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PHASE SPACE ANALYSIS APPLIED TO GEOPHYSICAL FLUIDS AND THERMOELASTICITY

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Introduction

Since it was used for the first time, the phase space analysis has become one of the most important tool in dealing with partial differential equations. The starting idea behind this theory is that there is a strong connection between space and frequency when one has to consider physical models and for this reason also mathematical analysis must take into account both variables. Fourier trasform is indeed the main tool in this context, since it is the bridge between the two visions. One of the theories that arose from this approach and that will be widely used in this thesis thanks to its flexibility is Littlewood-Paley ([LP]) decomposition, that is based on a dyadic partition of unity in the frequency space (see e.g. [AG] or [M]). first of all this theory allows to define Sobolev spaces in an equivalent way using the properties of the L^2 norms of the dyadic blocks, then it also gives an easy way to define interpolation spaces, such as Besov spaces, that will be repeatedly used in this work. This technique gives also origin to other fundamental instruments, like Bony's paraproduct that will be crucial to prove the inequalities necessary for one of the main results (see [B]).

The thesis is devoted to prove three theorems, the first two deals with existence and uniqueness of mild and weak solutions of Navier-Stokes equations for geophysical incompressible fluids, while the third concerns uniqueness for thermoelastic system. Here we give an overview of the three results.

We start from the topic discussed in chapter 2 and 3, namely Navier-Stokes equations for geophysical fluids, that we will refer to as Navier-Stokes-Coriolis equations, which are represented by the following system

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \Omega e_3 \times \mathbf{u} + \nabla p = 0 \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{1}$$

We focus our attention on mild solutions for this system, whose existence and uniqueness will be proved by using a suitable fix point theorem on the following equation, that can be obtained from system (1)

$$\mathcal{T}\mathbf{u} = \mathcal{G}(t)\mathbf{u_0} + B(\mathbf{u}, \mathbf{u}), \tag{2}$$

with

$$B(\mathbf{u}, \mathbf{v}) := -\int_0^t \mathcal{G}(t-\tau)\mathbb{P}\nabla \cdot (\mathbf{u} \otimes \mathbf{v})\mathbf{d}\tau,$$

and where $\mathcal{G}(t)$ is the following time dependant operator

$$\mathcal{G}(t)w = \mathfrak{F}^{-1}\left(\left(\cos\left(\Omega t \frac{\xi_3}{|\xi|}\right) I + \sin\left(\Omega t \frac{\xi_3}{|\xi|}\right) R(\xi)\right) e^{-\nu t|\xi|^2}\widehat{w}(\xi)\right),$$

See chapter 1 for precise definitions. Our purpose is to deal with an initial datum belonging to a low regular space: the idea comes from the result obtained in a similar framework on classical Navier-Stokes equations. Let us briefly describe this point. It is known that Navier-Stokes equations have a scaling property, which means that if $u(t,x)$ is a solution of N-S system then also

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$$

still represents a solution. Since a fixed point theorem will be used to prove existence of solutions, the norm of the spaces in which this theorem will be performed must be invariant with respect to this scaling (scaling invariance is also connected with other properties of the solutions like selfsimilarity, remarked in the pioneering work [L]).

The first invariant space used in studying mild solution for three dimensional NS equations is the Sobolev space $\dot{H}^{\frac{1}{2}}$ and this space was used by Fujita and Kato ([FK]) in 1964, while some years later Kato ([K]) generalized the previous result considering the space L^3.

Thanks to the characterization of Besov spaces, Cannone, Meyer and Planchon were able to obtain existence and uniqueness result for the space $\dot{B}^{\frac{3}{p}-1}_{p,\infty}$, providing a result in which highly oscillating initial data can be considered, since negative index of Besov space is allowed (see [CMP] and [CP]).

The final step was obtained by Koch and Tataru, who proved the sharpest result on this field ([KT]). They were able to enlarge the result in [CMP] to the space ∇BMO. The last space that could be considered is the largest scaling invariant space in $\mathbb{R}^3$, that is $\dot{B}^{-1}_{\infty,\infty}$, but for this space a counterexample of non existence has been proved.

We can suggestively resume all these results in the following picture, which properly underline the continuous imbeddings that link these spaces.

$$\underbrace{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}_{Fujita-Kato} \hookrightarrow \underbrace{L^3(\mathbb{R}^3)}_{Kato} \hookrightarrow \underbrace{\dot{B}^{\frac{3}{p}-1}_{\infty,p}(\mathbb{R}^3)}_{\substack{Cannone\\ Meyer\\ Planchon}} \hookrightarrow \underbrace{\nabla BMO(\mathbb{R}^3)}_{Koch-Tataru} \hookrightarrow B^{-1}_{\infty,\infty}(\mathbb{R}^3).$$

Before describing the technique of the proof, let us consider what can be said so far about Navier-Stokes-Coriolis equations: from [HS] we know that existence and uniqueness of mild solutions are known when the initial datum belongs to the Sobolev space $\dot{H}^{\frac{1}{2}}$. In chapter 2 we will obtain this result using the same technique used by Chemin in [C]. In the following chapter we try to follow the road represented above by proving two theorems, the first using L^3 space, the second using the Besov space $\dot{B}^{\frac{3}{p}-1}_{p,\infty}$, but limiting ourselves to $3 < p < 4$. We state here only the theorem concerning Besov space, since the first is analougous.

Theorem 1 *Let $3 < p < 4$. There exists a constant $c > 0$, depending on Ω/ν, such that if $\|u_0\|_{\mathcal{B}_{\dot{H}^{\frac{1}{2}},\dot{B}^{\frac{3}{p}-1}_{p,\infty}}} \leq c$, then there exist a unique fixed point $u(t) \in K_4$ of (2).*
Moreover it holds that

- *i)* $\|u(t)\|_{K_4} \leq 2\|\mathcal{G}(t)u_0\|_{K_4}$,
- *ii)* $u \in L^\infty(0,+\infty;\mathcal{B}_{\dot{H}^{\frac{1}{2}},\dot{B}^{\frac{3}{p}-1}_{p,\infty}})$,
- *iii)* $B(u,u) \in \mathcal{C}(0,+\infty;\dot{H}^{\frac{1}{2}})$ and $\lim_{t\to 0^+} \|B(u(t),u(t))\|_{\dot{H}^{\frac{1}{2}}} = 0$.

The space K_p is defined in this way

$$K_p := \left\{ f \in \mathcal{C}(]0,+\infty[;L^p)/\|f\|_{K_p} := \sup_{t\in[0,+\infty[} t^{\frac{1}{2}(1-\frac{3}{p})}\|f\|_{L^p} < +\infty \right\},$$

while the definition of the space $\mathcal{B}_{\dot{H}^{\frac{1}{2}},\dot{B}^{\frac{3}{p}-1}_{p,\infty}}$ is the following:

let $\chi_{B(0,1)}$ be the characteristic function of the ball $B(0,1)$, we define

$$\begin{aligned} f^{(1)}(t,x) &= \mathfrak{F}^{-1}(\chi_{B(0,1)}\widehat{f}) \\ f^{(2)}(t,x) &= \mathfrak{F}^{-1}((1-\chi_{B(0,1)})\widehat{f}), \end{aligned}$$

we say that $f \in \mathcal{B}_{\dot{H}^{\frac{1}{2}},\dot{B}^{\frac{3}{p}-1}_{p,\infty}}$ if

$$\|f\|_{\mathcal{B}_{\dot{H}^{\frac{1}{2}},\dot{B}^{\frac{3}{p}-1}_{p,\infty}}} = \|f^{(1)}\|_{\dot{H}^{\frac{1}{2}}} + \|f^{(2)}\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} < +\infty,$$

in the three dimensional case it holds that

$$\dot{H}^{\frac{1}{2}} \hookrightarrow \mathcal{B}_{\dot{H}^{\frac{1}{2}},\dot{B}^{\frac{3}{p}-1}_{p,\infty}},$$

so this theorem improves the result in [HS], moreover the choice of Besov spaces with negative index allows us to choose highly oscillating initial data. The idea of hybrid spaces is due to the work in [CMZ], where the authors obtain a result similar to ours, considering a Ω dependent space for initial datum.

The proof of this theorem consists essentially of three steps:

i) we first study the mapping property of $\mathcal{G}(t)$ acting on the initial datum, mainly focusing on its codomine;

ii) we prove a bilinear estimate for $B(\cdot,\cdot)$;

iii) we make use of a fix point theorem, that gives us the existence and uniqueness of a mild solution.

In chapter 4 we will briefly address the following anisotropic Navier-Stokes-Coriolis system

$$\begin{cases} \partial_t \mathbf{u} - \nu_h \Delta_h \mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + B(t,x_1,x_2)\times\mathbf{u} + \frac{1}{\rho}\nabla p = 0 \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0 \end{cases} \qquad (ANSC) \tag{3}$$

where the operator $\Delta_h = \partial^2_{x_1} + \partial^2_{x_2}$ is the horizontal laplacian. The system is called anisotropic since 0 vertical viscosity is considered. For this system we study here only existence: in fact we prove the following theorem

Theorem 2 *Let $u_0 \in B^{0,\frac{1}{2}}$ a divergence free vector fields. There exists $c > 0$ such that if $\|u_0\|_{B^{0,\frac{1}{2}}} \leq c\nu_h$ then there exists a global solution u of ANSC such that*

$$u \in L^\infty(\mathbb{R}_+; B^{0,\frac{1}{2}}) \quad \text{and} \quad \nabla_h u \in L^2(\mathbb{R}_+; B^{0,\frac{1}{2}}).$$

This theorem is the consequence of two results obtained in [MP] and [P]: in the first the system (ANSC) is studied in the anisotropic Sobolev space $H^{0,s}$, for $s > \frac{1}{2}$, while in the second anisotropic Navier-Stokes system (without Coriolis term) is studied in the anisotropic Besov space $B^{0,\frac{1}{2}}$, since the case $s = \frac{1}{2}$ can only be treated using this space instead of Sobolev one. This Besov space is defined as follows

$$\|u\|_{B^{0,\frac{1}{2}}} = \sum_{q\in\mathbb{Z}} 2^{\frac{q}{2}} \|\Delta_q^v u\|_{L^2} < +\infty,$$

where Δ_q^v is the vertical Littlewood-Paley decomposition. This result does not represent a proper generalization of [MP] since

$$B^{0,\frac{1}{2}} \hookrightarrow H^{0,\frac{1}{2}},$$

but it allows to go beyond the assumption $s > \frac{1}{2}$.

The proof relies on Ascoli-Arzelà theorem: first we will write a sequence of approximating systems of equations and build a sequence of solutions, then we will prove that there exists a subsequence converging to the solution of (3). Littlewood-Paley decomposition will play a crucial role in different points in this context, since it will be used to prove the boundness condition necessary to use Ascoli-Arzelà theorem.

The theorems in [MP] and [P] also deal with continuity and uniqueness of the solution. In our work we do not describe completely these two points, but we think that one can obtain a better result than the one above, since Coriolis term does not add any real difficulty and the proof in [P] can be repeated also in this case. So the theorem one could prove is the following:

Theorem 3 *Let $u_0 \in B^{0,\frac{1}{2}}$ a divergence free vector fields. There exists $c > 0$ such that if $\|u_0\|_{B^{0,\frac{1}{2}}} \leq c\nu_h$ then there exists a unique global solution u of ANSC such that*

$$u \in C_b(\mathbb{R}_+; B^{0,\frac{1}{2}}) \quad \text{and} \quad \nabla_h u \in L^2(\mathbb{R}_+; B^{0,\frac{1}{2}}).$$

The third result, shown in chapter 5, concerns the uniqueness of the solution for the following backward thermoelastic system

$$\begin{cases} \left.\begin{aligned} &\partial_t^2 u - \partial_x(a(t,x)\partial_x u) + \alpha(t,x)\partial_x\theta(t,x) + \beta(t,x)\theta(t,x) = f(t,x) \\ &\partial_t\theta + \partial_x(b(t,x)\partial_x\theta) + \delta(t,x)\partial_x\partial_t u(t,x) + \rho(t,x)\partial_t u(t,x) + \phi(t,x)u(t,x) = g(t,x) \end{aligned}\right\}]0,T[\times\mathbb{R} \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x),\ \theta(0,x) = \theta_0(x) \qquad \text{on } \mathbb{R}, \end{cases} \tag{4}$$

where low regularity assumptions for the coefficients $a(t,x)$ and $b(t,x)$ are considered, i.e.

$$\begin{aligned} a(t,x) &\in LL(\mathbb{R}_t^+, LL(\mathbb{R}_x)) \\ b(t,x) &\in \mathcal{C}^\mu(\mathbb{R}_t^+, \mathcal{C}^{1,\epsilon}(\mathbb{R}_x)), \end{aligned} \tag{5}$$

where LL refers to log-lipschitz space, while μ is a modulus of continuity satistfying Osgood condition, namely

$$\int_0^1 \frac{1}{\mu(t)} dt = +\infty,$$

and another technical condition, that we will refer to as $\star$ condition. These results improves the one by Koch and Lasiecka (see [KL]), where the same system was studied considering Lipschitz regularity in space and time for a and b. The combination of two Carleman inequalities, that we will describe below, used in [KL] is actually the same in this work. At the end of the chapter we will also introduce a non Lipschitz function space, whose modulus of continuity satisfies Osgood and $\star$ condition. For a description of the use of Carleman inequalities see [Ca].

The system (4) is represented by two partial differential equations coupled: the coupling is represented by the term $\delta(t,x)\partial_x\partial_t u(t,x)$ that represents the influence of the motion on the heat tranfer process (see for example [Bi] and [ChS]).

The principal part of these two equations are composed by a hyperbolic and a backward parabolic operator, writen in divergence form. Studying the uniqueness problem for these operators when low regular coefficients are considered is not a new issue, but it is just a new episode of a long story. The first and pioneering result on this topic is contained in the work by Colombini, De Giorgi and Spagnolo [CDGS] in 1979, where they first addressed the problem of uniqueness for hyperbolic operator with time dependent non-Lipschitz coefficient.

Being the source of a very fertile production of results, this work soon became a classical result, since even more than thirty years later some of the techniques used in [CDGS] are still performed in the result proved nowadays. All these results are based on an energy estimate for the hyperbolic operator, with loss of derivatives. Results on hyperbolic operators have been obtained by Tarama in [T], Colombini and Lerner [CL] and Colombini and Del Santo [CDS], just to give a (very) short list. The result in [CL] is in fact the one that inspired the result, since the regularity used here is just the same, but since we need a Carleman estimate for the energy, we also need to borrow some ideas from [CDS], where Log-Zygmund coefficients are considered.

For the backward parabolic operator our guide is the work by Del Santo and Prizzi ([DSP]) where a Carleman estimates for the backward parabolic operator is obtained when the coefficients are continuous in time with a modulus of continuity that satisfies Osgood condition. Due to the final combination of the two inequalities we will use in this work, we need one more hypotesis, the so called $\star$ condition, that we will describe precisely in the first chapter.

Since the system is linear in order to prove uniqueness it is sufficient to prove that the homogeneous system associated with the one above has only the trivial solution. In order to do this we first obtain two Carleman estimates involving the operators $\mathcal{P}u = \partial_t^2 u - \partial_x(a(t,x)\partial_x u)$ and $\mathcal{L}\theta = \partial_t\theta + \partial_x(b(t,x)\partial_x\theta)$ and then we combine this two inequalities to conclude the proof. The weights used in the Carleman estimates are defined using the properties of the modulus of continuity μ, that will be descibed in chapter 1.

The two inequalities are the following

$$\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} E(t)dt \leq \frac{C_0}{\left(\Phi'\left(\frac{\gamma T}{2}\right)\right)^2} \int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \|\mathcal{P}u\|^2_{H^{-\omega-\beta^* t}} dt$$

$$\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \|\mathcal{L}\theta\|^2_{H^{-1-\omega-\beta^* t}} dt \geq$$

$$\geq K \int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \left(\gamma\|\theta\|^2_{H^{-1-\omega-\beta^* t}} + \sqrt{\gamma}\|\theta\|^2_{H^{-\omega-\beta^* t}} + \frac{1}{(\Phi'(\gamma T))}\|\theta\|^2_{H^{1-\omega-\beta^* t}}\right) dt,$$

where the energy $E(t)$ is equivalent to $\|\partial_t u\|_{H^{-\omega-\beta^* t}} + \|u\|_{H^{-\omega-\beta^* t}} + \|u\|_{H^{1-\omega-\beta^* t}}$.

$\Phi(\cdot)$ is a function defined using μ, while ω belongs to $]0, \frac{1}{2}[$. The two inequalities hold for any γ greater than a fixed γ_0. As stated above for the proof of these two inequalities a very important role will be played by Bony's paraproduct, necessary to address difficulties arising with commutators.

The combination of the two inequalities allows us to prove that $E(t) \equiv 0$ for t beloging to a small time interval, this implies the following theorem

Theorem 4 *Let*

$$v \in H^2([0,T], L^2(\mathbb{R}_x)) \cap H^1([0,T], H^1(\mathbb{R}_x)) \cap L^2([0,T], H^2(\mathbb{R}_x))$$

$$\zeta \in H^1([0,T], L^2(\mathbb{R})) \cap L^2([0,T], H^2(\mathbb{R}_x))$$

solutions of (4)*, with* $f \equiv g \equiv u_0 \equiv u_1 \equiv \theta_0 \equiv 0$.

Then under the hypotesis (5) *we have that*

$$v \equiv \zeta \equiv 0.$$

Contents

Chapter 1

Background

1.1 Navier-Stokes-Coriolis equations

In this work we will study Navier-Stokes equations for geophysical fluids, namely the following system:

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \Omega e_3 \times \mathbf{u} + \nabla p = 0 \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{1.1}$$

where $\mathbf{u}$ is the velocity field, p the pressure, ν the viscosity and Ω the Coriolis parameter, which is proportional to the inverse of Rossby number.

This system models the behaviour of a fluid in the presence of the Coriolis force, which appears when one considers the dynamics in a rotating framework: the term $\Omega e_3 \times \mathbf{u}$ represents the consequence of rotation on the velocity field $\mathbf{u}$. For a detailed description of this system and all its features we refer to [CB].

Using the divergence free condition we can recast the system in the following way

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \Omega e_3 \times \mathbf{u} + \nabla p = 0 \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{1.2}$$

1.1.1 Semigroup computation

We start solving, in a formal way, the following linearized system.

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \Omega e_3 \times \mathbf{u} + \nabla p = 0 \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{1.3}$$

where we set $\nu = 1$ to fix ideas.

For convenience we now write this system component by component:

$$\begin{cases} \partial_t u_1 - \Delta u_1 - \Omega u_2 + \partial_x p = 0 \\ \partial_t u_2 - \Delta u_2 + \Omega u_1 + \partial_y p = 0 \\ \partial_t u_3 - \Delta u_3 \qquad\quad + \partial_z p = 0 \\ \partial_x u_1 + \partial_y u_2 + \partial_z u_3 = 0 \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{1.4}$$

We want to use Fourier transform, so we need the Fourier transform of p. To find it we take the divergence of the first three equation and we make use of the divergence free condition, thus obtaining:

$$\Delta p = \Omega(\partial_x u_2 - \partial_y u_1) = \Omega \omega_3,$$

in which we obtain the third component ω_3 of the vorticity vector field. Using Fourier transform we get:

$$\widehat{p} = -\frac{\Omega \widehat{\omega_3}}{|\xi|^2},$$

with $\widehat{\omega_3} = -i(\xi_1 \widehat{u_2} - \xi_2 \widehat{u_1})$. We write the system in frequency space and we obtain

$$\begin{cases} \partial_t \widehat{u_1} + |\xi|^2 \widehat{u_1} - \Omega \widehat{u_2} + i\xi_1 \dfrac{\Omega \widehat{\omega_3}}{|\xi|^2} = 0 \\ \partial_t \widehat{u_2} + |\xi|^2 \widehat{u_2} + \Omega \widehat{u_1} + i\xi_2 \dfrac{\Omega \widehat{\omega_3}}{|\xi|^2} = 0 \\ \partial_t \widehat{u_3} + |\xi|^2 \widehat{u_3} \qquad\quad + i\xi_3 \dfrac{\Omega \widehat{\omega_3}}{|\xi|^2} = 0 \\ \xi_1 \widehat{u_1} + \xi_2 \widehat{u_2} + \xi_3 \widehat{u_3} = 0. \end{cases} \tag{1.5}$$

We multiply the first equation by $i\xi_2$, the second by $i\xi_1$ and then we subtract the second from the first, obtaining

$$\partial_t \widehat{\omega_3} + |\xi|^2 \widehat{\omega_3} + i\Omega\xi_3 \widehat{u_3} = 0,$$

and so

$$\widehat{u_3} = i\frac{\partial_t \widehat{\omega_3} + |\xi|^2 \widehat{\omega_3}}{\Omega\xi_3}, \tag{1.6}$$

substituting on the third equation we obtain:

$$\partial_t (i\frac{\partial_t \widehat{\omega_3} + |\xi|^2 \widehat{\omega_3}}{\Omega\xi_3}) + |\xi|^2 i\frac{\partial_t \widehat{\omega_3} + |\xi|^2 \widehat{\omega_3}}{\Omega\xi_3} + \frac{i\Omega\xi_3}{|\xi|^2}\widehat{\omega_3} = 0.$$

Or equivalently:

$$\frac{(\partial_t + |\xi|^2)^2 \widehat{\omega_3}}{\Omega\xi_3} + \frac{\xi_3}{|\xi|^2}\Omega\widehat{\omega_3} = 0$$

Multiplying by $\Omega\xi_3$ and by $e^{|\xi|^2 t}$ and calling $f = ie^{|\xi|^2 t}\widehat{\omega_3}$ we can write

$$f'' + \Omega^2 \frac{\xi_3^2}{|\xi|^2} f = 0,$$

Writing explicitly the solution, we obtain

$$f(t) = A\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right) + B\sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)$$

and imposing the initial conditions we obtain

$$f(t) = (\xi_1 \widehat{u_2}(0) - \xi_2 \widehat{u_1}(0))\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right) + |\xi|\widehat{u_3}(0)\sin\left(\frac{\xi_3}{|\xi|}\Omega t\right).$$

Returning to $\widehat{\omega_3}(t)$ we find

$$\widehat{\omega_3}(t) = -ie^{-|\xi|^2 t}\left((\xi_1 \widehat{u_2}(0) - \xi_2 \widehat{u_1}(0))\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right) + |\xi|\widehat{u_3}(0)\sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)\right).$$

Returning now in (1.6) we obtain the expression for $\widehat{u_3}$:

$$\widehat{u_3} = e^{-|\xi|^2 t}\left(\widehat{u_3}(0)\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right) + \left(\frac{\xi_2}{|\xi|}\widehat{u_1}(0) - \frac{\xi_1}{|\xi|}\widehat{u_2}(0)\right)\sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)\right).$$

Similarly we can obtain the other components:

$$\widehat{\mathbf{u}}(t,\xi) = \cos\left(\frac{\xi_3}{|\xi|}\Omega t\right) e^{-|\xi|^2 t}\mathbf{I}\widehat{\mathbf{u}}_0(\xi) + \sin\left(\frac{\xi_3}{|\xi|}\Omega t\right) e^{-|\xi|^2 t}\mathbf{R}(\xi)\widehat{\mathbf{u}}_0(\xi), \tag{1.7}$$

where $\mathbf{I}$ s the identity matrix and

$$\mathbf{R}(\xi) = \frac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}.$$

The Stokes-Coriolis semigroup can be explicitly represented by:

$$\mathcal{G}(t)f = \mathfrak{F}^{-1}\left(\left(\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{I} + \sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{R}\right) e^{\nu t|\xi|^2}\widehat{f}\right),$$

1.1.2 Mild solution

Our goal is to study mild solution for Navier-Stokes-Coriolis, namely the fixed point of the map:

$$\mathcal{T}\mathbf{u} = \mathcal{G}(t)\mathbf{u_0} + B(\mathbf{u},\mathbf{u}),$$

with

$$B(\mathbf{u},\mathbf{v}) := -\int_0^t \mathcal{G}(t-\tau)\mathbb{P}\nabla\cdot(\mathbf{u}\otimes\mathbf{v})\mathbf{d}\tau,$$

where $\mathbb{P}$ is the Leray projector on divergence free vector field and $\mathcal{G}(t)$ is the semigroup associated with the linearized Stokes - Coriolis system.

We start working on $\mathcal{G}(t)$ defined by the linear problem

$$\begin{cases} \partial_t\mathbf{u} - \nu\Delta\mathbf{u} + \Omega e_3\times\mathbf{u} = f - \nabla p \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{1.8}$$

In the last section we obtained and wrote $\mathcal{G}(t)$ as

$$\mathcal{G}(t)w = \mathfrak{F}^{-1}\left(\left(\cos\left(\Omega t\frac{\xi_3}{|\xi|}\right)I + \sin\left(\Omega t\frac{\xi_3}{|\xi|}\right)R(\xi)\right)e^{-\nu t|\xi|^2}\widehat{w}(\xi)\right).$$

We obtain that

$$u(t) = \mathcal{G}(t)u_0 - \int_0^t \mathcal{G}(t-s)(\mathbb{P}(f(s)))ds$$

is solution of the problem (1.8), since $\mathcal{G}(t)u_0$ is the solution of

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \Omega e_3 \times \mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{1.9}$$

while Duhamel's principle tells us that $\int_0^t \mathcal{G}(t-s)(\mathbb{P}(f(s)))ds$ is the solution of

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \Omega e_3 \times \mathbf{u} = \mathbb{P}f \\ \mathbf{u}(0) = 0, \end{cases} \tag{1.10}$$

Now let us compute the divergence of $u(t)$, since $\nabla \cdot u_0 = 0$ and $\nabla \cdot (\mathbb{P}w) = 0$ we get

$$\begin{aligned} \nabla \cdot u(t) &= \nabla \cdot \left(\mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-s)(\mathbb{P}(f(s)))ds \right) \\ &= \mathcal{G}(t)(\nabla \cdot u_0) + \int_0^t \mathcal{G}(t-s)(\nabla \cdot (\mathbb{P}(f(s))))ds \\ &= 0, \end{aligned}$$

so $\nabla \cdot u = 0$ as we want.

Now, since

$$\mathbb{P}(\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \Omega e_3 \times \mathbf{u}) = \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \Omega e_3 \times \mathbf{u},$$

we obtain that

$$\mathbb{P}(\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \Omega e_3 \times \mathbf{u} - f) = 0,$$

and so

$$\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \Omega e_3 \times \mathbf{u} = \mathbb{P}f - \nabla p.$$

As a consequence of these computations the solutions of the problem (1.2) will be seen as the fixed point of

$$\mathbf{u}(t) = \mathcal{G}(t)\mathbf{u}_0 - \int_0^t \mathcal{G}(t-s)(\mathbb{P}(\nabla \cdot (\mathbf{u} \otimes \mathbf{u})))(s)ds, \tag{1.11}$$

with $\nabla \cdot \mathbf{u}_0 = 0$.

1.1.3 Existence of mild solutions

The strategy to show existence of mild solution is the following:

- choose a space $X(\mathbb{R}^3)$ for the initial data $\mathbf{u}_0$;
- choose a space $Y([0,T],\mathbb{R}^3)$ such that $\mathcal{G}(t)\mathbf{u}_0 \in Y$;
- prove that $B : Y \times Y \to Y$ is a bilinear bounded operator;
- use a fixed point argument.

The last point is represented by the following lemma:

Lemma 1 *Let $(Y, \|\cdot\|)$ be a Banach space and $B : Y \times Y \to Y$ a bilinear operator, such that*

$$\|B(y_1, y_2)\| \leq \eta \|y_1\| \|y_2\|$$

then for any $y_0 \in Y$ such that $4\eta\|y_0\| < 1$ the equation $y = y_0 + B(y, y)$ has a unique solution in the ball $B(0, \frac{1}{2\eta})$. In particular we have

$$\|y\| \leq 2\|y_0\|.$$

Proof. We start by defining the ball $B(0, R)$ with

$$R = \frac{1 - \sqrt{1 - 4\eta\|y_0\|}}{2\eta}.$$

We will prove that

$$T(y) = y_0 + B(y, y)$$

is a contraction in $B(0, R)$. First of all let us prove that $T : B(0, R) \mapsto B(0, R)$,

$$\|T(y)\| \leq \|y_0\| + \|B(y, y)\| \leq \|y_0\| + \eta\|y\|^2 \leq \|y_0\| + \eta R^2 = R,$$

where the last equality comes from the definition of R.

Now we consider y_1 and y_2 and we compute

$$\begin{aligned}\|T(y_1)-T(y_2)\| &= \|B(y_1, y_1-y_2)+B(y_1-y_2, y_2)\| \\ &\leq \eta(\|y_1\|+\|y_2\|)\|y_1-y_2\| \leq 2\eta R\|y_1-y_2\|,\end{aligned}$$

since $2\eta R < 1$ then we obtain that T is a contraction.

Thanks to Banach-Caccioppoli-Picard theorem we can state that the equation

$$y = y_0 + B(y, y)$$

has a unique solution in the ball $B(0, R)$. Moreover from this equation we get

$$\|y\| \leq \|y_0\| + \eta\|y\|^2$$

and so

$$\|y\|(1-\eta\|y\|) \leq \|y_0\|.$$

Now since $\|y\| \leq R \leq \dfrac{1}{2\eta}$, we finally obtain that

$$\|y\| \leq 2\|y_0\|.$$

To prove uniqueness in the ball $B(0, \frac{1}{2\eta})$ let $\widetilde{y} = y_0 + B(\widetilde{y}, \widetilde{y})$ be another solution, with $\widetilde{y} \in B(0, \frac{1}{2\eta})$. We have

$$\begin{aligned}\|y-\widetilde{y}\| &= \|B(y, y-\widetilde{y})+B(y-\widetilde{y}, \widetilde{y})\| \\ &\leq \eta(\|y\|+\|\widetilde{y}\|)\|y-\widetilde{y}\| \leq \eta(R+\frac{1}{2\eta})\|y-\widetilde{y}\| < \|y-\widetilde{y}\|.\end{aligned}$$

□

1.2 Littlewood-Paley theory

In this paragraph we briefly introduce Littlewood-Paley theory, showing some results concerning Sobolev and Hölder spaces

Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, $0 \le \varphi \le 1$, $\varphi_0(\xi) = 1$ if $|\xi| \le \frac{11}{10}$ and $\varphi_0(\xi) = 0$ if $|\xi| \ge \frac{19}{10}$.

For $\nu \in \mathbb{N}$, we consider $\varphi_\nu(\xi) = \varphi(2^{-\nu}\xi)$ and we define by $\widetilde{\varphi}_\nu(x)$ the inverse Fourier transform of $\varphi_\nu(\xi)$.

We define the following operators

$$\mathcal{S}_{-1}u = 0, \qquad \mathcal{S}_\nu u = \widetilde{\varphi}_\nu * u = \varphi_\nu(D)u, \qquad \text{for} \qquad \nu \ge 0,$$

$$\Delta_0 u = \mathcal{S}_0 u, \qquad \Delta_\nu u = \mathcal{S}_\nu u - \mathcal{S}_{\nu-1}u, \qquad \text{for} \qquad \nu \ge 1.$$

Remark.

The Littlewood-Paley decomposition can be defined in a “homogeneous” way as follows

$$\Delta_\nu u = \mathcal{S}_\nu u - \mathcal{S}_{\nu-1}u \quad \text{for } \nu \in \mathbb{Z}, \tag{1.12}$$

this definitionis will be used in the definition of homogenous spaces.

We now report three proposition without proof concerning characterization of Sobolev and Hölder spaces through Littlewood-Paley decomposition:

Proposition 1 *Let $s \in \mathbb{R}$. A tempered distribution u belongs to H^s if and only if the following two conditions hold:*

- *for every $\nu \ge 0$, $\Delta_\nu u \in L^2$*
- *the sequence $\delta_\nu = 2^{\nu s}\|\Delta_\nu u\|_{L^2}$ belongs to ℓ^2*

Moreover there exists $C_s > 1$ such that, for every $u \in H^s$,

$$\frac{1}{C_s}\|u\|_{H^s} \le \left(\sum_\nu \delta_\nu^2\right)^{\frac{1}{2}} \le C_s\|u\|_{H^s}. \tag{1.13}$$

Proposition 2 *Let $s \in \mathbb{R}$ and let $R > 2$. Let $(u_k)_k$ a sequence of L^2 functions such that*

i) supp $\widehat{u_0}$ is contained in $\{|\xi| \le R\}$ and supp $\widehat{u_k}$ is contained in $\{\frac{1}{R}2^k \le |\xi| \le R2^k\}$ for $k \ge 1$.

ii) the sequence $(\delta_k)_k$ with $\delta_k = 2^{ks}\|u_k\|_{L^2}$ is in ℓ^2.

Then $\sum_k u_k$ is a converging series in H^s with sum $u \in H^s$ and the norm in H^s of u is equivalent to the norm in ℓ^2 of $(\delta_k)_k$ (that is there exists C_s such that (1.13) holds.

If $s > 0$, it is sufficient to suppose that supp $\widehat{u}_k$ is contained in $\{|\xi| \leq R2^k\}$ for every $k \geq 1$.

Proposition 3 *A bounded function a belongs to $\mathcal{C}^{1,\epsilon}$ (with $0 < \epsilon < 1$), the space of Hölder functions of indices $1, \epsilon$, if and only if the sequence $(\alpha_k)_k$, with $\alpha_k = 2^{k(1+\epsilon)}\|\Delta_k a\|_{L^\infty}$, is in ℓ^∞.*

Moreover there exists $C_\epsilon > 1$, such that, for every $a \in \mathcal{C}^{1,\epsilon}$,

$$\frac{1}{C_\epsilon}\|a\|_{\mathcal{C}^{1,\epsilon}} \leq \sup_k \alpha_k \leq C_\epsilon \|a\|_{\mathcal{C}^{1,\epsilon}}. \tag{1.14}$$

Paraproduct

We now introduce Bony's paraproduct, giving some results that can be obtained using this tool. Let's start with a definition.

Definition. Let $a \in L^\infty$ and $u \in H^s$. Bony's paraproduct $T_a u$ is defined as follows

$$T_a u = \sum_{\nu=3}^{+\infty} S_{\nu-3} a \Delta_\nu u.$$

The following two propositions concern with some mapping properties of the paraproduct:

Proposition 4 *i) Let $s \in \mathbb{R}$ and $a \in L^\infty$. Then $T_a \colon H^s \to H^s$ and*

$$\|T_a u\|_{H^s} \leq C_s \|a\|_{L^\infty} \|u\|_{H^s}.$$

ii) Let $\epsilon \in]0,1[$, $a \in \mathcal{C}^{1,\epsilon}$, $s \in]0, 1+\epsilon]$. Then $u \mapsto au - T_a u$ maps H^{-s} in H^{1-s} and

$$\|au - T_a u\|_{H^{1-s}} \leq C_\epsilon \|a\|_{\mathcal{C}^{1,\epsilon}} \|u\|_{H^{-s}}.$$

Proof. Since the result of i) is classical and can be found in [B] we give only the proof of ii). We have

$$au - T_a u = \underbrace{\sum_{k=3}^{+\infty} \Delta_k a \mathcal{S}_{k-3} u}_{(A)} + \underbrace{\sum_{k=3}^{+\infty} (\sum_{\substack{j \geq 0 \\ |j-k| \leq 2}} \Delta_k a \Delta_j u)}_{(B)},$$

regarding (A) we can observe that the support of the Fourier transform of $\Delta_k a \mathcal{S}_{k-3} u$ is contained in $\{2^{k-2} \leq |\xi| \leq 2^{k+2}\}$ and one has

$$\begin{aligned} \|\Delta_k a \mathcal{S}_{k-3} u\|_{L^2} &\leq \|\Delta_k a\|_{L^\infty} \|\mathcal{S}_{k-3} u\|_{L^2} \\ &\leq C_\epsilon \|a\|_{\mathcal{C}^{1,\epsilon}} 2^{-k(1+\epsilon)} \sum_{j=0}^{k-3} 2^{js} \delta_j, \end{aligned}$$

where $(\delta_j)_j$ is in ℓ^2 and $\|(\delta_j)_j\|_{\ell^2} = C_s \|u\|_{H^{-s}}$. This yields

$$\|\Delta_k a \mathcal{S}_{k-3} u\|_{L^2} \leq C_\epsilon \|a\|_{\mathcal{C}^{1,\epsilon}} 2^{-k(1+\epsilon-s)} \sum_{j=0}^{k-3} 2^{-(k-j)s} \delta_j.$$

Now, defining $\widetilde{\delta_k} = \sum_{j=0}^{k-3} 2^{-(k-j)s} \delta_j$, as a consequence of Young inequality for convolution in ℓ^p, we have that $(\widetilde{\delta_k})_k \in \ell^2$ and

$$\|(\widetilde{\delta_k})_k\|_{\ell^2} \leq 2 \|(\delta_k)_k\|_{\ell^2}.$$

We easily obtain that $(A) \in H^{1-s}$ and

$$\|(A)\|_{H^{1-s}} \leq C_\epsilon \|a\|_{\mathcal{C}^{1,\epsilon}} \|u\|_{H^{-s}}.$$

Considering now (B) we have

$$(B) = \sum_{k=3}^{+\infty} (\Delta_k a \Delta_{k-2} u + \Delta_k a \Delta_{k-1} u + \Delta_k a \Delta_k u + \Delta_k a \Delta_{k+1} u + \Delta_k a \Delta_{k+2} u).$$

Every term can be treated analogously. We take the term $\sum_{k=3}^{+\infty} \Delta_k a \Delta_k u$ as an example. The support of its Fourier transform is contained in $\{|\xi| \leq 2^{k+2}\}$ and

$$\begin{aligned} \|\Delta_k a \Delta_k u\|_{L^2} &\leq \|\Delta_k a\|_{L^\infty} \|\Delta_k u\|_{L^2} \\ & C_\epsilon \|a\|_{\mathcal{C}^{1,\epsilon}} 2^{-k(1+\epsilon)} 2^{-ks} \delta_k, \end{aligned}$$

where as before $(\delta_j)_j \in \ell^2$ and $\|(\delta_j)_j\|_{\ell^2} = C_s \|u\|_{H^{-s}}$.

As a consequence $\sum_{k=3}^{+\infty} \Delta_k a \Delta_k u \in H^{1-s}$ and

$$\|\sum_{k=3}^{+\infty} \Delta_k a \Delta_k u\|_{H^{1-s}} \leq C_\epsilon \|a\|_{\mathcal{C}^{1,\epsilon}} \|u\|_{H^{-s}}.$$

□

Remark. In the last part, since the support of the Fourier transform of $\Delta_k a \Delta_k u$ is contained in a ball (and not in a ring) we can apply the second part of Proposition 2 and so it is necessary that $1 + \epsilon - s > 0$. On the contrary it is easy to see that the result of Proposition 4 is valid also for $s = 0$.

The next result will be necessary in dealing with the backward parabolic operator of the thermoelastic system in Chapter 5.

Proposition 5 *Let* $\epsilon \in]0, 1[$, $s \in]0, 1 + \epsilon]$, $a \in \mathcal{C}^{1,\epsilon}$. *Then*

$$\left(\sum_{\nu=0}^{+\infty} 2^{-2\nu s} \|\partial_x([\Delta_\nu, T_a]\partial_x u)\|_{L^2}^2 \right)^{\frac{1}{2}} \leq C_\epsilon \|a\|_{\mathcal{C}^{1,\epsilon}} \|u\|_{H^{1-s}},$$

for every $u \in H^{1-s}$.

($[A, B]$ *denotes the commutator of* A *and* B*, i.e.* $[A, B]u = A(B(u)) - B(A(u))$.*)*

Proof. We start by observing that

$$[\Delta_\nu, T_a]w = \sum_{K=3}^{+\infty} [\Delta_\nu, \mathcal{S}_{k-3} a] \Delta_k w,$$

so

$$\partial_x([\Delta_\nu, T_a]\partial_x u) = \partial_x\left(\sum_{K=3}^{+\infty}[\Delta_\nu, \mathcal{S}_{k-3}a]\Delta_k(\partial_x u)\right).$$

□

From the support of the Fourier transform of $[\Delta_\nu, \mathcal{S}_{k-3}]\Delta_k w$ we can obtain that $[\Delta_\nu, \mathcal{S}_{k-3}]\Delta_k w$ is identically zero if $|k-\nu| \geq 4$.

This allows us to infer that the sum in k reduces at most to 7 terms, so

$$\partial_x([\Delta_\nu, T_a]\partial_x u) = \partial_x([\Delta_\nu, \mathcal{S}_{\nu-6}a]\Delta_{\nu-3}(\partial_x u)) + \ldots + \partial_x([\Delta_\nu, \mathcal{S}_\nu a]\Delta_{\nu+3}(\partial_x u)).$$

The support of Fourier transform of each of these terms is contained in a ball proportional to 2^ν. We consider one of these terms: one has

$$\begin{aligned}\|\partial_x([\Delta_\nu, \mathcal{S}_{\nu-3}a]\Delta_\nu(\partial_x u))\|_{L^2} &\leq C2^\nu\|[\Delta_\nu, \mathcal{S}_{\nu-3}a]\partial_x(\Delta_\nu u)\|_{L^2} \\ &\leq C2^\nu\|a\|_{Lip}\|\Delta_\nu u\|_{L^2},\end{aligned}$$

where the first inequality comes from Bernstein's inequality, while the second derives from theorem 35 in Coifman and Meyer ([CM], see also [Ta]). The Proposition follows from this inequality and proposition 1.

1.3 Modulus of continuity

Let $\mu : [0,1] \to [0,1]$ continuous, concave and strictly increasing function, with $\mu(0) = 0$. We say thatμ is a *modulus of continuity*. Let $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$. We define $f \in \mathcal{C}^\mu(I, \mathbb{R})$ if $f \in L^\infty(I, \mathbb{R})$ and

$$\sup_{\substack{0 < |t-s| < 1 \\ t, s \in I}} \frac{|f(t) - f(s)|}{\mu(|t-s|)} < +\infty.$$

For example if $\mu(s) = s$ then $\mathcal{C}^\mu = Lip$.

In the following elementary proposition we collect some useful results:

Proposition 6 *Let μ be a modulus of continuity. Then*

- $\mu(s) \geq s\mu(1)$ for all $s \in [0,1]$;
- *the function* $s \mapsto \dfrac{\mu(s)}{s}$ *is decreasing in* $]0,1]$*;*
- *there exists* $\lim_{s\to 0^+} \dfrac{\mu(s)}{s}$*;*
- *the function* $\sigma \mapsto \dfrac{\mu(\frac{1}{\sigma})}{(\frac{1}{\sigma})}$ *is increasing in* $[1,+\infty]$*;*
- *the function* $\sigma \mapsto \dfrac{1}{\sigma^2\mu(\frac{1}{\sigma})}$ *is decreasing in* $[1,+\infty]$.

We remark that if $\sup_{s\in]0,1]} \dfrac{\mu(s)}{s} < +\infty$, then there exists $C > 0$ such that $\mu(s) \leq Cs$ for any $s \in [0,1]$: so $\mathcal{C}^\mu = Lip$. Moreover, if $C^\mu \neq Lip$, then $\lim_{s\to 0^+} \dfrac{\mu(s)}{s} = +\infty$.

We now define the so called *Osgood condition*.

Definition. (**Osgood condition**)

Let μ be a modulus of continuity. We say that μ satisfies the Osgood condition if:

$$\int_0^1 \frac{1}{\mu(s)} ds = +\infty.$$

From the modulus of continuity we will now define a weight function, that will play an important role in the study of the thermoelastic system.

Let μ be a modulus of continuity satisfying the Osgood condition, we define

$$\phi(t) = \int_{\frac{1}{t}}^1 \frac{1}{\mu(s)} ds,$$

$\phi \in \mathcal{C}^1$ and it is strictly increasing. From the Osgood condition we obtain that $\phi([1,+\infty[) = [0,+\infty[$ and $\phi'(t) = \frac{1}{(t^2\mu(\frac{1}{t}))} > 0$ for any $t \in [1,+\infty[$. We define:

$$\Phi(\tau) = \int_0^\tau \phi^{-1}(s) ds,$$

so $\Phi'(\tau) = \phi^{-1}(\tau)$, then

$$\lim_{\tau \to \infty} \Phi'(\tau) = +\infty.$$

Moreover

$$\Phi''(\tau) = (\Phi'(\tau))^2 \mu \left(\frac{1}{\Phi'(\tau)} \right), \tag{1.15}$$

for every $\tau \in [0, +\infty[$ and since $\sigma \mapsto \sigma\mu(\frac{1}{\sigma})$ is an increasing funtion on $[1, +\infty[$, we obtain that:

$$\lim_{\tau \to +\infty} \Phi''(\tau) = \lim_{\tau \to +\infty} (\Phi'(\tau))^2 \mu \left(\frac{1}{\Phi'(\tau)} \right) = +\infty. \tag{1.16}$$

To conclude this section we introduce a property of $\mu(s)$ that will be fundamental in the last chapter, where we will also give an example of a class of non - Lipschitz functions whose modulus of continuity satisfies this condition

Definition. ($\star$ **condition**)

Let μ be a modulus of continuity and let Φ defined as above: we say that μ satisfies the $\star$ condition if there exists $0 < a < 1$ such that

$$\lim_{s \to +\infty} \frac{\Phi'(s)}{(\Phi'(as))^2} = 0. \tag{$\star$}$$

We remark that since Φ' is an increasing function then, if μ satisfies this condition with a_0, the same condition holds for every $0 < a_0 < a < 1$.

1.4 Functional spaces

In this section we introduce some functional spaces used in this thesis, focusing in particular on the less known ones.

Besov spaces

We start by introducing Besov spaces, using the definition of the operators Δ_j given in section 1.2. We start with the inhomogeneous version: we say that $f \in B^s_{p,q}$ if

$$\|f\|_{B^s_{p,q}} = \left(\sum_{j\in\mathbb{N}} 2^{qjs}\|\Delta_j f\|^q_{L^p}\right)^{\frac{1}{q}} < +\infty,$$

while for the homogeneous version we say that $f \in \dot{B}^s_{p,q}$ if

$$\|f\|_{\dot{B}^s_{p,q}} = \left(\sum_{j\in\mathbb{Z}} 2^{qjs}\|\Delta_j f\|^q_{L^p}\right)^{\frac{1}{q}} < +\infty,$$

where in this second case the operators Δ_j are defined as in 1.12. We remark that in the case $p = q = 2$ we have that

$$B^s_{2,2} = H^s$$

where H^s is the Sobolev space of index s.

In this work we will also use an anistropic version of Besov spaces, namely the space $B^{0,s}$ whose norm is defined as follows

$$\|f\|_{B^{0,s}} = \sum_{j\in\mathbb{Z}} 2^{js}\|\Delta^v_j f\|_{L^2} < +\infty,$$

where the operators Δ^v_j define a monodimensional version of Littlewood-Paley decomposition.

The space $\widetilde{\mathbf{L^p}}$

In chapter 3 we will use the space $\widetilde{L^\infty}(\mathbb{R}^+, B^{0,\frac{1}{2}})$, so it is worth to define $\widetilde{L^p}$ spaces and remark its connections with the space L^p, using the space $B^{0,\frac{1}{2}}$ for example. We say that u belongs to $\widetilde{L^p}(\mathbb{R}^+, B^{0,\frac{1}{2}})$ if

$$\|u\|_{\widetilde{L^p}(\mathbb{R}^+,B^{0,\frac{1}{2}})} = \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\|\Delta^v_j u\|_{L^p(\mathbb{R}^+,L^2(\mathbb{R}))} < +\infty.$$

This definition slightly differs from the usual space $L^p(\mathbb{R}^+, B^{0,\frac{1}{2}})$, that is defined as follows

$$\|u\|_{L^p(\mathbb{R}^+,B^{0,\frac{1}{2}})} = \left\| \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \|\Delta_j^v u\|_{L^2(\mathbb{R}))} \right\|_{L^p(\mathbb{R}^+)} < +\infty,$$

and it is easy to prove that

$$\|u\|_{L^p(\mathbb{R}^+,B^{0,\frac{1}{2}})} \leq \|u\|_{\widetilde{L^p}(\mathbb{R}^+,B^{0,\frac{1}{2}})}.$$

Chapter 2

Mild solutions for the Navier-Stokes-Coriolis system: the $\dot{H}^{\frac{1}{2}}$ case

In this section we investigate the existence and uniqueness of mild solutions for the Navier-Stokes-Coriolis equations with Sobolev initial data. We consider the problem

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \Omega e_3 \times \mathbf{u} + \nabla p = 0 \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{2.1}$$

where $\mathbf{u}$ is the velocity field, p the pressure, ν the viscosity and Ω the Coriolis parameter.

Using the divergence free condition we can recast the system in the following way

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \Omega e_3 \times \mathbf{u} + \nabla p = 0 \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{2.2}$$

2.1 Technical results

In this section we investigate some properties of the solution of the linear problem in the case of $\dot{H}^s$ initial data.

Lemma 2 *Let $v_0 \in \dot{H}^s$ and $f \in L^2([0,T];\dot{H}^{s-1})$. We set*

$$v(t) = \mathcal{G}(t)v_0 + \int_0^t \mathcal{G}(t-s)f(s)ds.$$

Then

$$v \in \left(\bigcap_{p=2}^{+\infty} L^p([0,T];\dot{H}^{s+\frac{2}{p}})\right) \cap \mathcal{C}([0,T];\dot{H}^s).$$

Moreover

(i) $\|v(t)\|^2_{\dot{H}^s} + 2\nu \int_0^t \|\nabla v(t')\|^2_{\dot{H}^s} dt' = \|v_0\|^2_{\dot{H}^s} + 2\int_0^t < f(t'), v(t') >_s dt'$;

(ii) $\int_{\mathbb{R}^3} |\xi|^{2s} (\sup_{0\le t'\le T} |\widehat{v}(t')|)^2 d\xi \le 9\left(\|v_0\|_{\dot{H}^s} + \frac{1}{\sqrt{2\nu}}\|f\|_{L^2([0,T];\dot{H}^{s-1})}\right)^2$;

(iii) $\|v(t)\|_{L^p([0,T];H^{s+\frac{2}{p}})} \le \frac{3}{(\nu)^{\frac{1}{p}}}\left(\|v_0\|_{\dot{H}^s} + \frac{1}{\sqrt{\nu}}\|f\|_{L^2([0,T];\dot{H}^{s-1})}\right)$.

Proof. One has

$$\begin{aligned}\widehat{v}(t,\xi) &= \left(\cos\left(\Omega t\frac{\xi_3}{|\xi|}\right)Id + \sin\left(\Omega t\frac{\xi_3}{|\xi|}\right)R(\xi)\right)e^{-\nu|\xi|^2 t}\widehat{v}_0(\xi)\\ &\quad + \int_0^t \left(\cos\left(\Omega(t-s)\frac{\xi_3}{|\xi|}\right)Id + \sin\left(\Omega(t-s)\frac{\xi_3}{|\xi|}\right)R(\xi)\right)e^{-\nu|\xi|^2(t-s)}\widehat{f}(s,\xi)ds\end{aligned}$$

so we can write

$$\begin{aligned}|\widehat{v}(t,\xi)| &\le 3|\widehat{v}_0(\xi)| + \int_0^t 3e^{-\nu|\xi|^2(t-s)}|\widehat{f}(t-s,\xi)|ds\\ &\le 3\left(|\widehat{v}_0(\xi)| + \left(\int_0^t e^{-2\nu|\xi|^2(t-s)}ds\right)^{\frac{1}{2}}\left(\int_0^t |\widehat{f}(t-s,\xi)|^2 ds\right)^{\frac{1}{2}}\right)\\ &\le 3\left(|\widehat{v}_0(\xi)| + \frac{1}{\sqrt{2\nu|\xi|^2}}\|\widehat{f}(\cdot,\xi)\|_{L^2([0,t])}\right),\end{aligned}$$

and then

$$|\xi|^{2s}\left(\sup_{0\leq t'\leq t}|\widehat{v}(t',\xi)|\right)^2 \leq 9\left(|\widehat{v}_0(\xi)| + \frac{1}{\sqrt{2\nu|\xi|^2}}\|\widehat{f}(\cdot,\xi)\|_{L^2([0,t])}\right)^2|\xi|^{2s},$$

and finally we obtain (ii), in fact

$$\begin{aligned}\left(\int_{\mathbb{R}^3}|\xi|^{2s}\left(\sup_{0\leq t'\leq t}|\widehat{v}(t',\xi)|\right)^2 d\xi\right)^{\frac{1}{2}} &\leq 3\||\widehat{v}_0(\xi)| + \frac{1}{\sqrt{2\nu|\xi|^2}}\|\widehat{f}(\cdot,\xi)\|_{L^2([0,t])}\|_{\dot{H}^s}\\ &\leq 3\|\widehat{v}_0\|_{\dot{H}^s} + \frac{3}{\sqrt{2\nu}}\left(\int_{\mathbb{R}^3}\|\widehat{f}(\cdot,\xi)\|^2_{L^2[0,t]}|\xi|^{2s-2}d\xi\right)^{\frac{1}{2}}\\ &\leq 3\left(\|\widehat{v}_0\|_{\dot{H}^s} + \frac{1}{\sqrt{2\nu}}\|f\|_{L^2([0,t],\dot{H}^{s-1})}\right).\end{aligned}$$

Now we can deduce that $v \in \mathcal{C}([0,T],\dot{H}^s)$. First we observe that form (ii) just obtained, one has $v \in L^\infty([0,T],\dot{H}^s)$ in fact

$$\begin{aligned}\int_{\mathbb{R}^3}|\xi|^{2s}|\widehat{v}(t,\xi)|^2 d\xi &\leq \int_{\mathbb{R}^3}|\xi|^{2s}\left(\sup_{0\leq t\leq T}|\widehat{v}(t,\xi)|^2\right)d\xi\\ &\leq 9\left(\|\widehat{v}_0\|_{\dot{H}^s} + \frac{1}{\sqrt{2\nu}}\|f\|_{L^2([0,t],\dot{H}^{s-1})}\right)^2,\end{aligned}$$

the continuity is consequence of Lebesgue dominated convergence theorem, since

$$|\xi|^{2s}|\widehat{v}(t,\xi)|^2 \leq 18\left(|\widehat{v}_0(\xi)|^2 + \frac{1}{2\nu|\xi|^2}\int_0^T|\widehat{f}(s,\xi)|^2 ds\right)|\xi|^{2s}.$$

Let us now prove (i). Suppose first that $v \in L^2([0,T];\dot{H}^{s+1}) \cap \mathcal{C}([0,T],\dot{H}^s)$; since we have

$$\begin{cases}\partial_t v - \nu\Delta v + \Omega e_3 \times v = f - \nabla p\\ v(0) = v_0,\end{cases}$$

then

$$\partial_t\widehat{v} + \nu|\xi|^2\widehat{v} + \Omega e_3 \times \widehat{v} = \widehat{f} - \widehat{\nabla p},$$

multiplying by $\widehat{v}(\xi)|\xi|^{2s}$ and integrating with respect to ξ, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|v(t)\|_{\dot{H}^s}\right)^2+\nu\|\nabla v(t)\|^2_{\dot{H}^s}=<\widehat{f}(t),\widehat{v}(t)>_s,$$

and so

$$\|v(t)\|^2_{\dot{H}^s}-\|v_0\|^2_{\dot{H}^s}+2\nu\int_0^t\|\nabla v(t')\|^2_{\dot{H}^s}dt'=2\int_0^t<\widehat{f}(t'),\widehat{v}(t')>_s dt',\qquad(2.3)$$

Let now $v\in L^2([0,T];\dot{H}^s)$ with $v_0\in\dot{H}^s$ and $f\in L^2([0,T];\dot{H}^{s-1})$. We take a sequence $(v_{0,n})_n\in\mathcal{C}_0^\infty(\mathbb{R})$ with $v_{0,n}\to v_0$ in $\dot{H}^s$ and $(f_n)_n\in\mathcal{C}^\infty([0,T];\mathcal{C}_0^\infty(\mathbb{R}^3))$ with $f_n\to f$ in $L^2([0,T];\dot{H}^{s-1})$, in this way we obtain a sequence $(v_n)_n$ of solutions such that

$$v_n\to v\qquad\text{in}\qquad L^2([0,T];\dot{H}^s)$$

with $(v_n)_n$ bounded in $L^2([0,T];\dot{H}^{s+1})$.

In order to justify this last assertion, we use (2.3) obtaining

$$\|v(t)\|^2_{\dot{H}^s}+2\nu\int_0^t\|\nabla v(t')\|^2_{\dot{H}^s}dt'\leq\|v_0\|^2_{\dot{H}^s}+\int_0^t<\widehat{f}(t'),\widehat{v}(t')>_s dt'$$

and so

$$\begin{aligned}\|v(t)\|^2_{\dot{H}^s}+2\nu\int_0^t\|\nabla v(t')\|^2_{\dot{H}^s}dt' &\leq\|v_0\|^2_{\dot{H}^s}+\left(\int_0^t\|f(t')\|^2_{\dot{H}^{s-1}}dt'\right)^{\frac{1}{2}}\left(\int_0^t\|v(t')\|^2_{\dot{H}^{s+1}}dt'\right)^{\frac{1}{2}}\\ &\leq\|v_0\|^2_{\dot{H}^s}+C_\epsilon\left(\int_0^t\|f(t')\|^2_{\dot{H}^{s-1}}dt'\right)+\epsilon\left(\int_0^t\|v(t')\|^2_{\dot{H}^{s+1}}dt'\right)\end{aligned}$$

then

$$\|v(t)\|^2_{\dot{H}^s}+(2\nu-\epsilon)\int_0^t\|\nabla v(t')\|^2_{\dot{H}^s}dt'\leq\|v_0\|^2_{\dot{H}^s}+C_\epsilon\int_0^t\|f(t')\|^2_{\dot{H}^{s-1}}dt',$$

from which we obtain, applying to $v_{0,n}$ and f_n,

$$\|v_n(t)\|^2_{\dot{H}^s}+\nu\int_0^t\|\nabla v_n(t')\|^2_{\dot{H}^s}dt'\leq K,$$

from this we deduce that also $v \in L^2([0,T], \dot{H}^{s+1})$.

Now from (i) we deduce that

$$2\nu \int_0^T \|v(t)\|^2_{\dot{H}^{s+1}} dt \le \|v_0\|^2_{\dot{H}^s} + \frac{1}{\nu}\int_0^T \|f\|^2_{\dot{H}^{s-1}} + \nu \int_0^T \|v(t)\|^2_{\dot{H}^{s+1}} dt,$$

from which we obtain

$$\begin{aligned}\int_0^T \|v(t)\|^2_{\dot{H}^{s+1}} &\le \frac{1}{\nu}\left(\|v_0\|^2_{\dot{H}^s} + \frac{1}{\nu}\int_0^T \|f(t)\|^2_{\dot{H}^{s-1}} dt\right)\\ &\le \frac{1}{\nu}\left(\|v_0\|^2_{\dot{H}^s} + \frac{1}{\sqrt{\nu}}\left(\int_0^T \|f\|^2_{\dot{H}^{s-1}} dt\right)^{\frac{1}{2}}\right)^2,\end{aligned}$$

and so, finally

$$\|v\|_{L^2([0,T];\dot{H}^{s+1})} \le \frac{1}{\sqrt{\nu}}\left(\|v_0\|^2_{\dot{H}^s} + \frac{1}{\sqrt{\nu}}\|f\|_{L^2([0,T];\dot{H}^{s-1})}\|\right).$$

Resuming what we have just obtained:

$$\left(\int_{\mathbb{R}^3} (\sup_{0\le t\le T} |\widehat{v}(t',\xi|)|\xi|^{2s} d\xi\right)^{\frac{1}{2}} \le 3\left[\|v_0\|_{\dot{H}^s} + \frac{1}{\sqrt{\nu}}\|f\|_{L^2([0,T];\dot{H}^{s-1})}\right], \tag{2.4}$$

$$\|v\|_{L^2([0,T];\dot{H}^{s+1})} \le \frac{1}{\sqrt{\nu}}\left[\|v_0\|_{\dot{H}^s} + \frac{1}{\sqrt{\nu}}\|f\|_{L^2([0,T;\dot{H}^{s-1}])}\right]. \tag{2.5}$$

We now interpolate this last two inequalities. We have

$$|\widehat{v}(t,\xi)|^2|\xi|^{2s+\frac{4}{p}} = \left(|\widehat{v}(t,\xi)||\xi|^{2s+2}\right)^{\frac{2}{p}}\left(|\widehat{v}(t,\xi)|^2|\xi|^{2s}\right)^{1-\frac{2}{p}},$$

where

$$\left(|\widehat{v}(t,\xi)||\xi|^{2s+2}\right)^{\frac{2}{p}} \in L^{\frac{p}{2}},$$

$$\left(|\widehat{v}(t,\xi)|^2|\xi|^{2s}\right)^{1-\frac{2}{p}} \in L^{\frac{p}{p-2}},$$

consequently

$$\begin{aligned}\int_{\mathbb{R}^3}|\widehat{v}(t,\xi)|^2|\xi|^{2s+\frac4p}d\xi &\le \left(\int_{\mathbb{R}^3}|\widehat{v}(t,\xi)|^2|\xi|^{2s+2}d\xi\right)^{\frac2p}\left(\int_{\mathbb{R}^3}|\widehat{v}(t,\xi)|^2|\xi|^{2s}d\xi\right)^{1-\frac2p}\\ &\le \left(\int_{\mathbb{R}^3}|\widehat{v}(t,\xi)|^2|\xi|^{2s+2}d\xi\right)^{\frac2p}\left(\int_{\mathbb{R}^3}\sup_{0\le t\le t}|\widehat{v}(t,\xi)|^2|\xi|^{2s}d\xi\right)^{1-\frac2p},\end{aligned}$$

so that

$$\left(\int_{\mathbb{R}^3}|\widehat{v}(t,\xi)|^2|\xi|^{2s+\frac4p}d\xi\right)^{\frac p2} \le \int_{\mathbb{R}^3}|\widehat{v}(t,\xi)|^2|\xi|^{2s+2}d\xi\cdot\left(\int_{\mathbb{R}^3}\sup_{0\le t\le t}|\widehat{v}(t,\xi)|^2|\xi|^{2s}d\xi\right)^{\frac p2-1},$$

and thus

$$\|v\|^p_{L^p([0,T],\dot H^{s+\frac2p})}\le \|v\|^2_{L^2([0,T];\dot H^{s+1})}\cdot\left(\int_{\mathbb{R}^3}\sup_{0\le t\le T}(|\widehat{v}(t,\xi)|^2|\xi|^{2s})d\xi\right)^{\frac p2-1}.$$

Finally

$$\|v\|_{L^p([0,T],\dot H^{s+\frac2p})}\le \|v\|^{\frac2p}_{L^2([0,T];\dot H^{s+1})}\cdot\left[\left(\int_{\mathbb{R}^3}\sup_{0\le t\le T}(|\widehat{v}(t,\xi)|^2|\xi|^{2s})d\xi\right)^{\frac12}\right]^{1-\frac2p}.$$

Using (2.4) and (2.5) we obtain (*iii*). □

Let now $u,v\in L^4([0,T];\dot H^1)$, we want to estimate

$$\|\mathbb{P}(\nabla\cdot(u\otimes v))\|_{L^2([0,T];\dot H^{-\frac12})}=\left(\int_0^T\|\mathbb{P}(\nabla\cdot(u\otimes v)(t))\|^2_{\dot H^{-\frac12}}dt\right)^{\frac12}.$$

We use the following lemma

Lemma 3 *(Lemma 1.2.1 in [C])*
if $a,b\in\dot H^1$ *then* $\mathbb{P}(\nabla\cdot(a\otimes b))\in\dot H^{-\frac12}$ *and one has*

$$\|\mathbb{P}(\nabla\cdot(a\otimes b))\|_{\dot H^{-\frac12}}\le C\|a\|_{\dot H^1}\|b\|_{\dot H^1}.$$

Proof. From Corollary 1.2.1 in [C] we have that if $p \in]1,2]$ then $L^p(\mathbb{R}^3) \hookrightarrow \dot{H}^s$ with $s = 3\left(\frac{1}{2} - \frac{1}{p}\right)$ and so

$$L^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3).$$

Moreover we have

$$\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3).$$

Let $a, b \in \dot{H}^1$, then

$$\begin{aligned} |a|^{\frac{3}{2}} &\in L^4 \qquad \text{since} \qquad (|a|^{\frac{3}{2}})^4 = |a|^6 \in L^1, \\ |\partial_x b|^{\frac{3}{2}} &\in L^{\frac{4}{3}} \qquad \text{since} \qquad (|\partial_x b|^{\frac{3}{2}})^{\frac{4}{3}} = |\partial_x b|^2 \in L^1, \end{aligned}$$

so that

$$\begin{aligned} \|a\partial_x b\|_{L^{\frac{3}{2}}} &= \left(\| |a|^{\frac{3}{2}} |\partial_x b|^{\frac{3}{2}} \|_{L^1} \right)^{\frac{2}{3}} \\ &\leq \left(\left(\int |a|^6 \right)^{\frac{1}{4}} \left(\int |\partial_x b|^2 \right)^{\frac{3}{4}} \right)^{\frac{2}{3}} \\ &\leq \|a\|_{L^6} \|b\|_{\dot{H}^1} \\ &\leq C \|a\|_{\dot{H}^1} \|b\|_{\dot{H}^1}, \end{aligned}$$

and we obtain

$$\|\nabla \cdot (a \otimes b)\|_{\dot{H}^{-\frac{1}{2}}} \leq C \|a\|_{\dot{H}^1} \|b\|_{\dot{H}^1}.$$

The conclusion of the lemma is then a consequence of the fact that $\mathbb{P}$, having a 0-order homogeneous symbol, is continuous from $\dot{H}^s$ to $\dot{H}^s$. □

From this lemma we obtain

$$\int_0^T \|\mathbb{P}(\nabla \cdot (u \otimes v)(t))\|^2_{\dot{H}^{-\frac{1}{2}}} dt \leq C \int_0^T \|u\|^2_{\dot{H}^1} \|v\|^2_{\dot{H}^1} dt,$$

but we have that

$$t \longmapsto \|u\|^2_{\dot{H}^1}(t) \in L^2[0,T],$$

so

$$\int_0^T \|\mathbb{P}(\nabla \cdot (u \otimes v)(t))\|^2_{\dot{H}^{-\frac{1}{2}}} dt \leq C \left(\int_0^T \|u\|^4_{\dot{H}^1} dt \right)^{\frac{1}{2}} \left(\int_0^T \|v\|^4_{\dot{H}^1} dt \right)^{\frac{1}{2}},$$

from which we deduce that when $u, v \in L^4([0,T]; \dot{H}^1)$ we have $\mathbb{P}(\nabla \cdot (u \otimes v)) \in L^2([0,T]; \dot{H}^{-\frac{1}{2}})$.

We now define

$$B(u,v) = -\int_0^t \mathcal{G}(t-s)(\mathbb{P}(\nabla \cdot (u \otimes v)(s)))ds,$$

and we make use of Lemma 2 (with $v_0 = 0$) obtaining

$$B(u,v) \in \left(\bigcap_{p=2}^{+\infty} L^p([0,T]; \dot{H}^{\frac{1}{2}+\frac{2}{p}}) \right) \cap \mathcal{C}([0,T]; \dot{H}^{\frac{1}{2}}),$$

and one has

$$\|B(u,v)\|_{L^p([0,T];\dot{H}^{\frac{1}{2}+\frac{2}{p}})} \leq \frac{3}{\nu^{\frac{1}{p}+\frac{1}{2}}} \|\mathbb{P}(\nabla \cdot (u \otimes v))\|_{L^2([0,T];\dot{H}^{-\frac{1}{2}})},$$

and from Lemma 3

$$\|B(u,v)\|_{L^p([0,T];\dot{H}^{\frac{1}{2}+\frac{2}{p}})} \leq \frac{C}{\nu^{\frac{1}{p}+\frac{1}{2}}} \|u\|_{L^4([0,T];\dot{H}^1)} \|v\|_{L^4([0,T];\dot{H}^1)}.$$

Choosing $p = 4$,

$$\|B(u,v)\|_{L^4([0,T];\dot{H}^1)} \leq \frac{C}{\nu^{\frac{3}{4}}} \|u\|_{L^4([0,T];\dot{H}^1)} \|v\|_{L^4([0,T];\dot{H}^1)}, \tag{2.6}$$

and going back to the proof of Lemma 2 it is easy to see that C does not depend on T.

Finally let $v_0 \in \dot{H}^{\frac{1}{2}}$. We use again Lemma 2 (with $f = 0$), obtaining

$$\mathcal{G}(t)v_0 \in \left(\bigcap_{p=2}^{+\infty} L^p([0,T];\dot{H}^{\frac{1}{2}+\frac{2}{p}})\right) \cap \mathcal{C}([0,T];\dot{H}^{\frac{1}{2}}),$$

with

$$\|\mathcal{G}(t)v_0\|_{L^p([0,T];\dot{H}^{\frac{1}{2}+\frac{2}{p}})} \leq \frac{3}{\nu^{\frac{1}{p}}}\|v_0\|_{\dot{H}^{\frac{1}{2}}},$$

and again choosing $p = 4$ we get

$$\|\mathcal{G}(t)v_0\|_{L^4([0,T];\dot{H}^1)} \leq \frac{3}{\nu^{\frac{1}{4}}}\|v_0\|_{\dot{H}^{\frac{1}{2}}}. \tag{2.7}$$

2.2 Main theorem

We are ready to state and prove the main theorem of this section:

Theorem 5 *There exists $c > 0$ such that for every $u_0 \in \dot{H}^{\frac{1}{2}}$ with $\nabla \cdot u_0 = 0$, if $\|u_0\|_{\dot{H}^{\frac{1}{2}}} \leq c\nu$ then there exists a unique $u \in L^4([0,+\infty[;\dot{H}^1)$ solution of (1.11).*
Moreover

$$u \in \mathcal{C}_b([0,+\infty[;\dot{H}^{\frac{1}{2}}) \cap L^2([0,+\infty[;\dot{H}^{\frac{3}{2}}).$$

Proof. We use Lemma 1 with $Y = L^4([0,+\infty[;\dot{H}^1)$.

Posing

$$B(u,v) = -\int_0^t \mathcal{G}(t-s)(\mathbb{P}(\nabla \cdot (u \otimes v)(s)))ds,$$

we get from (2.6) that $\eta = \dfrac{C}{\nu^{\frac{3}{4}}}$,

then, posing $y = \mathcal{G}(t)u_0$ the condition $4\eta\|y\| < 1$ of the lemma becomes

$$4\eta\|\mathcal{G}(t)u_0\|_{L^4([0,+\infty[;\dot{H}^1)} < 1,$$

using (2.7) we get that this condition is satisfied if

$$4\eta\frac{3}{\nu^{\frac{1}{4}}}\|u_0\|_{\dot{H}^{\frac{1}{2}}} < 1,$$

and so, substituting the value of η, we get that if

$$\|u_0\|_{\dot{H}^{\frac{1}{2}}} < \frac{\nu}{12C},$$

we can use Lemma 1, thus obtaining existence and uniqueness in the ball of Y centered in 0, with radius $\dfrac{\nu^{\frac{3}{4}}}{2C}$.

The solution we found belongs to $L^4([0,+\infty[;\dot{H}^1)$. Applying Lemma 3 and Lemma 2 one has

$$u \in \mathcal{C}_b([0,+\infty[;\dot{H}^{\frac{1}{2}}) \cap L^2([0,+\infty[;\dot{H}^{\frac{3}{2}}).$$

Now we want to prove uniqueness of solution in $L^4([0,+\infty[;\dot{H}^1)$ (so far we have uniqueness only in the ball of radius $\dfrac{\nu^{\frac{3}{4}}}{2C}$); this can be obtained as a corollary of a stability result.

We can in fact show that if $u(t)$ and $v(t)$ are solutions of

$$\begin{aligned}
u(t) &= \mathcal{G}(t)u_0 - \int_0^t \mathcal{G}(t-s)(\mathbb{P}(\nabla\cdot(u\otimes u)(s)))ds\\
v(t) &= \mathcal{G}(t)v_0 - \int_0^t \mathcal{G}(t-s)(\mathbb{P}(\nabla\cdot(v\otimes v)(s)))ds\\
&\text{with } \nabla\cdot u_0 = \nabla\cdot v_0 = 0,
\end{aligned}$$

then

$$\begin{aligned}
\|u(t)-v(t)\|^2_{\dot{H}^{\frac{1}{2}}} &+\nu\int_0^t \|u(s)-v(s)\|^2_{\dot{H}^{\frac{3}{2}}}\\
&\leq \|u_0-v_0\|^2_{\dot{H}^{\frac{1}{2}}} \exp\left(\frac{C}{\nu^3}\int_0^t (\|u(s)\|^4_{\dot{H}^1}+\|v(s)\|^4_{\dot{H}^1})ds\right).
\end{aligned} \tag{2.8}$$

To show this last statement we suppose to have two solutions $u(t)$ and $v(t)$ as above, then

$$w(t) = u(t)-v(t) = \mathcal{G}(t)(u_0-v_0) - \int_0^t \mathcal{G}(t-s)(\mathbb{P}(\nabla\cdot((u\otimes u - v\otimes v)(s))ds,$$

so

$$w(t) = \mathcal{G}(t)w_0 - \int_0^t \mathcal{G}(t-s)(\mathbb{P}(\nabla \cdot (\frac{1}{2}(w \otimes (u+v) + (u+v) \otimes w)))(s))ds.$$

Now we have that $w_0 \in \dot{H}^{\frac{1}{2}}$, while

$$\mathbb{P}(\nabla \cdot (\frac{1}{2}(w \otimes (u+v) + (u+v) \otimes w))) \in L^2([0,+\infty[, \dot{H}^{-\frac{1}{2}}),$$

this last statement being the consequence of lemma 3 and of the fact that $u, v \in L^4([0,+\infty[, \dot{H}^1)$.

We can apply point (i) of Lemma 2 obtaining that

$$\begin{aligned} \|w(t)\|^2_{\dot{H}^{\frac{1}{2}}} & +2\nu \int_0^t \|\nabla w\|^2_{\dot{H}^{\frac{1}{2}}} ds \\ & = \|w_0\|^2_{\dot{H}^{\frac{1}{2}}} + 2\int_0^t < \mathbb{P}(\nabla \cdot (\frac{1}{2}(w \otimes (u+v) + (u+v) \otimes w))), w >_{\frac{1}{2}} ds. \end{aligned}$$

Now since

$$\begin{aligned} < f, g >_{\frac{1}{2}} & = \int \widehat{f}(\xi)\overline{\widehat{g}(\xi)}|\xi| d\xi \\ & = \int \widehat{f}(\xi)|\xi|^{-\frac{1}{2}}\overline{\widehat{g}(\xi)}|\xi|^{\frac{3}{2}} d\xi \\ & \leq \|f\|_{\dot{H}^{-\frac{1}{2}}} \|\nabla g\|_{\dot{H}^{\frac{1}{2}}}, \end{aligned}$$

we obtain

$$| < \mathbb{P}(\nabla \cdot (\frac{1}{2}(w \otimes (u+v) + (u+v) \otimes w))), w >_{\frac{1}{2}} | \leq C\|w\|_{\dot{H}^1}\|u+v\|_{\dot{H}^1}\|\nabla w\|_{\dot{H}^{\frac{1}{2}}},$$

where the last inequality comes from Lemma 3. We now can write

$$\begin{aligned} \Delta_w(t) & = \|w(t)\|^2_{\dot{H}^{\frac{1}{2}}} + 2\nu \int_0^t \|w(s)\|^2_{\dot{H}^{\frac{3}{2}}} ds \\ & \leq \|w_0\|^2_{\dot{H}^{\frac{1}{2}}} + C\int_0^t \|w\|_{\dot{H}^1}(\|u\|_{\dot{H}^1} + \|v\|_{\dot{H}^1})\|w\|_{\dot{H}^{\frac{3}{2}}} ds, \end{aligned}$$

and since $w(t) \in L^\infty([0,+\infty[,\dot{H}^{\frac{1}{2}}) \cap L^2([0,+\infty[,\dot{H}^{\frac{3}{2}})$ so

$$\|w(t)\|_{\dot{H}^1} \leq \|w(t)\|^{\frac{1}{2}}_{\dot{H}^{\frac{1}{2}}} \|w(t)\|^{\frac{1}{2}}_{\dot{H}^{\frac{3}{2}}},$$

that yields

$$\Delta_w(t) \leq \|w_0\|^2_{\dot{H}^{\frac{1}{2}}} + C\int_0^t \|w(s)\|^{\frac{1}{2}}_{\dot{H}^{\frac{1}{2}}}(\|u(s)\|_{\dot{H}^1} + \|v(s)\|_{\dot{H}^1})\|w(s)\|^{\frac{3}{2}}_{\dot{H}^{\frac{3}{2}}} ds,$$

and using $ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{\frac{4}{3}}$, we have

$$\Delta_w(t) \leq \|w_0\|^2_{\dot{H}^{\frac{1}{2}}} + \frac{C}{\nu^3}\int_0^t \|w(s)\|^2_{\dot{H}^{\frac{1}{2}}}(\|u(s)\|_{\dot{H}^1} + \|v(s)\|_{\dot{H}^1})^4 ds + \nu\int_0^t \|w(s)\|^2_{\dot{H}^{\frac{3}{2}}} ds$$

that finally yields

$$\begin{aligned}\|w(t)\|^2_{\dot{H}^{\frac{1}{2}}} &+ \nu\int_0^t \|w(s)\|^2_{\dot{H}^{\frac{3}{2}}} ds \\ &\leq \|w_0\|^2_{\dot{H}^{\frac{1}{2}}} + \frac{C}{\nu^3}\int_0^t \|w(s)\|^2_{\dot{H}^{\frac{1}{2}}}(\|u(s)\|_{\dot{H}^1} + \|v(s)\|_{\dot{H}^1})^4 ds\end{aligned}$$

The Gronwall's Lemma easily gives (2.8). □

Remark. It is easy to prove a similar result concerning local in time existence without any boundness condition on the norm of the initial datum.

Chapter 3

Mild solutions for the Navier-Stokes-Coriolis system: the L^3 and Besov case

3.1 The $\dot{H}^{\frac{1}{2}} - L^3$ case

We first deal with initial data belonging to a hybrid space, having different properties for high and low frequencies. Roughly speaking we will consider a function with $\dot{H}^{\frac{1}{2}}$ low frequencies and L^3 high frequencies. The precise definition of this space will be given below.

Mapping estimates for the semigroup

Let's start with some estimates on the semigroup $\mathcal{G}(t)$ in order to understand which spaces can be chosen to obtain the searched result. We start showing this proposition:

Proposition 7 *Let $u \in L^q(\mathbb{R}^3)$, with supp $\widehat{u} \subseteq (\mathbb{R}^3 - B(0,1))$, then*

$$t^{\frac{3}{2}(1-\frac{1}{r})} \|\mathcal{G}(t)u\|_{L^p} \leq C\|u\|_{L^q}$$

with $r > 1$ and $\dfrac{1}{q} + \dfrac{1}{r} = 1 + \dfrac{1}{p}$.

Proof. Let us introduce the function $\phi(\xi) \in \mathcal{C}^\infty(\mathbb{R}^3)$ defined as follows

$$\phi(\xi) = \begin{cases} 0 & \text{if } |\xi| < \frac{1}{2} \\ 1 & \text{if } |\xi| \geq 1 \end{cases}$$

we get

$$\begin{aligned} \mathcal{G}(t)u &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix\cdot\xi} \widehat{\mathcal{G}}(t,\xi)\phi(\xi)\widehat{u}(\xi) d\xi \\ &= \mathcal{F}^{-1}(\widehat{\mathcal{G}}(t,\xi)\phi(\xi)\widehat{u}(\xi)) \\ &= g(t,x) * u(x), \end{aligned}$$

with

$$g(t,x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix\cdot\xi} \widehat{\mathcal{G}}(t,\xi)\phi(\xi) d\xi,$$

so we have to prove that

$$\|g(t,\cdot)\|_{L^r} \leq Ct^{-\frac{3}{2}(1-\frac{1}{r})},$$

in fact, since $1 + \dfrac{1}{p} = \dfrac{1}{q} + \dfrac{1}{r}$, we have

$$\|\mathcal{G}(t)u\|_{L^p} = \|g * u\|_{L^p} \leq \|g(t,\cdot)\|_{L^r} \|u\|_{L^q}.$$

First of all

$$\begin{aligned} |g(t,x)| &\leq C \int_{\mathbb{R}^3} |\widehat{\mathcal{G}}(t,\xi)||\phi(\xi)| d\xi \\ &\leq C \int_{\mathbb{R}^3} e^{-\nu|\xi|^2 t} d\xi \\ &\leq \frac{C}{(\sqrt{\nu t})^3}, \end{aligned}$$

Moreover we can obtain another estimate for g: we consider $|x| \geq a > 0$, with a fixed and introduce the operator $\mathcal{L} = \dfrac{x \cdot \nabla_\xi}{i|x|^2}$; noting that $\mathcal{L}(e^{ix\cdot\xi}) = e^{ix\cdot\xi}$, we can write

$$g(t,x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi}(\mathcal{L}^*)^3(\widehat{\mathcal{G}}(t,\xi)\phi(\xi))d\xi,$$

We now have to estimate $|(\mathcal{L}^*)^3(\widehat{\mathcal{G}}(t,\xi)\phi(\xi))|$: we divide this term into two parts,

$$|(\mathcal{L}^*)^3(\widehat{\mathcal{G}}(t,\xi)\phi(\xi))| \leq \underbrace{|x|^{-3}\sum_{|\alpha|=3}\left|\partial_\xi^\alpha\left(\mathcal{G}(t,\xi)\right)||\phi(\xi)\right|}_{(A)} + \underbrace{|x|^{-3}\sum_{\substack{|\alpha|+|\beta|=3\\ |\alpha|<3}}|\partial_\xi^\alpha\left(\mathcal{G}(t,\xi)\right)||\partial_\xi^\beta\phi(\xi)|}_{(B)}.$$

It is easy to prove that the following estimate holds:

$$|\partial_\xi^\beta e^{-\nu|\xi|^2 t}| \leq C_\beta(\sqrt{\nu t})^{|\beta|}p_\beta(\sqrt{\nu t}\,|\xi|)e^{-\nu|\xi|^2 t},$$

where $p_\beta(\cdot)$ is a polinomial of degree $|\beta|$. On the other hand

$$\left|\cos\left(\Omega t\frac{\xi_3}{|\xi|}\right)Id + \sin\left(\Omega t\frac{\xi_3}{|\xi|}\right)R(\xi)\right| \leq C_0,$$

and, for $|\alpha| > 0$,

$$\left|\partial_\xi^\alpha\left(\cos\left(\Omega t\frac{\xi_3}{|\xi|}\right) + \sin\left(\Omega t\frac{\xi_3}{|\xi|}\right)R(\xi)\right)\right| \leq C_\alpha\left((\Omega t) + (\Omega t)^2 + ... + (\Omega t)^{|\alpha|}\right)|\xi|^{-|\alpha|}.$$

Now

$$\sum_{|\alpha|=3}\left|\partial_\xi^\alpha\left(\mathcal{G}(t,\xi)\right)||\phi(\xi)\right| \leq \underbrace{\sum_{\substack{|\alpha|+|\beta|=3\\ |\beta|<3}}\left|\partial_\xi^\alpha\cos\left(\Omega t\frac{\xi_3}{|\xi|}\right) + \sin\left(\Omega t\frac{\xi_3}{|\xi|}\right)R(\xi)\right|\left|\partial_\xi^\beta e^{-\nu|\xi|^2 t}\right|}_{(A_1)}$$
$$+ \underbrace{C_0\sum_{|\beta|=3}\left|\partial_\xi^\beta e^{-\nu|\xi|^2 t}\right|}_{(A_2)},$$

we have

$$|(A_2)| \leq C(\sqrt{\nu t})^3 p_3(\sqrt{\nu t}|\xi|)e^{-\nu|\xi|^2 t},$$

and consequently

$$|(A_2)| \leq C(\sqrt{\nu t})^3 e^{-\frac{\nu|\xi|^2 t}{2}},$$

(note that C does not depend on ν).

On the other hand

$$\begin{aligned}|(A_1)| \leq\ & C_1(\Omega t)|\xi|^{-1}\nu t\ p_2(\sqrt{\nu t}|\xi|)e^{-\nu t|\xi|^2} \\ & +C_2(\Omega t+\Omega^2 t^2)|\xi|^{-2}\sqrt{\nu t}\ p_1(\sqrt{\nu t}|\xi|)e^{-\nu t|\xi|^2} \\ & +C_3(\Omega t+\Omega^2 t^2+\Omega^3 t^3)|\xi|^{-3}e^{-\nu t|\xi|^2},\end{aligned}$$

and consequently

$$|(A_1)| \leq C(\Omega t+\Omega^2 t^2+\Omega^3 t^3)|\xi|^{-3}e^{\frac{-\nu t|\xi|^2}{2}},$$

where again C does not depend on ν and Ω.

Finally

$$|(A)| \leq |x|^{-3}C\left((\sqrt{\nu t})^3+(\Omega t+\Omega^2 t^2+\Omega^3 t^3)|\xi|^{-3}\right)e^{\frac{-\nu t|\xi|^2}{2}}.$$

Since $|\xi| \geq \frac{1}{2}$

$$\begin{aligned}(\Omega t+\Omega^2 t^2+\Omega^3 t^3)|\xi|^{-3} & \leq (\sqrt{\nu t})^3\left(\frac{\Omega}{\nu}\frac{1}{\sqrt{\nu t}}+\frac{\Omega^2}{\nu^2}\sqrt{\nu t}+\frac{\Omega^3}{t^3}(\sqrt{\nu t})^3\right)|\xi|^{-3} \\ & \leq C(\sqrt{\nu t})^3\left(\frac{\Omega}{\nu}\frac{1}{|\xi|\sqrt{\nu t}}+\frac{\Omega^2}{\nu^2}|\xi|\sqrt{\nu t}+\frac{\Omega^3}{t^3}(|\xi|\sqrt{\nu t})^3\right),\end{aligned}$$

finally a direct computation gives us

$$\int_{|\xi|\geq\frac{1}{2}} |(A)|d\xi \leq C_{\Omega/\nu}.$$

where we write $C_{\Omega/\nu}$ since this constant depends on the ratio $\dfrac{\Omega}{\nu}$.

A similar, or even easier computation leads to the same result for (B).

We can resume our estimates as follows

$$|g(t,x)| \le \begin{cases} \dfrac{C}{(\sqrt{\nu t})^3} & \text{if} \quad |x| < a \\ \dfrac{C_{\Omega/\nu}}{|x|^3} & \text{if} \quad |x| \ge a \end{cases},$$

now we choose $a = \sqrt{\nu t}$ obtaining

$$|g(t,x)| \le \begin{cases} \dfrac{C}{(\sqrt{\nu t})^3} & \text{if} \quad |x| < \sqrt{\nu t} \\ \dfrac{C_{\Omega/\nu}}{|x|^3} & \text{if} \quad |x| \ge \sqrt{\nu t} \end{cases},$$

finally obtaining, remembering that $r > 1$,

$$\begin{aligned} \|g(t,x)\|_{L^r}^r &= \int_{|x|\le\sqrt{\nu t}} |g(t,x)|^r dx + \int_{|x|>\sqrt{\nu t}} |g(t,x)|^r dx \\ &\le C\left(t^{-\frac{3}{2}r}\int_{|x|\le\sqrt{\nu t}} dx + \int_{|x|>\sqrt{\nu t}} |x|^{-3r} dx\right) \\ &\le Ct^{-\frac{3}{2}r+\frac{3}{2}} \end{aligned},$$

and so

$$\|g(t,x)\|_{L^r} \le Ct^{-\frac{3}{2}(1-\frac{1}{r})},$$

where C depends increasingly on $\frac{\Omega}{\nu}$. □

Remark. In particular, when $q = 3$ we get the following estimates:

$$\|\mathcal{G}(t)u\|_{L^p} \le Ct^{\frac{1}{2}(\frac{3}{p}-1)}\|u\|_{L^3},$$

where $p > 3$, because it must satisfy the relation $\dfrac{1}{p} = \dfrac{1}{r} - \dfrac{2}{3}$, where $r > 1$ (in this case $1 < r < \frac{3}{2}$).

We can also prove the following result:

Proposition 8 *Let $u \in \dot{H}^{\frac{1}{2}}$ and $p \geq 3$, then there exists C such that, for any $t > 0$,*

$$\|\mathcal{G}(t)u\|_{L^p} \leq Ct^{\frac{1}{2}(\frac{3}{p}-1)}\|u\|_{\dot{H}^{\frac{1}{2}}}$$

Proof. Since for $2s < 3$ we have

$$\dot{H}^s(\mathbb{R}^3) \hookrightarrow L^{\frac{6}{3-2s}}(\mathbb{R}^3),$$

we obtain that:

$$\|\mathcal{G}(t)u\|_{L^p} \leq C\|\mathcal{G}(t)u\|_{\dot{H}^{\frac{3}{2}-\frac{3}{p}}},$$

so, using the definition of $\|\cdot\|_{\dot{H}^s}$, we have

$$\begin{aligned}
\|\mathcal{G}(t)u\|^2_{\dot{H}^{\frac{3}{2}-\frac{3}{p}}} &\leq C\int_{\mathbb{R}^3} |\widehat{\mathcal{G}}(t,\xi)|^2|\xi|^{3-\frac{6}{p}}|\widehat{u}(\xi)|^2 d\xi \\
&\leq C\int_{\mathbb{R}^3} e^{-2\nu|\xi|^2 t}|\xi|^{2-\frac{6}{p}}|\xi||\widehat{u}_0(\xi)|^2 d\xi \\
&\leq C(\nu t)^{\frac{3}{p}-1}\int_{\mathbb{R}^3} (\nu|\xi|^2 t)^{1-\frac{3}{p}}e^{-2\nu|\xi|^2 t}|\xi||\widehat{u}_0(\xi)|^2 d\xi,
\end{aligned}$$

now, since there exists a constant K such that

$$(\nu|\xi|^2 t)^{1-\frac{3}{p}}e^{-2\nu|\xi|^2 t} \leq K,$$

we finally obtain that

$$\|\mathcal{G}(t)u\|_{\dot{H}^{\frac{3}{2}-\frac{3}{p}}} \leq C_\nu t^{\frac{1}{2}(\frac{3}{p}-1)}\|u\|_{\dot{H}^{\frac{1}{2}}},$$

this concludes the proof. □

Let now $\chi_{B(0,1)}$ be the characteristic function of the ball $B(0,1)$, from now on we will use the following notations:

$$f^{(1)}(t,x) = \mathfrak{F}^{-1}(\chi_{B(0,1)}f)$$
$$f^{(2)}(t,x) = \mathfrak{F}^{-1}((1-\chi_{B(0,1)})f),$$

The two last propositions suggest us to define the functional spaces $\mathcal{B}_{\dot{H}^{\frac{1}{2}},L^3}(\mathbb{R}^3)$ and K_p as follows:

we say that $u \in \mathcal{B}_{\dot{H}^{\frac{1}{2}},L^3}(\mathbb{R}^3)$ if

$$\|u\|_{\mathcal{B}_{\dot{H}^{\frac{1}{2}},L^3}} = \|u^{(1)}\|_{\dot{H}^{\frac{1}{2}}} + \|u^{(2)}\|_{L^3} < +\infty,$$

while

$$K_p := \left\{ f \in \mathcal{C}(]0,+\infty[;L^p) | \|f\|_{K_p} := \sup_{t\in[0,\infty[} t^{\frac{1}{2}(1-\frac{3}{p})}\|f\|_{L^p} < +\infty \right\}.$$

This definitions are suggested by the following remark, let $u_0 \in \mathcal{B}_{\dot{H}^{\frac{1}{2}},L^3}(\mathbb{R}^3)$ and $p > 3$, then we get

$$\begin{aligned}
\|\mathcal{G}(t)u\|_{K_p} &= \sup_{t\in[0,\infty[} t^{\frac{1}{2}(1-\frac{3}{p})}\|\mathcal{G}(t)u\|_{L^p} \\
&\leq \sup_{t\in[0,\infty[} t^{\frac{1}{2}(1-\frac{3}{p})}\|\mathcal{G}(t)u_0^{(1)}\|_{L^p} + \sup_{t\in[0,\infty[} t^{\frac{1}{2}(1-\frac{3}{p})}\|\mathcal{G}(t)u_0^{(2)}\|_{L^p} \\
&\leq C_1\|u_0^{(1)}\|_{\dot{H}^{\frac{1}{2}}} + C_2\|u_0^{(2)}\|_{L^3} = C\|u_0\|_{\mathcal{B}_{\dot{H}^{\frac{1}{2}},L^3}},
\end{aligned}$$

thus obtaining that

$$\mathcal{G}(t)u_0 \in K_p.$$

Bilinear estimates

We now choose the space K_4 as the space in which the fixed point argument works, thanks to the following

Proposition 9 *Let $u, v \in K_4$, then there exists a constant η such that*

$$\|B(u,v)\|_{K_4} \leq \eta\|u\|_{K_4}\|v\|_{K_4}.$$

In order to prove this proposition we need the following Lemma:

Lemma 4 *Let $u \in L^2$, then for any $s \geq -1$ there exist a constant C such that*

$$\|\mathcal{G}(t)\mathbb{P}\nabla u\|_{\dot{H}^s} \leq Ct^{-\left(\frac{s+1}{2}\right)}\|u\|_{L^2}$$

Proof.

$$
\begin{aligned}
\|\mathcal{G}(t)\mathbb{P}\nabla u\|_{\dot{H}^s} &\leq \left(\int_{\mathbb{R}^3} |\xi|^{2s}|\widehat{\mathcal{G}}(t,\xi)|^2|\widehat{\mathbb{P}}|^2|\xi|^2|\widehat{u}|^2 d\xi\right)^{\frac{1}{2}} \\
&\leq C\left(\int_{\mathbb{R}^3} |\xi|^{2s+2}e^{-2\nu|\xi|^2 t}|\widehat{u}|^2 d\xi\right)^{\frac{1}{2}} \\
&\leq C(\nu t)^{-\left(\frac{s+1}{2}\right)}\left(\int_{\mathbb{R}^3} (\nu|\xi|^2 t)^{s+1}e^{-2\nu|\xi|^2 t}|\widehat{u}|^2 d\xi\right)^{\frac{1}{2}} \\
&\leq C_{\nu,s}(t)^{-\left(\frac{s+1}{2}\right)}\|u\|_{L^2},
\end{aligned}
$$

where we used the fact that there exist a constant K, such that

$$
(\nu|\xi|^2 t)^{s+1}e^{-2\nu|\xi|^2 t} \leq K.
$$

□

We come now to the proof of the Proposition 9.

Proof. (Proposition 9) Since we have, by Sobolev imbeddings,

$$
\dot{H}^{\frac{3}{4}}(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)
$$

we can write, also using Lemma 4 and the definition of K_4,

$$
\begin{aligned}
\|B(u,v)\|_{K_4} &= \sup_{t\in[0,\infty[} t^{\frac{1}{8}}\|B(u,v)\|_{L^4} \\
&\leq \sup_{t\in[0,\infty[} t^{\frac{1}{8}}\int_0^t \|\mathcal{G}(t-s)\mathbb{P}\nabla(uv)(s)\|_{L^4} ds \\
&\leq C \sup_{t\in[0,\infty[} t^{\frac{1}{8}}\int_0^t \|\mathcal{G}(t-s)\mathbb{P}\nabla(uv)(s)\|_{\dot{H}^{\frac{3}{4}}} ds \\
&\leq C \sup_{t\in[0,\infty[} t^{\frac{1}{8}}\int_0^t (t-s)^{-\frac{7}{8}}\|uv(s)\|_{L^2} ds \\
&\leq C \sup_{t\in[0,\infty[} t^{\frac{1}{8}}\int_0^t (t-s)^{-\frac{7}{8}}\|u(s)\|_{L^4}\|v(s)\|_{L^4} ds \\
&\leq C\|u\|_{K_4}\|v\|_{K_4} \sup_{t\in[0,\infty[} t^{\frac{1}{8}}\int_0^t (t-s)^{-\frac{7}{8}}s^{-\frac{1}{4}} ds \\
&\leq \eta\|u\|_{K_4}\|v\|_{K_4},
\end{aligned}
$$

in the last line we use the fact that

$$\int_0^t (t-s)^{-\frac{7}{8}} s^{-\frac{1}{4}} ds \leq C t^{-\frac{1}{8}},$$

that is Lemma 6 in the Appendix, with $\alpha = \frac{7}{8}$ and $\beta = \frac{1}{4}$. □

Statement and proof of the main theorem

Now we are able to state and prove the theorem of existence and uniqueness:

Theorem 6 *There exists a constant $c > 0$, depending on the ratio Ω/ν, such that if $\|u_0\|_{\mathcal{B}_{\dot{H}^{\frac{1}{2}},L^3}} \leq c$, then there exist a unique fixed point $u(t)$ of (1.1.2), contained in a ball of K_4. Moreover it holds that*

$$\|u(t)\|_{K_4} \leq 2\|\mathcal{G}(t)u_0\|_{K_4}$$

$$u \in L^\infty(0,+\infty; \mathcal{B}_{\dot{H}^{\frac{1}{2}}, \dot{B}_{p,\infty}^{\frac{3}{p}-1}}) \qquad p > 3,$$

$$B(u,u) \in \mathcal{C}(0,+\infty; \dot{H}^{\frac{1}{2}}) \text{ and } \lim_{t\to 0^+} \|B(u(t),u(t))\|_{\dot{H}^{\frac{1}{2}}} = 0.$$

Proof. Proposition 9 tells us that

$$\|B(u,v)\|_{K_4} \leq \eta \|u\|_{K_4} \|v\|_{K_4},$$

moreover we have that

$$\|\mathcal{G}(t)u_0\|_{K_4} \leq C\|u_0\|_{\mathcal{B}_{\dot{H}^{\frac{1}{2}},L^3}} \leq Cc,$$

then if $c < \frac{1}{4\eta C}$, we can use Lemma 1 with $K_4 = Y$, $\mathcal{G}(t)u_0 = y_0$ and $u(t) = y$, thus proving existence and uniqueness in K_4.

We now want to reconstruct regularity of the solution, because we want to understand in which sense the initial datum is attained.

First of all we are interested in the term $B(u,u)$ that happens to be more regular, in fact, following the proof of Proposition 9

$$\begin{aligned}\|B(u,u)(t)\|_{\dot H^{\frac12}} &\le \int_0^t \|\mathcal{G}(t-s)\mathbb{P}\nabla u^2\|_{\dot H^{\frac12}} ds\\ &\le C\int_0^t (t-s)^{-\frac34}\|u^2\|_{L^2} ds\\ &\le C\|u\|_{K_4}^2 \int_0^t (t-s)^{-\frac34} s^{-\frac14} ds\\ &\le C\|u\|_{K_4}^2,\end{aligned}$$

where we used another time Lemma 6. This shows that $B(u,u)\in\mathcal{C}(]0,\infty[;\dot H^{\frac12})$, and also that

$$\|B(u,u)\|_{L^\infty([0,\infty[;\dot H^{\frac12})} \le C\|u\|_{K_4}^2.$$

The theorem gives us that

$$\|u\|_{K_4} \le 2\|\mathcal{G}(\tau)u_0\|_{K_4},$$

so

$$\|B(u,u)\|_{L^\infty([0,\infty[;\dot H^{\frac12})} \le 4C\|\mathcal{G}(\tau)u_0\|_{K_4}^2.$$

Moreover we can repeat all the reasonings made so far, restricting the time interval to $]0,t[$, so we can state

$$\|B(u,u)(t)\|_{\dot H^{\frac12}} \le 4C\|\mathcal{G}(\tau)u_0\|_{K_4(t)}^2,$$

where $K_4(t)$ is defined as K_4 but in the interval $]0,t[$, instead of $]0,\infty[$.

We now follow the idea in [C]: we take $\phi\in\mathcal{S}$ such that

$$\|u_0-\phi\|_{\mathcal{B}_{\dot H^{\frac12},L^3}} \le \epsilon,$$

we can write

$$\begin{aligned}\|\mathcal{G}(\tau)u_0\|_{K_4(t)} &\le \|\mathcal{G}(\tau)(u_0-\phi)\|_{K_4(t)} + \|\mathcal{G}(\tau)\phi\|_{K_4(t)}\\ &\le C\|u_0-\phi\|_{\mathcal{B}_{\dot H^{\frac12},L^3}} + \sup_{\tau\in[0,\infty[} \tau^{\frac18}\|\mathcal{G}(\tau)\phi\|_{L^4}\\ &\le C\epsilon + \sup_{\tau\in[0,t[} \tau^{\frac18}\|\mathcal{G}(\tau)\phi\|_{\dot H^{\frac34}}\\ &\le C\epsilon + t^{\frac18}\|\phi\|_{\dot H^{\frac34}},\end{aligned}$$

where we use Sobolev imbeddings and the fact that for $\phi \in \mathcal{S}$ for every τ it holds that

$$\|\mathcal{G}(\tau)\phi\|_{\dot{H}^{\frac{3}{4}}} \leq \|\phi\|_{\dot{H}^{\frac{3}{4}}},$$

so we obtain that

$$\lim_{t \to 0^+} \|B(u,u)(t)\|_{\dot{H}^{\frac{1}{2}}} = 0.$$

The problem now is that we cannot hope to obtain a $\dot{H}^{\frac{1}{2}}$ regularity also the solution $u(t)$, since $\mathcal{G}(t)u_0 \notin \dot{H}^{\frac{1}{2}}$, then for sure the solution will not assume the initial data in $\dot{H}^{\frac{1}{2}}$ sense. Nevertheless to obtain a satisfying result we can pass to Besov Spaces, in fact we can show the following

Proposition 10 *Let $\mathcal{C}$ be a dyadic ring. There exists two constants $c, C > 0$ depending only on ν such that if $u \in L^p$, $1 \leq p \leq \infty$ with supp $\widehat{u} \subseteq \lambda\mathcal{C}$, then if $t < \frac{1}{\Omega}$ we get for any $\lambda > 0$*

$$\|\mathcal{G}(t)u\|_{L^p} \leq Ce^{-c\lambda^2 t}\|u\|_{L^p}.$$

Proof. The proof follows from the argument in [CMZ], Proposition 3.1, with a slight modification. Let us define $\varphi(\xi) \in \mathcal{C}^\infty(\mathbb{R}^3 \setminus 0)$ equal to 1 near the ring $\mathcal{C}$. Similarly to what we did before we write

$$\begin{aligned} \mathcal{G}(t)u &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\widehat{\mathcal{G}}(t,\xi)\varphi(\lambda^{-1}\xi)\widehat{u}(\xi)d\xi \\ &= \mathcal{F}^{-1}(\widehat{\mathcal{G}}(t,\xi)\varphi(\lambda^{-1}\xi)\widehat{u}(\xi)) \\ &= g(t,x) * u(x), \end{aligned}$$

with

$$g(t,x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\widehat{\mathcal{G}}(t,\xi)\varphi(\lambda^{-1}\xi)d\xi,$$

so we have to prove that

$$\|g(t,\cdot)\|_{L^1} \leq Ce^{-c\lambda^2 t},$$

since

$$\|\mathcal{G}(t)u\|_{L^p} = \|g * u\|_{L^p} \leq \|g(t,\cdot)\|_{L^1}\|u\|_{L^p}.$$

For $|x| \leq \lambda^{-1}$ we can infer that

$$\int_{|x|\leq\lambda^{-1}} |g(t,x)|dx \leq C\int_{|x|\leq\lambda^{-1}}\int_{\mathbb{R}^3} |\varphi(\lambda^{-1}\xi)||\widehat{\mathcal{G}}(t,\xi)|d\xi dx \leq Ce^{-c\lambda^2 t}.$$

Dealing with $|x| > \lambda^{-1}$, we use the operator $\mathcal{L}$ used in Proposition 7, writing

$$g(t,x) = \int_{\mathcal{R}^3} e^{ix\cdot\xi}(\mathcal{L}^*)^4(\varphi(\lambda^{-1}\xi)\widehat{\mathcal{G}}(t,\xi))d\xi.$$

It is easy to prove the following two estimates, for $\alpha \leq 4$,

$$|\partial_\xi^\beta e^{-\nu|\xi|^2 t}| \leq C|\xi|^{-\beta}e^{-\frac{\nu}{2}|\xi|^2 t},$$

$$|\partial_\xi^\alpha (e^{\pm i\Omega\frac{\xi_3}{|\xi|}t})| \leq C|\xi|^{-\alpha}(1+\Omega t)^{|\alpha|} \leq 16C|\xi|^{-\alpha}$$

where in the last inequality we use the fact that $t \leq \frac{1}{\Omega}$.

Thanks to these two inequalities it is easy to prove that

$$|(\mathcal{L}^*)^4(\varphi(\lambda^{-1}\xi)\widehat{\mathcal{G}}(t,\xi))| \leq C|\lambda x|^{-4}e^{-\frac{\nu}{4}|\xi|^2 t},$$

which easily imply that

$$\int_{|x|>\frac{1}{\lambda}} |g(t,x)|dx \leq Ce^{-c\lambda^2 t}\lambda^3\int_{|x|>\frac{1}{\lambda}} |\lambda x|^{-4}dx \leq Ce^{-c\lambda^2 t},$$

that concludes the proof.

□

Now, since

$$\mathcal{B}_{\dot{H}^{\frac{1}{2}},L^3} \hookrightarrow \dot{B}^{\frac{3}{p}-1}_{\infty,p}$$

we are able to obtain the following estimates, for $t < \frac{1}{\Omega}$,

$$\begin{aligned}
\|\mathcal{G}(t)u_0\|_{\dot{B}^{\frac{3}{p}-1}_{\infty,p}} &= \sup_{j\in\mathbb{Z}} 2^{j(\frac{3}{p}-1)}\|\Delta_j\mathcal{G}(t)u_0\|_{L^p} \\
&\leq C\sup_{j\in\mathbb{Z}} 2^{j(\frac{3}{p}-1)}\|\Delta_j u_0\|_{L^p}\left(=\|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{\infty,p}}\right) \\
&\leq C\|u_0\|_{\mathcal{B}_{\dot{H}^{\frac{1}{2}},L^3}}.
\end{aligned}$$

Let now deal with $t > \frac{1}{\Omega}$, we consider as before two parts of $\mathcal{G}(t)u_0$, dividing low and high frequencies. For the low frequencies it can be easily shown that

$$\|\mathcal{G}(t)u_0^{(1)}\|_{\dot{H}^{\frac{1}{2}}} \leq C\|u_0^{(1)}\|_{\dot{H}^{\frac{1}{2}}},$$

and this result holds for any t. For the high frequencies, we get from Proposition 7, for any $j > 0$

$$\|\Delta_j\mathcal{G}(t)u_0^{(2)}\|_{L^p} \leq Ct^{\frac{1}{2}(\frac{3}{p}-1)}\|\Delta_j u_0^{(2)}\|_{L^3},$$

and so we can write

$$\begin{aligned}
\|\mathcal{G}(t)u_0^{(2)}\|_{\dot{B}^{\frac{3}{p}-1}_{\infty,p}} &= \sup_{j>0} 2^{j(\frac{3}{p}-1)}\|\Delta_j\mathcal{G}(t)u_0^{(2)}\|_{L^p} \\
&\leq Ct^{\frac{1}{2}(\frac{3}{p}-1)}\sup_{j>0} 2^{j(\frac{3}{p}-1)}\|\Delta_j u_0^{(2)}\|_{L^3} \\
&\leq C\Omega^{\frac{1}{2}(\frac{3}{p}-1)}\|u_0^{(2)}\|_{L^3},
\end{aligned}$$

since, choosing $p > 3$, we get $2^{j(\frac{3}{p}-1)} < 1$.

we can resume all these results stating that

$$\mathcal{G}(t)u_0 \in L^\infty\left(0,+\infty;\mathcal{B}_{\dot{H}^{\frac{1}{2}},\dot{B}^{\frac{3}{p}-1}_{\infty,p}}\right),$$

with $\mathcal{B}_{\dot{H}^{\frac{1}{2}},\dot{B}^{\frac{3}{p}-1}_{\infty,p}}$ defined as before, dividing low and high frequencies.

□

3.2 The case $\dot{H}^{\frac{1}{2}} - \dot{B}^{\frac{3}{p}-1}_{p,\infty}$

In this section we generalize the previous result choosing a more general space for initial data, dividing low and high frequencies as before, but allowing less regularity for high ones.

In particular we will consider Besov spaces with negative index instead of L^3: the choice is due to the fact that, for any $p > 3$ we get

$$L^3(\mathbb{R}^3) \hookrightarrow \dot{B}^{\frac{3}{p}-1}_{p,\infty}(\mathbb{R}^3),$$

thus generalizing the previous case.

More precisely we give the following definition:

Definition. We say that $u \in \mathcal{B}_{\dot{H}^{\frac{1}{2}}, \dot{B}^{\frac{3}{p}-1}_{p,\infty}}$ if

$$\|u^{(1)}\|_{\dot{H}^{\frac{1}{2}}} + \|u^{(2)}\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} < +\infty.$$

We will prove that also in this case a smallness condition on initial data gives existence and uniqueness of global solution.

The key estimate

Let now $3 < p < 4$, we take and we study as before the term $\mathcal{G}(t)u_0$, proving the following proposition:

Proposition 11 *Let* $3 < p < 4$ *and* $u_0 \in \mathcal{B}_{\dot{H}^{\frac{1}{2}}, \dot{B}^{\frac{3}{p}-1}_{p,\infty}}$, *then*

$$\mathcal{G}(t)u_0 \in K_4.$$

Proof. We only deal with high frequencies, since for the low ones nothing has changed. We write $\mathcal{G}(t) = \tilde{\mathcal{G}}(t)S(t)$, with

$$\tilde{\mathcal{G}}(t) = \left[\cos\left(R_3\Omega t\right) I + \sin\left(R_3\Omega t\right)\mathbf{R}\right] e^{\frac{\nu}{2}t\Delta} \qquad S(t) = e^{\frac{\nu}{2}t\Delta}$$

because we want to use the following

Proposition 12 *Let p be fixed in $1 \le p \le \infty$ and $\alpha > 0$ then there exist $C > 0$ such that then for every $f \in \mathcal{S}'(\mathbb{R}^3)$*

$$C^{-1} \sup_{j\in\mathbb{Z}} 2^{-j\alpha} \|\Delta_j f\|_{L^p} \le \sup_{t\ge 0} t^{\frac{\alpha}{2}} \|S(t)f\|_{L^p} \le C \sup_{j\in\mathbb{Z}} 2^{-j\alpha} \|\Delta_j f\|_{L^p}$$

Choosing $\alpha = 1 - \frac{3}{p}$ this proposition tell us that $f \in \dot{B}^{\frac{3}{p}-1}_{\infty,p} \Rightarrow S(t)f \in K_p$ and so we can use this result stating that $S(t)u_0^{(2)} \in K_p$, with $3 < p < 4$.

Since *supp* $\mathfrak{F}(S(t)u_0^{(2)}) \subseteq \mathcal{C}(B_1(0))$ we can use Proposition 7, applied to $\tilde{\mathcal{G}}(t)$, taking $r > 1$ and $1 + \frac{1}{4} = \frac{1}{r} + \frac{1}{p}$

$$\begin{aligned}
\|\mathcal{G}(t)u_0^{(2)}\|_{K_4} &= \|\tilde{\mathcal{G}}(t)S(t)u_0^{(2)}\|_{K_4} \\
&= \sup_{t\ge 0} t^{\frac{1}{8}} \|\tilde{\mathcal{G}}(t)S(t)u_0^{(2)}\|_{L^4} \\
&\le C \sup_{t\ge 0} t^{\frac{1}{8}} t^{-\frac{3}{2}(1-\frac{1}{r})} \|S(t)u_0^{(2)}\|_{L^p} \\
&\le C\|S(t)u_0^{(2)}\|_{K_p} \le C\|u_0^{(2)}\|_{\dot{B}^{\frac{3}{p}-1}_{\infty,p}},
\end{aligned}$$

where in the last line we use the fact that

$$\frac{1}{8} - \frac{3}{2}\left(1 - \frac{1}{r}\right) = \frac{1}{2}\left(1 - \frac{3}{p}\right).$$

□

Statement and proof of the theorem

Now we can use exactly the result obtained above for the bilinear part, since we are "landed" on K_4 as before, so we can directly state the following theorem:

Theorem 7 *Let $3 < p < 4$. There exists a constant $c > 0$, depending on Ω/ν, such that if $\|u_0\|_{\mathcal{B}_{\dot{H}^{\frac{1}{2}},\dot{B}^{\frac{3}{p}-1}_{p,\infty}}} \leq c$, then there exist a unique fixed point $u(t) \in K_4$ of (1.1.2). Moreover it holds that*

$$\|u(t)\|_{K_4} \leq 2\|\mathcal{G}(t)u_0\|_{K_4},$$

$$u \in L^\infty(0, +\infty; \mathcal{B}_{\dot{H}^{\frac{1}{2}},\dot{B}^{\frac{3}{p}-1}_{p,\infty}}) \qquad p > 3,$$

$$B(u,u) \in \mathcal{C}(0, +\infty; \dot{H}^{\frac{1}{2}}) \text{ and } \lim_{t\to 0^+} \|B(u(t), u(t))\|_{\dot{H}^{\frac{1}{2}}} = 0.$$

Proof. The proof is essentially the same as the one of the previous case, since Proposition 11 tells us that the linear part belongs to K_4, where we can apply Proposition 9 just as before.

Also in this case the solution is global and the recostruction of regularity can be obtained in the same way as above.

□

Chapter 4

Anisotropic Navier-Stokes-Coriolis system

In this chapter we will consider the following system:

$$\begin{cases} \partial_t \mathbf{u} - \nu_h \Delta_h \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + B(t, x_1, x_2) \wedge \mathbf{u} + \frac{1}{\rho} \nabla p = 0 \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0 \end{cases} \qquad (ANSC)$$

where the operator $\Delta_h = \partial^2_{x_1} + \partial^2_{x_2}$ is the horizontal laplacian.

We call this system Anisotropic Navier - Stokes - Coriolis, since zero vertical viscosity is considered. For this system the following theorem holds:

Theorem 8 (M. Majdoub - M. Paicu) *Let $s > \frac{1}{2}$ and $u_0 \in H^{0,s}$ a divergence free vector fields. There exists $c > 0$ such that if $\|u_0\|_{H^{0,s}} \leq c\nu_h$ then there exists a unique global solution u of ANSC such that*

$$u \in \mathcal{C}_b(\mathbb{R}_+; H^{0,s}) \quad \text{and} \quad \nabla_h u \in L^2(\mathbb{R}_+; H^{0,s}).$$

the proof of this theorem is in [MP]. We try to go beyond the assumption $s > \frac{1}{2}$, considering the following space:

$$\|u\|_{B^{0,\frac{1}{2}}} = \sum_{q \in \mathbb{Z}} 2^{\frac{q}{2}} \|\Delta^v_q u\|_{L^2} < +\infty,$$

this choice is motivated by the result in [P], where the case $s = \frac{1}{2}$ can only be treated using this space. This space is called anisotropic Besov space, since we consider only a vertical version of Littlewood - Paley decomposition in the definition. The use of this space does not represent a proper generalization of the previous result since

$$B^{0,\frac{1}{2}} \hookrightarrow H^{0,\frac{1}{2}}.$$

In the next sction we state and prove the main theorem for this functional space: the proof will be a combination of the proofs in the two articles cited above.

4.1 Main Theorem

We prove the following

Theorem 9 *Let $u_0 \in B^{0,\frac{1}{2}}$ a divergence free vector fields. There exists $c > 0$ such that if $\|u_0\|_{B^{0,\frac{1}{2}}} \leq c\nu_h$ then there exists a global solution u of ANSC such that*

$$u \in L^\infty(\mathbb{R}_+; B^{0,\frac{1}{2}}) \quad \text{and} \quad \nabla_h u \in L^2(\mathbb{R}_+; B^{0,\frac{1}{2}}).$$

Proof. We start conidering the following operators

$$\begin{aligned} J_n u &= \mathcal{F}^{-1}(\chi_{B(0,n)}\mathcal{F}u), \\ J_n^v u &= \mathcal{F}^{-1}(\chi_{\{\xi:|\xi_3|\leq n\}}\mathcal{F}u), \\ \tilde{J}_n u &= (J_n - J^v_{\frac{1}{n}})u, \end{aligned}$$

we look for v_n that satisfies the following system, which we refer to as $(ANSC^n)$,

$$\begin{cases} \partial_t v_n - \nu_h \tilde{J}_n \Delta_h v_n + \tilde{J}_n(\tilde{J}_n v_n \cdot \nabla \tilde{J}_n v_n) \\ \qquad +\tilde{J}_n(B(t,x_1,x_2) \wedge \tilde{J}_n v_n) - \tilde{J}_n \nabla \sum_{i,j} \partial_i \partial_j \Delta^{-1}(\tilde{J}_n v_n^i \tilde{J}_n v_n^j) \\ \qquad -\tilde{J}_n \nabla \Delta^{-1}(\nabla \cdot (B(t,x_1,x_2) \wedge \tilde{J}_n v_n) = 0 \\ \nabla \cdot v_n = 0 \\ v_n(0) = \tilde{J}_n \mathbf{u}_0, \end{cases}$$

where we used the fact that applying the divergence to (ANSC) we obtain

$$\Delta p = \sum \partial_i \partial_j (v^i v^j) + \nabla \cdot (B \wedge v)$$

and then we write the approximate version using the operator $\tilde{J}_n$ defined above.

Since $u_0 \in B^{0,\frac{1}{2}}$ then $\tilde{J}_n u_0 \in L^2$, moreover all terms appearing in the equation are continuous L^2 to L^2 or from $L^2 \times L^2$ to L^2, so the system is an equation in L^2. Thank to Cauchy - Lipschitz theorem we can state that there exists a unique solution local in time with values in L^2: for every n we call T_n the maximum life time of v_n.

Since $\tilde{J}_n^2 = \tilde{J}_n$ the function $\tilde{J}_n v_n$ is also a solution of the system above and from the uniqueness it holds that $v_n = \tilde{J}_n v_n$, so v_n satisfies

$$\begin{cases} \partial_t v_n - \nu_h \Delta_h v_n + \tilde{J}_n(v_n \cdot \nabla v_n) \\ \qquad + \tilde{J}_n(B(t,x_1,x_2) \wedge v_n) - \tilde{J}_n \nabla \sum_{i,j} \partial_i \partial_j \Delta^{-1}(v_n^i v_n^j) \\ \qquad - \tilde{J}_n \nabla \Delta^{-1}(\nabla \cdot (B(t,x_1,x_2) \wedge v_n) = 0 \\ \nabla \cdot v_n = 0 \\ v_n(0) = \tilde{J}_n \mathbf{u}_0, \end{cases} \tag{4.1}$$

from this equation we can obtain a L^2 energy estimate for v_n using the following facts:

1. $(\tilde{J}_n(v_n \cdot \nabla v_n), v_n)_{L^2} = 0$,
2. $(\tilde{J}_n(B \wedge v_n), v_n)_{L^2} = 0$,
3. $(\nabla p, v_n)_{L^2} = 0$.

The first assertion can be easily shown, in fact when $\nabla \cdot v_n = 0$ one has

$$(\tilde{J}_n(v_n \cdot \nabla v_n), v_n)_{L^2} = (\tilde{J}_n \nabla(v_n \otimes v_n), v_n)_{L^2}$$

so, integrating by parts, we obtain $\nabla \cdot v_n$, that is 0, on the second term.

The second assertion is true for geometrical reason, while the third can be obtained by divergence free condition.

The energy equation reads

$$\frac{1}{2}\frac{d}{dt}\|v_n(t)\|_{L^2}^2 + \nu_h \|\nabla_h v_n\|_{L^2}^2 = 0,$$

thanks to this equation we can assert that $T_n = +\infty$, since $\|v_n(t)\|_{L^2}$ remains bounded in $[0, T_n[$.

We now show a global estimate for the solution v_n in the space $B^{0,\frac{1}{2}}$. Then using classical argument on compactness and taking the limit in the approximation system $(ANSC^n)$, we will finally show the existence and uniqueness of a global solution to $(ANSC)$.

We start by applying the operator Δ_j^v to equation $(ANSC^n)$ (from now on we drop the suffix "n", to ease the notation).

$$(\Delta_j^v u)_t - \nu_h \Delta_h \Delta_j^v + \Delta_j^v (u \cdot \nabla) u + \Delta_j^v (B \times u) = -\nabla \Delta_j^v p, \tag{4.2}$$

Remark. Since $B = B(t, x_1, x_2)$

$$\Delta_j^v (B \times u) = B \times \Delta_j^v u,$$

so this vector is orthogonal to $\Delta_j^v u$.

thanks to this remark, taking the scalar product of 4.2 with $\Delta_j^v u$ and integrating in $\mathbb{R}^3$ we obtain:

$$\frac{1}{2} \frac{d}{dt} \|\Delta_j^v u\|_{L^2}^2 + \nu_h \|\nabla_h \Delta_j^v u\|_{L^2}^2 = -\left(\Delta_j^v (u \cdot \nabla) u, \Delta_j^v u\right)_{L^2},$$

and so

$$\frac{d}{dt} \|\Delta_j^v u\|_{L^2}^2 + 2\nu_h \|\nabla_h \Delta_j^v u\|_{L^2}^2 \leq 2 \left| \left(\Delta_j^v (u \cdot \nabla) u, \Delta_j^v u\right)_{L^2} \right|,$$

integrating in $[0, t]$

$$\|\Delta_j^v u(t)\|_{L^2}^2 + 2\nu_h \int_0^t \|\nabla_h \Delta_j^v u\|_{L^2}^2 ds \leq \|\Delta_j^v u_0\|_{L^2}^2 + C 2^{-j+1} a_j \|\nabla_h u\|_{\widetilde{L_t^2}(B^{0,\frac{1}{2}})}^2 \|u\|_{\widetilde{L_t^\infty}(B^{0,\frac{1}{2}})},$$

with $\sum_j a_j^{\frac{1}{2}} \leq 1$.

Where we used the following proposition ([MP]):

Proposition 13 *There exists $C > 0$ such that for any divergence free vector field v one has*

$$\int_0^T |(\Delta_j^v (v \cdot \nabla v)(t), \Delta_j^v v(t))_{L^2})| dt \leq C 2^{-j} a_j \|\nabla_h v\|_{\widetilde{L_T^2}(B^{0,\frac{1}{2}})}^2 \|v\|_{\widetilde{L_T^\infty}(B^{0,\frac{1}{2}})},$$

with

$$\sum_{j \in \mathbb{Z}} a_j^{\frac{1}{2}} \leq 1.$$

Taking the sup on $[0,T]$ we get

$$\|\Delta_j^v u\|^2_{L^\infty_T(L^2)}+2\nu_h\int_0^T\|\nabla\Delta_j^v u\|^2_{L^2}ds\leq C2^{-j}a_j\|\nabla_h u\|^2_{\widetilde{L^2_T}(B^{0,\frac12})}\|u\|_{\widetilde{L^\infty_T}(B^{0,\frac12})}+\|\Delta_j^v u_0\|^2_{L^2}$$

from which, remembering that for $a,b>0$ it holds that $\frac{1}{2}(a+b)^2\leq a^2+b^2\leq(a+b)^2$, we obtain

$$\|\Delta_j^v u\|_{L^\infty_T(L^2)}+\sqrt{2\nu_h}\left(\int_0^T\|\nabla\Delta_j^v u\|^2_{L^2}ds\right)^{\frac12}\leq\sqrt{2}\|\Delta_j^v u_0\|_{L^2}+\sqrt{2}C^{\frac12}2^{-\frac{j}{2}}a_j^{\frac12}\|\nabla_h u\|_{\widetilde{L^2_T}(B^{0,\frac12})}\|u\|^{\frac12}_{\widetilde{L^\infty_T}(B^{0,\frac12})}$$

We now multiply both side by $2^{\frac{j}{2}}$ and take the summation on j, obtaining

$$\|u\|_{\widetilde{L^\infty_T}(B^{0,\frac12})}+\sqrt{2\nu_h}\|\nabla_h u\|_{\widetilde{L^2_T}(B^{0,\frac12})}\leq\sqrt{2}\|u_0\|_{B^{0,\frac12}}+\sqrt{2}C\|\nabla_h u\|_{\widetilde{L^2_T}(B^{0,\frac12})}\|u\|^{\frac12}_{\widetilde{L^\infty_T}(B^{0,\frac12})}$$

And finally

$$\|u\|^2_{\widetilde{L^\infty_T}(B^{0,\frac12})}+2\nu_h\|\nabla_h u\|^2_{\widetilde{L^2_T}(B^{0,\frac12})}\leq 8\|u_0\|^2_{B^{0,\frac12}}+16C\|\nabla_h u\|^2_{\widetilde{L^2_T}(B^{0,\frac12})}\|u\|_{\widetilde{L^\infty_T}(B^{0,\frac12})}. \tag{4.3}$$

We now use this inequality to deduce global existence when the initial datum is small. Let

$$\|v_0\|_{B^{0,\frac12}}\leq c\nu_h,$$

with $c<\frac{1}{32\sqrt{2}C}$, with C as in 4.3.

Let v_n the regular solution of the approximated equation with the initial datum $\tilde{J}_n v_0$, let also T_n^* the maximum time for which

$$\|v_n(t)\|_{\widetilde{L^\infty_T}(B^{0,\frac12})}\leq 2\sqrt{2}c\nu_h,$$

for any $T<T_n^*$.

Using (4.3) we obtain

$$\|v_n\|^2_{\widetilde{L^\infty_T}(B^{0,\frac12})}+2\nu_h\|\nabla_h v_n\|^2_{\widetilde{L^2_T}(B^{0,\frac12})}\leq 8(c\nu_h)^2+\frac{16C}{16C}\nu_h\|\nabla_h v_n\|^2_{\widetilde{L^2_T}(B^{0,\frac12})}$$

and so

$$\|v_n\|_{\widetilde{L^\infty_T}(B^{0,\frac12})}\leq 2\sqrt{2}c\nu_h.$$

Since v_n is regular in time with values in $B^{0,\frac{1}{2}}$ and so the function

$$t \mapsto \|v_n\|_{\widetilde{L^\infty_T}(B^{0,\frac{1}{2}})}$$

is continuous, we can obtain $T_n^* = +\infty$.

Moreover one has

$$v_n \in \widetilde{L^\infty}(\mathbb{R}^+, B^{0,\frac{1}{2}}) \quad \text{with} \quad \nabla_h v_n \in \widetilde{L^2}(\mathbb{R}^+, B^{0,\frac{1}{2}}).$$

Now since $S_0^v v_n \in L_v^\infty(L_h^2)$ (see Appendix) and $(I - S_0^v)v_n \in L^2$ we obtain that $(v_n)_n$ is a bounded sequence in $L^\infty(\mathbb{R}^+, L^2_{loc})$ and moreover

$$(\partial_t v_n)_n \quad \text{is bounded in} \quad L^\infty(\mathbb{R}^+, H^{-N}_{loc}), \tag{4.4}$$

in fact

$$\partial_t v_n = -\widetilde{J}_n(v_n \cdot \nabla v_n) + \nu_h \Delta_h v_n - \widetilde{J}_n(B \times v_n) + \widetilde{J}_n \nabla \sum_{i,j} \partial_i \partial_j \Delta^{-1}(v_n^i v_n^j) + \widetilde{J}_n(\nabla \Delta div(B \times v_n)),$$

one has that there exists $C > 0$ such that

$$\|\partial_t v_n\|_{H^{-N}} = \sup_{\|\phi\|_{H_0^N} \leq 1} (\partial_t v_n, \phi)_{L^2} \leq 2C\|v_n\|^2_{L^2} + (\nu_h + 2\|B\|_{L^\infty})\|v_n\|_{L^2},$$

where we choose N such that $H_0^N \hookrightarrow L^\infty$ we obtain (4.4) (See Lemma 9 in the Appendix for details).

We now use Ascoli-Arzelà theorem, asseting that there exists a subsequence of v_n such that

$$v_{n_k} \to v \quad \text{in} \quad L^\infty_{loc}(\mathbb{R}^+; H^{-\sigma}_{loc}) \quad \forall \sigma > 0.$$

Moreover since v_n is bounded in $L^2_{loc}(\mathbb{R}, H^\epsilon_{loc})$, for any $\epsilon < \frac{1}{2}$, as a consequence of product law in Sobolev spaces, we have

$$v_n \otimes v_n \to v \otimes v \quad \text{in} \quad L^2_{loc}(H^{\epsilon - \sigma - \frac{3}{2}}) \quad \text{when } \sigma < \epsilon,$$

in particular

$$v_n \otimes v_n \to v \otimes v \quad \text{in} \quad \mathcal{D}',$$

passing t the limit in (4.1) we obtain a global solution of (ANSC), with $v \in \widetilde{L^\infty}(\mathbb{R}^+, B^{0,\frac{1}{2}})$ and $\nabla_h v \in \widetilde{L^2}(\mathbb{R}^+, B^{0,\frac{1}{2}})$ $\square$

Remark. We actually show that us the solutions have a better regularity since

$$\widetilde{L^\infty}(\mathbb{R}^+, B^{0,\frac{1}{2}}) \hookrightarrow L^\infty(\mathbb{R}^+, B^{0,\frac{1}{2}}) \text{ and } \widetilde{L^2}(\mathbb{R}^+, B^{0,\frac{1}{2}}) \hookrightarrow L^2(\mathbb{R}^+, B^{0,\frac{1}{2}})$$

Chapter 5

Backward uniqueness for thermoelastic system

In this chapter we will investigate the uniqueness of the following system:

$$\begin{cases} \left.\begin{array}{l} \partial_t^2 u - \partial_x(a(t,x)\partial_x u) + \alpha(t,x)\partial_x\theta(t,x) + \beta(t,x)\theta(t,x) = f(t,x) \\ \partial_t\theta + \partial_x(b(t,x)\partial_x\theta) + \delta(t,x)\partial_x\partial_t u(t,x) + \rho(t,x)\partial_t u(t,x) + \phi(t,x)u(t,x) = g(t,x) \end{array}\right\} \text{on }]0,T[\times\mathbb{R} \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x),\ \theta(0,x) = \theta_0(x) \qquad \text{on } \ \mathbb{R}, \end{cases} \tag{5.1}$$

where

$$\begin{array}{l} \bullet\ a \in LL([0,T], LL(\mathbb{R}_x), 0 < \lambda_0 \le a(t,x) \le \Lambda_0; \\ \bullet\ b \in \mathcal{C}^\mu([0,T], \mathcal{C}^{1,\epsilon}(\mathbb{R}_x)), 0 \le \mu_0 \le b(t,x) \le M_0, \\ \ \ \text{with } \mu \text{ that satisfies Osgood and } \star \text{ condition, defined in chapter 1;} \\ \bullet\ \delta \in L^\infty([0,T], \mathcal{C}^{1,\epsilon}(\mathbb{R}_x)); \\ \bullet\ \alpha, \beta, \rho, \phi \in L^\infty([0,T], \mathcal{C}^{0,\epsilon}(\mathbb{R}_x)). \end{array} \tag{5.2}$$

In order to prove uniqueness we consider the homogeneous system

$$\begin{cases} \left.\begin{array}{l} \partial_t^2 u - \partial_x(a(t,x)\partial_x u) + \alpha(t,x)\partial_x\theta(t,x) + \beta(t,x)\theta(t,x) = 0 \\ \partial_t\theta + \partial_x(b(t,x)\partial_x\theta) + \delta(t,x)\partial_x\partial_t u(t,x) + \rho(t,x)\partial_t u(t,x) + \phi(t,x)u(t,x) = 0 \end{array}\right\} \text{on }]0,T[\times\mathbb{R} \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x),\ \theta(0,x) = \theta_0(x) \qquad \text{on } \ \mathbb{R}, \end{cases} \tag{5.3}$$

5.1 Carleman estimate for the hyperbolic operator

In this section we will deal with the hyperbolic operator

$$\mathcal{P}u = \partial_t^2 u - \partial_x(a(t,x)\partial_x u),$$

proving an energy estimate.

In this section we consider $u \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with *supp* $u \subseteq [0, \frac{T}{2}] \times \mathbb{R}$. We start by applying to $\mathcal{P}u$ the Littlewood - Paley decomposition obtaining

$$\begin{aligned} \varphi_\nu(\mathcal{P}u) &= \partial_t^2 u_\nu - \partial_x(\varphi_\nu(a(t,x)\partial_x u)) \\ &= \partial_t^2 u_\nu - \partial_x([\varphi_\nu, a(t,x)]\,\partial_x u) - \partial_x(a(t,x)\partial_x u_\nu), \end{aligned} \tag{5.4}$$

where $[\cdot,\cdot]$ is the commutator.

We define a localized and approximated energy

$$e_{\nu,\epsilon}(t) = \int_{\mathbb{R}} a_\epsilon(t,x)|\partial_x u_\nu|^2 dx + \int_{\mathbb{R}} |\partial_t u_\nu|^2 dx + \int_{\mathbb{R}} |u_\nu|^2 dx,$$

where $a_\epsilon(t,x)$ is the mollified (w.r.t. time) version of $a(t,x)$. We prove the following lemma

Lemma 5 *Let $e_{\nu,\epsilon}(t)$ defined above and suppose $\epsilon = 2^{-\nu}$, then there exists $C > 0$ depending only on $\|a\|_{LL}$, λ_0 and Λ_0 such that setting $e_\nu = e_{\nu,2^{-\nu}}$*

$$\left|\frac{d}{dt}e_\nu(t)\right| \le C(\nu+1)e_\nu + C\|(Pu)_\nu\|_{L^2} e_\nu^{\frac{1}{2}} + 2^\nu e_\nu^{\frac{1}{2}} \|[\varphi_\nu, a(t,x)]\partial_x u\|_{L^2}$$

Proof.

$$\begin{aligned}
\frac{d}{dt}e_\nu(t) &= \int_{\mathbb{R}} \partial_t a_\epsilon \cdot |\partial_x u_\nu|^2 dx \ + 2Re \int_{\mathbb{R}} a_\epsilon \partial_x u_\nu \cdot \overline{\partial_t \partial_x u_\nu} dx \\
&\quad + 2Re \int_{\mathbb{R}} \partial_t u_\nu \cdot \overline{\partial_t^2 u_\nu} dx \ + \ 2Re \int_{\mathbb{R}} u_\nu \cdot \overline{\partial_t u_\nu} dx \\
&= \int_{\mathbb{R}} \partial_t a_\epsilon \cdot |\partial_x u_\nu|^2 dx \ + \ 2Re \int_{\mathbb{R}} a_\epsilon \cdot \partial_x u_\nu \cdot \overline{\partial_t \partial_x u_\nu} dx \ + \ 2Re \int_{\mathbb{R}} \partial_t u_\nu \cdot \overline{(\mathcal{P}u)_\nu} dx \\
&\quad + 2Re \int_{\mathbb{R}} \partial_t u_\nu \cdot \overline{\partial_x (a(t,x) \cdot \partial_x u_\nu)} dx \ + \ 2Re \int_{\mathbb{R}} \partial_t u_\nu \cdot \overline{\partial_x ([\varphi_\nu, a] \partial_x u)} dx \\
&\qquad + 2Re \int_{\mathbb{R}} u_\nu \cdot \overline{\partial_t u_\nu} dx \\
&= \int_{\mathbb{R}} \partial_t a_\epsilon \cdot |\partial_x u_\nu|^2 dx \ + \ 2Re \int_{\mathbb{R}} a_\epsilon \cdot \partial_x u_\nu \cdot \overline{\partial_t \partial_x u_\nu} dx \ + \ 2Re \int_{\mathbb{R}} \partial_t u_\nu \cdot \overline{(\mathcal{P}u)_\nu} dx \\
&\quad - 2Re \int_{\mathbb{R}} \partial_x \partial_t u_\nu \cdot \overline{(a(t,x) \cdot \partial_x u_\nu)} dx \ + \ 2Re \int_{\mathbb{R}} \partial_t u_\nu \cdot \overline{\partial_x ([\varphi_\nu, a] \partial_x u)} dx \\
&\qquad + 2Re \int_{\mathbb{R}} u_\nu \cdot \overline{\partial_t u_\nu} dx \\
&= \int_{\mathbb{R}} \partial_t a_\epsilon \cdot |\partial_x u_\nu|^2 dx \ + \ 2Re \int_{\mathbb{R}} (a_\epsilon - a) \cdot \partial_x u_\nu \cdot \overline{\partial_t \partial_x u_\nu} dx \ + \ 2Re \int_{\mathbb{R}} \partial_t u_\nu \cdot \overline{(\mathcal{P}u)_\nu} dx \\
&\quad + 2Re \int_{\mathbb{R}} \partial_t u_\nu \cdot \overline{\partial_x ([\varphi_\nu, a] \partial_x u)} dx \ + \ 2Re \int_{\mathbb{R}} u_\nu \cdot \overline{\partial_t u_\nu} dx \\
&\leq \int_{\mathbb{R}} \partial_t a_\epsilon \cdot |\partial_x u_\nu|^2 dx \ + \ 2 \int_{\mathbb{R}} |a_\epsilon - a| \cdot |\partial_x u_\nu| \cdot |\partial_t \partial_x u_\nu| dx \ + \ 2 \int_{\mathbb{R}} |\partial_t u_\nu| \cdot |(\mathcal{P}u)_\nu| dx \\
&\quad + 2 \int_{\mathbb{R}} |\partial_t u_\nu| \cdot |\partial_x ([\varphi_\nu, a] \partial_x u)| dx \ + \ 2 \int_{\mathbb{R}} |u_\nu| \cdot |\partial_t u_\nu| dx \\
&\leq C \log\left(\frac{1}{\epsilon}\right) e_\nu \ + \ C\epsilon \log\left(\frac{1}{\epsilon}\right) 2^\nu e_\nu \\
&\quad + C \|(\mathcal{P}u)_\nu\|_{L^2} e_\nu^{\frac{1}{2}} \ + \ 2^\nu \|[\varphi_\nu, a] \partial_x u\|_{L^2} e_\nu^{\frac{1}{2}} \ + \ 2e_\nu.
\end{aligned}$$

□

We now define the total energy choosing $\omega \in]0, \frac{1}{2}[$:

$$E(t) = \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t} 2^{-2\nu\omega} e_\nu(t)$$

Theorem 10 *(Carleman estimate for the Energy)*

Let E(t) be the energy defined above, then there exist $C_0, \gamma_0 > 0$ such that for any $\gamma > \gamma_0$ one has

$$\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} E(t)dt \leq \frac{C_0}{\left(\Phi'\left(\frac{\gamma T}{2}\right)\right)^2} \int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \|\mathcal{P}u\|^2_{H^{-\omega-\beta^* t}} dt$$

where $\Phi(\cdot)$ is the function defined in Chapter 1 starting from the modulus of continuity of $b(t,x)$ and $\beta^* = \beta \log 2$, with β as in the definition of $E(t)$.

Proof. We start by computing the time derivative of $E(t)$:

$$E'(t) = \sum_{\nu=0}^{+\infty} -2\beta(\nu+1)e^{-2\beta(\nu+1)t} 2^{-2\nu\omega} e_\nu(t) + \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t} e'_\nu(t) 2^{-2\nu\omega},$$

using the Lemma 5 we obtain the following inequality

$$\begin{aligned} E'(t) \leq \quad & \sum_{\nu=0}^{+\infty} -2\beta(\nu+1)e^{-2\beta(\nu+1)t}2^{-2\nu\omega} e_\nu(t) \\ & + \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t} C'(\nu+1) 2^{-2\nu\omega} e_\nu(t) \\ & + \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t} 2^{-2\nu\omega} C'' \|\mathcal{P}u\|_{L^2} e_\nu^{\frac{1}{2}}(t) \\ & + \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t} 2^{-2\nu\omega} 2^\nu e_\nu^{\frac{1}{2}}(t) \|[\varphi_\nu, a]\partial_x u\|_{L^2} \end{aligned} \tag{5.5}$$

We now focus our attention on the fourth term of the right hand side, we first write

$$[\varphi_\nu, a]\partial_x u = [\varphi_\nu, a] \left(\sum_{\mu=0}^{+\infty} \partial_x u_\mu \right) = \sum_{\mu=0}^{+\infty} ([\varphi_\nu, a]\psi_\mu) \partial_x u_\mu,$$

with $\psi_\mu = \varphi_{\mu-1} + \varphi_\mu + \varphi_{\mu+1}$, so we can write

$$\begin{aligned}\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}2^{-2\nu\omega}2^{\nu}e_{\nu}^{\frac{1}{2}}(t)\|[\varphi_\nu,a]\partial_x u\|_{L^2} &= \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}2^{-2\nu\omega}2^{\nu}e_{\nu}^{\frac{1}{2}}(t)\left\|\sum_{\mu=0}^{+\infty}([\varphi_\nu,a]\psi_\mu)\partial_x u_\mu\right\|_{L^2}\\ &\le \sum_{\mu,\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}2^{-2\nu\omega}2^{\nu}e_{\nu}^{\frac{1}{2}}(t)\,\|[\varphi_\nu,a]\psi_\mu\|_{\mathcal{L}(L^2)}\,\|\partial_x u_\mu\|_{L^2}\\ &\le C''\sum_{\mu,\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}2^{-2\nu\omega}2^{\nu}e_{\nu}^{\frac{1}{2}}(t)e_{\mu}^{\frac{1}{2}}(t)\,\|[\varphi_\nu,a]\psi_\mu\|_{\mathcal{L}(L^2)}\\ &\le C''\sum_{\mu,\nu=0}^{+\infty} K_{\mu,\nu}\left(2^{-\nu\omega}(\nu+1)^{\frac{1}{2}}2^{-\beta(\nu+1)t}e_{\nu}^{\frac{1}{2}}\right)\left(2^{-\mu\omega}(\mu+1)^{\frac{1}{2}}2^{-\beta(\mu+1)t}e_{\mu}^{\frac{1}{2}}\right)\end{aligned}$$

where

$$K_{\nu,\mu} = 2^{\nu}\|[\varphi_\nu,a]\psi_\mu\|_{\mathcal{L}(L^2)}e^{-\beta(\nu-\mu)t}(\nu+1)^{-\frac{1}{2}}(\mu+1)^{-\frac{1}{2}}2^{-(\nu-\mu)\omega},$$

now, since there exists $\bar{C} > 0$ such that

$$\sup_{\nu}\sum_{\mu=0}^{+\infty}|K_{\nu,\mu}| + \sup_{\mu}\sum_{\nu=0}^{+\infty}|K_{\nu,\mu}| \le \bar{C}$$

(see Lemma 11 in the appendix), we can use Schur's lemma and obtain

$$\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}2^{-2\nu\omega}2^{\nu}e_{\nu}^{\frac{1}{2}}(t)\|[\varphi_\nu,a]\partial_x u\|_{L^2} \le C_a\sum_{\nu=0}^{+\infty}2^{-2\nu\omega}(\nu+1)e^{-2\beta(\nu+1)t}e_{\nu},$$

returning to (5.5) we obtain

$$\begin{aligned}E'(t) &\le \sum_{\nu=0}^{+\infty}(-2\beta+C'+C_a)(\nu+1)2^{-2\nu\omega}e^{-2\beta(\nu+1)t}e_{\nu}\\ &+C''\sum_{\nu=0}^{+\infty}(e^{-\beta(\nu+1)t}2^{-\nu\omega}\|(\mathcal{P}u)_\nu\|_{L^2})(e^{-\beta(\nu+1)t}2^{-\nu\omega}e_{\nu}^{\frac{1}{2}}),\end{aligned}$$

we now choose β such that $C_a + C - 2\beta < 0$ (this will force us to choose a smaller T since βT has already been fixed) and we use Cauchy-Schwarz inequality to obtain

$$E'(t) \le C\|\mathcal{P}u\|_{H^{-\omega-\beta^* t}}(E(t))^{\frac{1}{2}},$$

and so

$$\frac{d}{dt}(E(t))^{\frac{1}{2}} \leq C\|\mathcal{P}u\|_{H^{-\omega-\beta^* t}} = g(t),$$

this yields

$$(E(t))^{\frac{1}{2}} - (E(0))^{\frac{1}{2}} = \int_0^t g(s)ds,$$

since $E(0) = 0$ we have

$$E(t) \leq \left(\int_0^t g(s)ds\right)^2.$$

From this last estimate we will obtain the Carleman estimate we are looking for: in order to do this we consider $0 \leq s \leq t \leq \frac{T}{2}$, we get, remembering the properties of Φ, defined in the first chapter:

$$\begin{aligned}
\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}E(t)dt &\leq \int_0^{\frac{T}{2}} \left(\int_0^t e^{\frac{1}{\gamma}(\Phi(\gamma(T-t)-\Phi(\gamma(T-s))}e^{\frac{1}{\gamma}\Phi(\gamma(T-s))}g(s)ds\right)^2 dt \\
&\leq \int_0^{\frac{T}{2}} \left(\int_{-\infty}^{+\infty} e^{-\Phi'(\frac{\gamma T}{2})|t-s|}e^{\frac{1}{\gamma}\Phi(\gamma(T-s))}g(s)\chi_{[0,\frac{T}{2}]}(s)ds\right)^2 dt \\
&\leq \int_0^{\frac{T}{2}} \left(e^{-\Phi'(\frac{\gamma T}{2})|t|} * e^{\frac{1}{\gamma}\Phi(\gamma(T-t))}g(t)\chi_{[0,\frac{T}{2}]}(t)\right)^2 dt,
\end{aligned}$$

now, since $\|f * g\|_{L^2}^2 \leq \|f\|_{L^1}^2\|g\|_{L^2}^2$, we finally obtain:

$$\begin{aligned}
\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}E(t)dt &\leq \frac{2}{\left(\Phi'\left(\frac{\gamma T}{2}\right)\right)^2}\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}g(t)^2 dt \\
&\leq \frac{C_0}{\left(\Phi'\left(\frac{\gamma T}{2}\right)\right)^2}\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\|\mathcal{P}u\|_{H^{-\omega-\beta^* t}}^2 dt
\end{aligned}$$

□

5.2 Carleman Estimate for parabolic operator

We now consider $\mathcal{L}\theta = \partial_t\theta + \partial_x(b(t,x)\partial_x\theta)$: let $\theta \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with $supp\ \theta \subseteq [0, \dfrac{T}{2}] \times \mathbb{R}$. We apply the Littlewood - Paley decomposition to this backward parabolic operator:

$$\begin{aligned}\varphi_\nu(\mathcal{L}\theta) &= (\mathcal{L}\theta)_\nu = \partial_t\theta_\nu + \partial_x(\varphi_\nu(b(t,x)\partial_x\theta)) \\ &= \partial_t\theta_\nu + \partial_x([\varphi_\nu, b(t,x)]\,\partial_x\theta) + \partial_x(b(t,x)\partial_x\theta_\nu),\end{aligned}$$

We now prove a Carleman estimate for this operator:

Theorem 11 *Let $\mathcal{L}$ defined above, β defined in Theorem 10 and ω used above. Then there exist K and $\gamma_0 > 0$ such that for every $\gamma > \gamma_0$ one has:*

$$\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\|\mathcal{L}\theta\|^2_{H^{-1-\omega-\omega-\beta^* t}}dt \geq$$

$$\geq K\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\left(\gamma\|\theta\|^2_{H^{-1-\omega-\beta^* t}} + \sqrt{\gamma}\|\theta\|^2_{H^{-\omega-\beta^* t}} + \frac{1}{(\Phi'(\gamma T))}\|\theta\|^2_{H^{1-\omega-\beta^* t}}\right)dt, \tag{5.6}$$

with $\beta^*\log 2 = \beta$.

Proof. We define:

$$\theta(t,x) = e^{-\frac{1}{\gamma}\Phi(\gamma(T-t))}w(t,x) \tag{5.7}$$

we obtain:

$$\partial_t\theta = e^{-\frac{1}{\gamma}\Phi(\gamma(T-t))}(\Phi'(\gamma(T-t))w + \partial_t w),$$

and so

$$e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\|\mathcal{L}\theta\|^2_{H^{-1-\omega-\beta^* t}} = \|\partial_t w + \partial_x(b(t,x)\partial_x w) + \Phi'(\gamma(T-t))w\|^2_{H^{-1-\omega-\beta^* t}},$$

so we have to estimate from below

$$\int_0^{\frac{T}{2}} \|\partial_t w + \partial_x(b(t,x)\partial_x w) + \Phi'(\gamma(T-t))w\|^2_{H^{-1-\omega-\beta^* t}}dt = (A).$$

We start by introducing the paraproduct T_b,

$$\begin{aligned}(A) \quad &\geq \frac{1}{2}\int_0^{\frac{T}{2}} \|\partial_t w + \partial_x(T_b\partial_x w) + \Phi'(\gamma(T-t))w\|^2_{H^{-1-\omega-\beta^* t}}dt \\ &- \int_0^{\frac{T}{2}} \|\partial_x((b-T_b)\partial_x w)\|^2_{H^{-1-\omega-\beta^* t}}dt,\end{aligned}$$

if we now define $-s = -1-\omega-\beta^* t$, we have $s \in]0, 1+\epsilon[$, so we can apply proposition 4 to obtain

$$\int_0^{\frac{T}{2}} \|\partial_x((b-T_b)\partial_x w)\|^2_{H^{-1-\omega-\beta^* t}} dt \le C_\epsilon^2 \|b\|^2_{\mathcal{C}^{1,\epsilon}} \int_0^{\frac{T}{2}} \|\partial_x w\|^2_{H^{-1-\omega-\beta^* t}} dt$$

so

$$\begin{aligned}(A) \quad &\ge \frac{1}{2}\int_0^{\frac{T}{2}} \|\partial_t w + \partial_x(T_b \partial_x w) + \Phi'(\gamma(T-t))w\|^2_{H^{-1-\omega-\beta^* t}} dt \\ &\quad - C_\epsilon^2 \|b\|^2_{\mathcal{C}^{1,\epsilon}} \int_0^{\frac{T}{2}} \|w\|^2_{H^{-\omega-\beta^* t}} dt,\end{aligned}$$

let now deal with the first term

$$\begin{aligned}&\int_0^{\frac{T}{2}} \|\partial_t w + \partial_x(T_b \partial_x w) + \Phi'(\gamma(T-t))w\|^2_{H^{-1-\omega-\beta^* t}} dt = \\ &= \int_0^{\frac{T}{2}} \sum_0^{+\infty} 2^{-2(1+\omega+\beta^* t)\nu} \|\Delta_\nu(\partial_t w + \partial_x(T_b \partial_x w) + \Phi'(\gamma(T-t))w)\|^2_{L^2} \\ &\ge \frac{1}{2}\int_0^{\frac{T}{2}} \sum_0^{+\infty} 2^{-2(1+\omega+\beta^* t)\nu} \|\partial_t w_\nu + \partial_x(T_b \partial_x w_\nu) + \Phi'(\gamma(T-t))w_\nu\|^2_{L^2} dt \\ &\quad - \int_0^{\frac{T}{2}} \sum_0^{+\infty} 2^{-2(1+\omega+\beta^* t)\nu} \|\partial_x\left([\Delta_\nu, T_b]\partial_x w\right)\|^2_{L^2} dt\end{aligned}$$

If we define $s = 1+\omega+\beta^* t$ we can apply proposition 5. We obtain

$$\int_0^{\frac{T}{2}} \sum_0^{+\infty} 2^{-2(1+\omega+\beta^* t)\nu} \|\partial_x\left([\Delta_\nu, T_b]\partial_x w\right)\|^2_{L^2} dt \le C_\epsilon^2 \|b\|^2_{\mathcal{C}^{1,\epsilon}} \int_0^{\frac{T}{2}} \|w\|^2_{H^{-\omega-\beta^* t}} dt$$

so we have

$$\begin{aligned}&\int_0^{\frac{T}{2}} \|\partial_t w + \partial_x(T_b \partial_x w) + \Phi'(\gamma(T-t))w\|^2_{H^{-1-\omega-\beta^* t}} dt \\ &\quad \ge \frac{1}{2}\int_0^{\frac{T}{2}} \sum_0^{+\infty} 2^{-2(1+\omega+\beta^* t)\nu} \|\partial_t w_\nu + \partial_x(T_b \partial_x w_\nu) + \Phi'(\gamma(T-t))w_\nu\|^2_{L^2} dt \\ &\qquad - C_\epsilon^2 \|b\|^2_{\mathcal{C}^{1,\epsilon}} \int_0^{\frac{T}{2}} \|w\|^2_{H^{-\omega-\beta^* t}} dt,\end{aligned}$$

finally we consider

$$
\begin{aligned}
\int_0^{\frac{T}{2}} \sum_0^{+\infty} 2^{-2(1+\omega+\beta^* t)\nu} \|\partial_t w_\nu + \partial_x(T_b \partial_x w_\nu) + \Phi'(\gamma(T-t)) w_\nu)\|_{L^2}^2 dt \\
\geq \frac{1}{2} \int_0^{\frac{T}{2}} \sum_0^{+\infty} 2^{-2(1+\omega+\beta^* t)\nu} \|\partial_t w_\nu + \partial_x(b(t,x)\partial_x w_\nu) + \Phi'(\gamma(T-t)) w_\nu\|_{L^2}^2 dt \\
- \int_0^{\frac{T}{2}} \sum_0^{+\infty} 2^{-2(1+\omega+\beta^* t)\nu} \|\partial_x((b(t,x) - T_b)\partial_x w_\nu)\|_{L^2}^2 dt,
\end{aligned}
$$

we can use proposition 4 and we have

$$
\|(b - T_b) f\|_{H^1} \leq \|f\|_{L^2}
$$

so

$$
\|\partial_x((b-T_b)\partial_x w_\nu)\|_{L^2}^2 \leq \widetilde{C} \|\partial_x w_\nu\|_{L^2}^2 \leq \widetilde{C} 2^\nu \|w_\nu\|_{L^2}^2,
$$

we obtain

$$
\int_0^{\frac{T}{2}} \sum_0^{+\infty} 2^{-2(1+\omega+\beta^* t)\nu} \|\partial_x((b-T_b)\partial_x w_\nu)\|_{L^2}^2 dt \leq \widetilde{C} \int_0^{\frac{T}{2}} \|w\|_{H^{-\omega-\beta^* t}}^2 dt
$$

so we can conclude this first part of the proof stating that

$$
\begin{aligned}
(A) \quad \geq \int_0^{\frac{T}{2}} \sum_0^{+\infty} 2^{-2(1+\omega+\beta^* t)\nu} \|\partial_t w_\nu + \partial_x(b(t,x)\partial_x w_\nu) + \Phi'(\gamma(T-t)) w_\nu\|_{L^2}^2 dt \\
-C \int_0^{\frac{T}{2}} \|w\|_{H^{-\omega-\beta^* t}} dt,
\end{aligned}
\tag{5.8}
$$

where C depends only by ϵ and by $\|b\|_{\mathcal{C}^{1,\epsilon}}$.

We now deal with the first term in (5.8): to do this we define

$$w = e^{\beta(\nu+1)t} v \tag{5.9}$$

so we can write

$$2^{-2(1+\omega+\beta^* t)\nu} \|\partial_t w_\nu + \partial_x(b(t,x)\partial_x w_\nu) + \Phi'(\gamma(T-t))w_\nu\|_{L^2}^2 =$$

$$= 2^{-2(1+\omega)\nu} \|\partial_t v_\nu + \partial_x(b(t,x)\partial_x v_\nu) + \Phi'(\gamma(T-t))v_\nu + \beta(\nu+1)v_\nu\|_{L^2}^2$$

$$\geq 2^{-2(1+\omega)\nu} \left(\frac{1}{2} \|\partial_t v_\nu + \partial_x(b(t,x)\partial_x v_\nu)\Phi'(\gamma(T-t))v_\nu\|_{L^2}^2 - \beta^2(\nu+1)^2\|v_\nu\|_{L^2}^2 \right).$$

We first consider the term

$$\|\partial_t v_\nu + \partial_x(b(t,x)\partial_x v_\nu) + \Phi'(\gamma(T-t))v_\nu\|_{L^2}^2$$

$$= \|\partial_t v_\nu\|_{L^2}^2 + \|\partial_x(b(t,x)\partial_x v_\nu) + \Phi'\left(\gamma(T-t)\right)v_\nu\|_{L^2}^2 +$$

$$+2Re(\partial_t v_\nu, \overline{\partial_x(b(t,x)\partial_x v_\nu) + \Phi'\left(\gamma(T-t)\right)v_\nu})_{L^2},$$

dealing with the last line we have

$$\int_0^{\frac{T}{2}} \Phi'\left(\gamma(T-t)\right) \underbrace{2Re(\partial_t v_\nu, \overline{v_\nu})}_{\partial_t(\|v_\nu\|^2)} = \int_0^{\frac{T}{2}} \gamma\Phi''\left(\gamma(T-t)\right)\|v_\nu\|_{L^2}^2 dt, \tag{5.10}$$

to deal with the other term, due to low regulaity in time for $b(t,x)$, we introduce its mollified version, namely

$$b_\epsilon(t,x) = (\phi_\epsilon * b)(t,x),$$

using this definition we can state that there exists $C, C' > 0$ such that (see [DSP]):

$$|b_\epsilon(t,x) - b(t,x)| \leq C\mu(\epsilon), \tag{5.11}$$

$$\left|\frac{d}{dt} b_\epsilon(t,x)\right| \leq C' \frac{\mu(\epsilon)}{\epsilon}, \tag{5.12}$$

for $\epsilon \in]0,1]$.

We can write

$$\int_0^{\frac{T}{2}} 2Re(\partial_t v_\nu, \overline{\partial_x(b(t,x)\partial_x v_\nu)})dt = -2Re\int_0^{\frac{T}{2}}(\partial_x\partial_t v_\nu, \overline{b(t,x)\partial_x v_\nu})dt$$
$$= -2Re\int_0^{\frac{T}{2}}(\partial_x\partial_t v_\nu, \overline{(b(t,x)-b_\epsilon(t,x))\partial_x v_\nu})dt - 2Re\int_0^{\frac{T}{2}}(\partial_x\partial_t v_\nu, \overline{b_\epsilon(t,x)\partial_x v_\nu})dt. \tag{5.13}$$

thanks to (5.11) we have:

$$\begin{aligned}
&\left|2Re\int_0^{\frac{T}{2}}(\partial_x\partial_t v_\nu, \overline{(b(t,x)-b_\epsilon(t,x))\partial_x v_\nu})dt\right| \\
&\le 2C\mu(\epsilon)\int_0^{\frac{T}{2}} \|\partial_x\partial_t v_\nu\|_{L^2}\|\partial_x v_\nu\|_{L^2}dt \\
&\le 2C\mu(\epsilon)\int_0^{\frac{T}{2}} \|\partial_t v_\nu\|_{L^2}2^{2(\nu+1)}\|v_\nu\|_{L^2}dt \\
&\le \frac{C}{N}\int_0^{\frac{T}{2}} \|\partial_t v_\nu\|^2_{L^2}dt + CN2^{4(\nu+1)}\mu^2(\epsilon)\int_0^{\frac{T}{2}} \|v_\nu\|^2_{L^2}dt,
\end{aligned} \tag{5.14}$$

for any $N > 0$. In the second line we used the fact that $\|\partial_x v_\nu\|_{L^2} \le 2^{\nu+1}\|v_\nu\|_{L^2}$ e $\|\partial_t\partial_x v_\nu\|_{L^2} \le 2^{\nu+1}\|\partial_t v_\nu\|_{L^2}$, and in the third Young's inequality with $\frac{1}{N}$.

Dealing with the last term in (5.13) using (5.12) we obtain:

$$\begin{aligned}
\left|2Re\int_0^{\frac{T}{2}}(\partial_x\partial_t v_\nu, \overline{b_\epsilon(t,x)\partial_x v_\nu})dt\right| &= \left|\int_0^{\frac{T}{2}}(\partial_x v_\nu, \overline{\partial_t b_\epsilon(t,x)\partial_x v_\nu})dt\right| \\
&\le C'\frac{\mu(\epsilon)}{\epsilon}\int_0^{\frac{T}{2}} \|\partial_x v_\nu\|^2_{L^2}dt \\
&\le C'2^{2(\nu+1)}\frac{\mu(\epsilon)}{\epsilon}\int_0^{\frac{T}{2}} \|v_\nu\|^2_{L^2}dt.
\end{aligned} \tag{5.15}$$

Combining (5.10),(5.14) and (5.15), choosing $N = C$ and using the fact that $\mu^2(\epsilon) \leq \mu(\epsilon)$ we obtain the for any $\nu \in \mathbb{N}$

$$\int_0^{\frac{T}{2}} \|\partial_t v_\nu + \partial_x(b(t,x)\partial_x v_\nu) + \Phi'(\gamma(T-t))v_\nu\|_{L^2}^2 dt \geq$$

$$\geq \int_0^{\frac{T}{2}} \|b(t,x)\partial_x^2 v_\nu + \Phi'(\gamma(T-t))v_\nu\|_{L^2}^2 + \gamma\Phi''(\gamma(T-t))\|v_\nu\|_{L^2}^2 \qquad (5.16)$$

$$-\left(C^2 2^{4(\nu+1)}\mu(\epsilon) + C' \cdot 2^{2(\nu+1)}\frac{\mu(\epsilon)}{\epsilon}\right)\|v_\nu\|_{L^2}^2 dt,$$

so, returning to our case

$$\int_0^{\frac{T}{2}} \|\partial_t v_\nu + \partial_x(b(t,x)\partial_x v_\nu) + \Phi'(\gamma(T-t))v_\nu + \beta(\nu+1)v_\nu\|_{L^2}^2 dt \geq$$

$$\geq \int_0^{\frac{T}{2}} \|b(t,x)\partial_x^2 v_\nu + \Phi'(\gamma(T-t))v_\nu\|_{L^2}^2 + \gamma\Phi''(\gamma(T-t))\|v_\nu\|_{L^2}^2$$

$$-\left(C^2 2^{4(\nu+1)}\mu(\epsilon) + C' \cdot 2^{2(\nu+1)}\frac{\mu(\epsilon)}{\epsilon} + 2\beta^2(\nu+1)^2\right)\|v_\nu\|_{L^2}^2 dt.$$

We now define $\epsilon = 2^{-2\nu}$. Since $\lim_{\nu\to\infty} \frac{\mu(2^{-2\nu})}{2^{-2\nu}} = +\infty$, we can write

$$2\beta^2(\nu+1)^2 \leq C_0 \frac{\mu(2^{-2\nu})}{2^{-2\nu}} 2^{2(\nu+1)},$$

so, using a convenient constant $\overline{C}$

$$\int_0^{\frac{T}{2}} \|\partial_t v_\nu + b(t,x)\partial_x^2 v_\nu + \Phi'(\gamma(T-t))v_\nu + \beta(\nu+1)v_\nu\|_{L^2}^2 dt \geq$$

$$\geq \int_0^{\frac{T}{2}} \|b(t,x)\partial_x^2 v_\nu + \Phi'(\gamma(T-t))v_\nu\|_{L^2}^2 + \gamma\Phi''(\gamma(T-t))\|v_\nu\|_{L^2}^2$$

$$-\overline{C} 2^{4(\nu+1)}\mu(2^{-2\nu})\|v_\nu\|_{L^2}^2 dt.$$

Let's first deal with $\nu = 0$. Since $t \in [0, \frac{T}{2}]$, knowing that $\lim_{\tau\to+\infty} \Phi''(\tau) = +\infty$, we can state that there exists $\gamma_0 > 0$ such that for any $\gamma > \gamma_0$ it holds that

$\Phi''(\gamma(T-t)) \geq 1$, then choosing $\epsilon = 1$ we can write

$$\begin{aligned}
&\int_0^{\frac{T}{2}} \|\partial_t v_0 + b(t,x)\partial_x^2 v_0 + \Phi'(\gamma(T-t))v_0 + \beta v_0\|_{L^2}^2 dt \\
&\geq \int_0^{\frac{T}{2}} \|b(t)\partial_x^2 v_0 + \Phi'(\gamma(T-t))v_0\|_{L^2}^2 + \gamma\Phi''(\gamma(T-t))\|v_0\|_{L^2}^2 \\
&\qquad - \underbrace{\left(16\overline{C}\mu(1)\right)}_{K} \|v_0\|_{L^2}^2 dt \qquad (5.17)\\
&\geq \int_0^{\frac{T}{2}} (\gamma - K)\|v_0\|_{L^2}^2 dt \\
&\geq \frac{\gamma}{2}\int_0^{\frac{T}{2}} \|v_0\|_{L^2}^2 dt,
\end{aligned}$$

where the last inequality holds for γ big enough.

Let now $\nu \geq 1$, we have

$$\begin{aligned}
&\int_0^{\frac{T}{2}} \|\partial_t v_\nu + \partial_x(b(t,x)\partial_x v_\nu) + \Phi'(\gamma(T-t))v_\nu + \beta(\nu+1)v_\nu\|_{L^2}^2 dt \\
&\geq \int_0^{\frac{T}{2}} \|\partial_x(b(t,x)\partial_x v_\nu) + \Phi'(\gamma(T-t))v_\nu\|_{L^2}^2 \\
&\qquad + \gamma\Phi''(\gamma(T-t))\|v_\nu\|_{L^2}^2 - \overline{C}2^{4\nu}\mu(2^{-2\nu})\|v_\nu\|_{L^2}^2 dt \\
&\geq \int_0^{\frac{T}{2}} \left(\|\partial_x(b(t,x)\partial_x v_\nu)\|_{L^2} - \Phi'(\gamma(T-t))\|v_\nu\|_{L^2}\right)^2 \\
&\qquad + \gamma\Phi''(\gamma(T-t))\|v_\nu\|_{L^2}^2 - \overline{C}2^{4\nu}\mu(2^{-2\nu})\|v_\nu\|_{L^2}^2 dt.
\end{aligned}$$

We can obtain the following estimate:

$$\begin{aligned}
\|\partial_x(b(t,x)\partial_x v_\nu)\|_{L^2}\,\|v_\nu\|_{L^2} &\geq |(\partial_x(b(t,x)\partial_x v_\nu), \overline{v_\nu})_{L^2}| \\
&\geq \left|\left(b(t)\partial_x v_\nu, \overline{\partial_x v_\nu}\right)_{L^2}\right| \\
&\geq \mu_0\|\partial_x v_\nu\|_{L^2}^2 \geq \frac{\mu_0}{4}2^{2\nu}\|v_\nu\|_{L^2}^2,
\end{aligned}$$

and so

$$\|\partial_x(b(t,x)\partial_x v_\nu)\|_{L^2} \geq \frac{\mu_0}{4}2^{2\nu}\|v_\nu\|_{L^2}. \tag{5.18}$$

We first suppose $\Phi'(\gamma(T-t)) \leq \frac{\mu_0}{8}2^{2\nu}$. From (5.18) we obtain that

$$\|\partial_x(b(t,x)\partial_x v_\nu)\|_{L^2} - \Phi'(\gamma(T-t))\|v_\nu\|_{L^2} \geq \frac{\mu_0}{8}2^{2\nu}\|v_\nu\|_{L^2},$$

and so, using another time the fact that $\Phi''(\gamma(T-t)) \geq 1$ we obtain that

$$\begin{aligned}
&\int_0^{\frac{T}{2}} \left(\|\partial_x(b(t,x)\partial_x v_\nu)\|_{L^2} - \Phi'(\gamma(T-t))\|v_\nu\|_{L^2}\right)^2 \\
&\qquad + \gamma\Phi''(\gamma(T-t))\|v_\nu\|^2_{L^2} - \overline{C}2^{4\nu}\mu(2^{-2\nu})\|v_\nu\|^2_{L^2}dt. \\
&\geq \int_0^{\frac{T}{2}} \left(\left(\frac{\mu_0}{8}2^{2\nu}\right)^2 + \gamma - \overline{C}2^{4\nu}\mu(2^{-2\nu})\right)\|v_\nu\|^2_{L^2}dt \\
&\geq \int_0^{\frac{T}{2}} \left(\left(\frac{1}{2}\left(\frac{\mu_0}{8}\right)^2 - \overline{C}\mu(2^{-2\nu})\right)2^{4\nu} + \frac{\gamma}{3}\right)\|v_\nu\|^2_{L^2}dt \\
&\qquad + \int_0^{\frac{T}{2}} \left(\frac{1}{2}\left(\frac{\mu}{8}\right)^2 2^{4\nu} + \frac{2}{3}\gamma\right)\|v_\nu\|^2_{L^2}dt.
\end{aligned}$$

Since $\lim_{\nu\to+\infty}\mu(2^{-2\nu}) = 0$, then there exists $\gamma_0 > 0$ such that

$$\left(\frac{1}{2}\left(\frac{\mu_0}{8}\right)^2 - \overline{C}\mu(2^{-2\nu})\right)2^{4\nu} + \frac{\gamma}{3} \geq 0,$$

for any $\gamma > \gamma_0$ and any $\nu \geq 1$. Thus there exists a constant $c > 0$ that does not depend on ν such that

$$\begin{aligned} &\int_0^{\frac{T}{2}} (\|\partial_x(b(t,x)\partial_x v_\nu)\|_{L^2} - \Phi'(\gamma(T-t))\|v_\nu\|_{L^2})^2 \\ &\qquad +\gamma\Phi''(\gamma(T-t))\|v_\nu\|^2_{L^2} - \overline{C}2^{4\nu}\mu(2^{-2\nu})\|v_\nu\|^2_{L^2}dt. \\ &\geq \int_0^{\frac{T}{2}} \left(\frac{1}{2}\left(\frac{\mu_0}{8}\right)^2 2^{4\nu} + \frac{2}{3}\gamma\right)\|v_\nu\|^2_{L^2}dt \\ &\geq \int_0^{\frac{T}{2}} \left(\frac{1}{4}\left(\frac{\mu_0}{8}\right)^2 2^{4\nu} + c\sqrt{\gamma}2^{2\nu} + \frac{\gamma}{2}\right)\|v_\nu\|^2_{L^2}dt, \end{aligned} \tag{5.19}$$

for any $\gamma > \gamma_0$, where in the last line we used the fact that

$$\frac{1}{4}\left(\frac{\mu_0}{8}\right)^2 2^{4\nu} + \frac{1}{6}\gamma \geq c\sqrt{\gamma}2^{2\nu}.$$

Moreover since $\Phi'(\tau)$ is an increasing function we can choose γ_0 in such a way that

$$\Phi'(\gamma T) \geq 1,$$

for any $\gamma > \gamma_0$. So we can write

$$\begin{aligned} &\int_0^{\frac{T}{2}} (\|\partial_x(b(t,x)\partial_x v_\nu)\|_{L^2} - \Phi'(\gamma(T-t))\|v_\nu\|_{L^2})^2 \\ &\qquad +\gamma\Phi''(\gamma(T-t))\|v_\nu\|^2_{L^2} - \overline{C}2^{4\nu}\mu(2^{-2\nu})\|v_\nu\|^2_{L^2}dt. \\ &\geq \int_0^{\frac{T}{2}} \left(\frac{\widetilde{c}}{\Phi'(\gamma T)}2^{4\nu} + c\sqrt{\gamma}2^{2\nu} + \frac{\gamma}{2}\right)\|v_\nu\|^2_{L^2}dt, \end{aligned}$$

with $\widetilde{c} = \frac{1}{4}\left(\frac{\mu_0}{8}\right)^2$.

Passing now to $\Phi'(\gamma(T-t)) \geq \frac{\mu_0}{8}2^{2\nu}$. First of all we can write

$$\Phi'(\gamma(T-t)) \geq \left(\frac{\mu_0}{8}\right)^2 \frac{2^{4\nu}}{\Phi'(\gamma T)}.$$

Using (1.16), the fact that we can choose $\mu_0 \leq 1$ and the properties of the modulus of continuity $\mu(s)$ one has:

$$\begin{aligned}\frac{1}{4}\Phi''(\gamma(T-t)) &= \frac{1}{4}\left(\Phi'(\gamma(T-t))\right)^2 \mu\left(\frac{1}{\Phi'(\gamma(T-t))}\right)\\ &\geq \frac{1}{4}\left(\frac{\mu_0}{8}\right)^2 2^{4\nu}\mu\left(\frac{8}{\mu_0}2^{-2\nu}\right) \geq \frac{1}{4}\left(\frac{\mu_0}{8}\right)^2 2^{4\nu}\mu(2^{-2\nu}).\end{aligned}$$

Moreover

$$\begin{aligned}\frac{1}{4}\Phi''(\gamma(T-t)) &= \frac{1}{4}\left(\Phi'(\gamma(T-t))\right)^2 \mu\left(\frac{1}{\Phi'(\gamma(T-t))}\right)\\ &\geq \frac{1}{4}\left(\frac{\mu_0}{8}\right)^2 \frac{2^{4\nu}}{\Phi'(\gamma T)}\left(\Phi'(\gamma(T-t))\right)\mu\left(\frac{1}{\Phi'(\gamma(T-t))}\right)\\ &\geq \widetilde{c}\frac{2^{4\nu}}{\Phi'(\gamma T)},\end{aligned}$$

where in the last line we used that $\widetilde{c}$ and the fact that

$$\left(\Phi'(\gamma(T-t))\right)\mu\left(\frac{1}{\Phi'(\gamma(T-t))}\right),$$

is an increasing function and so for any γ greater than a fixed γ_0 it is greater than 1.

In this way we obtain:

$$\begin{aligned}&\int_0^{\frac{T}{2}} \left(\|\partial_x(b(t,x)\partial_x v_\nu)\|_{L^2} - \Phi'(\gamma(T-t))\|v_\nu\|_{L^2}\right)^2\\ &\quad +\gamma\Phi''(\gamma(T-t))\|v_\nu\|^2_{L^2} - \overline{C}2^{4\nu}\mu(2^{-2\nu})\|v_\nu\|^2_{L^2}dt.\\ &\geq \int_0^{\frac{T}{2}}\left(\frac{\gamma}{2} + \left(\frac{\gamma}{4}\left(\frac{\mu_0}{8}\right)^2 - \overline{C}\right)\underbrace{2^{4\nu}\mu(2^{-2\nu})}_{\geq 2^{2\nu}} + \widetilde{c}\frac{2^{4\nu}}{\Phi'(\gamma T)}\right)\|v_\nu\|^2_{L^2}dt\\ &\geq \int_0^{\frac{T}{2}}\left(\frac{\gamma}{2} + c\gamma 2^{2\nu} + \widetilde{c}\frac{2^{4\nu}}{\Phi'(\gamma T)}\right)\|v_\nu\|^2_{L^2}dt.\end{aligned} \tag{5.20}$$

putting together (5.19) and (5.20) we obtain that there exist γ_0 and $K > 0$ such that

$$\begin{aligned}\int_0^{\frac{T}{2}} &\|\partial_t v_\nu + \partial_x(b(t,x)\partial_x v_\nu) + \Phi'(\gamma(T-t))v_\nu + \beta(\nu+1)v_\nu\|^2_{L^2}dt \\ &\geq K\int_0^{\frac{T}{2}}\left(\frac{\gamma}{2} + \sqrt{\gamma}2^{2\nu} + \frac{2^{4\nu}}{\Phi'(\gamma T)}\right)\|v_\nu\|^2_{L^2}dt,\end{aligned} \tag{5.21}$$

for any $\nu \geq 1$ and for any $\gamma > \gamma_0$. From (5.17) and (5.21) we finally obtain that there exist γ_0 and $K > 0$ such that

$$\begin{aligned}&\int_0^{\frac{T}{2}}\sum_\nu 2^{-2(1+\omega)\nu}\|\partial_t v_\nu + \partial_x(b(t,x)\partial_x v_\nu) + \Phi'(\gamma(T-t))v_\nu + \beta(\nu+1)v_\nu\|^2_{L^2}dt \\ &\geq K\int_0^{\frac{T}{2}}\sum_\nu 2^{-2(1+\omega)\nu}\left(\frac{\gamma}{2} + \sqrt{\gamma}2^{2\nu} + \frac{2^{4\nu}}{\Phi'(\gamma T)}\right)\|v_\nu\|^2_{L^2}dt \\ &= K\int_0^{\frac{T}{2}}\sum_\nu\left(\frac{\gamma}{2}2^{-2(1+\omega)\nu} + \sqrt{\gamma}2^{-2\omega\nu} + \frac{2^{2(1-\omega)\nu}}{\Phi'(\gamma T)}\right)\|v_\nu\|^2_{L^2}dt\end{aligned}$$

then using (5.7), (5.9) and the characterization of the norm $H^{-1-\omega-\beta^* t}$ one has that for any $\gamma > \gamma_0$:

$$\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\|\mathcal{L}\theta\|^2_{H^{-1-\omega-\beta^* t}}dt \geq$$

$$\geq K\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\left(\frac{\gamma}{2}\|\theta\|^2_{H^{-1-\omega-\beta^* t}} + \sqrt{\gamma}\|\theta\|^2_{H^{-\omega-\beta^* t}} + \frac{1}{\Phi'(\gamma T)}\|\theta\|^2_{H^{1-\omega-\beta^* t}}\right)dt,$$

and the proof is concluded. □

5.3 Statement and Proof of the main theorem

In this section we combine the two inequalities proved in the last two sections and proving the following theorem

Theorem 12 *Let* $v \in H^2([0,T], L^2(\mathbb{R}_x)) \cap H^1([0,T], H^1(\mathbb{R}_x)) \cap L^2([0,T], H^2(\mathbb{R}_x))$ *and* $\zeta \in H^1([0,T], L^2(\mathbb{R})) \cap L^2([0,T], H^2(\mathbb{R}_x))$ *solutions of* (5.3).

Under the hypotesis (5.2) *we have that*

$$v \equiv \zeta \equiv 0.$$

Proof. The two estimates obtained in the previous sections hold true for

$$u \in \mathcal{C}_0^\infty(\mathbb{R}^2) \text{ with } supp\ u \subseteq [0, \frac{T}{2}] \times \mathbb{R}$$

$$\theta \in \mathcal{C}_0^\infty(\mathbb{R}^2) \text{ with } supp\ \theta \subseteq [0, \frac{T}{2}] \times \mathbb{R}$$

.

Using a density argument the two inequalities still hold true for

$$u \in H^2([0,T], L^2(\mathbb{R}_x)) \cap H^1([0,T], H^1(\mathbb{R}_x)) \cap L^2([0,T], H^2(\mathbb{R}_x))$$

$$\theta \in H^1([0,T], L^2(\mathbb{R})) \cap L^2([0,T], H^2(\mathbb{R}_x))$$

if

$$u \equiv \theta \equiv 0 \text{ in } \left[\frac{T}{2}, T\right] \times \mathbb{R}$$

.

□

It is useful to define the following cut-off function

$$\chi(t) \in \mathcal{C}^\infty(\mathbb{R}) \qquad \text{with} \qquad \chi(t) = \begin{cases} 1 & \text{for} \quad t \leq \dfrac{T}{4} \\ 0 & \text{for} \quad t \geq \dfrac{T}{3}, \end{cases}$$

and consider two solution v e ζ of (5.3), then define

- $u(t,x) = \chi(t)v(t,x)$;
- $\theta(t,x) = \chi(t)\zeta(t,x)$.

multiplying both equations in (5.3) by $\chi(t)$, we obtain the following equations:

$$\mathcal{P}u = -\alpha(t,x)\partial_x\theta - \beta(t,x)\theta + \partial_t^2\chi(t)\cdot v + 2\partial_t\chi(t)\cdot\partial_t v \tag{5.22}$$

$$\begin{aligned}\mathcal{L}\theta = &-\delta(t,x)\partial_x\partial_t u + \delta(t,x)\partial_t\chi(t)\cdot\partial_x v(t,x) - \rho(t,x)\partial_t u \\ &-\phi(t,x)u + \partial_t\chi(t)\cdot\zeta + \rho(t,x)\partial_t\chi(t)\cdot v\end{aligned} \tag{5.23}$$

Let now start the series of inequalities that will lead us to the conclusion of the proof:

$$\begin{aligned}\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}E(t)dt &\leq \frac{1}{2(\Phi'(\frac{\gamma T}{2}))^2}\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\|\mathcal{P}u\|^2_{H^{-\omega-\beta^*t}}dt \\ &\leq \frac{1}{2\left(\Phi'(\frac{\gamma T}{2})\right)^2}\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\sum_{\nu=0}^{+\infty}e^{-2\beta(\nu+1)t}2^{-\nu\omega}\|(\mathcal{P}u)_\nu\|^2_{L^2}dt \\ &\leq \frac{1}{2\left(\Phi'(\frac{\gamma T}{2})\right)^2}\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\sum_{\nu=0}^{+\infty}e^{-2\beta(\nu+1)t}2^{-\nu\omega}\|\left(\alpha(t,x)\partial_x\theta\right)_\nu + \left(\beta(t,x)\theta\right)_\nu \\ &\qquad -\partial_t\chi(t)\cdot\partial_t v_\nu - \partial_t^2\chi(t)\cdot v_\nu\|^2_{L^2}dt,\end{aligned}$$

$$\begin{aligned}\leq \frac{3}{2\left(\Phi'(\frac{\gamma T}{2})\right)^2}\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\sum_{\nu=0}^{+\infty}e^{-2\beta(\nu+1)t}2^{-\nu\omega}\Big(&\|\left(\alpha(t,x)\partial_x\theta\right)_\nu\|^2_{L^2} + \|\left(\beta(t,x)\theta\right)_\nu\|^2_{L^2} \\ &+\|2\partial_t\chi(t)\cdot\partial_t v_\nu + \partial_t^2\chi(t)\cdot v_\nu\|^2_{L^2}\Big)dt,\end{aligned}$$

Thanks to the characterisation of Sobolev spaces we have

$$\frac{1}{C}\|\alpha(t,x)\partial_x\theta\|_{H^{-\omega-\beta^*t}} \leq \sum_{\nu=0}^{+\infty}e^{-2\beta(\nu+1)t}2^{-\nu\omega}\|\left(\alpha(t,x)\partial_x\theta\right)_\nu\|^2_{L^2} \leq C\|\alpha(t,x)\partial_x\theta\|^2_{H^{-\omega-\beta^*t}}.$$

and thanks to the theorem on multiplication of Hölder and Sobolev spaces we have

$$\|\alpha(t,x)\partial_x\theta\|_{H^{-\omega-\beta^*t}} \leq C\|\alpha\|_{\mathcal{C}^\epsilon}\|\partial_x\theta\|_{H^{-\omega-\beta^*t}}$$

It is not restrictive to suppose that $\beta^*t+\omega\leq\epsilon$ for any $t\in\left[0,\dfrac{T}{2}\right]$: otherwise we can choose ω and $T'<T$ in a proper way, repeating the same estimate above using

T' instead of T. This leads to the following inequality:

$$\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} E(t)dt \leq$$

$$\leq \frac{C_2 \Phi'(\gamma T)}{2\left(\Phi'(\frac{\gamma T}{2})\right)^2} \int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t} 2^{-\nu\omega} \left(\frac{1}{\Phi'(\gamma T)} \|\partial_x \theta_\nu\|_{L^2}^2 + \sqrt{\gamma}\|\theta_\nu\|_{L^2}^2 \right) dt$$

$$+ \frac{C_2}{2\left(\Phi'(\frac{\gamma T}{2})\right)^2} \int_{\frac{T}{4}}^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t} 2^{-\nu\omega} \|\partial_t \chi(t) \cdot \partial_t v_\nu + \partial_t^2 \chi(t) \cdot v_\nu\|_{L^2}^2 dt,$$

where for the last term we used the fact that $\partial_t \chi(t) \equiv \partial_t^2 \chi(t) \equiv 0$ on $\left[0, \frac{T}{4}\right]$.

If we first analyse the first integral on the right hand side using (5.6) and (5.23):

$$\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t} 2^{-\nu\omega} \left(\frac{1}{\Phi'(\gamma T)} \|\partial_x \theta_\nu\|_{L^2}^2 + \sqrt{\gamma}\|\theta_\nu\|_{L^2}^2 \right) dt \leq$$

$$\leq \frac{1}{K} \int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \left(\|(\mathcal{L}\theta)\|_{H^{-1-\omega-\beta^* t}}^2 \right) dt$$

$$\leq \frac{6}{K} \int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \left(\|\delta(t,x)\partial_x \partial_t u\|_{H^{-1-\omega-\beta^* t}}^2 + \|\phi(t,x) u\|_{H^{-\omega-\beta^* t}}^2 + \|\rho(t,x)\partial_t u\|_{H^{-\omega-\beta^* t}}^2 \right.$$

$$\left. + \|\partial_t \chi(t) \cdot \zeta\|_{L^2}^2 + \|\delta(t,x)\partial_t \chi(t) \partial_x v(t,x)\|_{L^2}^2 + \|\rho(t,x)\partial_t \chi(t) \cdot v\|_{L^2}^2 \right) dt$$

For what concerns the second and third term of this last formula the same estimates used before hold, while concerning the first term we can write:

$$\begin{aligned}
\|\delta(t,x)\partial_x \partial_t u\|_{H^{-1-\omega-\beta^* t}} &= \sup_{\|v\|_{H^{1+\omega+\beta^* t}}=1} \delta(t,x)\partial_x \partial_t u(v) \\
&= \left| \int \delta(t,x)\partial_x \partial_t u \ v \ dx \right| \\
&= \left| \int \partial_t u \ \partial_x(\delta(t,x)v) dx \right| \\
&\leq \|\partial_t u\|_{H^{-\omega-\beta^* t}} \|\partial_x \delta(t,x) v + \delta(t,x)\partial_x v\|_{H^{\beta^* t+\omega}} \\
&\leq \|\partial_t u\|_{H^{-\omega-\beta^* t}} \|\delta\|_{\mathcal{C}^{1+\epsilon}} \|v\|_{H^{1+\beta^* t+\omega}} \\
&\leq \|\partial_t u\|_{H^{-\omega-\beta^* t}} \|\delta\|_{\mathcal{C}^{1+\epsilon}},
\end{aligned}$$

also in this case we suppose that $\beta^* t + \omega \le \epsilon$, using the same argument as before otherwise.

We can resume these last passages by saying that there exists C_3 such that

$$\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \left(\|\delta(t,x)\partial_x\partial_t u\|^2_{H^{-1-\omega-\beta^* t}} + \|\phi(t,x)u\|^2_{H^{-\omega-\beta^* t}} + \|\rho(t,x)\partial_t u\|^2_{H^{-\omega-\beta^* t}} \right) dt$$

$$\le C_3 \int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} E(t) dt,$$

where we used the definition of energy. Now there exists C_4 such that

$$\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} E(t) dt \le C_4 \frac{\Phi'(\gamma T)}{\left(\Phi'(\frac{\gamma T}{2})\right)^2} \int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} E(t) dt$$
$$+ \frac{C_2}{2\left(\Phi'(\frac{\gamma T}{2})\right)^2} \int_{\frac{T}{4}}^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t} 2^{-\nu\omega} \|\partial_t\chi(t)\cdot\partial_t v_\nu + \partial_t^2\chi(t)\cdot v_\nu\|^2_{L^2} dt$$
$$+ C_4 \frac{\Phi'(\gamma T)}{\left(\Phi'(\frac{\gamma T}{2})\right)^2} \int_{\frac{T}{4}}^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \left(\|\partial_t\chi(t)\cdot\zeta\|^2_{L^2} + \|\delta(t,x)\partial_t\chi(t)\partial_x v(t,x)\|^2_{L^2} \right.$$
$$\left. + \|\rho(t,x)\partial_t\chi(t)\cdot v\|^2_{L^2} \right) dt$$

where we used another time the fact that $\partial_t\chi(t) \equiv 0$ su $\left[0, \dfrac{T}{4}\right]$.

We now take advantage of the hypotesis that μ satisfies $\star$- condition and we suppose to fix ideas that it holds with $a = \frac{1}{2}$, from this we obtain

$$\lim_{\gamma\to+\infty} \frac{\Phi'(\gamma T)}{\left(\Phi'(\frac{\gamma T}{2})\right)^2} = 0$$

thus we can write that for any γ greater than a fixed γ_0 we have

$$\frac{1}{2}\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}E(t)dt \leq$$
$$\frac{C_2}{2\left(\Phi'(\frac{\gamma T}{2})\right)^2}\int_{\frac{T}{4}}^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\|\partial_t\chi(t)\cdot\partial_t v+\partial_t^2\chi(t)\cdot v\|_{L^2}^2 dt$$
$$+C_4\frac{\Phi'(\gamma T)}{\left(\Phi'(\frac{\gamma T}{2})\right)^2}\int_{\frac{T}{4}}^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\left(\ \|\partial_t\chi(t)\cdot\zeta\|_{L^2}^2+\|\rho(t,x)\partial_t\chi(t)\cdot v\|_{L^2}^2\right.$$
$$\left.+\|\delta(t,x)\partial_t\chi(t)\partial_x v(t,x)\|_{L^2}^2\ \right)dt$$

Consider now the following estimate:

$$e^{\frac{2}{\gamma}(\Phi(\gamma(\frac{7}{8}T)))}\int_0^{\frac{T}{8}} E(t)dt \leq \int_0^{\frac{T}{8}} e^{\frac{2}{\gamma}(\Phi(\gamma(T-t)))}E(t)dt,$$

from which we can conclude that:

$$\frac{1}{2}\int_0^{\frac{T}{8}} E(t)dt \leq \frac{\widetilde{C}}{\left(\Phi'(\frac{\gamma T}{2})\right)^2}e^{\frac{2}{\gamma}(\Phi(\frac{3}{4}\gamma T)-\Phi(\frac{7}{8}\gamma T))}\int_{\frac{T}{4}}^{\frac{T}{2}}\|\partial_t\chi(t)\partial_t v+\partial_t^2\chi(t)v\|_{L^2}^2 dt$$
$$+\frac{\Phi'(\gamma T)}{\left(\Phi'(\frac{\gamma T}{2})\right)^2}e^{\frac{2}{\gamma}(\Phi(\frac{3}{4}\gamma T)-\Phi(\frac{7}{8}\gamma T))}\int_{\frac{T}{4}}^{\frac{T}{2}}\|\partial_t\chi(t)\cdot\zeta\|_{L^2}^2+\|\rho(t,x)\partial_t\chi(t)\cdot v\|_{L^2}^2$$
$$+\|\delta(t,x)\partial_t\chi(t)\partial_x v(t,x)\|_{L^2}^2 dt$$

then since

$$\lim_{\gamma\to+\infty}\frac{2}{\gamma}\left(\Phi\left(\frac{3}{4}\gamma T\right)-\Phi\left(\frac{7}{8}\gamma T\right)\right)=-\infty,$$

we obtain that the term on the right hand side goes to 0 when $\gamma\to+\infty$; this yields $E(t)\equiv 0$ in $[0,\frac{T}{8}]$ and so $u\equiv 0$ in $[0,\frac{T}{8}]$. With this result we can state that $\theta\equiv 0$ in the same interval. Repeating the same argument a certain number of times we can completely cover the interval $[0,T]$, obtaining that $u\equiv\theta\equiv 0$ in $[0,T]$. This concludes the proof of theorem 12 .

5.4 $\mathbf{Log_K}$-Lipschitz functions

In this last paragraph we study a function class whose modulus of continuity satisfies both Osgood and $\star$ conditions.

Let $K > 1$, we define the class $log_K - Lip$ as follows: we say that $f : \mathbb{R} \to \mathbb{R}$ belongs to this class if $f \in L^\infty$ and there exists $C > 0$ such that

$$\frac{|f(x) - f(y)|}{|x-y|\,(|\ln|x-y|\,|)^{1-\frac{1}{K}}} \leq C,$$

for any $x, y \in \mathbb{R}$ such that $0 < |x - y| < \frac{1}{2}$.

This set of continuous function is a sort of medium class between Lip and $log - Lip$. Choosing $K = 1$ we retrieve the class Lip while taking $K = \infty$ we obtain the class $log - Lip$;

moreover the following inclusions hold:

$$Lip(\mathbb{R}) \subseteq log_K - Lip(\mathbb{R}) \subseteq log - Lip(\mathbb{R}).$$

Let us compute $\Phi'(\tau)$ coming from the modulus of continuity of this class, namely $\mu(s) = s|\ln s|^{1-\frac{1}{K}}$. For the sake of simplicity we write $\alpha = 1 - \dfrac{1}{K}$: following the definitions given in the first chapter we compute

$$\begin{aligned}
\phi(t) &= \int_{\frac{1}{t}}^{1} \frac{1}{\mu(s)} ds \\
&= \int_{\frac{1}{t}}^{1} \frac{1}{s(-\ln s)^\alpha} ds \\
&= \frac{1}{1-\alpha}(-\ln s)^{1-\alpha}\Big|_{1}^{\frac{1}{t}} \\
&= \frac{1}{1-\alpha}\,(\ln(t))^{1-\alpha} \\
&= K\,(\ln(t))^{\frac{1}{K}}\,,
\end{aligned}$$

where in the last passage we substitute α with $1 - \dfrac{1}{K}$.

Now we look for $\phi^{-1}(\tau) = \Phi'(\tau)$: the inverse of the function $\phi(t)$ is

$$\phi^{-1}(\tau) = e^{\left(\frac{\tau}{K}\right)^K},$$

and finally we verify the existence of $0 < a < 1$ such that

$$\lim_{\tau\to\infty} \frac{\Phi'(\tau)}{(\Phi'(a\tau))^2} = 0.$$

To show this last point we use the function $\Phi'(\tau)$ just computed, obtaining :

$$\lim_{\tau\to\infty} \frac{e^{\left(\frac{\tau}{K}\right)^K}}{e^{2\left(\frac{a\tau}{K}\right)^K}} = \lim_{\tau\to\infty} e^{\left(\frac{\tau}{K}\right)^K(1-2a^K)} = 0,$$

that is equivalent to the request $1 - 2a^K < 0$ and so $a > \left(\frac{1}{2}\right)^{\frac{1}{K}}$.

So for any $K > 1$, $log_K - Lip$ function satisfy Osgood and $\star$ conditions, with a that gets closer to 1 as K increases.

As a final remarks we stress that in the last part of the proof of theorem 12 we restrict the time interval to $\left[0, \frac{T}{2}\right]$, using the following fact:

$$\frac{1}{\Phi'(\gamma(T-t))} \le \frac{1}{\Phi'(\gamma\frac{T}{2})},$$

where we use that $\Phi'(t)$ is an increasing function. Then we supposed that the modulus of continuity μ satisfied $\star$ condition with $a = \frac{1}{2}$, since we can obtain that

$$\lim_{\gamma\to+\infty} \frac{\Phi'(\gamma T)}{\left(\Phi'(\gamma\frac{T}{2})\right)^2} = 0.$$

In the case of a $log_K - Lip$ function with an arbitrary $K > 1$ we can restrict the interval to $[0, (1-a)T]$ with $(\frac{1}{2})^{\frac{1}{K}} < a < 1$ in such a way to allow the following estimate

$$\frac{1}{\Phi'(\gamma(T-t))} \leq \frac{1}{\Phi'(\gamma(aT))},$$

that lead us to the conclusion of the proof as we did before in the particular case $a = \frac{1}{2}$.

Appendix

Lemma 6 *Let $0 \leq \alpha, \beta < 1$, then there exists $C > 0$ such that*

$$\int_0^t (t-s)^{-\alpha} s^{-\beta} ds \leq C t^{-\alpha-\beta+1}$$

Proof. With the change of variable $\sigma = \frac{s}{t}$ we obtain

$$\begin{aligned} \int_0^t (t-s)^{-\alpha} s^{-\beta} ds &= \int_0^1 t^{-\alpha}(1-\sigma)^{-\alpha} t^{-\beta} \sigma^{-\beta} t d\sigma \\ &= t^{-\alpha-\beta+1} \int_0^1 (1-\sigma)^{-\alpha} \sigma^{-\beta} d\sigma \\ &\leq C t^{-\alpha-\beta+1} \end{aligned}$$

□

Lemma 7 *let $f \in B^{0,\frac{1}{2}}$ one has*

$$S_0^v f \in L_v^\infty(L_h^2).$$

Proof.

$$\begin{aligned} \|S_0^v f\|_{L_v^\infty(L_h^2)} &= \|\sum_{j<0} \Delta_j^v f\|_{L_v^\infty(L_h^2)} \\ &\leq \sum_{j<0} \|\Delta_j^v f\|_{L_h^2(L_v^\infty)} \\ &\leq \sum_{j<0} 2^{\frac{j}{2}} \|\Delta_j^v f\|_{L^2} \\ &\leq \|f\|_{B^{0,\frac{1}{2}}}, \end{aligned}$$

where in the third line we used Berstein inequality in one variable. □

Lemma 8 *Let $f \in B^{0,\frac{1}{2}}$ one has*

$$(I - S_0^v)f \in L^2.$$

Proof.

$$\begin{aligned}\|(I - S_0^v)f\|_{L^2} &= \|\sum_{j>0} \Delta_j^v f\|_{L^2} \leq \sum_{j>0} \|\Delta_j^v f\|_{L^2} \\ &\leq \sum_{j>0} 2^{\frac{j}{2}} \|\Delta_j^v f\|_{L^2} \\ &\leq \|f\|_{B^{0,\frac{1}{2}}},\end{aligned}$$

where we added the term $2^{\frac{j}{2}}$ since we are working in $j > 0$. □

Lemma 9 (proof of 4.4)

$$\|\partial_t v_n\|_{H^{-N}} = \sup_{\|\phi\|_{H_0^N} \leq 1} (\partial_t v_n, \phi)_{L^2} \leq 2C\|v_n\|_{L^2}^2 + (\nu_h + 2\|B\|_{L^\infty})\, \|v_n\|_{L^2},$$

where $N > 0$ is great enough such that

$$H_0^N \hookrightarrow L^\infty.$$

Proof. We have that

$$\partial_t v_n = -\widetilde{J}_n(v_n \cdot \nabla v_n) + \nu_h \Delta_h v_n - \widetilde{J}_n(B \times v_n) + \widetilde{J}_n \nabla \sum_{i,j} \partial_i \partial_j \Delta^{-1}(v_n^i v_n^j) + \widetilde{J}_n(\nabla \Delta div(B \times v_n)),$$

Starting from the first term:

$$\begin{aligned}|(\widetilde{J}_n \nabla(v_n \otimes v_n), \phi)_{L^2}| &\leq C\|\widetilde{J}_n(n_n \otimes v_n)\|_{L^1} \|\nabla \phi\|_{L^\infty} \\ &\leq \|v_n\|_{L^2}^2 \|\phi\|_{H_0^N}.\end{aligned}$$

For the second term we have:

$$\nu_h |(\Delta_h v_n, \phi)_{L^2}| \leq \nu_h \|v_n\|_{L^2} \|\Delta_h \phi\|_{L^2} \leq \nu_h \|v_n\|_{L^2} \|\phi\|_{H_0^N}.$$

For the third term one has:

$$|(\widetilde{J}_n(B \times v_n), \phi)_{L^2}| \leq \|B\|_{L^\infty} \|v_n\|_{L^2} \|\phi\|_{L^2} \leq \|B\|_{L^\infty} \|v_n\|_{L^2} \|\phi\|_{H_0^N}.$$

The fourth term:

$$\begin{aligned}|(\widetilde{J}_n \nabla \sum_{i,j} \partial_i \partial_j \Delta^{-1}(v_i v_j), \phi)_{L^2}| &\leq C\|v^2\|_{L^1} \|P(D)\phi\|_{L^\infty} \\ &C\|v\|_{L^2}^2 \|\phi\|_{H_0^N},\end{aligned}$$

where $P(D) = \sum_{i,j} \nabla \partial_i \partial_j \Delta^{-1}$.

Finally:

$$\begin{aligned} |(\widetilde{J}_n \nabla \Delta^{-1} div(B \times v_n), \phi)_{L^2}| &\leq \|\widetilde{J}_n (B \times v_n)\|_{L^2} \|\mathcal{Z}(D)\phi\|_{L^2} \\ &\leq \|B\|_{L^\infty} \|v_n\|_{L^2} \|\phi\|_{H_0^N}. \end{aligned}$$

With this last inequality the lemma is proved. □

We state this lemma for which we give no proof:

Lemma 10 *There exist two continuous decreasing functions* $\vartheta_1,\ \vartheta_2\ :\ ]0,1] \mapsto\ ]0,+\infty[$, *with* $\lim_{c\to 0^+} \vartheta_j = +\infty$ *for* $j = 1,\ 2$, *such that, for all* $c \in]0,1]$ *and for all* $m \geq 1$,

$$\sum_{j=1}^{m} e^{cj} j^{-\frac{1}{2}} \leq \vartheta_1(c) e^{cm} m^{-\frac{1}{2}}, \qquad \sum_{j=m}^{+\infty} e^{-cj} j^{\frac{1}{2}} \leq \vartheta_2(c) e^{-cm} m^{\frac{1}{2}}.$$

The following lemma clarifies the proof of theorem 10:

Lemma 11 *Let*

$$K_{\nu,\mu} = 2^\nu \|[\varphi_\nu, a]\psi_\mu\|_{\mathcal{L}(L^2)} e^{-\beta(\nu-\nu)t} (\nu+1)^{-\frac{1}{2}} (\mu+1)^{-\frac{1}{2}} 2^{-(\nu-\mu)\vartheta},$$

(see Theorem 10), then there exists $\bar{C} > 0$ *such that*

$$\sup_\nu \sum_{\mu=0}^{+\infty} |K_{\nu,\mu}| + \sup_\mu \sum_{\nu=0}^{+\infty} |K_{\nu,\mu}| \leq \bar{C}$$

Proof.

We will make use of the following inequality (see [CL], Prop. 3.6)

$$\|[\varphi_\nu, a]\psi_\mu\|_{\mathcal{L}(L^2)} \leq \begin{cases} C''' 2^{-\nu}(\nu+1) & \text{if } |\nu-\mu| \leq 2 \\ C''' 2^{\max\{\nu,\mu\}} \max\{\nu+1, \mu+1\} & \text{if } |\nu-\mu| \geq 3. \end{cases}$$

We choose T in theorem 10 such that $\beta T \leq \frac{\vartheta \log 2}{2}$. We start by fixing μ and divide the estimate into two parts

$$\sum_{\nu=0}^{+\infty}|K_{\nu,\mu}| = \sum_{\nu=0}^{\mu}|K_{\nu,\mu}| + \sum_{\nu=\mu+1}^{+\infty}|K_{\nu,\mu}|,$$

we start with the first term

$$\begin{aligned}
\sum_{\nu=0}^{\mu}|K_{\nu,\mu}| &\leq 2^{(\mu+1)\vartheta}(\mu+1)^{-\frac{1}{2}}e^{\beta(\mu+1)t}\sum_{\nu=0}^{\mu}\|[\varphi_\nu, a]\psi_\mu\|_{\mathcal{L}(L^2)}2^{\nu}2^{-(\nu+1)\vartheta}e^{-\beta(\nu+1)t}(\nu+1)^{\frac{1}{2}} \\
&\leq C2^{(\mu+1)}2^{(\mu+1)\vartheta}(\mu+1)^{\frac{1}{2}}e^{\beta(\mu+1)t}\sum_{\nu=0}^{\mu}(\nu+1)^{-\frac{1}{2}}e^{-\beta(\nu+1)t}e^{-(\nu+1)\vartheta\log 2}e^{(\nu+1)\log 2} \\
&\leq C2^{(\mu+1)}2^{(\mu+1)\vartheta}(\mu+1)^{\frac{1}{2}}e^{\beta(\mu+1)t}\sum_{\nu=0}^{\mu}(\nu+1)^{-\frac{1}{2}}e^{(\nu+1)((1-\vartheta)\log 2-\beta t)} \\
&\leq C2^{(\mu+1)}2^{(\mu+1)\vartheta}(\mu+1)^{\frac{1}{2}}e^{\beta(\mu+1)t}\vartheta_1((1-\vartheta)\log 2-\beta t)(\mu+1)^{-\frac{1}{2}}e^{((1-\vartheta)\log 2-\beta t)(\mu+1)} \\
&\leq C\vartheta_1(1-\frac{3}{2}\vartheta\log 2) \leq \mathcal{K}_1,
\end{aligned}$$

since from $\beta T \leq \frac{\vartheta\log 2}{2}$ we have $(1-\vartheta)\log 2 - \beta t > 1 - \frac{3}{2}\vartheta\log 2 > 0$ and we can use lemma 10.

We then consider the second term

$$\begin{aligned}
\sum_{\nu=\mu+1}^{+\infty}|K_{\nu,\mu}| &\leq Ce^{\beta(\mu+1)t}(\mu+1)^{-\frac{1}{2}}2^{\mu\vartheta}\sum_{\nu=\mu+1}^{+\infty}2^{-\nu\vartheta}(\nu+1)2^{-\nu}2^{\nu}e^{-\beta(\nu+1)t}(\nu+1)^{-\frac{1}{2}} \\
&\leq Ce^{\beta(\mu+1)t}(\mu+1)^{-\frac{1}{2}}2^{(\mu+1)\vartheta}\sum_{\nu=\mu+1}^{+\infty}e^{-\beta(\nu+1)t}(\nu+1)^{\frac{1}{2}}2^{-(\nu+1)\vartheta} \\
&\leq Ce^{\beta(\mu+1)t}(\mu+1)^{-\frac{1}{2}}2^{(\mu+1)\vartheta}\sum_{\nu=\mu+1}^{+\infty}e^{-(\nu+1)(\beta t+\vartheta\log 2)}(\nu+1)^{\frac{1}{2}} \\
&\leq Ce^{\beta(\mu+1)t}(\mu+1)^{-\frac{1}{2}}2^{(\mu+1)\vartheta}\vartheta_2(\beta t+\vartheta\log 2)e^{-(\mu+1)(\beta t+\vartheta\log 2)}(\mu+1)^{\frac{1}{2}} \\
&\leq C\vartheta_2(\beta t+\vartheta\log 2) \leq \mathcal{K}_2.
\end{aligned}$$

We now fix ν and divide the estimate into two parts as before

$$\sum_{\mu=0}^{+\infty}|K_{\nu,\mu}| = \sum_{\mu=0}^{\nu}|K_{\nu,\mu}| + \sum_{\mu=\nu+1}^{+\infty}|K_{\nu,\mu}|,$$

starting with the first term we have

$$\begin{aligned}\sum_{\mu=0}^{\nu}|K_{\nu,\mu}| &\le C2^{\nu}2^{-\nu}(\nu+1)^{\frac{1}{2}}e^{-\beta(\nu+1)t}2^{-(\nu+1)\vartheta}\sum_{\mu=0}^{\nu}2^{(\mu+1)\vartheta}e^{\beta(\mu+1)t}(\mu+1)^{-\frac{1}{2}}\\ &\le C(\nu+1)^{\frac{1}{2}}e^{-\beta(\nu+1)t}2^{-(\nu+1)\vartheta}\sum_{\mu=0}^{\nu}e^{(\mu+1)(\beta t+\vartheta\log 2)}(\mu+1)^{-\frac{1}{2}}\\ &\le C(\nu+1)^{\frac{1}{2}}e^{-\beta(\nu+1)t}2^{-(\nu+1)\vartheta}\vartheta_1(\vartheta\log 2)e^{(\nu+1)(\beta t+\vartheta\log 2)}(\nu+1)^{-\frac{1}{2}}\\ &\le \vartheta_1(\vartheta\log 2)\le \mathcal{K}_3,\end{aligned}$$

while for the second part we have

$$\begin{aligned}\sum_{\mu=\nu+1}^{+\infty}|K_{\nu,\mu}| &\le C2^{(\nu+1)}e^{-\beta(\nu+1)t}2^{-(\nu+1)\vartheta}(\nu+1)^{-\frac{1}{2}}\sum_{\mu=\nu+1}^{+\infty}2^{-(\mu+1)}(\mu+1)^{\frac{1}{2}}e^{\beta(\mu+1)t}2^{(\mu+1)\vartheta}\\ &\le C2^{(\nu+1)}e^{-\beta(\nu+1)t}2^{-(\nu+1)\vartheta}(\nu+1)^{-\frac{1}{2}}\sum_{\mu=\nu+1}^{+\infty}e^{-(\mu+1)((1-\vartheta)\log 2-\beta t)}(\mu+1)^{\frac{1}{2}}\\ &\le C2^{(\nu+1)}e^{-\beta(\nu+1)t}2^{-(\nu+1)\vartheta}(\nu+1)^{-\frac{1}{2}}\vartheta_2(1-\frac{3}{2}\vartheta\log 2)e^{-(\nu+1)((1-\vartheta)\log 2-\beta t)}(\nu+1)^{\frac{1}{2}}\\ &\le C\vartheta_2(1-\frac{3}{2}\vartheta\log 2)\le \mathcal{K}_4.\end{aligned}$$

□

Bibliography

[AG] S. Alinhac and P. Gérard, Pseudo-differential operators and the Nash-Moser theorem, Advanced studies in mathematics, vol 82 (2007), American Mathematical Society.

[B] J. M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Annales Scientifiques de l' école Normale Supérieure **14** (1981), 209–246.

[Bi] M. A. Biot, Thermoelasicity and irreversible thermodynamics, *J. Appl. Phys.* **27** (1956), 240-253.

[C] J. Y. Chemin, Localization in Fourier Space and Navier-Stokes System, Phase space analysis of partial differential equations, Proceedings, CRM series (2004), 53-136.

[Ca] T. Carleman, Sur un probléme d'unicité pur les systemes d'équations aux derivées partielles a deux variables indépendantes, *Ark. Mat. Astr. Fys* **26B** (1939), 1–9.

[CB] B. Cushman-Roisin and J.M. Beckers, Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects, International Geophysics Series **101**, Elsevier, (2011).

[CDGS] F. Colombini, E. De Giorgi and S. Spagnolo, Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **6** (1979), no. 3, 511-559.

[CDS] F. Colombini and D. Del Santo, A note on hyperbolic operators with Log-Zygmund coefficients, *J. Math. Sci. Univ. Tokyo* **16** (2009), 95-11.

[ChS] P. Chadwick and I. N. Sneddon, Plane Waves in an elastic solid conducting heat, *J. Mech. and Phys. of Solids* **6** (1958), 223-230.

[CL] F. Colombini and N. Lerner, Hyperbolic operators with non-Lipschitz coefficients, *Duke Math. J.* **77** (1995), 657–698.

[CM] R. Coifman and Y. Meyer, Au dela des operateurs pseudodifferentiels, Asterisque, No. 57, Soc. Math. de France, 1978.

[CMP] M. Cannone, Y. Meyer and F. Planchon, Solutions auto-similaires des équations de Navier-Stokes, Exposé n. VIII, Séminaire "Équations aux dérivées partielles", École Polytechnique (Janvier 1994).

[CMZ] Q. Chen, C. Miao and Z. Zhang, Global well-posedness for the 3D rotating Navier-Stokes equations with highly oscillating initial data, arXiv:0910.3064v2, 2010, 31 march.

[CP] M. Cannone and F. Planchon, Self-similar solutions of the Navier-Stokes equations in $\mathbb{R}^3$, *Comm. Part. Diff. Eq.*, **21** (1996), 179-193.

[DSP] D. Del Santo and M. Prizzi, Backward uniqueness for parabolic operators whose coefficients are non-Lipschitz continuous in time, *J. Math. Purés Appl.* **84** (2005), 471–491.

[FK] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, *Arch. Rat. Mech. Anal.* **16** (1964), 269-315.

[HS] M. Hieber and Y. Shibata, the Fujita-Kato approach to the Navier-Stokes equations in the rotational framework., *Math. Z.*, 265 (2) (2009), 481-491.

[K] T. Kato, Strong L^p solutions for the Navier-Stokes equations in $\mathbb{R}^m$ with applications to weak solutions, *Math. Zeit.* **187** (1984), 471-480.

[KL] H. Koch and I. Lasiecka, Backward uniqueness in linear thermoelasticity with time and space variable coefficients, *Fuctional Analysis and Evolution Equations. The Günter Lumer volume* (2007), 389–403.

[KT] H. Koch and D. Tataru, Well-posedness for the Navier-Stokes equations, *Adv. Math.* **157** (2001), no. 1, 22–35.

[L] J. Leray, Sur le mouvements d'un liquide visqueux emplissant l'espace, *Acta Math.* **63** (1934), 193-248.

[LP] J. E. Littlewood and R. Paley, Theorems on Fourier series and power series II, Proc. London Math. Soc. **42** (1937), 52-89.

[M] G. Metivier, Para-differential Calculus and Applications to the Cauchy Problem for Nonlinear System, 5 CRM series, Edizioni della Normale, Pisa, (2008).

[MP] M. Majdoub and M. Paicu, Uniform Local Existence for Inhomogeneous Rotating Fluid Equations, *J Dyn Diff Equat* **21** (2009), 21-44.

[P] M. Paicu, Equation anisotrope de Navier - Stokes dans des espace critiques, *Rev. Mat. Iberoamericana* **21**, no. 1, (2005), 179 - 235.

[T] S. Tarama, Energy estimate for wave equations with coefficients in some Besov type class, *Electron. J. Differ. Equ.* (2007), Paper no. **85**, 12 p., (electronic).

[Ta] M. E. Taylor, Pseudodifferential Operators and Nonlinear PDE. Progress in Mathematics **100**, Birkhäuser, Boston, 1991.