Rend. Istit. Mat. Univ. Trieste
Vol. XXXVII, 145–154 (2005)

An Orlicz Extension of Some New Sequence Spaces

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Summary. - *The aim of this note is to introduce and study a new concept of lacunary σ-convergence with respect to an Orlicz function and examine some properties of the resulting sequence spaces.*

1. Introduction and Background

Let w denote the set of all real and complex sequences $x = (x_k)$. By l_∞ and c, we denote the Banach spaces of bounded and convergent sequences $x = (x_k)$ normed by $||x|| = \sup_n |x_n|$, respectively. A linear functional L on l_∞ is said to be a Banach limit [1] if it has the following properties:

1. $L(x) \geq 0$ if $x \geq 0$ (i.e. $x_n \geq 0$ for all n),
2. $L(e) = 1$ where $e = (1, 1, \ldots)$,
3. $L(Dx) = L(x)$, where the shift operator D is defined by $D(x_n) = x_{n+1}$.

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This research was completed while the first author was a Fulbright scholar at Indiana University, Bloomington, IN, U.S.A., during the fall and spring semesters of 2003-2004.
Keywords: Orlicz function, Invariant Mean, Almost convergence, Lacunary sequence.
AMS Subject Classification: Primary, 40H05; Secondary 40C05.

Let B be the set of all Banach limits on l_∞. A sequence x is said to be almost convergent to a number L if $L(x) = L$ for all $L \in B$. Lorentz [8] has shown that

$$\hat{c} = \left\{ x \in l_\infty : \lim_m t_{m,n}(x) \text{ exists uniformly in } n \right\}$$

where

$$t_{m,n}(x) = \frac{x_n + x_{n+1} + x_{n+2} + \cdots + x_{n+m}}{m+1}.$$

Shaefer [14] defines the σ-convergence as follows: Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or a σ-mean if and only if

1. $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_n \geq 0$ for all n;
2. $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$ and
3. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

Let V_σ denote the set of bounded sequences which have unique σ-mean. If $x \in V_\sigma$ and $\phi(x) = l$, then we write $l = V_\sigma - \lim x$. In case σ is the translation mapping $n \to n+1$, σ-mean reduces to the unique Banach limit and V_σ reduces to $\hat{c}$. We denote by V_σ the space of σ-convergent sequences. It is known that $x \in V_\sigma$ if and only if

$$\frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)}$$

has a limit as $m \to \infty$, uniformly in n.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1}$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$.The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

Recently Das and Mishra [3] have introduced the space AC_θ of lacunary almost convergent sequences as follows:

$$AC_\theta = \left\{ x = (x_k) : \begin{array}{l} \lim_r \frac{1}{h_r} \sum_{k \in I_r} (x_{k+n} - L) = 0, \\ \text{for some } L \text{ uniformly in } n \end{array} \right\}.$$

Note that in the special case where $\theta = 2^r$, we have $AC_\theta = \hat{c}$.

Quite recently, concept of lacunary σ-convergent was introduced and studied by Savas [13] which is a generalization of the idea of lacunary almost convergence due to Das and Mishra [3]. If $x \in V_\sigma^\theta$ denotes the set of all lacunary σ-convergent sequences, then Savas [13] defined

$$V_\sigma^\theta = \left\{ x = (x_k) : \lim_r t_{r,n}(x) = L, \text{ uniformly in } n \text{ for some } L \right\}$$

where

$$t_{r,n}(x) = \frac{1}{h_r} \sum_{k \in I_r} x_{\sigma^k(n)}.$$

Note that for $\sigma(n) = n + 1$, the space V_σ^θ is the same as AC_θ. We write $V_\sigma^\theta = V_{\sigma_0}^\theta$ whenever $L = 0$.

Recall in [6] that an Orlicz function $M : [0, \infty) \to [0, \infty)$ is continuous, convex, non decreasing function defined by $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called the modulus function which is defined and characterized by Ruckle [11].

Lindeastrauss and Tzafriri [7] used the concept of Orlicz function to construct the following sequence space:

$$L_M = \left\{ x \in w : \sum_k M\left(\frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space L_M with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_k M\left(\frac{|x_k|}{\rho} \right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space.

An Orlicz function M is said to satisfy Δ_2-condition of all values of u, if there exists a constant $K > 0$ such that

$$M(2u) \leq KM(u), \text{ for all } u \geq 0.$$

The Δ_2-condition is equivalent to

$$M(lu) \leq KlM(u)$$

for all values of u and $l \geq 1$.

In the present paper, we introduce and study some properties of the following three sequence spaces that are defined using the Orlicz function.

Let M be an Orlicz function then

$$V_{\sigma_0}^{\theta}(M) = \left\{ x = (x_k) : \begin{array}{l} \lim_{r\to\infty} M\left(\frac{|t_{r,n}(x)|}{\rho}\right) = 0 \\ \text{uniformly in } n \text{ for some } \rho > 0 \end{array} \right\},$$

$$V_{\sigma}^{\theta}(M) = \left\{ x = (x_k) : \begin{array}{l} \lim_{r\to\infty} M\left(\frac{|t_{r,n}(x) - le|}{\rho}\right) = 0 \\ \text{uniformly in } n \text{ for some } \rho > 0 \end{array} \right\},$$

and

$$V_{\sigma_\infty}^{\theta}(M) = \left\{ x = (x_k) : \sup_{r,n} M\left(\frac{|t_{r,n}(x)|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

If $t_{r,n}(x)$ is replaced by x, then we have the following sequence spaces:

$$c_0(M) = \left\{ x = (x_k) : \lim_k M\left(\frac{|x_k|}{\rho}\right) = 0 \text{ for some } \rho > 0 \right\},$$

$$c(M) = \left\{ x = (x_k) : \begin{array}{l} \lim_k M\left(\frac{|x_k - l|}{\rho}\right) = 0 \\ \text{for some } l > 0 \text{ and } \rho > 0 \end{array} \right\},$$

and

$$l_\infty(M) = \left\{ x = (x_k) : \sup_k M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

It is easy to see that $V_{\sigma_0}^{\theta}$, V_{σ}^{θ} and $V_{\sigma_\infty}^{\theta}$ are linear spaces over the complex field. With consider of the above sequence spaces we now present the following theorem.

2. Main Result

THEOREM 2.1. *The linear spaces* $V_{\sigma_0}^{\theta}(M)$, $V_{\sigma}^{\theta}(M)$ *and* $V_{\sigma_\infty}^{\theta}(M)$ *are Banach spaces with the norm*

$$||x|| = \inf\left\{\rho > 0 : \sup_{r,n} M\left(\frac{|t_{r,n}(x)|}{\rho}\right) \leq 1\right\}.$$

Proof. It is clear that the spaces are normed spaces with the above norm. We shall only establish that $V_{\sigma_\infty}^{\theta}(M)$ is a Banach space. The others can be established in a manner similar to $V_{\sigma_\infty}^{\theta}(M)$. Let $(x_k^i)_k$ be a Cauchy sequence in $V_{\sigma_\infty}^{\theta}(M)$. Let $s, x_0 > \epsilon$ be fixed such that $M\left(\frac{sx_0}{2}\right) \geq 1$. Then for each $\frac{\epsilon}{x_0 s} > 0$ there exists a positive integer N such that for all $i, j \geq N$

$$\left|\left|x^i - x^j\right|\right| \leq \frac{\epsilon}{x_0 s}.$$

The definition of norm above implies that for all $i, j \geq N$

$$\sup_{r,n} M\left(\frac{\left|t_{r,n}(x^i - x^j)\right|}{||x^i - x^j||}\right) \leq 1, \tag{1}$$

since $||x^i - x^j||$ is positive so we can substitute ρ for $||x^i - x^j||$. Thus

$$M\left(\frac{\left|t_{r,n}(x^i - x^j)\right|}{||x^i - x^j||}\right) \leq 1 \text{ for all } r, n \geq 0 \text{ and for all } i, j \geq N.$$

Since $M\left(\frac{sx_0}{2}\right) \geq 1$ we have

$$M\left(\frac{\left|t_{r,n}(x^i - x^j)\right|}{||x^i - x^j||}\right) \leq M\left(\frac{sx_0}{2}\right) \text{ for all } r, n.$$

This implies that

$$\left|t_{r,n}(x^i - x^j)\right| \leq \frac{x_0 s}{2}\frac{\epsilon}{x_0 s} = \frac{\epsilon}{2} \text{ for all } r, n.$$

In particular $|t_{1,n}(x^i - x^j)| = |x_{\sigma(n)}^i - x_{\sigma(n)}^j| \to 0$ as $i, j \to \infty$ for each fixed n. Hence (x^i) is a Cauchy sequence in the complex plane.

Therefore for each $\epsilon (0 < \epsilon < 1)$, there exists a positive integer N such that $\left|t_{r,n}(x^i - x^j)\right| < \epsilon$, for all $i, j \geq N$ and for all r, n. Using the continuity of M and taking the limit as $j \to \infty$ we have that

$$\sup_{r \geq N} M\left(\frac{\left|t_{r,n}(x_k^i - x_k)\right|}{\rho}\right) \leq 1.$$

Taking the infimum over such ρ's we get the following for all n

$$\inf\left\{\rho > 0 : \sup_{r \geq N} M\left(\frac{\left|t_{r,n}(x_k^i - x_k)\right|}{\rho}\right) \leq 1\right\} < \epsilon,$$

for all $i \geq N$. Since $x^i \in V_{\sigma_\infty}^{\theta}(M)$ and M is an Orlicz function, it follows that $x \in V_{\sigma_\infty}^{\theta}(M)$. This completes the proof.

□

Theorem 2.2. *Let M be an Orlicz function that satisfies the Δ_2-condition then*

$$c_0(M) \subset V_{\sigma_0}^{\theta}(M),$$

$$c(M) \subset V_{\sigma}^{\theta}(M),$$

and

$$l_\infty(M) \subset V_{\sigma_\infty}^{\theta}(M).$$

The verification of this theorem is routine and thus omitted.

It is quite natural to expect that the spaces $V_{\sigma_0}^{\theta}(M)$, $V_{\sigma}^{\theta}(M)$ and $V_{\sigma_\infty}^{\theta}(M)$ can be extended to $V_{\sigma_0}^{\theta}(M,p)$, $V_{\sigma}^{\theta}(M,p)$ and $V_{\sigma_\infty}^{\theta}(M,p)$ in manner similar to the extension of c, c_0, and l_∞ to $c(p)$, $c_0(p)$, and $l_\infty(p)$ respectively (see, Simons [15] and Moddox [9]).

In this section of this paper, we study the spaces $V_{\sigma_0}^{\theta}(M,p)$, $V_{\sigma}^{\theta}(M,p)$ and $V_{\sigma_\infty}^{\theta}(M,p)$ which are defined below:

Let M be Orlicz function, $p = (p_r)$ be any sequence of positive real numbers.

$$V_{\sigma_0}^{\theta}(M,p) = \left\{x = (x_k) : \begin{array}{c} \lim_{r\to\infty}\left(M\left(\frac{|t_{r,n}(x)|}{\rho}\right)\right)^{p_r} = 0 \\ \text{uniformly in } n \text{ for some } \rho > 0 \end{array}\right\},$$

$$V_{\sigma}^{\theta}(M,p) = \left\{x = (x_k) : \begin{array}{c} \lim_{r\to\infty}\left(M\left(\frac{|t_{r,n}(x) - Le|}{\rho}\right)\right)^{p_r} = 0 \\ \text{uniformly in } n \text{ for some } L, \rho > 0 \end{array}\right\},$$

and

$$V^{\theta}_{\sigma_\infty}(M,p)=\left\{x=(x_k):\begin{array}{l}\sup_{r,n}\left(M\left(\frac{|t_{r,n}(x)|}{\rho}\right)\right)^{p_r}<\infty\\ \text{for some } \rho>0\end{array}\right\}.$$

If $p=p_r$ is a constant sequence, i.e., $p_r=p$ for all r, then we write $V^{\theta}_{\sigma_0}(M,p)=V^{\theta}_{\sigma_0}(M)$, $V^{\theta}_{\sigma}(M,p)=V^{\theta}_{\sigma}(M)$ and $V^{\theta}_{\sigma_\infty}(M,p)=V^{\theta}_{\sigma_\infty}(M)$. If we let $\sigma(n)=n+1$, the spaces $V^{\theta}_{\sigma_0}(M,p)$, $V^{\theta}_{\sigma}(M,p)$ and $V^{\theta}_{\sigma_\infty}(M,p)$ reduce to the following sequence spaces:

$$\hat{V}_0(M,p)=\left\{x=(x_k):\begin{array}{l}\lim_{r\to\infty}\left(M\left(\frac{|d_{r,n}(x)|}{\rho}\right)\right)^{p_r}=0\\ \text{uniformly in } n \text{ for some } \rho>0\end{array}\right\},$$

$$\hat{V}(M,p)=\left\{x=(x_k):\begin{array}{l}\lim_{r\to\infty}\left(M\left(\frac{|d_{r,n}(x)-Le|}{\rho}\right)\right)^{p_r}=0\\ \text{uniformly in } n \text{ for some } L,\ \rho>0\end{array}\right\},$$

and

$$\hat{V}_\infty(M,p)=\left\{x=(x_k):\begin{array}{l}\sup_{r,n}\left(M\left(\frac{|d_{r,n}(x)|}{\rho}\right)\right)^{p_r}<\infty\\ \text{for some } \rho>0\end{array}\right\}$$

where

$$d_{r,n}(x)=\frac{1}{h_r}\sum_{k\in I_r}x_{k+n},$$

We now consider the following theorem:

THEOREM 2.3. *The linear spaces* $V^{\theta}_{\sigma_0}(M,p)$, $V^{\theta}_{\sigma}(M,p)$ *and* $V^{\theta}_{\sigma_\infty}(M,p)$ *are paranormed spaces with*

$$g(x)=\inf\left\{\rho^{\frac{p_n}{H}}:\left\{sup_r\left(M\left(\frac{|t_{r,n}(x)|}{\rho}\right)\right)^{p_r}\right\}^{\frac{1}{H}}\leq 1, n=1,2\}\right\},$$

where $H=\max\{1,\sup_r p_r$.

Proof. This can be proved by using the techniques similar to those used in Theorem 1 and Theorem 2 in [10]. □

THEOREM 2.4. *Let* p_k *and* q_k *be two sequences of real numbers such that* $0 < p_k \leq q_k < \infty$ *for each* k. *Then*

$$V_{\sigma_0}^{\theta}(M,p) \subseteq V_{\sigma_0}^{\theta}(M,q).$$

Proof. Let $x \in V_{\sigma_0}^{\theta}(M,p)$. Then there exists some $\rho > 0$ such that

$$\lim_{r\to\infty} \left(M \left(\frac{|t_{r,n}(x)|}{\rho} \right) \right)^{p_r} = 0, \text{ uniformly in } n.$$

This implies that

$$M \left(\frac{|t_{r,n}(x)|}{\rho} \right) \leq 1$$

for sufficiently large n. Since M is non-decreasing, we obtain the following

$$\lim_{r\to\infty} \left(M \left(\frac{|t_{r,n}(x)|}{\rho} \right) \right)^{q_r} \leq \lim_{r\to\infty} \left(M \left(\frac{|t_{r,n}(x)|}{\rho} \right) \right)^{p_r} = 0$$

uniformly in n that is $x \in V_{\sigma_0}^{\theta}(M,q)$. This completes the proof. □

THEOREM 2.5. *(1) Let* $0 < \inf p_k \leq p_k \leq 1$. *Then*

$$V_{\sigma_0}^{\theta}(M,p) \subseteq V_{\sigma_0}^{\theta}(M).$$

(2) Let $0 < p_k \leq \sup_k p_k \leq \infty$. *Then*

$$V_{\sigma_0}^{\theta}(M) \subseteq V_{\sigma_0}^{\theta}(M,p).$$

Proof. (1) Let $x \in V_{\sigma_0}^{\theta}(M,p)$, that is $\lim_{r\to\infty} \left(M \left(\frac{|t_{r,n}(x)|}{\rho} \right) \right)^{p_r} = 0$, uniformly in n. Since $0 < \inf p_k \leq p_k \leq 1$ we have

$$\lim_{r\to\infty} \left(M \left(\frac{|t_{r,n}(x)|}{\rho} \right) \right) \leq \lim_{r\to\infty} \left(M \left(\frac{|t_{r,n}(x)|}{\rho} \right) \right)^{p_r} = 0$$

uniformly in n and hence $x \in V_{\sigma_0}^{\theta}(M)$.

(2) Let $p_k \geq 1$ for each k and $\sup_k p_k < \infty$. Let $x \in V_{\sigma_0}^{\theta}(M)$, then for each $\epsilon (0 < \epsilon < 1)$ there exists a positive integer N such that

$M\left(\frac{|t_{r,n}(x)|}{\rho}\right) \leq \epsilon$ for all $r \geq N$ and for all n. Since $0 < p_k \leq \sup p_k < \infty$, we have for all n.

$$\lim_{r\to\infty}\left(M\left(\frac{|t_{r,n}(x)|}{\rho}\right)\right)^{p_r} \leq \lim_{r\to\infty}\left(M\left(\frac{|t_{r,n}(x)|}{\rho}\right)\right) \leq \epsilon < 1.$$

Therefore $x \in V^{\theta}_{\sigma_0}(M,p)$. This completes the proof. □

Acknowledgement The authors wishes to thank the referee for his constructive comments and suggestions.

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Received November 23, 2004.