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A Remark on Partial Regularity of Minimizers of Quasiconvex Integrals of Higher Order

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Summary. - *We prove partial regularity of minimizers of strictly quasiconvex functionals of the form* $I(u) = \int_\Omega F(D^k u)\,dx$, *where the integrand grows as* $|\xi|^p$.

1. Introduction

In this paper we study the partial regularity for minimizers of functionals of the following type

$$I(u) = \int_\Omega F(D^k u)\,dx\,, \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$, F is a C^2 function, $u \in W^{k,p}(\Omega;\mathbb{R}^N)$ with $p > 1$ and $k > 1$. Many authors investigated the case $k = 1$ assuming F either convex (see for instance [11]) or more generally quasiconvex. In particular the major breakthrough in the latter framework is due to Evans [7] who proved the regularity of minimizers under the following hypothesis:

$$|F(\xi)| \le C(1 + |\xi|^p), \qquad p \ge 2\,; \tag{i}$$

$$|D^2 F(\xi)| \le C(1 + |\xi|^{p-2})\,; \tag{ii}$$

$$\int_\Omega \left[F(\xi) + \nu(1 + |D\phi(y)|^2)^{\frac{p-2}{2}}|D\phi|^2\right]\,dy \le \int_\Omega F(\xi + D\phi(y))\,dy\,, \tag{iii}$$

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for all $\xi \in M^{m\times N}$ and $\phi \in C_0^1(\Omega;\mathbb{R}^N)$ with $\nu > 0$. However it should be mentioned that many natural examples of quasiconvex integrals satisfing (i) and (iii) but not (ii) can be found in literature, see for instance (2) below. Anyway in [2] it is shown that Evans' result still holds if growth condition (ii) is dropped.
The first examples of genuine quasiconvex functionals with subquadratic growth i.e. when $p \in (1,2)$ have been discovered few years later (see e.g. [14]). As much as concerns the regularity of minimizers in the subquadratic case we refer to [5];

The aim of this paper is to prove the $C^{k,\gamma}$ partial regularity of minimizers of functional (1), assuming F a C^2 integrand with polinomial growth

$$|DF(\xi)| \leq L(1+|\xi|^2)^{\frac{p-1}{2}}, \tag{H1}$$

verifying the condition

$$\int_\Omega \left[F(\xi) + \nu(1+|D^k\phi(y)|^2)^{\frac{p-2}{2}}|D^k\phi|^2\right] dy \leq \int_\Omega F(\xi + D^k\phi(y))\, dy\,, \tag{H2}$$

for any $\xi \in M^{m\times N}$ and $\phi \in C_0^k(\Omega;\mathbb{R}^N)$, with $\nu > 0$. It is well known that when $k = 1$ quasiconvexity, i.e. (H2) with $\nu = 0$, together with (i) is equivalent to (H1), see [13]. On the other hand if $k > 1$ this equivalence has not yet been proved except in the special case $k = 2$ (see [12]).
In view of applications, the regularity results obtained in this paper applies to integral functionals such as

$$I(u) = \int_\Omega |D^2u(y)|^2 + \sqrt{\det(D^2u(y))}\, dy, \tag{2}$$

where $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$.
As a common feature in this framework partial regularity is achieved through the decay estimate for the excess function, which if $p = 2$, is given by

$$E(x,R) = \fint_{B_R} |D^ku - (D^ku)_{x,R}|^2\, dy\,.$$

The key point in the proof is to show that if at some point x $E(x,R)$ is sufficiently small then $E(x,\rho)$ decays like ρ^2 as ρ goes to zero.

2. Notation and preliminary results

In the following Ω is an open bounded subset of $\mathbb{R}^n$, $B_R(x)$ the ball $\{y \in \mathbb{R}^n : |y - x| < R\}$ and

$$(u)_{x,R} = \fint_{B_R(x)} u(y)\, dy,$$

where u is an integrable function. Here and in the following when no confusion arises we may denote by B_R the ball of center 0 and radious R. Given $p > 1$ and $u \in W^{k,p}(\Omega; \mathbb{R}^N)$ with $k \geq 1$, $P(y) = P_u(x, R; y)$ stands for the unique polynomial of degree $k - 1$ such that

$$\int_{B_R(x)} D^l(u(y) - P(y))\, dy = 0 \quad l = 1, \dots, k-1.$$

Its coefficients depend on x, R and obviously also the derivatives of u; for more details on such polynomial see for example [9] pages 79-80. Whenever it is not ambigous we may omit the dependence of P on R, x and on u. The dimension of the simmetric space in which $D^k u$ takes value is $m = \binom{n+k}{k-1}$.

Let V be the function defined by

$$V(\xi) = (1 + |\xi|^2)^{\frac{p-2}{4}} \xi \tag{3}$$

Lemma 2.1. *Let $p > 1$ and let $V : \mathbb{R}^m \to \mathbb{R}^m$ be the function defined by (3). Then for any ξ, $\eta \in \mathbb{R}^m$ and $t > 0$ we have*

$$|V(t\xi)| \leq \max\{t, t^{\frac{p}{2}}\} |V(\xi)|\,; \tag{4}$$

$$|V(\xi + \eta)| \leq c(|V(\xi)| + |V(\eta)|)\,; \tag{5}$$

$$|V(\xi) - V(\eta)| \leq c |V(\xi - \eta)|\,; \tag{6}$$

$$\max\{|\xi|,\, |\xi|^{\frac{p}{2}}\} \leq |V(\xi)| \leq c \max\{|\xi|,\, |\xi|^{\frac{p}{2}}\} \quad \textit{if} \quad p \geq 2; \tag{6a}$$

$$c \min\{|\xi|,\, |\xi|^{\frac{p}{2}}\} \leq |V(\xi)| \leq \min\{|\xi|,\, |\xi|^{\frac{p}{2}}\} \quad \textit{if} \quad p < 2; \tag{6b}$$

$$c|\xi - \eta| \leq \frac{|V(\xi) - V(\eta)|}{(1 + |\xi|^2 + |\eta|^2)} \leq C|\xi - \eta|, \tag{8}$$

with constants c and C depending only on p and m. Moreover for every $M > 0$, there exists $C(p, M)$ such that

$$|V(\xi - \eta)| \leq c(p, M) |V(\xi) - V(\eta)| \quad \textit{if} \quad \min\{|\xi|, |\eta|\} \leq M. \tag{9}$$

Proof. If $p < 2$ we refer to lemma 1.1 [5], in the other case note that

$$|\xi|(1 + |\xi|^{\frac{p-2}{2}}) \leq |V(\xi)| \leq c(p)|\xi|(1 + |\xi|^{\frac{p-2}{2}}),$$

so (4)–(6a) follow easily; (8) can be proved in the same way as in [4], and (9) follows from the previous one (see [5]). □

We are now interested in a Sobolev-Poincaré type inequality on balls for the function V. In [5] the authors were able to prove such inequality but paying something in terms of enlarging the radius of the integration domain. By induction it is easy to extend the result to higher order derivatives.

PROPOSITION 2.2. *Let $p > 1$, $2 \geq \alpha > \max\{\frac{2}{p}, 1\}$, and $u \in W^{k,p}(B_{3^k R}(x), \mathbb{R}^N)$. If P is the polynomial of degree $k-1$ s.t.*

$$\int_{B_{3^l R}(x)} D^l(u-P)\,dy = 0 \qquad l = 0, \ldots, k-1, > \tag{10}$$

then $\sigma = \sigma(n, \alpha, p) > 0$ and $c = c(p, n, k)$ exist such that

$$\left(\fint_{B_{3^l R}(x)} \left| V\left(\frac{D^l(u-P)}{R^{k-l}} \right) \right|^{2(1+\sigma)} dy \right)^{\frac{1}{2(1+\sigma)}} \leq c \left(\fint_{B_{3^m R}(x)} \left| V\left(\frac{D^m(u-P)}{R^{k-m}} \right) \right|^{\alpha} dy \right)^{\frac{1}{\alpha}}, \tag{11}$$

for $l = 0, \ldots, k-1$ and $m = l+1, \ldots, k$.

Proof. Setting $v(y) = \frac{u(x+Ry)-P(x+Ry)}{R^k}$, we may assume $R = 1$, $x = 0$. For $l \geq 0$ and $m = l+1$ then the result is Theorem 2.4 in [5], which, in a slightly modified version, also applies to the case $p \geq 2$. Suppose then $m > 1$, let us prove it for $m+1$. Letting $w = D^m v$, we have

$$
\begin{aligned}
\left(\fint_{B_{3l}} |V(D^l v)|^{2(1+\sigma)} dy\right)^{\frac{1}{2(1+\sigma)}} &\leq \left(\fint_{B_{3m}} |V(D^m v)|^{\alpha} dy\right)^{\frac{1}{\alpha}} \\
&\leq c\left(\fint_{B_{3m}} |V(D^m v)|^{2(1+\sigma)} dy\right)^{\frac{1}{2(1+\sigma)}} \\
&= c\left(\fint_{B_{3m}} |V(w)|^{2(1+\sigma)} dy\right)^{\frac{1}{2(1+\sigma)}} \\
&\leq c\left(\fint_{B_{3m}} |V(Dw)|^{\alpha} dy\right)^{\frac{1}{\alpha}} \leq c\left(\fint_{B_{3m}} |V(D^{m+1} v)|^{\alpha} dy\right)^{\frac{1}{\alpha}}.
\end{aligned}
$$

□

The next lemma is a straightforward generalization of Lemma 3.1, Chap. 5 in [9] using (4) in place of homogenity.

LEMMA 2.3. *Let* $h : [r/2, r] \to [0, \infty]$ *be a bounded function such that for all* $r/2 \leq t < s \leq r$

$$
h(t) \leq \theta h(s) + \sum_{l=0}^{m} c_l \int_{B_r} \left| V\left(\frac{g_l(y)}{(s-t)^l}\right)\right|^2 dy \,,
$$

where $g_l \in L^p(B_r)$, $0 < \theta < 1$, $c_l \geq 0$ *and at least one is non zero. Then a constant* $C = C(\theta)$ *exists such that*

$$
h(r/2) \leq C \sum_{l=0}^{m} c_l \int_{B_r} \left| V\left(\frac{g_l(y)}{r^l}\right)\right|^2 dy \,.
$$

From the previous two lemmas we can obtain the next higher integrability result.

PROPOSITION 2.4. *Let* $g : \mathbb{R}^{mN} \to \mathbb{R}$ *be a continuous function satisfying*

$$
|g(\xi)| \leq L|V(\lambda\xi)|^2, \tag{12}
$$

$$
\nu \int_{\Omega} |V(\lambda D^k \phi)|^2 \, dy \leq \int_{\Omega} g(D^k \phi) \, dy, \tag{13}
$$

for all $\phi \in W_0^{k,p}(\Omega;\mathbb{R}^N)$ *and for some* L, ν, $\lambda > 0$. *Let* $p > 1$ *and* $u \in W^{k,p}(\Omega;\mathbb{R}^N)$ *be such that*

$$\int_\Omega g(D^k u)\,dy \le \int_\Omega g(D^k u + D^k \phi)\,dy \qquad \forall \phi \in W_0^{k,p}(\Omega;\mathbb{R}^N), \quad (14)$$

then there exist c, $\delta > 0$ *depending neither on* λ *nor on* u *such that if* $B_R(x) \subset \Omega$

$$-\!\!\!\!\!\!\int_{B_{\frac{R}{2}}(x)} |V(\lambda D^k u)|^{2(1+\delta)}\,dy \le c \left(-\!\!\!\!\!\!\int_{B_R(x)} |V(\lambda D^k u)|^2\,dy \right)^{1+\delta}. \quad (15)$$

Proof. As usual we may assume $x = 0$. Fixed $B_{3^k r} \subset\subset \Omega$, let $\frac{r}{2} \le t < s \le r$ and $\vartheta \in C_0^k(B_s)$ be a cut–off function, $0 \le \vartheta \le 1$, $\vartheta = 1$ on B_t and $|D^l \vartheta| \le \frac{c}{(s-t)^l}$. Set

$$\phi_1 = \vartheta(u - P)\,; \qquad \phi_2 = (1-\vartheta)(u-P)\,,$$

where P is the polynomial of the Sobolev-Poincaré inequality associated to u, see (10) with $R = r$. Since $D^k u = D^k \phi_1 + D^k \phi_2$, from (13), (14), (12) and the same argument of Lemma II.4 in [2], we get

$$\begin{aligned}
\nu \int_{B_s} |V(\lambda D^k \phi_1)|^2\,dy &\le \int_{B_s} g(D^k \phi_1)\,dy = \int_{B_s} g(D^k u - D^k \phi_2)\,dy \\
&\le \int_{B_s \setminus B_t} g(D^k \phi_2)\,dy + \int_{B_s \setminus B_t} g(D^k u - D^k \phi_2) - g(D^k u)\,dy \\
&\le L \int_{B_s \setminus B_t} \Big\{ |V(\lambda D^k \phi_2)|^2 + |V(\lambda(D^k u - D^k \phi_2))|^2 + |V(\lambda D^k u)|^2 \Big\}\,dy.
\end{aligned}$$

By this inequality and (5) we obtain

$$\begin{aligned}
\int_{B_t} |V(\lambda D^k u)|^2\,dy \le \int_{B_s} |V(\lambda D^k \phi_1)|^2\,dy &\le c \int_{B_s \setminus B_t} |V(\lambda D^k u)|^2\,dy \\
&\quad + c \int_{B_r} \sum_{l=1}^{k} \left| V\left(\lambda D^{k-l} \left(\frac{u-P}{(s-t)^l} \right) \right) \right|^2 dy.
\end{aligned}$$

"Filling the hole" by adding

$$c \int_{B_t} |V(\lambda D^k u)|^2\,dy$$

to both sides and dividing by $c+1$ we get by lemma 2.3

$$\int_{B_{\frac{r}{2}}} |V(\lambda D^k u)|^2 \leq c \sum_{l=1}^{k} \int_{B_r} \left| V\left(\lambda \frac{D^{k-l}(u-P)}{r^l}\right)\right|^2 dy,$$

and now using the Hölder and the Sobolev-Poincaré inequalities

$$⨍_{B_{\frac{r}{2}}} |V(\lambda D^k u)|^2 \leq \left(⨍_{B_{3^k r}} |V(\lambda D^k u)|^{\alpha} \right)^{\frac{2}{\alpha}},$$

where $\max\{\frac{p}{2}, 1\} < \alpha < 2$. By a modified version of the Gehring Lemma due to Giaquinta Modica (see [9] Proposition 1.1, Chap 5), we get the result. □

Next Proposition is a mere adaptation of the one obtained in [5] regarding linear elliptic system of the second order. We recall that the theory valid for second order linear elliptic systems can be extended to higher order ones (see [9] note 5 pag. 76).

Proposition 2.5. *Let $u \in W^{k,1}(\Omega;\mathbb{R}^n)$ satisfy the linear system*

$$\int_{\Omega} A^{i,j}_{\alpha\beta} D_\alpha u^i D_\beta \phi^j \, dy = 0, \qquad \forall \phi \in C^k_0(\Omega;\mathbb{R}^n),$$

where α, β are multiindeces of lenght k, $A^{ij}_{\alpha\beta}$ are constants and satisfy the Legendre-Hadamard condition:

$$A^{ij}_{\alpha\beta}\lambda^i\lambda^j v_\alpha v_\beta \geq \nu|\lambda|^2|v|^{2k} \quad \text{for any} \quad \lambda \in \mathbb{R}^N,\ v \in \mathbb{R}^n.$$

Then u is C^∞ and for all $B_R(x) \in \Omega$

$$\sup_{B_{\frac{R}{2}}(x)} |D^k u| \leq \frac{c}{R^n} \|u\|_{W^{k,1}(B_R(x);\mathbb{R}^N)}. \tag{16}$$

Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a locally integrable function. The Hardy–Littlewood maximal function $M(f)$ is defined by

$$M(f)(x) = \sup_r ⨍_{B_r(x)} |f(y)|\, dy.$$

If $q > 1$, such function is a continuous operator from L^q to L^q. The next result is proved in [5].

Lemma 2.6. *Let $p > 1$ and $\alpha > \max\{1, \frac{2}{p}\}$, then there exists $c = c(\alpha, p, n)$ such that if $f : \mathbb{R}^n \to \mathbb{R}^N$ is measurable, then*

$$\int_{\mathbb{R}^n} |V(M(f))|^\alpha \, dy \le c \int_{\mathbb{R}^n} |V(f)|^\alpha \, dy$$

The following Lemma is a slightly modified version for higher derivatives of the approximation result proved in [3].

Lemma 2.7. *Let $u \in W^{k,p}(\mathbb{R}^n; \mathbb{R}^m)$, with $p > 1$. For every $K > 0$, set*

$$H_K = \{y \in \mathbb{R}^n : |M(D^k u)(y)| \le K\}.$$

Then a function $w \in W^{k,\infty}(\mathbb{R}^n; \mathbb{R}^N)$ exists such that

$$\|D^k w\|_\infty \le cK;$$
$$w = u \quad on \quad H_K;$$
$$|\mathbb{R}^n \setminus H_K| \le \frac{\|D^k u\|_p^p}{K^p},$$

with c depending on n, N, p.

We need also a selection Lemma. For the proof we refer to Eisen [6].

Lemma 2.8. *Let G be a measurable subset of $\mathbb{R}^n$, with meas$(G) < +\infty$. Assume (M_h) is a sequence of measurable subsets of G such that, for some $\varepsilon > 0$ the following estimate holds*

$$\text{meas} M_h \ge \varepsilon \qquad \textit{for all} \quad h \in \mathbb{N}.$$

Then a subsequence (M_{h_k}) exists such that $\cap_k M_{h_k} \ne \emptyset$.

3. Proof of the main result

Before stating the main Theorem we need a technical lemma, see [2]. Remark also that the proof of Lemma 2 in [2] holds for $1 < p < 2$ as well.

LEMMA 3.1. *Let* $f\colon \mathbb{R}^n \to \mathbb{R}$ *be a* C^2 *function satisfying for any* $\xi \in \mathbb{R}^n$

$$|Df(\xi)| \le L(1+|\xi^2|)^{\frac{p-1}{2}},$$

with $p>1$. *Then for any* $M>0$ *there exist a constant* $c=c(M,p,L)$ *such that if we set for any* $\lambda>0$ *and* $A\in\mathbb{R}^n$ *with* $|A|\le M$

$$f_{A,\lambda}(\xi) = \lambda^{-2}[f(A+\lambda\xi) - f(A) - \lambda Df(A)\xi]\,,$$

then

$$|f_{\lambda,A}(\xi)| \le c(1+|\lambda\xi|^2)^{\frac{p-2}{2}}|\xi|^2.$$

We recall that $u\in W^{k,p}(\Omega;\mathbb{R}^N)$ is a minimum point for

$$I(v) = \int_\Omega F(D^k v(y))\,dy \tag{17}$$

if

$$I(u) \le I(u+\phi) \tag{18}$$

for every $\phi \in W_0^{k,p}(\Omega;\mathbb{R}^N)$.

THEOREM 3.2. *Let* $u \in W^{k,p}(\Omega;\mathbb{R}^N)$, $p>1$ *be a solution of the minimum problem (18) and* F *be a* C^2 *function verifying (H1) and (H2). Then, there exists an open set* $\Omega_0 \Subset \Omega$ *with* $meas(\Omega\setminus\Omega_0)=0$ *such that* $u\in C^{k,\gamma}_{loc}(\Omega_0;\mathbb{R}^N)$ *for any* $\gamma<1$.

REMARK 3.3. In the non quadratic case it is convenient to use as excess function

$$E(x,R) = \fint_{B_R(x)} |V(D^k u) - V((D^k u)_{x,R})|^2\,dy\,. \tag{19}$$

Indeed when $2\le p<\infty$ it can be easily verified that

$$\begin{aligned} c\fint_{B_R(x)} &|D^k u - (D^k u)_{x,R}|^p + |D^k u - (D^k u)_{x,R}|^2\,dy \le E(x,R)\\ &\le C\fint_{B_R(x)} |D^k u - (D^k u)_{x,R}|^p + |D^k u - (D^k u)_{x,R}|^2\,dy, \end{aligned}$$

for some constants $C\ge c>0$

Proposition 3.4. *Let F and u satisfy the hypotheses of Theorem 3.2. Given $M>0$, a constant C_M exists such that, for all $0<\tau<\frac{1}{3^k+1}$, there exists $\varepsilon=\varepsilon(\tau,M)$ such that, if*

$$|(D^k u)_{x,R}| \le M \qquad \text{and} \qquad E(x,R)\le \varepsilon, \tag{20}$$

then

$$E(x,\tau R)\le C_M \tau^2 E(x,R). \tag{21}$$

The proof passes through various steps and follows closely the one due to Acerbi–Fusco [2] (and [5] in the subquadratic case) where in order to avoid Caccioppoli estimate, which would require a control on the second order derivatives of F, they make use of the higher integrability result for minima of non coercive functionals.

Proof. Fixed M and τ we will determine C_M later. By contradiction assume there are x_h and R_h with $B_{R_h}(x_h)\subset\Omega$ for each h such that

$$|(D^k u)_{x_h R_h}|, \le M\,, \qquad \lim_h E(x_h,R_h)=0\,, \tag{22}$$

$$E(x_h,\tau R_h) > C_M\tau^2 E(x_h,R_h). \tag{23}$$

Let $A_h=(D^k u)_{x_h R_h}$, $\lambda_h^2=E(x_h,R_h)$ and let P be the polynomial such that

$$\int_{B_{R_h}(x_h)} D^l(u-P)=0 \qquad l=0,\dots,k.$$

Step 1. *Blow-up.* If we set

$$v_h(y)=\frac{u(x_h+R_h y)-P(x_h+R_h y)}{R_h^k\lambda_h},$$

then

$$\int_{B_1} D^l v_h = 0 \qquad l=0,\dots,k \tag{24}$$

and

$$D^k v_h = \frac{D^k u(x_h+R_h y)-A_h}{\lambda_h}.$$

Since from (9) and (22)

$$\frac{1}{\lambda_h^2}\fint_{B_1}|V(\lambda_h D^k v_h)|^2\,dy = \frac{1}{\lambda_h^2}\fint_{B_{R_h}(x_h)}|V(D^k u)-(D^k u)_{x_h R_h})|^2\,dy$$
$$\leq \frac{c(M)}{\lambda_h^2}\fint_{B_{R_h}(x_h)}|V(D^k u)-V((D^k u)_{x_h R_h})|^2\,dy = c(M)\,, \quad (25)$$

(6a) and (25) imply that

$$\int_{B_1}\left\{|D^k v_h|^2+\lambda_h^{p-1}|D^k v_h|^p\right\}dy \leq c \qquad \text{if } 2\leq p<\infty, \quad (26)$$
$$\int_{B_1}|D^k v_h|^p\,dy \leq c \qquad \text{if } 1\leq p<2. \quad (27)$$

In particular

$$\fint_{B_1}|D^k v_h|^q \leq c, \quad (28)$$

where, here and in the following, $q=\min\{2,p\}$. Hence by (24), possibly passing to not relabelled subsequences

$$v_h \rightharpoonup v \quad W^{k,q}(B_1;\mathbb{R}^N)\,, \qquad A_h\to A\,, \qquad |A|\leq M\,. \quad (29)$$

Step 2. *v solves a linear system.* Since u is a minimum point for $I(\cdot)$, rescaling the Euler equation in terms of v_h, we get

$$\int_{B_1}\frac{\partial F}{\partial \xi_\alpha^i}(A_h+\lambda_h D^k v_h)D_\alpha\phi^i\,dy=0, \qquad \forall\phi\in C_0^k(B_1;\mathbb{R}^N),$$

where $|\alpha|=k$. Here and in the sequel we use the convetion that repeted indeces are summed up.

$$\frac{1}{\lambda_h}\int_{B_1}\left[\frac{\partial F}{\partial \xi_\alpha^i}(A_h+\lambda_h D^k v_h)-\frac{\partial F}{\partial \xi_\alpha^i}(A_h)\right]D_\alpha\phi^i\,dy=0.$$

Splitting B_1 as

$$E_h^+\cup E_h^- = \{y\in B_1:\lambda_h|D^k v_h|\geq 1\}\cup\{y\in B_1:\lambda_h|D^k v_h|<1\}\,,$$

from (28), we get

$$|E_h^+| \le \int_{B_1} \lambda_h^q |D^k v_h|^q \le c\lambda_h^q. \tag{30}$$

Hence

$$\frac{1}{\lambda_h}\int_{E_h^+} |\frac{\partial F}{\partial \xi_\alpha^i}(A_h+\lambda_h D^k v_h) - \frac{\partial F}{\partial \xi_\alpha^i}(A_h)||D_\alpha \phi^i|\,dy$$
$$\le \frac{c(L,M)}{\lambda_h}||D^k\phi||_\infty \int_{E_h^+}(1+\lambda_h^{p-1}|D^k v_h|^{p-1})\,dy$$
$$\le c(L,M)||D^k\phi||_\infty \left\{ \frac{|E_h^+|}{\lambda_h} + \lambda_h^{p-2}|E_h^+|^{\frac{1}{p}}\left(\int_{B_1}|D^k v_h|^p\right)^{\frac{p-1}{p}} \right\}.$$

From (H1) and (20) we claim that the last term of the latter inequality vanishes as $h \to \infty$. Indeed for $2 \le p < \infty$ we have that $q = \min\{p,2\} = 2$ so the right hand side of the previous estimate reduces to

$$c(L,M)||D^k\phi||_\infty \lambda_h \left\{1 + \left(\int_{B_1}\lambda_h^{p-2}|D^k v_h|^p\right)^{\frac{p-1}{p}}\right\},$$

because of (30) and this goes to zero by (26). Similarly, for $1 < p < 2$, we have that $q = p$ and

$$c(L,M)||D^k\phi||_\infty \lambda_h^{p-1}\left\{1 + \left(\int_{B_1}|D^k v_h|^p\right)^{\frac{p-1}{p}}\right\} \to 0$$

because of (30) and (27). We have then

$$\frac{1}{\lambda_h}\int_{E_h^-}[\frac{\partial F}{\partial \xi_\alpha^i}(A_h+\lambda_h D^k v_h) - \frac{\partial F}{\partial \xi_\alpha^i}(A_h)]D_\alpha\phi^i\,dy$$
$$= \int_{E_h^-} dy \int_0^1 [\frac{\partial^2 F}{\partial \xi_\alpha^i \partial \xi_\beta^j}(A_h + s\lambda_h D^k v_h) - \frac{\partial^2 F}{\partial \xi_\alpha^i \partial \xi_\beta^j}(A_h)]D_\beta v_h^j D_\alpha \phi^i\,ds$$
$$+ \int_{E_h^-}\frac{\partial^2 F}{\partial \xi_\alpha^i \partial \xi_\beta^j}(A_h)D_\beta v_h^j D_\alpha\phi^i\,dy = I_h + II_h.$$

We may suppose $\lambda_h D^k v_h \to 0$ a.e., so that, as $D^2 F$ is continuous, $I_h \to 0$. On the other hand (30) implies that the characteristic functions $1_{E_h^-} \to 1$ in L^μ for all $\mu < \infty$. Then $II_h \to \int_{B_1} D^2F(A) D^k v D^k \phi \, dy$. Finally v satysfies

$$\int_{B_1} \frac{\partial^2 F}{\partial \xi_\alpha^i \partial \xi_\beta^j}(A) D_\alpha v^i D_\beta \phi^j = 0.$$

It is well known that (H1) and (H2) yield

$$c(\nu, M)|\lambda^2||v|^{2k} \leq \frac{\partial^2 F}{\partial \xi_\alpha^i \partial \xi_\beta^j}(A) \lambda^i \lambda^j v_\alpha v_\beta \leq c(M)|\lambda|^2 |v|^{2k},$$

hence from Proposition 2.5 v is C^∞. Furthermore from the theory of linear elliptic system (see [9] Theorem 2.1 chap 3), and (16) if $0 < \tau < 1/2$ we have

$$\begin{aligned} -\!\!\!\!\!\!\int_{B_\tau} |D^k v - (D^k v)_\tau|^2 \, dy &\leq c(M)\tau^2 -\!\!\!\!\!\!\int_{B_{\frac{1}{2}}} |D^k v - (D^k v)_{\frac{1}{2}}|^2 \, dy \\ &\leq c(M)\tau^2 \sup |D^k v|^2 \leq c(M)\tau^2 \|D^k v\|^2_{W^{k,q}(B_1;\mathbb{R}^N)} \leq c^*(M)\tau^2 \end{aligned} \tag{31}$$

Step 3. *Higher integrability.* If we set

$$F_h(\xi) = \lambda_h^{-2}[F(A_h + \lambda_h \xi) - F(A_h) - \lambda_h DF(A_h)\xi] \tag{32}$$

from (H1), (H2) and Lemma 3.1 it follows that

$$|F_h(\xi)| \leq \frac{c(M)}{\lambda_h^2} |V(\lambda_h \xi)|^2, \tag{33}$$

$$\int_{B_1} F_h(D^k \phi) \, dy \geq \frac{\nu}{\lambda_h^2} \int_{B_1} |V(\lambda_h D^k \phi)|^2, \tag{34}$$

for every $\phi \in C_0^k(B_1; \mathbb{R}^N)$. For any $0 < r < 1$, set

$$I_r^h(w) = \int_{B_r} F_h(D^k w(y)) \, dy.$$

One can then easily verify that v_h is a minimum for $I_r^h(\cdot)$. Hence, by Proposition 2.4 applied to $g(\xi) = \lambda_h^2 F_h(\xi)$ and inequalities (25) and (9)

$$\begin{aligned}
\fint_{B_{\frac{1}{2}}} |V(\lambda_h D^k v_h)|^{2(1+\delta)} dy &\le c \left(\fint_{B_1} |V(\lambda_h D^k v_h|^2 \right)^{1+\delta} dy \\
&= \left(\fint_{B_{R_h}(x_h)} |V(D^k u - A_h)\, dy|^2 \right)^{1+\delta} \\
&\le c(M) \left(\int_{B_{R_h}(x_h)} |V(D^k u) - V(A_h)|^2 \right)^{1+\delta} \le c\lambda_h^{2(1+\delta)}.
\end{aligned} \tag{35}$$

From (6a) and the very same argument that yields (28), it follows that $(D^k v_h)_h$ is bounded in $L^{q(1+\delta)}(B_{\frac{1}{2}}; \mathbb{R}^{mN})$.

Step 4. *Upper bound.* Fixed $r < \frac{1}{3^k}$, it is not restrictive to assume that

$$\limsup_h [I_r^h(v_h) - I_r^h(v)]$$

exists. We claim that this limit is 0. Choose $s < r$ and take $\vartheta \in C_0^k(B_r)$ such that $0 \le \vartheta \le 1$, $\vartheta = 1$ on B_s and $|D^l \vartheta| \le c/(r-s)^l$ for $l = 1, \dots, k$. If we set $\overline{q} = \max\{2, p\}$ and $\phi_h = (v - v_h)\vartheta$, by (4), (5), (33) and the minimality of v_h, it follows

$$\begin{aligned}
I_r^h(v_h) - I_r^h(v) &\le I_r^h(v_h + \phi_h) - I_r^h(v) \\
&= \int_{B_r \setminus B_s} [F_h(D^k v_h + D^k \phi_h) - F_h(D^k v)]\, dy \\
&\le \frac{c}{\lambda_h^2} \int_{B_r \setminus B_s} [|V(\lambda_h D^k v)|^2 + |V(\lambda_h D^k((v_h - v)\vartheta) + D^k v_h)|^2]\, dy \\
&\le \frac{c}{\lambda_h^2} \int_{B_r \setminus B_s} \Big(|V(\lambda_h D^k v)|^2 + |V(\lambda_h D^k v_h)|^2 \\
&\qquad + \sum_{l=0}^{k-1} \frac{1}{(r-s)^{\overline{q}(k-l)}} |V(\lambda_h D^l (v_h - v)|^2 \Big)\, dy\,.
\end{aligned}$$

Now since v is smooth on B_1, it follows that

$$\frac{1}{\lambda_h^2}\int_{B_r\setminus B_s}|V(\lambda_h D^k v|^2\,dy \le c\left\{1+\left(\sup_{B_r}|D^k v|\right)^{\overline{q}}\right\}(r-s)$$

because of (4) if $2 \le p < \infty$ and inequality $|V(\xi)| \le |\xi|$ if $1 < p < 2$. From (35) we get

$$\begin{aligned}
&\int_{B_r\setminus B_s}|V(\lambda_h D^k v_h)|^2\,dy\\
&\qquad\le\left(\int_{B_r\setminus B_s}|V(\lambda_h D^k v_h)|^{2(1+\delta)}\right)^{\frac{1}{1+\delta}}|B_r\setminus B_s|^{\frac{\delta}{1+\delta}}\\
&\qquad\le c\lambda_h^2(r-s)^{\frac{\delta}{1+\delta}}.
\end{aligned}$$

As much as concerns the remaining summands we distiguish now three cases.

A) $2 < p < \infty$

Denoting by P_h the polynomial of degree $k-1$ such that $\int_{B_1} D^l(P_h - (v_h)) = 0$ for $l < k$ and setting

$$p^* = \begin{cases}\frac{np}{n-lp}, & \text{if} \quad p < \frac{n}{l},\\ r > p, & \text{if} \quad p \ge \frac{n}{l},\end{cases}$$

then $\mu \in (0,1)$ exists such that $1/p = \mu/p^* + (1-\mu)/2$. From (6a) we deduce that

$$\begin{aligned}
&\frac{1}{\lambda_h^2}\int_{B_r\setminus B_s}|V(\lambda_h D^l(v-v_h))|^2\,dy\\
&\le c\int_{B_1}|D^l(v-v_h)|^2\,dy + c\int_{B_1}\lambda_h^{p-2}|D^l(v-v_h)|^p\,dy = I_h + II_h.
\end{aligned} \tag{36}$$

Now

$$
\begin{aligned}
II_h &\leq \lambda_h^{p-2}\left(\int_{B_1}|D^l(v-v_h)|^2\right)^{\frac{p(1-\mu)}{2}}\left(\int_{B_1}|D^l(v-v_h)|^{p^*}\right)^{\frac{p\mu}{p^*}}\\
&\leq c\lambda_h^{p-2}\left(\int_{B_1}|D^l((v-v_h)-P_h)|^{p^*}\right)^{\frac{p\mu}{p^*}}\\
&\qquad + c\lambda_h^{p-2}\left(\int_{B_1}|D^l(P_h)|^{p^*}\right)^{\frac{p\mu}{p^*}}\\
&\leq c\lambda_h^{p-2}\left(\int_{B_1}|D^k v_h|^q\right)^{\mu}\overset{(26)}{\leq} c\lambda_h^{(p-2)(1-\mu)},
\end{aligned}
$$

where we have used the Sobolev-Poincaré inequality, and the fact that $D^l v_h$ converges to $D^l v$ strongly L^2 for $l < k$. As $\mu < 1$, then $\lim_h I_h + II_h = 0$.

B) $p = 2$

It reduces to I_h.

C) $1 < p < 2$

Denoting by $\tilde{P}_h$ the polynomial of Proposition 2.2 with $R = 3^{-k}$, $x = 0$ and $u = v - v_h$, let θ be such that $\frac{1}{2} = \theta + \frac{1-\theta}{2(1+\sigma)}$. Then since

$V(\xi) \leq |\xi|$ if $p < 2$, from the smoothness of v and from (25), we get

$$\begin{aligned}
&\int_{B_r \setminus B_s} |V(\lambda_h D^l(v - v_h))|^2 \, dy \\
&\quad \leq \left(\int_{B_r \setminus B_s} |V(\lambda_h D^l(v - v_h))| \, dy \right)^{2\theta} \\
&\qquad \times \left(\int_{B_r \setminus B_s} |V(\lambda_h D^l(v - v_h))|^{2(1+\sigma)} \, dy \right)^{\frac{1-\theta}{1+\sigma}} \\
&\quad \leq c\lambda_h^{2\theta} \left(\int_{B_r \setminus B_s} |D^l(v - v_h)| \, dy \right)^{2\theta} \\
&\times \left\{ \left(\int_{B_{3^{l-k}}} |V(\lambda_h D^l((v - v_h) - \tilde{P}_h))|^{2(1+\sigma)} \, dy \right)^{\frac{1-\theta}{1+\sigma}} \right\} + \lambda_h^{2(1-\theta)} \\
&\quad \leq c\lambda_h^{2\theta} \left(\int_{B_r \setminus B_s} |D^l(v - v_h)| \, dy \right)^{2\theta} \\
&\qquad \times \left(\int_{B_1} |V(\lambda_h D^k(v_h - v))|^2 \, dy + |\lambda_h|^2 \right)^{1-\theta} \\
&\quad \leq c\lambda_h^2 \left(\int_{B_1} |D^l(v - v_h)| \, dy \right)^{2\theta}.
\end{aligned}$$

where we used the fact that

$$\int_{B_1} |\tilde{P}_h|^{2(1+\sigma)} \, dy \leq c \|v_h - v\|^{2(1+\sigma)}_{W^{k-1,p}(B_1;\mathbb{R}^N)} \leq c.$$

Finally letting s go to r we prove the claim.
Step 5. *Lower bound.* We claim that if $t < r < \frac{1}{3^k+1}$ then

$$\limsup_h \frac{1}{\lambda_h^2} \int_{B_t} |V(\lambda_h(D^k v - D^k v_h))|^2 \, dy \leq c \limsup_h [I_r^h(v_h) - I_r^h(v)].$$

Let $\varphi \in C_0^k(B_{3^{-k}})$ be a cut-off function between $B_{\frac{1}{3^k+1}}$ and $B_{\frac{1}{3^k}}$, $|D^l \varphi| < c$ for $l \leq k$. Set

$$\tilde{v}_h = \varphi v_h; \qquad \tilde{v} = \varphi v$$

We may assume that the exponent δ given by the higher integrability estimate (35) is always less or equal to the one, σ, provided by the Sobolev-Poincaré inequality. Therefore, by (5)

$$\int_{\mathbb{R}^n} |V(\lambda_h D^k \tilde{v}_h)|^{2(1+\sigma)} dy \leq c \int_{B_{3^{-k}}} \sum_{l=0}^{k} |V(\lambda_h D^l v_h)|^{2(1+\sigma)} dy.$$

Now the summand corresponding to $l = k$ is bounded by $c\lambda_h^{2(1+\delta)}$ because of (35) whereas, for $l = 0, \ldots, k-1$ we have that

$$\begin{aligned}\int_{B_{3^{-k}}} |V(\lambda_h D^l v_h)|^{2(1+\sigma)} dy &\leq c \left(\int_{B_{3^{-k}}} |V(\lambda_h D^l v_h)|^{2(1+\delta)} dy \right)^{\frac{1+\delta}{1+\sigma}} \\ &\leq c \left\{ \int_{B_{3^{l-k}}} |V(\lambda_h D^l (v_h - P_h))|^{2(1+\sigma)} dy \right. \\ &\qquad \left. + \int_{B_{3^{l-k}}} |V(\lambda_h D^l P_h)|^{2(1+\sigma)} dy \right\}^{\frac{1+\delta}{1+\sigma}},\end{aligned}$$

where P_h is the polynomial associated to v_h with $R = 3^{l-k}$ (see Proposition 2.2). Then, on account either of (4) if $2 \leq p < \infty$, or of $|V(\xi)| \leq |\xi|$ if $1 < p < 2$, it follows from the definition of P_h that

$$\int_{B_{3^{l-k}}} |V(\lambda_h D^l P_h)|^{2(1+\sigma)} dy \leq c\lambda_h^{2(1+\sigma)}.$$

As to the other terms, Proposition 2.2 yields that

$$\int_{B_{3^{l-k}}} |V(\lambda_h D^l (v_h - P_h))|^{2(1+\sigma)} dy \leq c \left(\int_{B_1} |V(\lambda_h D^k v_h)|^2 \right)^{1+\delta},$$

and from inequality (25) we finally get

$$\int_{\mathbb{R}^n} |V(\lambda_h D^k \tilde{v}_h)|^{2(1+\sigma)} dy \leq c\lambda_h^{2(1+\delta)}.$$

Now Lemma 2.6 implies that for any h

$$\begin{aligned}\frac{1}{\lambda_h} \Big[&\|V(\lambda_h D^k \tilde{v}_h)\|_{L^{2(1+\delta)}(\mathbb{R}^n;\mathbb{R}^N)} \\ &\qquad + \|V(\lambda_h M(D^k \tilde{v}_h))\|_{L^{2(1+\delta)}(\mathbb{R}^n;\mathbb{R}^N)} \Big] \leq c,\end{aligned}$$

so if we fix $\varepsilon > 0$, there exists $\eta > 0$ such that if $G \in \mathbb{R}^n$ is measurable with $|G| < \eta$ then

$$\frac{1}{\lambda_h^2}\left[\int_G |V(\lambda_h D^k \tilde{v}_h)|^2\,dy + \int_G |V(\lambda_h M(D^k \tilde{v}_h))|^2\,dy\right] \leq \varepsilon. \tag{37}$$

By the continuity of the maximal function in L^q spaces, there exists $K_0 = K_0(\varepsilon) > 1$ such that, setting $S_h = \{y : |M(D^k \tilde{v}_h)(y)| > K\}$ for $K > K_0$, then

$$|S_h| < \eta \qquad \text{for any } h. \tag{38}$$

Notice that for $t \in (0, \infty)$, $(1 + t^2)^{\frac{p-2}{2}} t^2$ is increasing for every $1 < p < \infty$ so from the definition of S_h and from the previous inequalities

$$|S_h|(1 + \lambda_h^2 K^2)^{\frac{p-2}{2}} K^2 \leq \frac{1}{\lambda_h^2}\int_{S_h} |V(\lambda_h M(D^k \tilde{v}_h))|^2 dy \leq \varepsilon,$$

hence for h large enough

$$|S_h| \leq \frac{2\varepsilon}{K^2}. \tag{39}$$

By the approximation Lemma 2.7 a sequence w_h in $W^{k,\infty}(\mathbb{R}^n; \mathbb{R}^N)$ exists such that

$$w_h = \tilde{v}_h \quad \text{on} \quad \mathbb{R}^n \setminus S_h, \qquad \|D^k w_h\|_\infty \leq cK. \tag{40}$$

Furthermore, $w_h \overset{*}{\rightharpoonup} w$ in $W^{k,\infty}(\mathbb{R}^n; \mathbb{R}^N)$. Let us now consider

$$\begin{aligned} I_r^h(v_h) - I_r^h(v) &= [I_r^h(\tilde{v}_h) - I_r^h(w_h)] + [I_r^h(w_h) - I_r^h(w)] \\ &\quad + [I_r^h(w) - I_r^h(v)] = R_1^h + R_2^h + R_3^h. \end{aligned} \tag{41}$$

By (37) and (38) we have

$$\frac{1}{\lambda_h^2}\int_{S_h} |V(\lambda_h D^k \tilde{v}_h)|^2 \leq \varepsilon,$$

and for every $K > K_0(\varepsilon)$, there is $h_0 = h_0(\varepsilon)$ such that

$$h \geq h_0 \Longrightarrow \frac{1}{\lambda_h^2}|V(\lambda_h D^k w_h)|^2 = (1 + \lambda_h^2 |D^k w_h|^2)^{\frac{p-2}{2}} |D^k w_h|^2 \leq 2K^2.$$

Hence,

$$\limsup_h \frac{1}{\lambda_h^2}\int_{S_h} |V(\lambda_h D^k w_h)|^2 \le \limsup_h 2K^2|S_h| \le 4\varepsilon$$

by (39). Finally from the previous estimates we have

$$\begin{aligned}\limsup_h |R_1^h| &\le \limsup_h \int_{S_h\cap B_r} [F_h(D^k\tilde v_h) - F_h(D^k w_h)]\, dy \\ &\le \limsup_h \frac{c}{\lambda_h^2}\int_{S_h} |V(\lambda_h D^k \tilde v_h)|^2 + |V(\lambda_h D^k w_h)|^2 dy \le c\varepsilon. \quad (42)\end{aligned}$$

Fix now $t < s < r$, and let ϑ be a cut-off function between B_s and B_r just as in previous step. Setting

$$\phi_h = \vartheta(w_h - w),$$

$$\begin{aligned}R_2^h &= [I_r^h(w_h) - I_r^h(w+\psi_h)] \\ &\quad + [I_r^h(w+\phi_h) - I_r^h(w) - I_r^h(\phi_h)] + [I_r^h(\phi_h)] \\ &= R_4^h + R_5^h + R_6^h\,,\end{aligned}$$

by (5) we have

$$\begin{aligned}|R_4^h| &\le \int_{B_r\setminus B_s} |F_h(D^k w_h) - F_h(D^k w + D^k\phi_h)|\, dy \\ &\le \frac{c}{\lambda_h^2}\int_{B_r\setminus B_s} \Bigg[|V(\lambda_h D^k w_h)|^2 + |V(\lambda_h D^k w)|^2 \\ &\qquad + \sum_{l=0}^{k-1} \frac{1}{(r-s)^{\overline{q}(k-l)}} |V(\lambda_h D^l(w - w_h))|^2\Bigg]\, dy.\end{aligned}$$

By a similar argument employed for R_1^h and by (40), since $D^l w_h \to D^l w$ uniformly for $l < k$,

$$\limsup_h |R_4^h| \le c(K)(r-s).$$

Notice that from the definition of F_h (see (32))

$$R_5^h = \int_{B_r} dy \int_0^1\int_0^1 D^2F(A_h + s\lambda_h D^k w_h + t\lambda_h D^k\phi_h) D^k w D^k \phi_h\, ds\, dt,$$

and, no matter what K is, this quantity goes to 0 since $D^2F(A_h + s\lambda_h D^k w_h + t\lambda_h D^k\phi_h)$ uniformly converges to $D^2F(A)$.
On the other hand from (34)

$$\begin{aligned} R_6^h = \int_{B_r} F_h(D^k\phi_h) &\geq \frac{\nu}{\lambda_h^2}\int_{B_r} |V(\lambda_h D^k\phi_h)|^2 dy \\ &\geq \frac{\nu}{\lambda_h^2}\int_{B_s} |V(\lambda_h D^k(w_h - w))|^2 dy. \end{aligned}$$

Therefore, possibly passing to subsequences, we may suppose $\lim_h R_2^h$ exists and

$$\begin{aligned} \lim_h R_2^h \geq \limsup_h \frac{\nu}{\lambda_h^2}\int_{B_s} |V(\lambda_h(D^k w_h - D^k w))|^2 dy \\ - c(K)(r-s)\,. \end{aligned} \tag{43}$$

To deal with R_3^h we use a technique introduced in [1]: setting $S = \{y \in B_r : v(y) \neq w(y)\}$ and $\overline{S} = S \cap \{y \in B_r : v(y) = \lim_h v_h(y)\}$, then $|S| = |\overline{S}|$. Arguing by contraddiction, we now prove that

$$|S| \leq \frac{3\varepsilon}{K^2}. \tag{44}$$

Were this inequality false, then by (39) for h large enough we would have

$$|\overline{S} \setminus S_h| > \frac{\varepsilon}{K^2}\,,$$

but by Lemma 2.8 there is a $\overline{y} \in B_r$ such that

$$\overline{y} \in \overline{S} \setminus S_h \qquad \text{for infinitely many } h.$$

Passing to this subsequence, we get

$$v(\overline{y}) = \lim_h v_h(\overline{y}) = \lim_h w_h(\overline{y}) = w(\overline{y}),$$

hence $\overline{y} \in S$ and (44) is proved. Now, since $D^k v = D^k w$ a.e. in $B_r \setminus S$, we have

$$\begin{aligned} |R_3^h| &\leq \int_{B_r\cap S} |F_h(D^k w) - F_h(D^k v)|\, dy \\ &\leq \frac{c}{\lambda_h^2}\int_{B_r\cap S} |V(\lambda_h D^k v)|^2 + |V(\lambda_h D^k w)|^2\, dy \end{aligned}$$

and by the very same argument used to prove (42) we get

$$\limsup_h |R_3^h| \leq c\varepsilon.$$

By this inequality, (41), (42) and (43) we deduce

$$\begin{aligned}\lim_h [I_r^h(v_h) - I_r^h(v)] \geq \limsup_h \frac{\nu}{\lambda_h^2}\int_{B_s} |V(\lambda_h(D^k w_h - D^k w))|^2\,dy \\ - c(K)(r-s) - c\varepsilon\,.\end{aligned}$$

On the other hand

$$\begin{aligned}\frac{1}{\lambda_h^2}\int_{B_t} &|V(\lambda_h(D^k v - D^k v_h))|^2 dy \\ &\leq \frac{c}{\lambda_h^2}\int_{B_s} |V(\lambda_h(D^k w - D^k w_h))|^2 dy \\ &+ \frac{c}{\lambda_h^2}\int_{B_s\cap S_h} |V(\lambda_h(D^k w_h - D^k v_h))|^2 dy \\ &+ \frac{c}{\lambda_h^2}\int_{B_s\cap S} |V(\lambda_h(D^k v - D^k w))|^2 dy \\ &= R_7^h + R_8^h + R_9^h.\end{aligned}$$

Arguing as for R_1^h, we get

$$R_8^h \leq \frac{1}{\lambda_h^2}\int_{B_s\cap S_h} |V(\lambda_h D^k \tilde{v}_h)|^2 \leq c\varepsilon\,,$$

and similarly

$$R_9^h \leq \frac{1}{\lambda_h^2}\int_{B_s\cap S} |V(\lambda_h D^k \tilde{v})|^2 \leq c\varepsilon\,.$$

From these last estimates we finally conclude that

$$\begin{aligned}\limsup_h \frac{1}{\lambda_h^2}\int_{B_t} |V(\lambda_h(D^k v - D^k v_h))|^2 dy \leq c[\limsup_h I_r^h(v_h) - I_r^h(v)] \\ + c\varepsilon + c(K)(r-s),\end{aligned}$$

and letting $s \to r$ and then $\varepsilon \to 0^+$ we obtain the result.
Step 6. *Conclusion of the proof.* If $0 < \tau < \frac{1}{3^k+1}$ then by the previous two steps

$$\lim_h \frac{1}{\lambda_h^2} \int_{B_r} |V(\lambda_h(D^k v - D^k v_h))|^2 = 0$$

hence by (5), (6) and (31)

$$\begin{aligned}
\limsup_h \frac{E(x_h, \tau R_h)}{\lambda_h^2} &= \lim_h \frac{1}{\lambda_h^2} \fint_{B_{\tau R_h}(x_h)} |V(D^k u) - V((D^k u)_{x_h, \tau r_h})|^2 \\
&\le \limsup_h \frac{1}{\lambda_h^2} \fint_{B_\tau} |V(\lambda_h(D^k v - (D^k v_h)_\tau))|^2 \\
&\le \limsup_h \frac{c}{\lambda_h^2} \fint_{B_\tau} [|V(\lambda_h(D^k v_h - D^k v))|^2 \\
&\quad + |V(\lambda_h(D^k v - (D^k v)_\tau))|^2 + V(\lambda_h((D^k v)_\tau - (D^k v_h)_\tau))|^2]\, dy \\
&\le c^*(M)\tau^2 + \lim_h |(D^k v)_\tau - (D^k v_h)_\tau|^{\overline{q}},
\end{aligned}$$

and as $D^k v_h \rightharpoonup D^k v$ in $L^q(B_1; \mathbb{R}^{mN})$,

$$\limsup_h \frac{E(x_h, \tau R_h)}{\lambda_h^2} \le c^*(M)\tau^2,$$

which contraddicts (24) if we choose $C_M = c^*(M)$ □

The proof of Theorem 3.2 follows from the previuos Proposition by a standard argument see [5] Theorem 3.2.

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