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On the Topology of the Character Variety of a Free Group

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Summary. - *We investigate the topology of the space of characters of a free group into $SL_2\mathbb{R}$, $SL_2\mathbb{C}$, SO_2 and SU_2.*

We investigate the topology of the character varieties for representations of a free group into $SO_2, SU_2, SL_2\mathbb{R}$ and $SL_2\mathbb{C}$. This work is a summary of the PhD thesis of the first author written under the direction of the second author. Since Bratholdt has not continued in mathematics, the second author has prepared this report so that the work is not lost.

Let F_n denote the free non-abelian group of rank n, and G a linear group. In this paper we will be concerned with the cases $G = SL_2K$ and SU_2K where K is one of the fields $\mathbb{R}$ or $\mathbb{C}$ and $SU_2\mathbb{R} \equiv SO_2$. The space $Hom(F_n, G)$ with the compact-open topology is homeomorphic to G^n. The quotient $Hom(F_n, G)/G$ by the action of G by conjugacy is Hausdorff for the case $G = SU_2K$ but not for $G = SL_2K$. The character variety $X(F_n, G)$ is the set of characters $\chi(\rho)$ as ρ varies over $Hom(F_n, G)$. It is an affine algebraic set which we refer to as a variety ($X(F_n, SU_2(\mathbb{C}))$ is a *real* algebraic set). The topology of the space of representations modulo conjugacy is closely related to that of the corresponding character variety.

There is a natural map $X : Hom(F_n, G)/G \to X(F_n, G)$ which is a homeomorphism for $G = SU_2$. For $G = SL_2\mathbb{C}$ and $n > 1$ the restriction to the subspace of conjugacy classes of irreducible representations is a homeomorphism onto a dense open subset. There is

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an element of $SL_2\mathbb{C}$ whose action on $\mathbb{H}^3$ restricts to an orientation reversing isometry of $\mathbb{H}^2$. From this one sees that the normalizer of $SL_2\mathbb{R}$ in $SL_2\mathbb{C}$ contains $SL_2\mathbb{R}$ as a subgroup of index 2. Thus the map X is generically $2:1$ for $G = SL_2\mathbb{R}$ or SO_2. From this it is easy to see that

THEOREM 1. *$X(F_n, SO_2) \cong T^n/\mathbb{Z}_2$ where T^n is the n-torus and the involution sends*
$[exp(i\theta_1), \cdots, exp(i\theta_n)]$ to $[exp(-i\theta_1), \cdots, exp(-i\theta_n)]$.
In particular for $n = 2$ we get $X(F_2, SO_2)$ is homeomorphic to S^2.

These representation spaces and character varieties have been studied by a number of people, usually from an algebraic perspective [2],[3],[4],[5],[6],[7]. We are concerned with the topology of the character varieties. It is well known that $X(F_2, SL_2\mathbb{C}) = \mathbb{C}^3$.

THEOREM 2. *For $K = \mathbb{R}$ or $\mathbb{C}$ there is a strong deformation retraction of $X(F_n, SL_2K)$ onto $X(F_n, SU_2K)$.*

THEOREM 3. *$X(F_3, SU_2)$ is homeomorphic to S^6.*

THEOREM 4. *For $n \geq 4$ the space $X(F_n, SU_2)$ is not a manifold. In fact, for $n \geq 4$, the character of an abelian representation with image not contained in the center of SU_2 has a neighborhood in $X(F_n, SU_2)$ homeomorphic to $cone(\mathbb{C}P^{n-2}) \times \mathbb{R}^n$*

The group S^3 of unit quaternions is isomorphic to SU_2. The standard Riemannian metric on S^3 is bi-invariant for the Lie group structure on S^3, and so conjugacy by an element of S^3 gives an isometry of S^3 which fixes the identity element I of S^3 and every such orientation preserving isometry is given by some conjugacy.

A representation $\rho : F_n \longrightarrow S^3$ is uniquely determined by an n-tuple $(x_1, \cdots, x_n)$ of points in S^3 given by the images under ρ of an ordered basis of F_n. We thus see that the problem of determining $Hom(F_n, S^3)/S^3$ is equivalent to the classification, up to orientation preserving isometry of S^3 fixing the identity, of n-tuples of points in S^3. If we define $x_0 = I$ this is equivalent to the classification, up to the action of $Isom_+(S^3)$, of $(n+1)$-tuples $(x_0, \cdots, x_n)$ of points in S^3. We regard S^3 as the unit sphere in Euclidean space $\mathbb{R}^4$ centered on the origin, and write $< x, y >$ for the standard inner product on $\mathbb{R}^4$. Then we identify $Isom_+(S^3)$ with $SO(4)$. Thus we have:

Lemma 5. *The quotient $Hom(F_n, S^3)/S^3$ by conjugacy is homeomorphic to the quotient of $(S^3)^{n+1}$ by the diagonal action of $SO(4)$ acting standardly on each S^3 factor.*

By a **tetrahedron** in S^3 we mean a 4-tuple $A = (x_1, \cdots, x_4)$ of points, called *vertices*, in S^3. We will regard A as a 4×4 matrix with rows x_i. Observe that the edges of the tetrahedron are not specified, and that we allow two or more vertices to coincide. We regard the vertices as marked and are interested in the classification of such tetrahedra up to (orientation preserving) isometry of S^3 which preserves the marking.

Given a tetrahedron A then $B = A \cdot A^t$ is a symmetric matrix with entries $< x_i, x_j >$. The matrix B determines the spherical tetrahedron A up to right multiplication by an element of $O(4)$. Define $\mathcal{Q}$ to be the set of all such matrices $A \cdot A^t$.

Let $Sym(4)$ denote the space of real symmetric 4×4 matrices. There is a continuous map

$$\sigma : S^3 \times S^3 \times S^3 \times S^3 \longrightarrow Sym(4)$$

given by $\sigma(A) = A \cdot A^t$. Furthermore σ descends to a well defined continuous map

$$\overline{\sigma} : S^3 \times S^3 \times S^3 \times S^3/O(4) \longrightarrow Sym(4).$$

We will show that this map is a homeomorphism onto a compact 6-dimensional cell. First we show that $\overline{\sigma}$ is injective.

Suppose we are given two n-tuples $(x_1, \cdots, x_n)$ and $(x'_1, \cdots, x'_n)$ of points in $\mathbb{R}^n$ such that for all i, j we have $< x_i, x_j > = < x'_i, x'_j >$. It is elementary to show that there is T in $O(n)$ with $T(x_i) = x'_i$ for all $1 \leq i \leq n$. Furthermore we may choose T to be orientation preserving if and only if

$$det[x_1, \cdots, x_n] \cdot det[x'_1, \cdots, x'_n] \geq 0.$$

Applying this to tetrahedra gives the injectivity of $\overline{\sigma}$. Now the domain of $\overline{\sigma}$ is compact and the image is Hausdorff thus $\overline{\sigma}$ is a homeomorphism onto its image.

Define $Sym_1(4)$ to be the subspace of $Sym(4)$ consisting of those symmetric matrices which have every diagonal entry equal to 1.

Thus $Sym_1(4)$ is an affine 6-dimensional subspace of the vector space $Sym(4)$. Define $\mathcal{Q} \subset Sym_1(4)$ to be the subspace of matrices which correspond to positive semi-definite quadratic forms.

LEMMA 6. *The image of $\overline{\sigma}$ is $\mathcal{Q}$.*

Proof. First observe since the rows of A are unit vectors it follows that $B = A \cdot A^t$ has every diagonal entry equal to 1. This matrix is symmetric. We show next that B is positive semi-definite. There is an orthonormal basis $\mathcal{B}$ which diagonalizes B. Thus there is $P \in O(4)$ with $D = P^t A A^t P$ diagonal. Now $D = C \cdot C^t$ where $C = P^t AP$. Thus the rows of C are orthogonal vectors and the diagonal entries of D are the squares of the lengths of these row vectors. Hence D is a non-negative diagonal matrix. Let q be the quadratic form corresponding to B with respect to the standard basis of $\mathbb{R}^4$. Then the matrix of q with respect to $\mathcal{B}$ is D and hence q is positive semi-definite. We have now established that the image of $\overline{\sigma}$ is contained in $\mathcal{Q}$.

It remains to show surjectivity. Given B in $\mathcal{Q}$ we must find a real matrix A such that $A \cdot A^t = B$. The symmetric matrix B determines a quadratic form q on $\mathbb{R}^4$. There is an orthonormal change of coordinates on $\mathbb{R}^4$ which diagonalizes this quadratic form. Let D be the matrix of q in this basis, thus $B = P^t DP$ where $P \in O(4)$ is the change of basis matrix. Since q is positive semi-definite there is a non-negative diagonal square root $C = \sqrt{D}$, thus $D = C \cdot C^t$. Define $A = P^t CP$ which is the matrix with respect to the standard basis of the quadratic form with matrix C in the new basis. Then $A \cdot A^t = P^t C \cdot C^t P = B$ as required. □

LEMMA 7. *$\mathcal{Q}$ is a compact 6-dimensional cell.*

Proof. Observe that $\mathcal{Q}$ is a convex subset of $Sym_1(4)$ since the set of positive semi-definite quadratic forms is convex. Since $\mathcal{Q}$ is the continuous image of a compact set under the map σ it is compact and clearly has dimension 6. Thus $\mathcal{Q}$ is a compact convex 6-dimensional cell contained in $Sym_1(4)$. □

COROLLARY 8. *The set of isometry classes of non-degenerate marked spherical tetrahedra is an open convex set, $int(\mathcal{Q})$, with respect to the cosines of the lengths of the edges.*

The proof of 3 follows from 6 and the following:

COROLLARY 9. *The quotient of* $(S^3)^4$ *by the diagonal action of* $SO(4)$ *acting standardly on each factor is homeomorphic to* S^6.

Proof. Consider the map

$$\phi : (S^3)^4 \longrightarrow \mathcal{Q} \times \mathbb{R}$$

given by $\phi(A) = (\sigma(A), det(A))$. It is clear that this factors through a map

$$\overline{\phi} : (S^3)^4/SO(4) \longrightarrow \mathcal{Q} \times \mathbb{R}.$$

We will show that $\overline{\phi}$ is a homeomorphism onto its image. Indeed, the proof follows from what has been done before together with the observation that an orientation reversing isometry, τ, sends a tetrahedron A to another, $\tau(A)$, with $det(\tau(A)) = -det(A)$.

Assertion $det(A) = 0$ if and only if $\sigma(A) \in \partial\mathcal{Q}$.

Assuming this, the diagonal action of $O(4)$ on $(S^3)^4$ induces an involution on $(S^3)^4/SO(4)$ corresponding to some reflection. Let τ be the involution on $\mathcal{Q} \times \mathbb{R}$ which is trivial on the first factor, and is given by multiplication by -1 on the second factor. These involutions correspond via the map $\overline{\phi}$.

Consider the map

$$\theta : (S^3)^4/O(4) \longrightarrow \mathcal{Q} \times [0, \infty)$$

given by $\theta(A) = (\sigma(A), |det(A)|)$. By 7 this is a homeomorphism onto its image which is thus a compact 6-dimensional cell C. By the assertion $\partial C = C \cap (\mathcal{Q} \times 0)$.

Thus the image of $\overline{\phi}$ is $C \cup \tau(C)$. Furthermore $C \cap \tau(C) = \partial C$ thus C is the union of two 6-cells identified along their common boundary and is thus homeomorphic to a 6-sphere.

It remains to prove the assertion. Observe that $det(A) = 0$ if and only if $A \cdot A^t$ in not strictly positive definite. Thus the assertion is equivalent to the assertion that $\mathcal{Q} - \partial\mathcal{Q}$ is the subspace consisting of strictly positive definite quadratic forms. Since being strictly positive definite is an open condition it follows that strictly positive definite forms are in the interior of $\mathcal{Q}$. For the converse, observe that $\partial\mathcal{Q}$

equals the frontier of $\mathcal{Q}$ in $Sym_1(4)$. Given $q \in \mathcal{Q}$ the matrix of q in the standard basis may be written as $I + M$ where M is symmetric with every diagonal entry 0. If q is not strictly positive definite then $det(q) = det(I + M) = 0$ thus M has an eigenvalue of -1. For $\epsilon > 0$ the matrix $I + (1 + \epsilon)M$ corresponds to a form in $Sym_1(4)$ which has an eigenvalue of $-1 - \epsilon$. Thus q is in the frontier of $\mathcal{Q}$. □

Proof of 4

Every abelian subgroup of SU_2 is conjugate to a diagonal subgroup (here we are identifying $S^3 \cong SU_2$) so suppose that $\rho \in Hom(F_n, SU_2)$ is a diagonal representation. If there is an element α of F_n with $\rho(\alpha) \neq \pm I$ then we may choose a basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ of F_n with $\rho(\alpha_i) \neq \pm I$ for every $1 \leq i \leq n$.

By choosing a small neighborhood N in $X(F_n, SU_2)$ of $\chi(\rho)$, we may suppose that for all $\chi(\rho')$ in N that $\rho'(\alpha_i) \neq \pm I$ for all $1 \leq i \leq n$. We can vary ρ' in its conjugacy class so that $\rho'(\alpha_1)$ is diagonal. Furthermore the choice of ρ' is unique up to conjugacy by an element of $Normalizer(S^1)$ where S^1 is the subgroup of SU_2 consisting of diagonal matrices. Now $Normalizer(S^1) \cong O_2$, thus if we set

$$N_1 = \{\rho' : \chi(\rho') \in N \quad \& \quad \rho'(\alpha_1) \in S^1\}$$

then $N_1/(O_2 - conjugacy)$ is homeomorphic to N. We choose a parameterization Ψ of N_1

$$\Psi : N_1 \longrightarrow (D^2)^{n-1} \times (S^1)^n$$

as described below. We will show that the image of Ψ consists of two components, $N_1^{\pm}$, each homeomorphic to $(D^2)^{n-1} \times \mathbb{R}^n$.

The epimorphism $SU_2 \longrightarrow SO_3$ gives an action of SU_2 on S^2. Under this action, the subgroup S^1 of SU_2 is the double cover of the group SO_2 acting on S^2 by Euclidean rotations around a fixed axis. We denote by p one of the points on S^2 lying on this axis, and thus fixed by this SO_2 action.

Fix $\epsilon \in (0, \pi/2)$ and identify D^2 with the disc in S^2 of radius ϵ centered on p. For each k the isometry $\rho(\alpha_k)$ fixes p. Thus if the neighborhood N_1 is chosen sufficiently small then for all $\rho' \in N_1$ and for each k the isometry $\rho'(\alpha_k) \neq \pm I$ is a rotation which fixes a unique point in D^2. Given ρ' in N_1 we obtain a point in $(D^2)^{n-1}$ by taking the endpoints of the axes of $\rho'(\alpha_k)$ which lie in D^2 for $2 \leq k \leq n$.

For $1 \leq k \leq n$ define $\theta_k \in S^1$ to be the angle of rotation of $\rho(\alpha_k)$. Now choose $\epsilon_k > 0$ small enough that the points ± 1 in S^1 do not lie in $(\theta_k - \epsilon_k, \theta_k + \epsilon_k)$. By choosing N_1 small enough we can ensure that for $\rho' \in N_1$ the angle of rotation of $\rho'(\alpha_k)$ lies in $(\theta_k - \epsilon_k, \theta_k + \epsilon_k) \sqcup (-\theta_k - \epsilon_k, -\theta_k + \epsilon_k)$. This assigns to ρ' a point in $(S^1)^n$.

It is now clear that the image of Ψ is as asserted and that $N_1/O_2 = N_1^+/SO_2$. The action of SO_2 on each D^2 factor is by rotation fixing p, and hence the quotient of $(D^2)^{n-1}$ by the diagonal action of SO_2 is $cone(\mathbb{C}P^{n-2})$. To see this we can identify D^2 with the unit disc in the $\mathbb{C}$ and then $\partial(D^2)^{n-1}/SO_2$ is homeomorphic to $\mathbb{C}P^{n-2}$. Now $(D^2)^{n-1} = cone(\partial(D^2)^{n-1})$ hence $(D^2)^{n-1}/SO_2 \cong cone(\mathbb{C}P^{n-2})$. The action of SO_2 on the $(S^1)^n$ factor is trivial, hence $N_1^+/(SO_2 - conjugacy) \cong cone(\mathbb{C}P^{n-2}) \times \mathbb{R}^n$ as claimed.

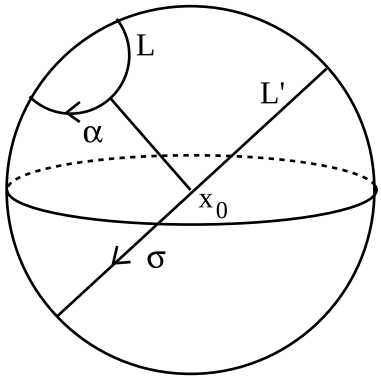
We conclude with an outline of the proof of 2. The proof is based on:

THEOREM 10. *Let $Irrep_n \subset Hom(F_n, SL_2\mathbb{C})$ be the subspace consisting of irreducible representations. Then there is a continuous map $C : Irrep_n \to \mathbb{H}^3$ such that for every $\rho \in Irrep_n$ and $A \in SL_2\mathbb{C}$ we have $C(A \cdot \rho \cdot A^{-1}) = A(C(\rho))$.*

Proof. In [1] Bestvina introduced the notion of the *center*, x_0, of an irreducible representation, ρ, of a finitely generated group G into $PSL_2(\mathbb{C})$. One fixes a finite generating set S of G and then considers the partial orbit $\rho(S)x$ of an arbitrary point x in $\mathbb{H}^3$. If ρ is irreducible then there is an x which minimizes the diameter of $\rho(S)x$. Furthermore the subset of $\mathbb{H}^3$ consisting of such points is compact. The center of mass of the convex hull of this set is the *center* of ρ. ∎

Theorem 11. *There is a strong deformation retraction* $W : SL_2\mathbb{C} \to SU_2$ *called* **winching** *which is equivariant with respect to the action of* SU_2 *by conjugation.*

Proof. We identify hyperbolic 3-space with the ball model, and identify $SL_2\mathbb{C}$ with the isometries of the ball in such a way that SU_2 is the stabilizer of the center of the ball, which we will denote by x_0.



Suppose that $A \in SL_2\mathbb{C}$ represents a loxodromic isometry α with axis L. Define L' to be the geodesic containing x_0 obtained by parallel transportation of L along the shortest geodesic from x_0 to L. Thus if L contains x_0 then $L' = L$. Let ϕ be the isometry given by this parallel transport; thus $\phi(L) = L'$. Define $\sigma = \phi \circ \alpha \circ \phi^{-1}$. Thus σ has the same complex translation length as α.

Now define τ to be the isometry with the same axis and rotational part as σ but zero translational part. This determines a unique element of PSU_2. Observe that $\pm trace(\tau) = Im(trace(A))$. We now choose the matrix $W(A)$ in SU_2 corresponding to τ such that the imaginary parts of the traces of A and $W(A)$ are equal.

In the remaining case that $trace(A) = 2\epsilon$ with $\epsilon = \pm 1$ define $W(A) = \epsilon I$. It is obvious that $W(B \cdot A \cdot B^{-1}) = B \cdot W(A) \cdot B^{-1}$ for $B \in SU_2$ and $A \in SL_2\mathbb{C}$. We leave it as an exercise to check that W is continuous and a strong deformation retraction. □

Consider the subspace $Irrep_n(x_0) \subset Irrep_n$ consisting of those representations with center x_0 in the ball model. There is a map $\mathcal{W} : Irrep_n(x_0) \to Hom(F_n, SU_2)$ which is obtained by applying the winch retraction to each element of the standard basis of F_n. Thus $\mathcal{W}(\rho) = (W(\rho(\alpha_1)), \cdots, W(\rho(\alpha_n)))$ where $\alpha_1, \cdots, \alpha_n$ is a fixed basis of F_n. This map is homotopic through maps into $Hom(F_n, SL_2\mathbb{C})$ to the inclusion $Irrep_n(x_0) \hookrightarrow Hom(F_n, SL_2\mathbb{C})$. Furthermore it follows from 11 that $\mathcal{W}$ is equivariant with respect to the action of SU_2 by conjugacy.

Now taking characters embeds $Irrep_n(x_0)$ as a dense subset $X(Irrep_n(x_0)) \subset X(F_n, SL_2\mathbb{C})$. Furthermore two irreducible representations have the same character if and only if they are conjugate. If ρ, ρ' are conjugate representations in $Irrep_n(x_0)$ then the conjugacy fixes x_0 (since this is the unique center of both representations and using 10). Hence they are conjugate by an element of SU_2. It follows that $\mathcal{W}$ induces a map $X(\mathcal{W}) : X(Irrep_n(x_0)) \to X(F_n, SU_2)$. Furthermore this map extends to a strong deformation retraction $X(F_n, SL_2\mathbb{C}) \to X(F_n, SU_2)$. This ends the sketch of the proof of 2.

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