n$. This allows us to compute

$$(d\sigma(X)f)(z) = \int_{\mathbb{C}^n} b_X(z, w) f(w) e^{-\lambda |w|^2/2} d\mu_\lambda(w)$$

and to get the desired result. \square

We can also give formulas for the Berezin symbols of $\sigma(k)$ for $k \in S$ and $d\sigma(X)$ for $X \in \mathfrak{s}$. We immediately obtain the following proposition.

PROPOSITION 6.6. 1. *Let $k = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in S$. we have*

$$S_\lambda(\sigma(k))(z) = (\operatorname{Det} P)^{-1/2} \times \exp\left(\frac{\lambda}{4} (z(\bar{Q}P^{-1}z) + 2\bar{z}(P^{-1} - I_n)z - \bar{z}(P^{-1}Q\bar{z}))\right).$$

2. *Let $X = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \in \mathfrak{s}$. Then we have*

$$S_\lambda(d\sigma(X))(z) = -\frac{1}{2} \operatorname{Tr}(A) + \frac{\lambda}{4} z(\bar{B}z) - \frac{\lambda}{2} (Az)\bar{z} - \frac{\lambda}{4} \bar{z}(B\bar{z}).$$

7. Complex Weyl symbols of metaplectic operators

In this section, we compute $W_0(\sigma(k))$ for $k \in S$ and $W_0(d\sigma(X))$ for $X \in \mathfrak{s}$. We begin with two technical lemmas. The first one is a variant of [21, Theorem 3, p. 258].

LEMMA 7.1. *Let A, B, D be $n \times n$ complex matrices such that $A^t = A, D^t = D$. Let $M = \begin{pmatrix} A & B^t \\ B & D \end{pmatrix}$, $U = \begin{pmatrix} I_n & iI_n \\ I_n & -iI_n \end{pmatrix}$ and $N = U^t M U$. Assume that $\text{Re}(N)$ is positive definite. Let $u, v \in \mathbb{C}^n$. Then we have*

$$\begin{aligned} & \int_{\mathbb{C}^n} \exp(- (w(Aw) + \bar{w}(D\bar{w}) + 2\bar{w}(Bw))) \exp(uw + v\bar{w}) dm(w) \\ &= \pi^n (\text{Det } N)^{-1/2} \exp\left(\frac{1}{4} \begin{pmatrix} u & v \end{pmatrix} M^{-1} \begin{pmatrix} u \\ v \end{pmatrix}\right). \end{aligned}$$

Proof. Write $w = x + iy$ with $x, y \in \mathbb{R}^n$. Then $\begin{pmatrix} w \\ \bar{w} \end{pmatrix} = U \begin{pmatrix} x \\ y \end{pmatrix}$. We have

$$w(Aw) + \bar{w}(D\bar{w}) + 2\bar{w}(Bw) = (w, \bar{w}) M \begin{pmatrix} w \\ \bar{w} \end{pmatrix} = (x, y) N \begin{pmatrix} x \\ y \end{pmatrix}$$

and $uw + v\bar{w} = \begin{pmatrix} u & v \end{pmatrix} U \begin{pmatrix} x \\ y \end{pmatrix}$.

The result then follows from the well-known equality

$$\int_{\mathbb{R}^n} \exp(-xAx + zx) dx = (\text{Det } A)^{-1/2} \pi^{n/2} \exp\left(\frac{1}{4}z(A^{-1}z)\right)$$

for $z \in \mathbb{C}^n$ and A a $n \times n$ symmetric complex matrix such that $\text{Re}(A)$ is definite positive. \square

LEMMA 7.2. *1. Let a, b, p be $n \times n$ complex matrices such that $\begin{pmatrix} -a & I_n + p^t \\ I_n + p & d \end{pmatrix}$ is invertible with inverse matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then we have*

$$\begin{aligned} & \begin{pmatrix} a & I_n - p^t \\ p - I_n & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & p^t - I_n \\ I_n - p & d \end{pmatrix} \\ &= \begin{pmatrix} 4\delta - a & 3I_n - 4\gamma - p^t \\ 3I_n - 4\beta - p & 4\alpha + d \end{pmatrix}. \end{aligned}$$

2. Take $a = \bar{Q}P^{-1}$, $d = P^{-1}Q$ and $p = P^{-1}$ with $k = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in S$. Let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then we have

$$\frac{1}{2}J(k - I_{2n})(k + I_{2n})^{-1} = \begin{pmatrix} \delta & \frac{1}{2}I_n - \gamma \\ \frac{1}{2}I_n - \beta & \alpha \end{pmatrix}.$$

3. Let $k = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in S$. Then

$$\text{Det} \begin{pmatrix} -\bar{Q}P^{-1} & I_n + (P^t)^{-1} \\ I_n + P^{-1} & P^{-1}Q \end{pmatrix} = (-1)^n (\text{Det } P)^{-1} \text{Det}(k + I_{2n}).$$

Proof. 1. By writing

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} -a & I_n + p^t \\ I_n + p & d \end{pmatrix} = \begin{pmatrix} -a & I_n + p^t \\ I_n + p & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = I_{2n}$$

we obtain the series of equations

$$\begin{aligned} \alpha a &= \beta(I_n + p) - I_n; & \gamma a &= \delta(I_n + p); \\ \beta d &= -\alpha(I_n + p^t); & \delta d &= I_n - \gamma(I_n + p^t); \\ \alpha \alpha &= (I_n + p^t)\gamma - I_n; & \alpha \beta &= (I_n + p^t)\delta; \\ d\gamma &= -(I_n + p)\alpha; & d\delta &= I_n - (I_n + p)\beta. \end{aligned}$$

By using these equations, we obtain firstly

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & -I_n + p^t \\ I_n - p & d \end{pmatrix} = \begin{pmatrix} 2\beta - I_n & -2\alpha \\ 2\delta & I_n - 2\gamma \end{pmatrix}$$

and, secondly,

$$\begin{pmatrix} a & I_n - p^t \\ -I_n + p & d \end{pmatrix} \begin{pmatrix} 2\beta - I_n & -2\alpha \\ 2\delta & I_n - 2\gamma \end{pmatrix} = \begin{pmatrix} 4\delta - a & 3I_n - 4\gamma - p^t \\ 3I_n - 4\beta - p & 4\alpha + d \end{pmatrix}.$$

2-3. First we have

$$\begin{aligned} & \begin{pmatrix} -\bar{Q}P^{-1} & I_n + (P^t)^{-1} \\ I_n + P^{-1} & P^{-1}Q \end{pmatrix} \begin{pmatrix} -P & -Q \\ 0 & I_n \end{pmatrix} \\ &= \begin{pmatrix} \bar{Q} & \bar{Q}P^{-1}Q + I_n + (P^t)^{-1} \\ -I_n - P & -Q \end{pmatrix} \\ &= \begin{pmatrix} \bar{Q} & I_n + \bar{P} \\ -I_n - P & -Q \end{pmatrix} = J(k + I_{2n}) \end{aligned}$$

since

$$\bar{Q}P^{-1}Q + (P^t)^{-1} = (\bar{Q}P^{-1}QP^t + I_n)(P^t)^{-1} = (\bar{Q}Q^t + I_n)(P^t)^{-1} = \bar{P}.$$

On the one hand, passing to the determinant, we obtain 3 and, on the other hand, we deduce that

$$\begin{pmatrix} -P & -Q \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} J(k + I_{2n}).$$

This implies that

$$\begin{aligned}
 \begin{pmatrix} \delta & \frac{1}{2}I_n - \gamma \\ \frac{1}{2}I_n - \beta & \alpha \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} - J \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} J \\
 &= \frac{1}{2} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} - J \begin{pmatrix} -P & -Q \\ 0 & I_n \end{pmatrix} (k + I_{2n})^{-1} \\
 &= \left(\frac{1}{2} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} (k + I_{2n}) - J \begin{pmatrix} -P & -Q \\ 0 & I_n \end{pmatrix} \right) (k + I_{2n})^{-1} \\
 &= \frac{1}{2} J(k - I_{2n})(k + I_{2n})^{-1}.
 \end{aligned}$$

□

We denote by $\text{Arg}(z)$ the principal argument of $z \in \mathbb{C}$.

PROPOSITION 7.3. *Let $k = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in S$. Then we have*

$$W_0(\sigma(k))(z) = c_n(k) \exp\left(\frac{\lambda}{2} \begin{pmatrix} z & \bar{z} \end{pmatrix} J(k - I_{2n})(k + I_{2n})^{-1} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)$$

where

$$\begin{aligned}
 c_n(k) &= 2^n (\text{Det}(I_{2n} + k))^{-1/2} \text{ if } \text{Det}(I_{2n} + k) > 0; \\
 c_n(k) &= -i2^n |\text{Det}(I_{2n} + k)|^{-1/2} \text{ if } \text{Det}(I_{2n} + k) < 0 \text{ and } \text{Arg}(\text{Det}(P)) \in]0, \pi[; \\
 c_n(k) &= i2^n |\text{Det}(I_{2n} + k)|^{-1/2} \text{ if } \text{Det}(I_{2n} + k) < 0 \text{ and } \text{Arg}(\text{Det}(P)) \in]-\pi, 0[.
 \end{aligned}$$

Proof. Let $k = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in S$. Recall that $W_0(\sigma(k))$ is given by

$$W_0(\sigma(k))(z) = \left(\frac{\lambda}{\pi}\right)^n \int_{\mathbb{C}^n} b_k(z+w, z-w) \exp\left(\frac{\lambda}{2} (-z\bar{z} - w\bar{w} + z\bar{w} - \bar{z}w)\right) dm(w)$$

where the kernel of $\sigma(k)$ is

$$b_k(z, w) = (\text{Det } P)^{-1/2} \exp\left(\frac{\lambda}{4} (z(\bar{Q}P^{-1}z) + 2\bar{w}(P^{-1}z) - \bar{w}(P^{-1}Q\bar{w}))\right).$$

Then we get

$$\begin{aligned}
 W_0(\sigma(k))(z) &= \left(\frac{\lambda}{\pi}\right)^n (\text{Det } P)^{-1/2} \\
 &\times \exp\left(\frac{\lambda}{4} (z(\bar{Q}P^{-1}z) + 2\bar{z}(P^{-1} - I_n)z - \bar{z}(P^{-1}Q\bar{z}))\right) \\
 &\times \int_{\mathbb{C}^n} \exp\left(\frac{\lambda}{4} (w\bar{Q}P^{-1}w - \bar{w}(P^{-1}Q\bar{w}) - 2\bar{w}(I_n + P^{-1})w)\right) \\
 &\times \exp\left(\frac{\lambda}{2} ((\bar{Q}P^{-1}z + ((P^t)^{-1} - I_n)\bar{z})w + ((I_n - P^{-1})z + P^{-1}Q\bar{z})\bar{w})\right) dm(w)
 \end{aligned}$$

The integral $I(k)$ in the preceding formula can be evaluated by using Lemma 7.1 with

$$M = \frac{\lambda}{4} \begin{pmatrix} -\bar{Q}P^{-1} & I_n + (P^t)^{-1} \\ I_n + P^{-1} & P^{-1}Q \end{pmatrix}$$

and

$$u = \frac{\lambda}{2}(\bar{Q}P^{-1}z + ((P^t)^{-1} - I_n)\bar{z}); v = \frac{\lambda}{2}((I_n - P^{-1})z + P^{-1}Q\bar{z}).$$

Observing that, by Lemma 7.2, we have

$$\begin{aligned} & (u \ v)M^{-1} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \lambda \begin{pmatrix} z & \bar{z} \end{pmatrix} \begin{pmatrix} \bar{Q}P^{-1} & I_n - (P^t)^{-1} \\ -I_n + P^{-1} & P^{-1}Q \end{pmatrix} \begin{pmatrix} -\bar{Q}P^{-1} & I_n + (P^t)^{-1} \\ I_n + P^{-1} & P^{-1}Q \end{pmatrix}^{-1} \\ & \quad \times \begin{pmatrix} \bar{Q}P^{-1} & -I_n + (P^t)^{-1} \\ I_n - P^{-1} & P^{-1}Q \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \\ &= \lambda \begin{pmatrix} z & \bar{z} \end{pmatrix} \begin{pmatrix} 4\delta - \bar{Q}P^{-1} & 3I_n - 4\gamma - (P^t)^{-1} \\ 3I_n - 4\beta - P^{-1} & 4\alpha + P^{-1}Q \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}, \end{aligned}$$

we find that

$$\begin{aligned} I(k) &= \pi^n (\text{Det } U^t M U)^{-1/2} \\ & \quad \times \exp \left(\frac{\lambda}{4} \begin{pmatrix} z & \bar{z} \end{pmatrix} \begin{pmatrix} 4\delta - \bar{Q}P^{-1} & 3I_n - 4\gamma - (P^t)^{-1} \\ 3I_n - 4\beta - P^{-1} & 4\alpha + P^{-1}Q \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \right). \end{aligned}$$

Then we get

$$\begin{aligned} W_0(\sigma(k))(z) &= \lambda^n (\text{Det } P)^{-1/2} (\text{Det } U^t M U)^{-1/2} \\ & \quad \times \exp \left(\lambda \begin{pmatrix} z & \bar{z} \end{pmatrix} \begin{pmatrix} \delta & \frac{1}{2}I_n - \gamma \\ \frac{1}{2}I_n - \beta & \alpha \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \right). \end{aligned}$$

Note that

$$\text{Det } U^t M U = (-1)^n 2^{2n} \text{Det}(M) = 2^{2n} (\lambda/4)^{2n} \text{Det}(k + I_{2n}) (\text{Det}(P))^{-1}.$$

Finally, by Lemma 7.2 again, we obtain

$$W_0(\sigma(k))(z) = c_n(k) \exp \left(\frac{\lambda}{2} \begin{pmatrix} z & \bar{z} \end{pmatrix} J(k - I_{2n})(k + I_{2n})^{-1} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \right)$$

where $c_n(k) = 2^n (\text{Det } P)^{-1/2} ((\text{Det}(k + I_{2n}) (\text{Det } P)^{-1})^{-1/2}$. The result hence follows by taking into account the fact that, since $k = U^{-1}gU$ with $g \in Sp(n, \mathbb{R})$, we have $\text{Det}(I_{2n} + k) = \text{Det}(I_{2n} + g) \in \mathbb{R}$. \square

PROPOSITION 7.4. *Let $X = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in \mathfrak{s}$. Then we have*

$$W_0(d\sigma(X))(z) = \frac{\lambda}{4}(z(\bar{B}z) - \bar{z}(B\bar{z}) - 2(Az)\bar{z}).$$

Proof. There are different ways to prove this result. For instance, one can differentiate $W_0(\sigma(k))(z)$ or one can use the integral formula for $W_0(d\sigma(X))$, see Proposition 3.1. However, the fastest method is based on the formula $W_0 = B_\lambda^{-1/2}S_\lambda$, see Section 3. Since $B_\lambda = \exp(\Delta/2\lambda)$ where $\Delta = 4 \sum_{k=1}^n \partial^2/\partial z_k \partial \bar{z}_k$, see Section 2, we have $B_\lambda^{-1/2} = \exp(-\frac{1}{\lambda} \sum_{k=1}^n \partial^2/\partial z_k \partial \bar{z}_k)$. By using the formula for $S_\lambda(d\sigma(X))$ given in Proposition 6.6, we get

$$-\frac{1}{\lambda} \sum_{k=1}^n \partial^2/\partial z_k \partial \bar{z}_k(S_\lambda(d\sigma(X))) = \frac{1}{2} \sum_{k=1}^n (Ae_k)e_k = \frac{1}{2} \text{Tr}(A)$$

hence

$$(B_\lambda^{-1/2}S_\lambda(d\sigma(X)))(z) = \frac{\lambda}{4}(z(\bar{B}z) - \bar{z}(B\bar{z}) - 2(Az)\bar{z}).$$

□

8. Applications

Here we recover some known results about the classical Weyl symbols of the metaplectic representation operators of $Sp(n, \mathbb{R})$ and about the computation of some star-exponentials. We take $\lambda = 1$.

8.1. Weyl symbols of metaplectic representation operators for $Sp(n, \mathbb{R})$

The metaplectic representation σ of S can be translated to $Sp(n, \mathbb{R})$ as follows, see [21, Chapter IV]. For each $g \in Sp(n, \mathbb{R})$, we define $\sigma'(g) = \mathcal{B}^{-1}\sigma(UgU^{-1})\mathcal{B}$ where $U = \begin{pmatrix} I_n & iI_n \\ I_n & -iI_n \end{pmatrix}$. Then we can deduce from Proposition 7.3 a formula for $W_1(\sigma'(g))$, $g \in Sp(n, \mathbb{R})$. Recall that W_1 is the inverse of the classical Weyl correspondence \mathcal{W} , see Section 2.

PROPOSITION 8.1. *Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$. Then we have*

$$W_1(\sigma'(g))(x, y) = c'_n(g) \exp\left(-i \begin{pmatrix} x & y \end{pmatrix} J(g - I_{2n})(g + I_{2n})^{-1} \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

where

$$\begin{aligned} c'_n(g) &= c_n(UgU^{-1}) = 2^n (\text{Det}(I_{2n} + g))^{-1/2} \quad \text{if } \text{Det}(I_{2n} + g) > 0; \\ c'_n(g) &= -i2^n |\text{Det}(I_{2n} + g)|^{-1/2} \\ &\quad \text{if } \text{Det}(I_{2n} + g) < 0 \text{ and } \text{Arg}(\text{Det}(A + D + i(C - B))) \in]0, \pi[; \\ c'_n(g) &= i2^n |\text{Det}(I_{2n} + g)|^{-1/2} \\ &\quad \text{if } \text{Det}(I_{2n} + g) < 0 \text{ and } \text{Arg}(\text{Det}(A + D + i(C - B))) \in]-\pi, 0[. \end{aligned}$$

Proof. Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $k = UgU^{-1} = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix}$. By Proposition 3.4 and Proposition 7.3, we have, for each $(x, y) \in \mathbb{R}^{2n}$,

$$\begin{aligned} W_1(\sigma'(g))(x, y) &= W_1(\mathcal{B}^{-1}\sigma(k)\mathcal{B})(x, y) = W_0(\sigma(k))(z) \\ &= c_n(k) \exp\left(\frac{1}{2} \begin{pmatrix} z & \bar{z} \end{pmatrix} J(k - I_{2n})(k + I_{2n})^{-1} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) \end{aligned}$$

where $z = x + iy \in \mathbb{C}^n$. Now, since $\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = U \begin{pmatrix} x \\ y \end{pmatrix}$ and $k = UgU^{-1}$, we obtain

$$\begin{aligned} \frac{1}{2} \begin{pmatrix} z & \bar{z} \end{pmatrix} J(k - I_{2n})(k + I_{2n})^{-1} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} U^t J U (g - I_{2n})(g + I_{2n})^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= -i \begin{pmatrix} x & y \end{pmatrix} J (g - I_{2n})(g + I_{2n})^{-1} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

Moreover, we have $\text{Det}(k + I_{2n}) = \text{Det}(g + I_{2n})$, $P = \frac{1}{2}(A + D + i(C - B))$ and the rest of the proposition is just a reformulation of the discussion on the value of $c_n(k)$, see Proposition 7.3. \square

Similar formulas involving Cayley transform can be found in [16, 17, 18, 20, 25]. In the excellent book [17], it seems that a factor 2^n must be added in Formula (3.71) to make it consistent with Formula (3.80).

The expression of $W_1(\sigma'(g))$ takes a more simple form when $g = \exp(X)$, where $X \in sp(n, \mathbb{R})$.

COROLLARY 8.2. *Let $X \in sp(n, \mathbb{R})$. Then we have*

$$\begin{aligned} W_1(\sigma'(\exp(X)))(x, y) \\ &= (\text{Det}(\cosh(\frac{1}{2}X)))^{-1/2} \exp\left(-i \begin{pmatrix} x & y \end{pmatrix} J \tanh(\frac{1}{2}X) \begin{pmatrix} x \\ y \end{pmatrix}\right). \end{aligned}$$

Proof. We just apply Proposition 8.1 to $g = \exp(X)$, $X \in sp(n, \mathbb{R})$. We have

$$\text{Det}(g + I_{2n}) = 2^{2n} \text{Det}(\exp(\frac{1}{2}X)) \text{Det}(\cosh(\frac{1}{2}X)) = 2^{2n} \text{Det}(\cosh(\frac{1}{2}X)) \geq 0$$

hence $c'_n(g) = (\text{Det}(\cosh(\frac{1}{2}X)))^{-1/2}$ and $(g + I_{2n})(g - I_{2n})^{-1} = \tanh(\frac{1}{2}X)$. The result follows. \square

We can also deduce the computation of $W_1(\sigma'(X))$, $X \in sp(n, \mathbb{R})$, from Proposition 7.4.

PROPOSITION 8.3. *Let $X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in sp(n, \mathbb{R})$. Then we have*

$$W_1(d\sigma'(X))(x, y) = \frac{1}{2}i(2y(Ax) + y(By) - x(Cx)) = -\frac{1}{2}i \begin{pmatrix} x & y \end{pmatrix} JX \begin{pmatrix} x \\ y \end{pmatrix}.$$

Proof. Let $X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in sp(n, \mathbb{R})$. Define $Y = UXU^{-1} = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in \mathfrak{s}$. Then we have

$$P = \frac{1}{2}(A + D + i(-B + C)) \quad ; \quad Q = \frac{1}{2}(A - D + i(B + C)).$$

Now, we have

$$\begin{aligned} W_1(d\sigma'(X))(x, y) &= W_1(\mathcal{B}^{-1}d\sigma(Y)\mathcal{B})(x, y) = W_0(d\sigma(Y))(x + iy) \\ &= \frac{1}{4}((x + iy)\bar{Q}(x + iy) - (x - iy)Q(x - iy) - 2(P(x + iy))(x - iy)). \end{aligned}$$

By replacing P and Q with the expressions given above, the result follows from a tedious but easy calculation. \square

REMARK 8.4. Let $X \in sp(n, \mathbb{R})$. Then $M = \frac{1}{2}JX$ is a symmetric matrix and

$$q_M(x, y) = \begin{pmatrix} x & y \end{pmatrix} M \begin{pmatrix} x \\ y \end{pmatrix}$$

is a quadratic form which is the Weyl symbol of the operator $id\sigma'(X)$.

On the other hand, by Corollary 8.2, we have, for each $t \in \mathbb{R}$ and each $(x, y) \in \mathbb{R}^{2n}$,

$$\begin{aligned} W_1(\exp(d\sigma'(tX)))(x, y) \\ = (\text{Det}(\cosh(\frac{1}{2}tX)))^{-1/2} \exp\left(-i \begin{pmatrix} x & y \end{pmatrix} J \tanh(\frac{1}{2}tX) \begin{pmatrix} x \\ y \end{pmatrix}\right). \end{aligned}$$

By the identity theorem for analytic functions, we can take $t = i$ in the preceding equality. This gives

$$\begin{aligned} W_1(\exp(d\sigma'(iX)))(x, y) \\ = (\text{Det}(\cos(\frac{1}{2}X)))^{-1/2} \exp\left(\begin{pmatrix} x & y \end{pmatrix} J \tan(\frac{1}{2}X) \begin{pmatrix} x \\ y \end{pmatrix}\right) \\ = (\text{Det}(\cos(JM)))^{-1/2} \exp\left(-\begin{pmatrix} x & y \end{pmatrix} J \tan(JM) \begin{pmatrix} x \\ y \end{pmatrix}\right). \end{aligned}$$

Then we recover a result of [27] about the Weyl symbol of the exponential of an operator whose Weyl symbol is a quadratic form $q_M(x, y)$.

8.2. Star exponentials

The preceding results can be reformulated in terms of star exponentials for the Moyal star product. Let us recall that the notion of star product was introduced in [4] in order to interpret quantum mechanics as deformation of classical mechanics. Roughly speaking, a star product on a Poisson manifold M is a formal deformation $(f, g) \rightarrow f * g = f \cdot g + t\{f, g\} + \sum_{l \geq 2} t^l C_l(f, g)$ of the pointwise multiplication of $C^\infty(M)$. An important problem is then the computation of the star exponential $\exp_*(f) = \sum_{l \geq 0} \frac{1}{l!} f^{*,l}$ for some functions f , see [2, 22]. Note that such computations are usually done by solving some differential systems, see [5, 6]. Here we only consider the Moyal star product on \mathbb{R}^{2n} defined as follows.

Take coordinates (p, q) on $\mathbb{R}^{2n} \cong \mathbb{R}^n \times \mathbb{R}^n$ and let $x = (p, q)$. Then one has $x_i = p_i$ for $1 \leq i \leq n$ and $x_i = q_{i-n}$ for $n+1 \leq i \leq 2n$. For $u, v \in C^\infty(\mathbb{R}^{2n})$, define $P^0(u, v) := uv$,

$$P^1(u, v) := \sum_{k=1}^n \left(\frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k} - \frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k} \right) = \sum_{1 \leq i, j \leq n} \Lambda^{ij} \partial_{x_i} u \partial_{x_j} v$$

(the Poisson brackets) and, more generally, for $l \geq 2$,

$$P^l(u, v) := \sum_{1 \leq i_1, \dots, i_l, j_1, \dots, j_l \leq n} \Lambda^{i_1 j_1} \Lambda^{i_2 j_2} \dots \Lambda^{i_l j_l} \partial_{x_{i_1} \dots x_{i_l}}^l u \partial_{x_{j_1} \dots x_{j_l}}^l v.$$

Then the Moyal product $*$ is the following formal deformation of the pointwise multiplication of $C^\infty(\mathbb{R}^{2n})$

$$u * v := \sum_{l \geq 0} \frac{t^l}{l!} P^l(u, v)$$

where t is a formal parameter.

Let us restrict $*$ to polynomials on \mathbb{R}^{2n} (this is sufficient for our purpose) and take $t = -i/2$. Then $*$ induces an associative product on the polynomials also denoted by $*$.

On the other hand, the classical Weyl correspondence \mathcal{W} (see Section 3) can be extended to polynomials [26]. More precisely, if $f(p, q) = u(p)q^\alpha$ where u is a polynomial on \mathbb{R}^n then we have

$$(\mathcal{W}(f)\varphi)(p) = \left(i \frac{\partial}{\partial s} \right)^\alpha \left(u(p + \frac{1}{2}s) \varphi(p + s) \right) \Big|_{s=0},$$

see [34]. Hence if f is a polynomial then $\mathcal{W}(f)$ is a differential operator with polynomial coefficients. Moreover, we can show that $*$ corresponds to the composition of operators in the Weyl quantization, that is, for each polynomials

f_1, f_2 on \mathbb{R}^{2n} , we have $\mathcal{W}(f_1 * f_2) = \mathcal{W}(f_1)\mathcal{W}(f_2)$. Equivalently, we also have $W_1(A_1) * W_1(A_2) = W_1(A_1 A_2)$ for each differential operators A_1, A_2 with polynomial coefficients.

Let M be a real, symmetric $(2n) \times (2n)$ matrix. Let q_M be the quadratic form on \mathbb{R}^{2n} associated with M . Let $X = -2JM \in sp(n, \mathbb{R})$. Then by Proposition 8.3 we have $W_1(d\sigma'(X)) = -iq_M$, hence

$$\exp_*(-iq_M) = \exp_*(W_1(d\sigma'(X))) = W_1(\exp(d\sigma'(X))) = W_1(\sigma'(\exp(X)))$$

and

$$\exp_*(-iq_M)(x, y) = (\text{Det}(\cosh(JM))^{-1/2} \exp \left(i \begin{pmatrix} x & y \end{pmatrix} J \tanh(JM) \begin{pmatrix} x \\ y \end{pmatrix} \right).$$

This is precisely the equation given in [6, Theorem 1]. In particular, for $q_M(x, y) = t(x^2 + y^2)$, $t \in \mathbb{R}$, we have $JM = tJ$, $\cosh(JM) = \cosh(t)I_{2n}$, $\tanh(JM) = \tanh(t)J$ and the preceding formula becomes

$$\exp_*(-it(x^2 + y^2)) = (\cosh(t))^{-n} \exp(-i \tanh(t)(x^2 + y^2)).$$

Up to normalization, we recover the formula of [5, Proposition 1].

Note that, in general, computations of star exponentials involve special functions, see [5, 22, 24].

Strangely enough, it seems that the connection between the computation of the star exponential of a quadratic form (for the Moyal product) and the computation of the Weyl symbol of the exponential of a differential operator whose Weyl symbol is a quadratic form has not been mentioned in the literature.

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