PRODUCTS OF SEQUENTIALLY COMPACT SPACES (*)

by Petr Simon (in Praha)(**)

Sommario. - *Si prova che $\mathfrak{h}$ è il minimo numero cardinale tale che il prodotto di ogni famiglia di $< \mathfrak{h}$ spazi sequenzialmente compatti è sequenzialmente compatto.*

Summary. - *We shall show that $\mathfrak{h}$ is the minimal cardinal such that the product of every family consisting of $< \mathfrak{h}$ sequentially compact spaces is sequentially compact.*

Recall that a topological space is called *sequentially compact*, if every infinite set in it contains a convergent sequence.

In 1984, Eric van Douwen asked the following question in his Handbook article: Let $\mu = \min\{\kappa :$ some product of κ sequentially compact spaces is not sequentially compact$\}$.
Can μ be expressed as a set theoretically defined cardinal? [vD, 6.10]

By that time, the answer was already known: Jan Pelant, Peter Nyikos and myself had discovered it independently. However, the joint paper by us [NPS] has not been published. Jan Pelant intended to include the theorem as a remark into another paper; that one, joint with A. Šostak, unfortunately, was even never written. My own paper [S] contains the more difficult inequality, however, as the topics considered there were different, it is in an implicit form only. The answer, namely $\mu = \mathfrak{h}$, appeared in Jerry Vaughan's article in Open Problems [V]. For the proof, there are two references in [V], the first one to unpublished [NPS] and the second one to [FV].

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(**) Indirizzo dell'Autore: Matematický Ústav University Karlovy, Sokolovská 83, 18600 Praha 8 (Repubblica Ceca).

But Frič and Vojtáš show only that $\mu \geq \mathfrak{h}$ and for the opposite inequality they simply state that it was proved by Pelant, Nyikos and Simon. So the vicious circle is closed and still there is no published proof of the result.

During the Trieste International conference of topology, O. T. Alas and V. V. Uspenskij asked me to present the proof somewhere. The present note results, in fact, from their request.

The cardinal number $\mathfrak{h}$ was introduced in [BPS]. We shall recall its definition and the main statement concerning it.

We use the standard notation. $\mathbb{N}$ denotes the set of all natural numbers, ω is the first infinite cardinal. If X is a set, then $[X]^\omega = \{Y \subseteq X : |Y| = \omega\}$ and $[X]^{<\omega} = \{Y \subseteq X : |Y| < \omega\}$.

Two infinite subsets A, B of $\mathbb{N}$ are called *almost disjoint*, if $A \cap B$ is finite. A is *almost contained* in B ($A \subseteq^* B$) if $A \setminus B$ is finite. A family $\mathcal{A}$ of subsets of $\mathbb{N}$ is a *MAD family*, if all members of $\mathcal{A}$ are infinite, any two members of $\mathcal{A}$ are almost disjoint and $\mathcal{A}$ is maximal with respect to these two properties. A MAD family $\mathcal{A}$ refines a MAD family $\mathcal{B}$ if for every $A \in \mathcal{A}$ there is some $B \in \mathcal{B}$ such that A is almost contained in B.

Definition. ([BPS]) A collection Θ consisting of MAD families on $\mathbb{N}$ is called *splitting*, if for every $M \in [\mathbb{N}]^\omega$ there is some $\mathcal{A} \in \Theta$ such that $M \cap A$ and $M \cap B$ are infinite for two distinct $A, B \in \mathcal{A}$.

Define $\quad \mathfrak{h} = \min\{|\Theta| : \Theta \text{ is splitting}\}$.

Notice that the well known splitting number $\mathfrak{s}$ may be defined in a quite analogous way: $\mathfrak{s} = \min\{|\Theta| : \Theta \text{ is splitting and every } \mathcal{A} \in \Theta \text{ satisfies } |\mathcal{A}| = 2\}$. The number $\mathfrak{s}$ is related to van Douwen's question, too: $\mathfrak{s} = \min\{\kappa :$ any product of κ sequentially compact spaces is not sequentially compact$\}$ (here, the spaces are assumed to contain at least two points, of course).

More information concerning $\mathfrak{h}$ can be found in [BPS]. Here we remark only that $\mathfrak{h}$ is a regular cardinal and $\mathfrak{t} \leq \mathfrak{h} \leq \min\{\mathfrak{s}, \mathfrak{b}, \mathrm{cf}(\mathfrak{c})\}$. The numbers $\mathfrak{t}, \mathfrak{s}, \mathfrak{b}$ as well as a lot of other so-called small cardinals are studied in [vD] and [V].

The rest of the paper will be devoted to the proof of the following

Theorem. $\mu = \mathfrak{h}$.

Proof. $\mu \geq \mathfrak{h}$: Let $\tau < \mathfrak{h}$, let X_α ($\alpha \in \tau$) be sequentially compact spaces and let $X = \prod\{X_\alpha : \alpha \in \tau\}$. We use the standard notation: $\pi_\alpha : X \longrightarrow X_\alpha$ is the projection mapping, for $x \in X$, $x = \langle x(\alpha) : \alpha \in \tau \rangle$, $\pi_\alpha(x) = x(\alpha)$.

Let $\langle x_n : n \in \mathbb{N}\rangle$ be a sequence in X; we need to find some convergent subsequence. For $\alpha \in \tau$, let $\mathcal{M}_\alpha$ be a family of all $M \in [\mathbb{N}]^\omega$ such that the sequence $\langle x_n(\alpha) : n \in M\rangle$ converges in X_α. Let $\mathcal{Q}_\alpha$ be a maximal family of members of $\mathcal{M}_\alpha$ such that any two sets in $\mathcal{Q}_\alpha$ are almost disjoint. Then $\mathcal{Q}_\alpha$ is a MAD family: If not, then some $C \in [\mathbb{N}]^\omega$ is almost disjoint with all members of $\mathcal{Q}_\alpha$. Since $\mathcal{Q}_\alpha$ is maximal, no infinite subset of the set C can belong to $\mathcal{M}_\alpha$. So no subsequence of $\langle x_n(\alpha) : n \in C\rangle$ converges, which contradicts the assumption that the space X_α is sequentially compact.

Since $\tau < \mathfrak{h}$, there must be an infinite set $T \subseteq \mathbb{N}$ which meets only one member of $\mathcal{Q}_\alpha$ (call it Q_α) in an infinite set. Since $\mathcal{Q}_\alpha$ is MAD, we have $T \setminus Q_\alpha$ finite. Let $x(\alpha) = \lim\langle x_n(\alpha) : n \in Q_\alpha\rangle$. It remains to show that the point $x = \langle x(\alpha) : \alpha \in \tau\rangle$ is the limit of the sequence $\langle x_n : n \in T\rangle$.

To this end, consider an arbitrary basic open set U containing x: $U = \prod\{U_\alpha : \alpha \in \tau\}$, where every U_α is a neighborhood of $x(\alpha)$ and for some finite set $F \subseteq \tau$, $U_\alpha = X_\alpha$ whenever $\alpha \in \tau \setminus F$. For every $\alpha \in F$, there is some $n_\alpha \in \mathbb{N}$ such that $x_n(\alpha) \in U_\alpha$ for all $n \in Q_\alpha$, $n \geq n_\alpha$. Since T is infinite and $T \setminus Q_\alpha$ is finite, there is some $m_\alpha \geq n_\alpha$ such that $x_n(\alpha) \in U_\alpha$ whenever $n \in T$, $n \geq m_\alpha$. Thus if $n \in T$, $n \geq \max\{m_\alpha : \alpha \in F\}$, then $x_n \in U$, which was to be proved.

$\mu \leq \mathfrak{h}$: We shall find a family $\{X_\alpha : \alpha \in \mathfrak{h}\}$ of compact, sequentially compact spaces and an infinite set $N \subseteq \prod\{X_\alpha : \alpha \in \mathfrak{h}\}$ containing no convergent sequence. Since this will prove that $\prod\{X_\alpha : \alpha \in \mathfrak{h}\}$ is not sequentially compact, the inequality will follow.

Let $\{\mathcal{A}_\alpha : \alpha \in \mathfrak{h}\}$ be a splitting collection of MAD families on $\mathbb{N}$. For $\alpha \in \mathfrak{h}$, let X_α be the Stone space of a Boolean algebra $\mathcal{B}_\alpha \subseteq \mathcal{P}(\mathbb{N})$, generated by $\mathcal{A}_\alpha \cup [\mathbb{N}]^{<\omega}$. Stating in different words, X_α is the one-point compactification of the space $\Psi(\mathbb{N}, \mathcal{A}_\alpha)$. (Recall that the underlying set of the space $\Psi(\mathbb{N}, \mathcal{A}_\alpha)$ is the set $\mathbb{N} \cup \mathcal{A}_\alpha$, all points $n \in \mathbb{N}$ are isolated and the family $\{\{A\} \cup (A \setminus K) : K \in [\mathbb{N}]^{<\omega}\}$ is a local base at $A \in \mathcal{A}_\alpha$). The space X_α is obviously sequentially compact: If T is an infinite subset of X_α, then either $T \cap \mathbb{N}$ is infinite and then, by the maximality of $\mathcal{A}_\alpha$, some $A \cap T$ is infinite and then the sequence $\langle n : n \in A \cap T\rangle$ converges to $A \in \mathcal{A}_\alpha$, or $T \cap \mathcal{A}_\alpha$ is infinite and then $\langle A : A \in T \cap \mathcal{A}_\alpha\rangle$ converges to the point, which compactifies the space $\Psi(\mathbb{N}, \mathcal{A}_\alpha)$.

Let $X = \prod\{X_\alpha : \alpha \in \mathfrak{h}\}$ and consider the set $N = \{x_n : n \in \mathbb{N}\}$, where $x_n \in X$ is defined by $x_n(\alpha) = n$ for all $\alpha \in \mathfrak{h}$. Let $M \in [\mathbb{N}]^\omega$ be arbitrary and let us consider the sequence $\langle x_n : n \in M\rangle$. According to the definition of $\mathfrak{h}$, there is some $\alpha \in \mathfrak{h}$ and two members $A, B \in \mathcal{A}_\alpha$ such that both sets $A \cap M$ and $B \cap M$ are infinite. Then in the space X_α, the sequence

$\langle n : n \in A \cap M \rangle$ converges to A while the sequence $\langle n : n \in B \cap M \rangle$ converges to B. So $\pi_\alpha[\{x_n : n \in M\}]$ has at least two cluster points in X_α, hence $\langle x_n : n \in M \rangle$ cannot converge in X. Since M was arbitrary, this shows that X is not sequentially compact. $\diamondsuit$

References

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