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Half-unknotted 2-orbifolds in orientable spherical 3-orbifolds

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Dedicated to Bruno Zimmermann on his 70th birthday

Abstract. *If an embedding of a 2-orbifold in an orientable spherical 3-orbifold splits the 3-orbifold into two parts such that at least one part is a handlebody orbifold, then we call it half-unknotted. We will give different kinds of algebraic conditions on the embedding such that it is half-unknotted. The results will be applied to questions about extendable actions on surfaces. As an example, we will show that embeddings realizing the maximum order of extendable cyclic actions on genus $g > 1$ surfaces must be unknotted.*

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1. Introduction

In this paper we will work in the piecewise linear category (or smooth category), namely all manifolds, orbifolds, maps will be piecewise linear (or smooth).

Let Σ_g denote the genus g orientable closed surface and V_g denote the genus g orientable handlebody. Then, Σ_0 is the two-dimensional sphere S^2, Σ_1 is the two-dimensional torus T^2, V_0 is the three-dimensional ball B^3, and V_1 is the solid torus. Let S^3 be the three-dimensional sphere. For an embedded Σ_g in S^3 the following result is well known. It is also called Alexander's theorem.

Theorem 1.1. *Every embedded Σ_g in S^3 splits S^3 into two parts. If $g = 0$, then each part is homeomorphic to V_0; if $g = 1$, then at least one part is homeomorphic to V_1; if $g \geq 2$, then it is possible that neither part is homeomorphic to V_g.*

For a given embedding $e : \Sigma_g \hookrightarrow S^3$, if the image of Σ_g splits S^3 into two handlebodies V_g, then we call e *unknotted*; if at least one part is homeomorphic to V_g, then we call e *half-unknotted*; otherwise, we call e *totally-knotted*. Then, Theorem 1.1 can be reformulated as: an embedding from Σ_g to S^3 must be unknotted if $g = 0$, must be half-unknotted if $g \leq 1$, and can be totally-knotted if $g \geq 2$.

REMARK 1.2: If e is unknotted, then it gives a Heegaard splitting of S^3. By a well known result of Waldhausen (see [7]), any two Heegaard splittings of S^3 of the same genus are isotopic. Hence, such e is essentially unique. Generally, if e is not unknotted, then it is called *knotted*.

The goal of this paper is to obtain a similar statement in the case of orbifolds. And the results will be applied to questions about extendable actions on surfaces. The theory of orbifolds has been developed by many authors (see [1, 2, 6]). And extendable actions on surfaces were defined and studied in [9].

The objects corresponding to Σ_g, V_g and S^3 will be orientable closed 2-orbifolds, orientable handlebody orbifolds and orientable spherical 3-orbifolds, which have the forms Σ_g/G, V_g/G and S^3/G, respectively. In each case, G is a finite group acting on the manifold, and the G-action is orientation-preserving.

In this paper, we always assume that: $\mathcal{F}$ is an orientable closed 2-orbifold; $\mathcal{O}$ is an orientable spherical 3-orbifold; p is the orbifold covering map from S^3 to $\mathcal{O}$; and $\hat{e}$ is an orbifold embedding from $\mathcal{F}$ to $\mathcal{O}$. We will identify $\mathcal{F}$ with $\hat{e}(\mathcal{F})$.

DEFINITION 1.3. *For an embedding $\hat{e} : \mathcal{F} \hookrightarrow \mathcal{O}$, suppose that $\mathcal{F}$ splits $\mathcal{O}$ into $\mathcal{O}_1$ and $\mathcal{O}_2$. If both $\mathcal{O}_1$ and $\mathcal{O}_2$ are handlebody orbifolds, then we call $\hat{e}$ unknotted; if at least one of $\mathcal{O}_1$ and $\mathcal{O}_2$ is a handlebody orbifold, then we call $\hat{e}$ half-unknotted; otherwise, we call $\hat{e}$ totally-knotted.*

It is known that every embedded $\mathcal{F}$ in $\mathcal{O}$ splits $\mathcal{O}$ into two parts (Lemma 2.1). Hence, we can always say if $\hat{e}$ is unknotted, or half-unknotted, or totally-knotted. Note that different from the manifold case, when $\hat{e}$ is unknotted, the two parts $\mathcal{O}_1$ and $\mathcal{O}_2$ may be non-homeomorphic (in the orbifold meaning).

The underlying space of $\mathcal{F}$ is always an orientable closed surface. Let $\hat{g}$ denote its genus. Let n denote the number of singular points contained in $\mathcal{F}$. Compared with Theorem 1.1, we have the following result.

THEOREM 1.4. *A π_1-surjective embedding from $\mathcal{F}$ to $\mathcal{O}$ must be unknotted if $\hat{g} = 0$, $n \leq 3$, must be half-unknotted if $\hat{g} = 0$, $n \leq 5$ or $\hat{g} = 1$, $n \leq 1$, and can be totally-knotted if $\hat{g} = 0$, $n \geq 6$, or $\hat{g} = 1$, $n \geq 2$, or $\hat{g} \geq 2$.*

In Theorem 1.4, "the embedding $\mathcal{F} \hookrightarrow \mathcal{O}$ is π_1-surjective" is equivalent to "the pre-image $p^{-1}(\mathcal{F})$ in S^3 is connected" (see Lemma 2.10 in [10]). Surely this should be the most interesting case. Clearly, if $\mathcal{F}$ is Σ_g and $\mathcal{O}$ is S^3, then $n = 0$, and Theorem 1.4 becomes Theorem 1.1.

It is known that every embedded $\mathcal{F}$ in $\mathcal{O}$ is compressible (Lemma 2.1). Hence $\mathcal{F}$ is compressible in $\mathcal{O}_1$ or $\mathcal{O}_2$. If $\mathcal{F}$ is compressible on each side, then Theorem 1.4 can be improved, and we have the following result.

THEOREM 1.5. *A π_1-surjective embedding from $\mathcal{F}$ to $\mathcal{O}$ such that $\mathcal{F}$ is compressible on each side must be unknotted if $\hat{g} = 0$, $n = 4$ or $\hat{g} = 1$, $n \leq 1$, and must be half-unknotted if $\hat{g} = 1$, $n = 2$.*

Theorem 1.4 and 1.5 can be naturally related to extendable actions on surfaces. For an embedded Σ_g in S^3, a G-action on Σ_g is called *extendable* if the group G can also act on S^3 leaving Σ_g invariant and its restriction on Σ_g is the given action. In [10], the case in Theorem 1.4 when $\hat{g} = 0$, $n \leq 4$ is given. It plays a central role in the classification of orientation-preserving extendable finite group actions on Σ_g with order $|G| > 4g - 4$. By Theorem 1.4, it is hopeful to classify such actions, or at least to get all the relations between $|G|$ and g, when $|G| > 2g - 2$.

By [8], for general extendable finite group actions on Σ_g, where elements in the group may reverse the orientation of Σ_g or S^3, if the order reaches the maximum for a given g, then the corresponding embedding must be unknotted. By [11], the maximum order of extendable finite cyclic group actions on Σ_g is $4g + 4$ when g is even, and $4g - 4$ when g is odd. And an action can realize the maximum order only when its generator reverses the orientation of Σ_g and preserves the orientation of S^3. By combining Theorem 1.5 with this result, we have the following result.

THEOREM 1.6. *Given $g > 1$, if an extendable cyclic group action on Σ_g has order reaching the maximum, then its corresponding embedding must be unknotted.*

Philosophically, the above results mean that the most symmetric surfaces in our space should be topologically simple. Note that for orientation-preserving actions of arbitrary finite groups this is not always the case; in fact, by [9, 10], if $g = 21$ or $g = 481$, then the maximum order in the orientation-preserving case is reached only for knotted embeddings.

We will prove Theorem 1.4, 1.5 and 1.6 in section 2. In section 3, we will give various examples of π_1-surjective embeddings, which are totally-knotted, as supplements to the theorems.

2. Conditions on half-unknotted embeddings

In this section, we give several conditions on the embedded $\mathcal{F}$ in $\mathcal{O}$ which imply that $\mathcal{F}$ is half-unknotted. The underlying space of $\mathcal{F}$ and $\mathcal{O}$ will be denoted by $|\mathcal{F}|$ and $|\mathcal{O}|$, respectively. For results about discal 3-orbifolds, spherical 3-orbifolds and handlebody orbifolds, one can see [1, 2, 3, 5], as well as [10, 11]. Part of the following results can also be found in these literatures.

LEMMA 2.1. *$\mathcal{F}$ splits $\mathcal{O}$ into two parts and it is compressible in one of them.*

Proof. Since $\mathcal{F}$ and $\mathcal{O}$ are orientable, $|\mathcal{F}|$ is two sided in $|\mathcal{O}|$. Because $\pi_1(\mathcal{O})$ is a finite group, $\pi_1(|\mathcal{O}|)$ is also finite. If $\mathcal{F}$ does not split $\mathcal{O}$, then there exists a simple closed curve C in $|\mathcal{O}|$ such that $C \cap |\mathcal{F}|$ is exactly 1 point. Then there exists a map $f : |\mathcal{O}| \to S^1$ such that $f_* : \pi_1(|\mathcal{O}|) \to \pi_1(S^1)$ is surjective. This is a contradiction. Hence, $\mathcal{F}$ must split $\mathcal{O}$ into two parts.

Suppose that $\mathcal{F}$ splits $\mathcal{O}$ into two 3-orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$. Then $p^{-1}(\mathcal{F})$ divides S^3 into several components $M_1, M_2, \cdots, M_m$. Each $p(M_i)$ will be either $\mathcal{O}_1$ or $\mathcal{O}_2$. And if $\partial M_i \cap \partial M_j \neq \emptyset$ and $M_i \neq M_j$, then $p(M_i) \neq p(M_j)$.

If $\mathcal{F}$ is a spherical 2-orbifold, then $p^{-1}(\mathcal{F})$ is a disjoint union of 2-spheres. By the irreducibility of S^3 and B^3, there exists a M_i such that $M_i \cong B^3$. Then one of $\mathcal{O}_1$ and $\mathcal{O}_2$ is a discal 3-orbifold. Hence, $\mathcal{F}$ is compressible in $\mathcal{O}_1$ or $\mathcal{O}_2$.

If $\mathcal{F}$ is not spherical, then $F = p^{-1}(\mathcal{F})$ is a disjoint union of homeomorphic closed surfaces of genus $g \geq 1$ in S^3. Let $F_1, F_2, \cdots, F_n$ be the components of F. Since F_1 is compressible in S^3, there exists a compression disk D_1 of F_1 such that D_1 intersects F transversely. Then, $D_1 \cap F$ consists of some circles. Assume that C_1 is an innermost circle in D_1 and $C_1 \subset F_i$. It bounds a disk D'_1 in D_1. If D'_1 is not a compression disk of F_i, then C_1 bounds a disk D' in F_i. Then, $D_1 \cap D'$ and C_1 can be removed by surgeries such that D_1 becomes a compression disk D_2 of F_1, where $D_2 \cap F$ has less components than $D_1 \cap F$.

Hence, there exists a compression disk D of some F_j such that $D \cap F = \partial D$, by induction. Suppose that $D \subset M_i$. Since $\pi_1(\mathcal{O})$ acts on S^3 and preserves F, by the equivariant Dehn's Lemma (see [4]), there is an equivariant compression disk in M_i, whose orientation is preserved by the action. Then, the image of the disk in $\mathcal{O}$ is a compression disk of $\mathcal{F}$. Hence, $\mathcal{F}$ is compressible in $\mathcal{O}_1$ or $\mathcal{O}_2$. □

LEMMA 2.2. *If* $|\mathcal{F}| \cong S^2$ *and* $\mathcal{F}$ *contains not more than* 3 *singular points, then* $\mathcal{F}$ *is spherical and it bounds a discal 3-orbifold in* $\mathcal{O}$.

Proof. Since $|\mathcal{F}| \cong S^2$ and $\mathcal{F}$ has not more than 3 singular points, every simple closed curve in $\mathcal{F}$ bounds a discal 2-orbifold in $\mathcal{F}$. Hence, $\mathcal{F}$ has no compression disk. By Lemma 2.1, $\mathcal{F}$ is spherical and it bounds a discal 3-orbifold in $\mathcal{O}$. □

LEMMA 2.3. *Let* D *be a discal 2-orbifold in* $\mathcal{O}$ *such that* $D \cap \mathcal{F} = \partial D$ *and* ∂D *cuts* $\mathcal{F}$ *into* F_1 *and* F_2. *If* $p^{-1}(\mathcal{F})$ *is connected,* $|F_2| \cong B^2$, *and* $F_2 \cup D$ *has not more than* 3 *singular points, then* $F_2 \cup D$ *bounds a discal 3-orbifold* B *with* $B \cap F_1 = \partial D$.

Proof. By Lemma 2.2, $F_2 \cup D$ bounds a discal 3-orbifold B. Then, $B \cap F_1 = \partial D$ or $F_1 \subset B$. If $F_1 \subset B$, then $\mathcal{F} \subset B$. Since $p^{-1}(\mathcal{F})$ is connected, $p^{-1}(B)$ is connected. Hence, $p^{-1}(B)$ is a 3-ball, and $\overline{S^3 - p^{-1}(B)}$ is also a 3-ball. Then, $B' = \overline{\mathcal{O} - B}$ is a discal 3-orbifold bounded by $F_2 \cup D$, and $B' \cap F_1 = \partial D$. □

LEMMA 2.4. *If* $p^{-1}(\mathcal{F})$ *is connected,* $|\mathcal{F}| \cong S^2$, *and* $\mathcal{F}$ *has precisely* 4 *singular points, then* $\mathcal{F}$ *bounds a handlebody orbifold in* $\mathcal{O}$.

Proof. Since $\mathcal{F}$ has precisely 4 singular points, $\mathcal{F}$ is not spherical. By Lemma 2.1, $\mathcal{F}$ has a compression disk D in $\mathcal{O}$. Because $|\mathcal{F}| \cong S^2$, ∂D cuts $\mathcal{F}$ into D_1 and D_2, where $|D_1| \cong |D_2| \cong B^2$ and each of D_1 and D_2 contains 2 singular points. Since $p^{-1}(\mathcal{F})$ is connected, by Lemma 2.3, $D_i \cup D$ bounds a discal 3-orbifold B_i in $\mathcal{O}$ with $B_i \cap D_j = \partial D$, where $i, j = 1, 2$ and $i \neq j$. Then, $B_1 \cap B_2 = D$, and $B_1 \cup B_2$ is a handlebody orbifold bounded by $\mathcal{F}$. □

Lemma 2.5. *If $|\mathcal{F}| \cong S^2$, $\mathcal{F}$ has precisely 4 singular points and has a compression disk D with 1 singular point, then $\mathcal{F}$ bounds a handlebody orbifold in $\mathcal{O}$.*

Proof. Since $|\mathcal{F}| \cong S^2$, ∂D cuts $\mathcal{F}$ into D_1 and D_2, and $|D_1 \cup D| \cong |D_2 \cup D| \cong S^2$. Because $\mathcal{F}$ has precisely 4 singular points and D contains 1 singular point, each of $D_1 \cup D$ and $D_2 \cup D$ has precisely 3 singular points. By Lemma 2.2, $D_i \cup D$ bounds a discal 3-orbifold B_i in $\mathcal{O}$, $i = 1, 2$. If $B_1 \cap B_2 = D$, then $B_1 \cup B_2$ is a handlebody orbifold bounded by $\mathcal{F}$; otherwise, $D_2 \subset B_1$ or $D_1 \subset B_2$.

If $D_2 \subset B_1$, then let Υ be the singular set of B_1. We can assume that

$$\begin{aligned}
|B_1| &= \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\},\\
|D_1| &= \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq 0\},\\
|D| &= \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \leq 0\},\\
\Upsilon &= \{(0, t, |t|) \mid |t| \leq \sqrt{2}/2\} \cup \{(0, 0, t) \mid -1 \leq t \leq 0\}.
\end{aligned}$$

Suppose that D_2 intersects the yz-plane transversely, then the intersection consists of an arc A from $(0, -1, 0)$ to $(0, 1, 0)$ and some circles. Since $D_2 \cap \Upsilon$ consists of 2 points and $A \cap \Upsilon \neq \emptyset$, any circle cannot intersect Υ. Hence, circles can be removed by the irreducibility of the 3-ball, and D_2 is isotopic to D_1. Hence, $\mathcal{F} = D_1 \cup D_2$ bounds a handlebody orbifold in B_1. If $D_1 \subset B_2$, the proof is similar. □

Lemma 2.6. *If $p^{-1}(\mathcal{F})$ is connected, $|\mathcal{F}| \cong S^2$, and $\mathcal{F}$ has precisely 5 singular points, then $\mathcal{F}$ bounds a handlebody orbifold in $\mathcal{O}$.*

Proof. Since $\mathcal{F}$ has precisely 5 singular points, $\mathcal{F}$ is not spherical. By Lemma 2.1, $\mathcal{F}$ has a compression disk D in $\mathcal{O}$. Because $|\mathcal{F}| \cong S^2$, ∂D cuts $\mathcal{F}$ into D_1 and D_2, where $|D_1| \cong |D_2| \cong B^2$. Suppose that D_1 contains 2 singular points, D_2 contains 3 singular points. If D does not contain singular points, then by the same proof of Lemma 2.4, $\mathcal{F}$ bounds a handlebody orbifold; otherwise, $D_2 \cup D$ is not spherical, and by Lemma 2.1, it has a compression disk D' in $\mathcal{O}$. In what follows, we assume that every compression disk of $\mathcal{F}$ contains a singular point.

We can assume that $\partial D' \subset D_2$ and D' intersects $\mathcal{F}$ transversely. Then, $D' \cap \mathcal{F}$ consists of some circles. If $D' \cap D_1 \neq \emptyset$, then there exists an innermost circle C_1 in D_1. If C_1 bounds a discal 2-orbifold D_1' in D_1, then by Lemma 2.2, D_1' and the discal 2-orbifold in D' bounded by C_1 have the same number of singular points. So C_1 can be removed by surgeries. Hence, we can assume that all circles in $D' \cap D_1$ are parallel to $\partial D_1 = \partial D$. Then, in each case of $D' \cap D_1 = \emptyset$ and $D' \cap D_1 \neq \emptyset$, we have that D' contains a singular point.

Then, by Lemma 2.5, $D_2 \cup D$ bounds a handlebody orbifold H. Because $p^{-1}(\mathcal{F})$ is connected and $D_1 \cup D$ has 3 singular points, by Lemma 2.3, $D_1 \cup D$ bounds a discal 3-orbifold B with $B \cap D_2 = \partial D$. Either $B \cap H = D$ or $B \subset H$. If $B \subset H$, by a similar argument as Lemma 2.5, $\overline{H - B}$ is a handlebody orbifold bounded by $\mathcal{F}$; otherwise, $B \cup H$ is a handlebody orbifold bounded by $\mathcal{F}$. □

Lemma 2.7. *If $p^{-1}(\mathcal{F})$ is connected, $|\mathcal{F}| \cong T^2$, and $\mathcal{F}$ contains at most* 1 *singular point, then $\mathcal{F}$ bounds a handlebody orbifold in $\mathcal{O}$.*

Proof. Because $|\mathcal{F}| \cong T^2$, $\mathcal{F}$ is not spherical. By Lemma 2.1, $\mathcal{F}$ has a compression disk D in $\mathcal{O}$. Since $\mathcal{F}$ has at most 1 singular point, ∂D is an essential simple closed curve in $|\mathcal{F}| \cong T^2$. Let D' be a compression disk of $\mathcal{F}$ which is parallel to D, then $\partial D \cup \partial D'$ cuts $\mathcal{F}$ into two parts, denoted by A_1 and A_2. Suppose that $A_1 \cup D \cup D'$ bounds the discal 3-orbifold $B_1 \cong D \times I$. Then, $A_2 \cup D \cup D'$ has not more than 3 singular points and $|A_2 \cup D \cup D'| \cong S^2$. Hence, by Lemma 2.2, $A_2 \cup D \cup D'$ bounds a discal 3-orbifold B_2. Either $B_1 \cap B_2 = D \cup D'$ or $B_1 \subset B_2$.

If $B_1 \cap B_2 = D \cup D'$, then $B_1 \cup B_2$ is a handlebody orbifold bounded by $\mathcal{F}$.

If $B_1 \subset B_2$, then $\mathcal{F} \subset B_2$. Because $p^{-1}(\mathcal{F})$ is connected, $p^{-1}(B_2)$ is connected. Hence, $p^{-1}(B_2)$ is a 3-ball. So $\overline{S^3 - p^{-1}(B_2)}$ is also a 3-ball and $B_2' = \overline{\mathcal{O} - B_2}$ is a discal 3-orbifold. Then, $B_1 \cup B_2'$ is a handlebody orbifold bounded by $\mathcal{F}$. □

Lemma 2.8. *If $|\mathcal{F}| \cong T^2$, $\mathcal{F}$ has at most* 1 *singular point and has a compression disk D with* 1 *singular point, then $\mathcal{F}$ bounds a handlebody orbifold in $\mathcal{O}$.*

Proof. Let D, D', A_1, A_2, B_1, B_2 be as in the proof of Lemma 2.7. We only need to show that if $B_1 \subset B_2$, then $\mathcal{F}$ also bounds a handlebody orbifold. Since B_1 is a regular neighbourhood of a singular arc and B_2 is a discal 3-orbifold, if $B_1 \subset B_2$, then the singular set of B_2 must be the singular arc in B_1, and $\overline{B_2 - B_1}$ is a solid torus, which is bounded by $\mathcal{F}$. □

Lemma 2.9. *If $p^{-1}(\mathcal{F})$ is connected, $|\mathcal{F}| \cong S^2$, $\mathcal{F}$ has precisely* 4 *singular points and is compressible on each side, then $\mathcal{F}$ bounds a handlebody orbifold on each side.*

Proof. By Lemma 2.4, we can assume that $\mathcal{F}$ bounds a handlebody orbifold $\mathcal{O}_1$ in $\mathcal{O}$. Let $\mathcal{O}_2$ denote the other side of $\mathcal{F}$, then $\mathcal{F}$ has a compression disk D in $\mathcal{O}_2$, by the assumption. Then, by the same proof of Lemma 2.4 we have discal 3-orbifolds B_1 and B_2 in $\mathcal{O}_2$. Hence $\mathcal{O}_2 = B_1 \cup B_2$ is also a handlebody orbifold. □

LEMMA 2.10. *If $p^{-1}(\mathcal{F})$ is connected, $|\mathcal{F}| \cong T^2$, $\mathcal{F}$ has at most 1 singular point and is compressible on each side, then $\mathcal{F}$ bounds a handlebody orbifold on each side.*

Proof. By Lemma 2.7, we can assume that $\mathcal{F}$ bounds a handlebody orbifold $\mathcal{O}_1$ in $\mathcal{O}$. Let $\mathcal{O}_2$ denote the other side of $\mathcal{F}$, then $\mathcal{F}$ has a compression disk D in $\mathcal{O}_2$, by the assumption. Then, by the proof of Lemma 2.7 there exist discal 3-orbifolds B_1 and B_2 in $\mathcal{O}_2$ such that $\mathcal{O}_2 = B_1 \cup B_2$ is a handlebody orbifold. □

LEMMA 2.11. *If $p^{-1}(\mathcal{F})$ is connected, $|\mathcal{F}| \cong T^2$, $\mathcal{F}$ has precisely 2 singular points and is compressible on each side, then $\mathcal{F}$ bounds a handlebody orbifold in $\mathcal{O}$.*

Proof. Because $|\mathcal{F}| \cong T^2$, $\mathcal{F}$ is not spherical. Hence, $\mathcal{F}$ has a compression disk on each side. In what follows, we divide the proof into two cases.

Case 1: There exists a compression disk D of $\mathcal{F}$ such that ∂D is an essential simple closed curve in $|\mathcal{F}| \cong T^2$. We can further assume that all such D contains a singular point, for if D does not have singular points, then by an argument similar to Lemma 2.7, there will be a handlebody orbifold bounded by $\mathcal{F}$.

Let D', A_1 and A_2 be as in the proof of Lemma 2.7. Suppose that $A_1 \cup D \cup D'$ bounds the discal 3-orbifold $B \cong D \times I$. Then, $A_2 \cup D \cup D'$ has 4 singular points and $|A_2 \cup D \cup D'| \cong S^2$. Hence, $A_2 \cup D \cup D'$ is not spherical, and by Lemma 2.1, it has a compression disk D_1.

If D_1 contains a singular point, then by Lemma 2.5, $A_2 \cup D \cup D'$ will bound a handlebody orbifold H. Either $B \cap H = D \cup D'$ or $B \subset H$. If $B \cap H = D \cup D'$, then $B \cup H$ is a handlebody orbifold bounded by $\mathcal{F}$. If $B \subset H$, then the singular set of H consists of two singular arcs, and B is the regular neighbourhood of one singular arc. Hence, $\overline{H - B}$ is a handlebody orbifold bounded by $\mathcal{F}$.

Otherwise, D_1 does not contain singular points. We can assume that $\partial D_1 \subset A_2$ and D_1 intersects $\mathcal{F}$ transversely. If $D_1 \cap A_1 \neq \emptyset$, then by Lemma 2.2, there exists a circle in it bounding a disk in A_1. All such circles can be removed by surgeries. Then, D_1 becomes a compression disk D_1' of $\mathcal{F}$. Since D_1' does not contain singular points, $\partial D_1'$ is trivial in $|\mathcal{F}| \simeq T^2$. Note that $D_1' \cap D = \emptyset$.

Assume that $\partial D_1'$ cuts $\mathcal{F}$ into S and T such that $|S \cup D_1'| \cong S^2$, where S has 2 singular points, and $T \cup D_1'$ is a T^2. Since $p^{-1}(\mathcal{F})$ is connected, by Lemma 2.3, $S \cup D_1'$ bounds a discal 3-orbifold B_1 such that $B_1 \cap T = \partial D_1'$. Since D contains 1 singular point, by Lemma 2.8, $T \cup D_1'$ bounds a handlebody orbifold H_1, which is a solid torus or a regular neighbourhood of a singular circle.

Either $B_1 \cap H_1 = D_1'$ or $B_1 \subset H_1$. If $B_1 \cap H_1 = D_1'$, then $B_1 \cup H_1$ is a handlebody orbifold bounded by $\mathcal{F}$. Otherwise, $\mathcal{F} \subset H_1$ and H_1 is a regular neighbourhood of a singular circle. Since $p^{-1}(\mathcal{F})$ is connected, $p^{-1}(H_1)$ is connected. Hence, $p^{-1}(H_1)$ is a solid torus. The pre-image of the singular circle in H_1 is a knot in S^3. By the positive solution of the Smith Conjecture, it must be trivial. Hence, $\overline{S^3 - p^{-1}(H_1)}$ is also a solid torus, and $H_1' = \overline{\mathcal{O} - H_1}$ is a handlebody orbifold. Then, $B_1 \cup H_1'$ is a handlebody orbifold bounded by $\mathcal{F}$.

Case 2: For any compression disk D of $\mathcal{F}$, ∂D is trivial in $|\mathcal{F}| \cong T^2$.

Let D be a compression disk of $\mathcal{F}$. Assume that ∂D cuts $\mathcal{F}$ into S and T such that $|S \cup D| \cong S^2$, where S has 2 singular points, and $|T \cup D| \cong T^2$. Since $p^{-1}(\mathcal{F})$ is connected, by Lemma 2.3, $S \cup D$ bounds a discal 3-orbifold B with $B \cap T = \partial D$. Since $T \cup D$ is not spherical, by Lemma 2.1, it has a compression disk D_1.

We can assume that $\partial D_1 \subseteq T$ and D_1 intersects $\mathcal{F}$ transversely. If $D_1 \cap S = \emptyset$, then D_1 is a compression disk of $\mathcal{F}$ such that ∂D_1 is essential in $|\mathcal{F}| \cong T^2$, which is a contradiction. Hence, $D_1 \cap S \neq \emptyset$. Then, by a similar argument as in the proof of Lemma 2.6, we can assume that all circles in $D_1 \cap S$ are parallel to $\partial S = \partial D$. By surgeries, we can further assume that all circles in $D_1 \cap S$ bound disjoint discal 2-orbifolds in D_1, which are all parallel to D in B.

If D_1 contains a singular point, then by Lemma 2.8, $T \cup D$ bounds a handlebody orbifold H. Since the compression disk of $T \cup D$ in H has a nontrivial boundary, it must intersect S. Hence, $B \subset H$ and $D_1 \subset H$. If D contains a singular point, by the proof of Lemma 2.7 and 2.8, H contains a singular vertex of degree 3, and D_1 will contain at least 2 singular points; otherwise, H is a regular neighbourhood of a singular circle, and by previous arguments, $\overline{\mathcal{O} - H}$ will be a handlebody orbifold which does not contain B. In each case we get a contradiction.

Hence, D_1 does not contain singular points. Then, D does not contain singular points. By above arguments, $T \cap D$ cannot bound handlebody orbifolds. Suppose that $\mathcal{F}$ cuts $\mathcal{O}$ into $\mathcal{O}_1$ and $\mathcal{O}_2$, and $B \subset \mathcal{O}_1$. Then, $D \subset \mathcal{O}_1$ and $D_1 \subset \mathcal{O}_2 \cup B$. By the assumption, there exists another compression disk D' of $\mathcal{F}$ in $\mathcal{O}_2$. Then, $\partial D'$ is trivial in $|\mathcal{F}| \cong T^2$, and D' does not contain singular points.

Claim: There exists a compression disk D_1' of $T \cup D$ such that $\partial D_1' \subseteq T$ and D_1' intersects $\mathcal{F}$ transversely, but $D_1' \cap S$ has less components than $D_1 \cap S$.

The claim gives a contradiction and finishes the proof. We prove it as below.

Since $\partial D'$ bounds a disk in $|\mathcal{F}|$, which contains the 2 singular points, it can be thought as the boundary of a regular neighborhood of an arc joining the 2 points. The circles in $D_1 \cap \mathcal{F}$ other than ∂D_1 are all parallel to ∂D in S. Hence, we can assume that D' intersects D_1 transversely in $\mathcal{O}_2$ such that any bi-gon bounded by $\partial D' \cup (D_1 \cap \mathcal{F})$ in $|\mathcal{F}|$ contains 1 singular point. We can also assume that $D' \cap D_1$ consists of arcs. Otherwise, an innermost circle in D' can be removed by surgeries, and D_1 will become a disk D_2 with $\partial D_2 = \partial D_1$

and $D_2 \cap S \subseteq D_1 \cap S$.

If $D' \cap D_1 = \emptyset$, then we can assume that $\partial D' = \partial D$. For $p^{-1}(\mathcal{F})$ is connected, by Lemma 2.3, $S \cup D'$ bounds a discal 3-orbifold B' with $B' \cap T = \partial D'$. Hence, $B' \subset \mathcal{O}_2$ and $|B' \cup B| \cong B^3$, which contains a singular circle. Then, by Lemma 2.2, $\overline{\mathcal{O} - B' \cup B}$ is a B^3. Since $D_1 \cap (D \cup D') = \emptyset$, we have $D_1 \cap S = \emptyset$.

If $D' \cap D_1 \neq \emptyset$, then consider an outermost arc A_0 of $D' \cap D_1$ in D'. Let A' be an arc in $\partial D'$ such that $\partial A' = \partial A_0 = A' \cap D_1$. Then, $A' \cup A_0$ bounds a disk D'_0 in D'. Note that $D_1 \cap \mathcal{O}_2$ is a punctured disk, where $D_1 \cap S$ gives the punctures and A_0 is a proper arc. Let C_1 and C_2 be the innermost and outermost circle of $D_1 \cap S$ in S, respectively. According to the position of A' in $\mathcal{F}$, there are several cases:

Case (A): The two points in $\partial A'$ belong to the same circle in $D_1 \cap \mathcal{F}$.

(A1): $\partial A' \subset \partial D_1$. Then, we can assume that $A' \subset T$. Let A be an arc in ∂D_1 such that $\partial A = \partial A'$. Then, $A \cup A_0$ bounds a disk D_0 in D_1, and $A \cup A'$ bounds the disk $D_0 \cup D'_0$. The circle $A \cup A'$ is essential in $|\mathcal{F}|$. Otherwise, by cutting $|\mathcal{F}|$ along ∂D_1, we will find a bi-gon without singular points. Hence, $D_0 \cup D'_0$ gives a compression disk D'_1 of $T \cup D$ such that $D'_1 \cap S \subseteq D_1 \cap S$. Note that there are two choices of A. For one of them we will have $D'_1 \cap S \subset D_1 \cap S$.

(A2): $\partial A' \subset C_1$ and $A' \subset S$. Let A be an arc in C_1 such that $\partial A = \partial A'$. Then, $A \cup A_0$ bounds a disk D_0 in D_1, and $A \cup A'$ bounds the disk $D_0 \cup D'_0$. But on the other hand, $A \cup A'$ bounds a discal 2-orbifold in S, which has precisely 1 singular point. By Lemma 2.2, this is impossible in a spherical 3-orbifold.

(A3): $\partial A' \subset C_2$ and $A' \cap \partial D \neq \emptyset$. There exists an arc A in C_2 such that $A \cup A_0$ bounds a disk D_0 in D_1, and D_0 does not contain the disk in D_1 bounded by C_2. Then, the circle $A \cup A'$ bounds the disk $D_0 \cup D'_0$, and it must be essential in $|\mathcal{F}|$. Since C_2 is parallel to ∂D, we can move $A \cup A'$ into T. Then, $D_0 \cup D'_0$ becomes a compression disk D'_1 of $T \cup D$, and we have $D'_1 \cap S \subset D_1 \cap S$.

Since $\partial A'$ can not belong to other circles in $D_1 \cap \mathcal{F}$. Case (A) is finished.

Case (B): The two points of $\partial A'$ belong to different circles in $D_1 \cap \mathcal{F}$.

(B1): The two points belong to different circles in $D_1 \cap S$. Since $A' \cup A_0$ bounds the disk D'_0, the arc A_0 can be moved to A' along D'_0. Then, it can be moved into B, and the components of $D_1 \cap S$ can be reduced by surgeries.

(B2): The two points belong to ∂D_1 and C_2, respectively. Since $A' \cup A_0$ bounds the disk D'_0, the arc A_0 can be moved to A' along D'_0. Then, it can be moved into $\mathcal{O}_1$. After the movement, $D_1 \cap \mathcal{O}_1$ will be the union of the disks in $D_1 \cap B$ and a regular neighborhood of A_0. Remove this neighborhood and the disk bounded by C_2 from D_1. Then, we can get a disk D'_1. Since C_2 is parallel to ∂D, we can move $\partial D'_1$ into T. Then, D'_1 is a compression disk of $T \cup D$ and $D'_1 \cap S \subset D_1 \cap S$.

Then, Case (B) is finished. And we have finished the proof of Lemma 2.11. □

Proof of Theorems 1.4 and 1.5. The unknotted and half-unknotted parts of Theorem 1.4 are consequences of Lemma 2.2, 2.4, 2.6, 2.7. The totally-knotted part of Theorem 1.4 can be obtained from the examples in section 3. Theorem 1.5 is a consequence of Lemma 2.9, 2.10 and 2.11. □

Corollary 2.12. *Given $g > 1$, if an extendable G-action on Σ_g preserves both the orientations of Σ_g and S^3 and $|G| > 2g-2$, then Σ_g bounds a handlebody in S^3.*

Proof. The extendable G-action corresponds to an orbifold pair $\mathcal{F} \subset \mathcal{O}$, where $\mathcal{F}$ is Σ_g/G and $\mathcal{O}$ is S^3/G. Assume that the singular points of $\mathcal{F}$ have indices $q_1, \cdots, q_n$. Then, by the Riemann-Hurwitz formula, the Euler characteristic of $\mathcal{F}$ satisfies

$$\chi(\mathcal{F}) = 2 - 2\hat{g} - \sum_{i=1}^{n}(1 - \frac{1}{q_i}) = \frac{2-2g}{|G|} > -1.$$

Hence, $\hat{g} = 0$, $n < 6$, or $\hat{g} = 1$, $n = 1$. Since $\mathcal{O}$ is spherical, by Lemma 2.2, $n > 3$ when $\hat{g} = 0$. Hence, the only possible solutions of $(\hat{g}, n)$ are $(0,4)$, $(0,5)$ and $(1,1)$. Then, by Theorem 1.4, $\mathcal{F}$ bounds a handlebody orbifold in $\mathcal{O}$. Hence, Σ_g bounds a handlebody in S^3. □

Proof of Theorem 1.6. By results in [11], the maximum order of extendable cyclic group actions on Σ_g is $4g+4$ when g is even, and $4g-4$ when g is odd. Moreover, a generator of the group action that realizes the maximum order must reverse the orientation of Σ_g and preserve the orientation of S^3. So it exchanges the two sides of Σ_g. Suppose that h is such a generator.

Let G be the group generated by h^2. Then, the G-action on Σ_g is extendable and it preserves both the orientations of Σ_g and S^3. When g is even, $|G| = 2g+2$. Then, by Corollary 2.12, Σ_g bounds a handlebody in S^3. Since h exchanges the two sides of Σ_g, the embedding of Σ_g is unknotted.

If g is odd, then $|G| = 2g-2$. Let $\mathcal{F} \subset \mathcal{O}$ be the orbifold pair corresponding to G. By the Riemann-Hurwitz formula, $\chi(\mathcal{F}) = -1$. Then, $\hat{g} = 0$, $n \leq 6$, or $\hat{g} = 1$, $n = 2$. By Lemma 2.2, $n > 3$ when $\hat{g} = 0$. Since G is cyclic, the singular set of $\mathcal{O}$ consists of circles. Hence, the singular points in $\mathcal{F}$ can be paired and n is even. Then, the only possible solutions of $(\hat{g}, n)$ are $(0,4)$, $(0,6)$ and $(1,2)$.

If $(\hat{g}, n)$ is $(0,6)$, then all the singular points have index 2, and the action is an involution on Σ_2. But g is odd, so $(\hat{g}, n)$ is $(0,4)$ or $(1,2)$. Since h exchanges the two sides of Σ_g, $\mathcal{F}$ is compressible on each side. Then, by Theorem 1.5, $\mathcal{F}$ bounds a handlebody orbifold in $\mathcal{O}$, and Σ_g bounds a handlebody in S^3. Since h exchanges the two sides of Σ_g, the embedding of Σ_g is unknotted. □

3. Examples of totally-knotted embeddings

In this section, we give several examples of the orbifold pairs $\mathcal{F} \subset \mathcal{O}$, where the embedding of $\mathcal{F}$ in $\mathcal{O}$ is π_1-surjective and totally-knotted. By these examples, we can finish the proof of Theorem 1.4.

Example 3.1: There exists a totally-knotted π_1-surjective embedding $\mathcal{F} \subset \mathcal{O}$ such that $(\hat{g}, n)$ is $(0, 6)$. Figure 1 shows an embedded Σ_2 in $S^3 = \mathbb{R}^3 \cup \{\infty\}$, obtained as the boundary of a closed 3-ball to which a knotted 1-handle is added, and from whose interior a regular neighbourhood of a knotted arc is removed. The dashed line indicates the axis of a π-rotation τ, which keeps the Σ_2 invariant.

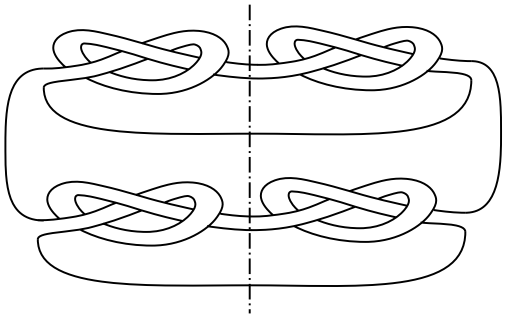

Figure 1: An involution of Σ_2 with 6 fixed points

Let $\mathcal{F} = \Sigma_2/\tau$ and $\mathcal{O} = S^3/\tau$, then $|\mathcal{F}| \cong S^2$ and $|\mathcal{O}| \cong S^3$. The singular set of $\mathcal{O}$ is a circle of index 2. It intersects $\mathcal{F}$ at 6 points. Because each side of Σ_2 can be obtained from the closed complement of a nontrivial knot by adding one handle, it cannot be a handlebody. Hence, $\mathcal{F}$ is totally-knotted in $\mathcal{O}$.

Example 3.2: There exists a totally-knotted π_1-surjective embedding $\mathcal{F} \subset \mathcal{O}$ such that $(\hat{g}, n)$ is $(1, 2)$. Figure 2 gives an embedded Σ_2 in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. The dashed line indicates the axis of a π-rotation τ, which keeps the Σ_2 invariant.

Let $\mathcal{F} = \Sigma_2/\tau$ and $\mathcal{O} = S^3/\tau$, then $|\mathcal{F}| \cong T^2$ and $|\mathcal{O}| \cong S^3$. The singular set of $\mathcal{O}$ is a circle of index 2. It intersects $\mathcal{F}$ at 2 points. One side of Σ_2 is a boundary connected sum of two copies of the closed trefoil knot complement. The other side of Σ_2 is the closed complement of the graph shown in Figure 3.

Let Γ denote this graph. Let M denote the closed complement of Γ in S^3. The fundamental group of M has the following presentation

$$\pi_1(M) = \langle x, y, z \mid y^{-1}xyx^{-1}z^{-1}xz \rangle,$$

where, up to conjugation, x is a meridian of the arc of Γ while y and z are meridians of the two circles of Γ, respectively. The map $x \mapsto (1, 2, 3)$, $y \mapsto (1, 2)$,

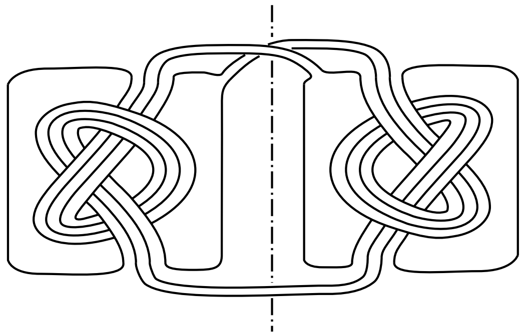

Figure 2: An involution of Σ_2 with 2 fixed points

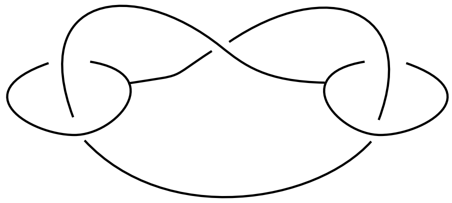

Figure 3: A knotted graph in S^3

$z \mapsto (1,2)$ gives an epimorphism from $\pi_1(M)$ to the permutation group $\mathbb{S}_3$. So x is nontrivial in $\pi_1(M)$. Then, the epimorphism from $\pi_1(M)$ to $\mathbb{Z}*\mathbb{Z}$ mapping x to the identity has a nontrivial kernel. Because the free group $\mathbb{Z}*\mathbb{Z}$ is hopfian, $\pi_1(M)$ is not isomorphic to $\mathbb{Z}*\mathbb{Z}$. Hence, M is not a handlebody, and $\mathcal{F}$ is totally-knotted in $\mathcal{O}$.

By Lemma 2.11, $\mathcal{F}$ cannot be compressible on each side. Hence, $\mathcal{F}$ is incompressible in M/τ. By the equivariant Dehn's Lemma, Σ_2 is incompressible in M. Up to isotopy, we see that Σ_2 has only one compression disk in S^3, which is the disk of the boundary connected sum. It gives the unique compression disk of $\mathcal{F}$, whose boundary is trivial in $|\mathcal{F}|$.

Example 3.3: The above two examples can be modified to give totally-knotted π_1-surjective embeddings $\mathcal{F} \subset \mathcal{O}$ such that $(\hat{g}, n)$ is $(0,7)$ or $(1,3)$. Figure 4 shows an embedded Σ_4 in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. The two dashed lines indicate the axes of two π-rotations, which keep the Σ_4 invariant. They generate a group $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Let $\mathcal{F} = \Sigma_4/G$, $\mathcal{O} = S^3/G$, then $|\mathcal{F}| \cong T^2$ and $|\mathcal{O}| \cong S^3$. The singular set of $\mathcal{O}$ is a θ-curve with all three edges of index 2. It intersects $\mathcal{F}$ at 3 points. One side of Σ_4 is a boundary connected sum of four copies of the closed trefoil knot complement. The other side of Σ_4 is a boundary connected sum of two copies of

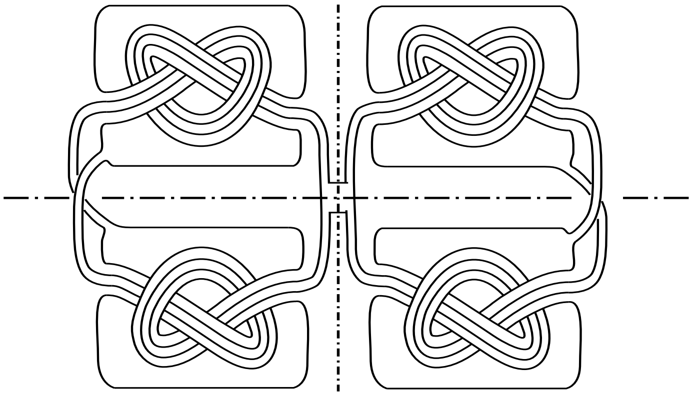

Figure 4: A $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action on Σ_4

the closed complement of Γ in Figure 3. By Example 3.2, $\mathcal{F}$ is totally-knotted in $\mathcal{O}$.

The example when $(\hat{g}, n)$ is $(0, 7)$ can be obtained similarly from Example 3.1.

Example 3.4: There exists a totally-knotted π_1-surjective embedding $\mathcal{F} \subset \mathcal{O}$ such that $(\hat{g}, n)$ is $(2, 1)$. Figure 5 gives an embedded Σ_{11} in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. The dashed line and circle indicate the axes of two π-rotations, which keep the Σ_{11} invariant.

The group G generated by the rotations is isomorphic to the dihedral group $\mathbb{D}_4$. It contains five π-rotations. In Figure 6, the left picture gives four axes, where the lines and the circles are in the dual position. There is another π-rotation τ whose axis passes through the intersections and infinity. The middle picture shows the axes in S^3/τ, and the right picture shows the singular set of the 3-orbifold S^3/G.

Let $\mathcal{F} = \Sigma_{11}/G$ and $\mathcal{O} = S^3/G$, then $|\mathcal{F}| \cong \Sigma_2$ and $|\mathcal{O}| \cong S^3$. The singular set of $\mathcal{O}$ will intersect $\mathcal{F}$ at 1 point. One side of Σ_{11} is a boundary connected sum of V_3 and eight copies of the closed trefoil knot complement. The other side of Σ_{11} is a boundary connected sum of V_3 and four copies of the closed complement of Γ in Figure 3. By Example 3.2, $\mathcal{F}$ is totally-knotted in $\mathcal{O}$.

Proof of the totally-knotted part of Theorem 1.4. By Theorem 1.1, we can assume that $n > 0$. Then, the required examples can be obtained from previous examples by adding handles to one side of the surface in S^3 equivariantly. If the handle does not intersect any rotation axes, then $\hat{g}$ will increase by 1. If the handle intersects a rotation axis in an arc, then n will increase by 2 for each such intersection. Hence, all cases can be obtained. Moreover, $\mathcal{F}$ can be compressible on each side if $(\hat{g}, n)$ is not $(1, 2)$. □

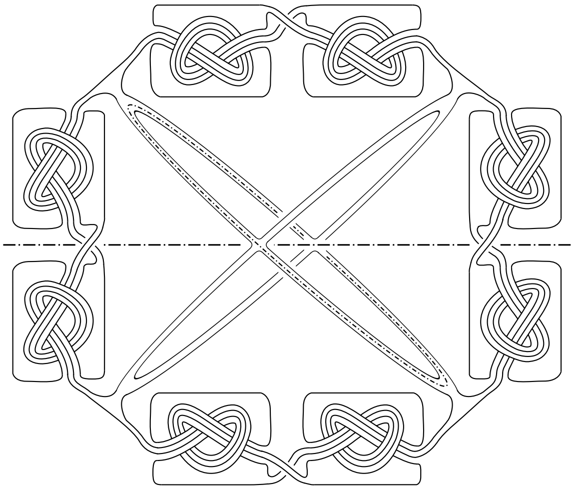

Figure 5: A $\mathbb{D}_4$-action on Σ_{11}

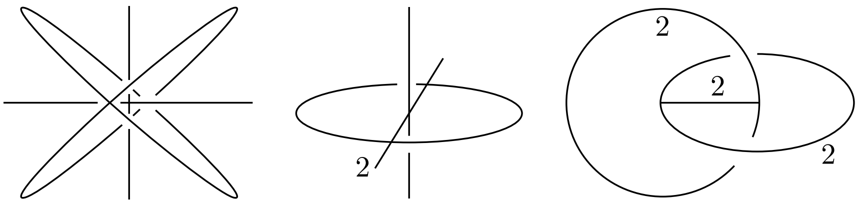

Figure 6: The coverings from S^3 to $\mathcal{O}$

Remark 3.5: When $(\hat{g}, n)$ is $(0,4)$ or $(0,5)$ or $(1,2)$, there exists a half-unknotted π_1-surjective embedding $\mathcal{F} \subset \mathcal{O}$ which is knotted, where $\mathcal{F}$ can be compressible on each side if $(\hat{g}, n)$ is not $(0,4)$. At present, it is not known if $\mathcal{F}$ can be knotted in $\mathcal{O}$ when $(\hat{g}, n)$ is $(1,1)$. See [9, 10] for more examples.

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