Rend. Istit. Mat. Univ. Trieste
Volume 48 (2016), 209–219
DOI: 10.13137/2464-8728/13157

Asymptotic formulas for steady state voltage potentials in the presence of thin inhomogeneities. The complex case

Elena Beretta and Elisa Francini

Dedicated to Giovanni Alessandrini on the occasion of his 60th birthday

Abstract. *We consider a conducting body with complex valued admittivity containing a finite number of well separated thin inclusions. We derive an asymptotic formula for the boundary values of the potential in terms of the width of the inclusions.*

Keywords: Thin inhomogeneities, complex coefficients.
MS Classification 2010: 35J25.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain representing the region occupied by a conducting body and consider, at a fixed frequency ω, the complex valued admittivity background

$$\gamma_0 = \sigma_0 + i\omega\varepsilon_0 \text{ in } \Omega,$$

where σ_0 and ε_0 are real valued functions representing the electrical conductivity and permittivity, respectively.

Let $\Sigma_j \subset\subset \Omega$, for $j = 1, \dots, N$ be a collection of simple, regular curves and consider, for ϵ sufficiently small, a neighborhood of Σ_j given by

$$D_\epsilon^j = \{x \in \Omega \,:\, dist(x, \Sigma_j) < \epsilon\},$$

representing a thin inhomogeneity of admittivity

$$\gamma_j = \sigma_j + i\omega\varepsilon_j.$$

Let $\psi \in H^{1/2}(\partial\Omega)$ represents a complex valued boundary current and let

u_0 be the background potential which satisfies

$$\begin{cases} \operatorname{div}(\gamma_0 \nabla u_0) = 0 & \text{in} \quad \Omega \\ \gamma_0 \frac{\partial u_0}{\partial \nu} = \psi & \text{on} \quad \partial\Omega, \end{cases}$$

where ν is the unit outer normal to $\partial\Omega$.

Let

$$\gamma_\epsilon = \gamma_0 + \sum_{j=1}^{N} (\gamma_j - \gamma_0) \chi_{D_\epsilon^j}$$

and consider the perturbed complex-valued potential u_ϵ solution to

$$\begin{cases} \operatorname{div}(\gamma_\epsilon \nabla u_\epsilon) = 0 & \text{in} \quad \Omega \\ \gamma_\epsilon \frac{\partial u_\epsilon}{\partial \nu} = \psi & \text{on} \quad \partial\Omega. \end{cases}$$

The main goal of the paper is to obtain an asymptotic expansion for the boundary values $(u_\epsilon - u_0)_{|\partial\Omega}$ as $\epsilon \to 0$.

The formula we derive is analogue to the one obtained in [3] in the case of constant real valued conductivities σ_0 and σ_1 ($\omega = 0$).

More precisely, we show that for $y \in \partial\Omega$ and $\epsilon \to 0$,

$$(u_\epsilon - u_0)(y) = \epsilon v(y) + o(\epsilon).$$

where

$$v(y) = \sum_{j=1}^{N} 2 \int_{\Sigma_j} (\gamma_0 - \gamma_j)(x) \mathbb{M}^j(x) \nabla u_0(x) \nabla_x N(x, y) \, d\sigma_x.$$

Here $N(x, y)$ is the Neumann function corresponding to the operator $\operatorname{div}(\gamma_0 \nabla \cdot)$ and $\mathbb{M}^j$ is a two by two matrix with complex valued entries.

It is well known that this type of expansion can be used in order to solve the inverse problem of detecting the curves $\Sigma_j, j = 1, \ldots, N$ from boundary measurements. In fact, in [1] the authors show that for the conductivity equation, it is possible to detect finitely many segments from knowledge on the boundary of the first order term v appearing in the expansion. Moreover they show the continuous dependence of the segments from the boundary measurement v is Lipschitz stable. A similar result has been obtained in the case of the system of linearized elasticity, for the case $N = 1$, in [2].

2. Main assumptions and results

For $j = 1, \ldots, N$, let Σ_j be a simple, regular $C^{2,\alpha}$ curve with $\alpha \in (0, 1)$ and assume there exists a contant $K > 1$ such that, in a neighborhood of radius

K^{-1} of each point in Σ_j, the curve is the graph of a $C^{2,\alpha}$ function and

$$\begin{gathered}\|\Sigma_j\|_{C^{2,\alpha}} \leq K, \quad dist(\Sigma_j, \partial\Omega) \geq K^{-1}, \\ K^{-1} \leq \mathcal{L}(\Sigma_j) \leq K, \quad dist(\Sigma_j, \Sigma_k) \geq K^{-1} \text{ if } j \neq k,\end{gathered} \tag{1}$$

where $\mathcal{L}$ denotes the length.

On each curve Σ_j we fix an continuous orthonormal system $(n_j(x), t_j(x))$ such that $n_j(x)$ is a normal direction to Σ_j at its point x and $\tau_j(x)$ is a tangent direction.

Assume $\gamma_0, \gamma_j : \Omega \to \mathbb{C}$ such that $\gamma_0 \in C^{1,\alpha}(\Omega)$, $\gamma_j \in C^{\alpha}(\Omega)$ with

$$\|\gamma_0\|_{C^{1,\alpha}(\Omega)}, \|\gamma_j\|_{C^{\alpha}(\Omega)} \leq K, \tag{2}$$

and, furthermore, assume there exists $c_0 > 0$ such that

$$\sigma_j \geq c_0, \text{ for } j = 0, 1, \dots, N. \tag{3}$$

Consider finally a complex valued flux $\psi \in H^{-1/2}(\partial\Omega)$ satisfying the compatibility condition

$$\int_{\partial\Omega} \psi = 0. \tag{4}$$

Then, under the above assumptions, there exist unique weak solutions u_0 and u_ϵ in $H^1(\Omega)$ to

$$\begin{cases} \operatorname{div}(\gamma_0 \nabla u_0) = 0 & \text{in} \quad \Omega, \\ \gamma_0 \frac{\partial u_0}{\partial \nu} = \psi & \text{on} \quad \partial\Omega, \\ \int_{\partial\Omega} u_0 = 0, & \end{cases} \tag{5}$$

and

$$\begin{cases} \operatorname{div}(\gamma_\epsilon \nabla u_\epsilon) = 0 & \text{in} \quad \Omega, \\ \gamma_\epsilon \frac{\partial u_\epsilon}{\partial \nu} = \psi & \text{on} \quad \partial\Omega, \\ \int_{\partial\Omega} u_\epsilon = 0. & \end{cases} \tag{6}$$

We also introduce the Neumann function N solution to

$$\begin{cases} \operatorname{div}(\gamma_0 \nabla N(\cdot, y)) = \delta_y & \text{in} \quad \Omega, \\ \gamma_0 \frac{\partial N(\cdot, y)}{\partial \nu} = \frac{1}{\mathcal{L}(\partial\Omega)} & \text{on} \quad \partial\Omega, \\ \int_{\partial\Omega} N(\cdot, y) = 0, & \end{cases} \tag{7}$$

It is well known that under assumptions (2) and (3) there exists a unique solution to (7) (see [5]).

We are now ready to state our main result.

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^2$ be bounded smooth domain and $\{\Sigma\}_{j=1}^N \subset\subset \Omega$ a set of curves satisfying* (1), *let γ_0 and γ_j (for $j = 1, \dots, N$) be admittivities satisfying* (2) *and* (3) *and let u_0 and u_ϵ be solutions to* (5) *and* (6), *respectively. Then, for $y \in \partial\Omega$ and $\epsilon \to 0$,*

$$(u_\epsilon - u_0)(y) = 2\epsilon \sum_{j=1}^N \int_{\Sigma_j} (\gamma_0(x) - \gamma_j(x))\mathbb{M}^j(x)\nabla u_0(x) \cdot \nabla N(x, y)\, d\sigma_x + o(\epsilon),$$

where

$$\mathbb{M}^j(x) = \frac{\gamma_0(x)}{\gamma_j(x)}\, n_j(x) \otimes n_j(x) + \tau_j(x) \otimes \tau_j(x).$$

3. Proof of Theorem 2.1

We will perform the proof in the case $N = 1$. Since the curves are well separated one from each other, the same argument will work for the case of multiple inclusions.

A complex valued equation as

$$\operatorname{div}(\gamma\nabla u) = 0$$

can be interpreted as a two by two system for real valued functions. In fact, denoting by

$$u^1 = \Re u \text{ and } u^2 = \Im u,$$

we have that the function $u = (u^1, u^2) : \Omega \to \mathbb{R}^2$ satisfies the system

$$\frac{\partial}{\partial x_k}\left(a_{ij}^{hk}\frac{\partial u^j}{\partial x_k}\right) = 0 \text{ for } i = 1, 2$$

where, for $i, j, h, k = 1, 2$,

$$a_{ij}^{hk} = \delta_{hk}\delta_{ij}\Re\gamma - \delta_{hk}(\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1})\Im\gamma.$$

If

$$\Re\gamma \geq c_0 > 0,$$

then

$$a_{ij}^{hk}\xi_h^i\xi_k^j \geq c_0|\xi|^2$$

which corresponds to strong ellipticity. For this reason we can apply to our equations the results that hold for strongly elliptic systems.

We first establish some key energy estimates.

LEMMA 3.1. *There exists a constant $C = C(K, c_o, \Omega)$ such that*

$$\|u_\epsilon - u_0\|_{H^1(\Omega)} \leq C|D_\epsilon^1|^{1/2}\|\psi\|_{H^{-1/2}(\partial\Omega)}.$$

Proof. Since u_0 and u_ϵ are solutions to (5) and (6), then $w_\epsilon = u_\epsilon - u_0$ is weak solution to

$$\begin{cases} \operatorname{div}(\gamma_\epsilon \nabla w_\epsilon) &= \operatorname{div}\big((\gamma_0-\gamma_1)\chi_{D^1_\epsilon}\nabla u_0\big) \text{ in } \Omega, \\ \gamma_\epsilon \frac{\partial w_\epsilon}{\partial \nu} &= 0 \text{ on } \partial\Omega, \\ \int_{\partial\Omega} u_\epsilon &= 0. \end{cases}$$

Hence, for every $\phi \in H^1(\Omega)$,

$$\int_\Omega \gamma_\epsilon \nabla w_\epsilon \cdot \nabla \phi = \int_\Omega (\gamma_0-\gamma_1)\chi_{D^1_\epsilon}\nabla u_0 \cdot \nabla\phi. \tag{8}$$

By choosing $\phi = \overline{w}_\epsilon \in H^1(\Omega)$, we get

$$\int_\Omega \gamma_\epsilon |\nabla w_\epsilon|^2 = \int_\Omega (\gamma_0-\gamma_1)\chi_{D^1_\epsilon}\nabla u_0 \cdot \nabla\overline{w}_\epsilon.$$

Now, by (3), we have

$$\|\nabla w_\epsilon\|_{L^2(\Omega)} \le C|D^1_\epsilon|^{1/2}\,\|\gamma_0-\gamma_1\|_{L^\infty(\Omega)} \sup_{D^1_\epsilon}|\nabla u_0|.$$

By interior regularity results (see [4, Theorem2.1, Chapter 2]),

$$\sup_{D^1_\epsilon}|\nabla u_0| \le C\|\psi\|_{H^{-1/2}(\partial\Omega)},$$

so that

$$\|\nabla w_\epsilon\|_{L^2(\Omega)} \le C|D^1_\epsilon|^{1/2}\|\psi\|_{H^{-1/2}(\partial\Omega)}$$

where $C = C(K, c_0)$.

Finally, since

$$\int_{\partial\Omega} w_\epsilon = 0,$$

by Poincaré inequality we have

$$\|w_\epsilon\|_{L^2(\Omega)} \le C\|\nabla w_\epsilon\|_{L^2(\Omega)}$$

with $C = C(\Omega)$ and we obtain

$$\|u_\epsilon - u_0\|_{H^1(\Omega)} \le |D^1_\epsilon|^{1/2}\|\psi\|_{H^{1/2}(\partial\Omega)},$$

which ends the proof. □

We will also make use of some key regularity results for elliptic systems with discontinuous coefficients due to [6] (that extend the one established in [7] for scalar elliptic equations). We state here a simplified version of Proposition 5.1 of [6].

Let c and M be two positive constants with $M > 2K$ and denote by $Q_{c,M}$ the set of points $x \in \Omega$ such that $dist(x, \partial\Omega) > M^{-1}$ and such that there is a square of size c centered at x that intersects ∂D_ϵ^1 in at most two cartesian curves whose $C^{1,\alpha}$ norms are bounded by M, i.e. there exists a coordinate system at x such that $\partial D_\epsilon^1 \cap [-c, c]^2$ consists in graphs of at most two functions $h_- < h_+$ with $\|h_\pm\|_{C^{1,\alpha}} \leq M$. Let us denote by $\Omega_M = \{x \in \Omega \,:\, dist(x, \partial\Omega) > \frac{1}{2M}\}$.

LEMMA 3.2. *Let $\beta \in (0, 1/4)$ and $M > 2K$. there exists a constant C depending on α, K, c_0 and M such that if $\epsilon \in (0, \frac{1}{3K})$, $0 < c < \frac{1}{6K}$ and $u_\epsilon \in H^1(\Omega)$ is a solution of*

$$\operatorname{div}(\gamma_\epsilon \nabla u) = 0 \ \textit{in}\ \Omega,$$

then

$$\|u_\epsilon\|_{C^{1,\beta}(Q_{c,M} \cap \overline{D}_\epsilon^1)} \leq \frac{C}{c^{1+\beta}} \|u_\epsilon\|_{L^2(\Omega)} \tag{9}$$

and

$$\|u_\epsilon\|_{C^{1,\beta}(Q_{c,M} \setminus D_\epsilon^1)} \leq \frac{C}{c^{1+\beta}} \|u_\epsilon\|_{L^2(\Omega)}. \tag{10}$$

LEMMA 3.3. *There exists $\eta > 0$ such that, if $u \in H^1(\Omega)$ is a solution to the complex valued equation*

$$\operatorname{div}(\gamma \nabla u) = f \ \textit{in}\ \Omega,$$

where $\gamma : \Omega \to \mathbb{C}$, $\gamma \in L^\infty(\Omega)$ such that

$$\Re\gamma \geq c_0 > 0$$

and $f \in H^{-1,2+\eta}(\Omega)$, then $u \in H^{1,2+\eta}_{loc}(\Omega)$ and, given B_ρ and $B_{2\rho}$ concentric disks contained in Ω,

$$\|\nabla u\|_{L^{2+\eta}(B_\rho)} \leq C \left(\|f\|_{H^{-1,2+\eta}(B_{2\rho})} + \rho^{\frac{2}{2+\eta}-1} \|\nabla u\|_{L^2(B_{2\rho})} \right).$$

For the proof see [4, Chapter 2, Section 10].

Proof of Theorem 2.1. Take $y \in \partial\Omega$. Then, by the definition of the Neumann function, it is easy to see that

$$(u_\epsilon - u_0)(y) = \int_{D_\epsilon^1} (\gamma_0 - \gamma_1) \nabla u_\epsilon \cdot \nabla N(\cdot, y).$$

We prove the theorem in the more interesting and complicated case when Σ_1 is an open curve. In fact, in this case, ∂D_ϵ^1 has derivatives (near the endpoints of Σ_1) that degenerate as ϵ tends to zero. This implies that the regularity estimates of Lemma 3.2 cannot be applied near the endpoints of Σ_1.

Let P_1 and Q_1 be the endpoints of Σ_1, let $\theta \in (0,1)$ to be chosen later, and define

$$D_\epsilon^{1,\theta} = \left\{ x + \mu n(x) \,:\, x \in \Sigma_1,\, dist(x, P_1 \cup Q_1) > \epsilon^\theta,\, \mu \in (\epsilon, \epsilon) \right\}.$$

It is easy to see that there exists a constant $M > 2K$ and depending only on K, such that

$$D_\epsilon^{1,\theta} \subset Q_{\frac{\epsilon\beta}{4},M} \cap D_\epsilon^1.$$

An application of Lemma 3.2 thus gives

$$\|\nabla u_\epsilon\|_{C^\beta(\overline{D_\epsilon^{1,\theta}})} \leq C\epsilon^{-\theta(1+\beta)} \|u_\epsilon\|_{L^2(\Omega)} \leq C\epsilon^{-\theta(1+\beta)} \|\psi\|_{H^{-1/2}(\partial\Omega)} \tag{11}$$

where C does not depend on ϵ.

Then,

$$\begin{aligned}
&\int_{D_\epsilon^1} (\gamma_0 - \gamma_1)(x) \nabla u_\epsilon(x) \cdot \nabla N(x,y)\, dx \\
&\quad = \int_{D_\epsilon^{1,\theta}} (\gamma_0 - \gamma_1)(x) \nabla u_\epsilon(x) \cdot \nabla N(x,y)\, dx \\
&\qquad + \int_{D_\epsilon^1 \setminus D_\epsilon^{1,\theta}} (\gamma_0 - \gamma_1)(x) \nabla u_\epsilon(x) \cdot \nabla N(x,y)\, dx := I_1 + I_2.
\end{aligned}$$

Let us estimate I_2 first.

$$\begin{aligned}
|I_2| \quad \leq \quad & \left| \int_{D_\epsilon^1 \setminus D_\epsilon^{1,\theta}} (\gamma_0 - \gamma_1) \nabla (u_\epsilon - u_0) \cdot \nabla N(\cdot, y) \right| \\
& + \left| \int_{D_\epsilon^1 \setminus D_\epsilon^{1,\theta}} (\gamma_0 - \gamma_1) \nabla u_0 \cdot \nabla N(\cdot, y) \right|.
\end{aligned}$$

Observe that, since $\gamma_0 \in C^{1,\alpha}(\Omega)$, by interior regularity results ([4, Theorem 2.1, Chapter 2], by [5] and by (2) we get

$$\|\nabla u_0\|_{L^\infty(D_\epsilon^1)} \leq C \|\psi\|_{H^{-1/2}(\partial\Omega)}, \tag{12}$$

$$|\nabla N(x,y)| \leq \frac{C}{|x-y|},$$

and, since $y \in \partial\Omega$

$$\|\nabla N(\cdot, y)\|_{L^\infty(D_\epsilon^1)} \leq C. \tag{13}$$

Hence, by (12), (13) and Lemma 3.1,

$$|I_2| \leq C \left| D_\epsilon^1 \setminus D_\epsilon^{1,\theta} \right| \|\psi\|_{H^{-1/2}(\partial\Omega)} \leq C\epsilon^{1+\theta} \|\psi\|_{H^{-1/2}(\partial\Omega)}.$$

We now define

$$\Sigma_\eta^\theta = \left\{ x + \eta n_1(x) \,:\, x \in \Sigma_1,\, dist(x, P_1 \cup Q_1) > \epsilon^\theta \right\}.$$

Due to the regularity of Σ_1, if we denote by $d\sigma_x^\eta$ the arclength measure on Σ_η^θ and by $d\sigma_x$ the arclength measure on Σ_1, we have

$$d\sigma_x^\eta = (1 + O(\eta)) d\sigma_x.$$

For every point $x + \eta n(x) \in D_\epsilon^{1,\theta}$, let $x_\epsilon = x + \epsilon n(x)$. By (11),

$$\begin{aligned} |\nabla u_\epsilon(x + \eta n(x)) - \nabla u_\epsilon(x_\epsilon)| &\leq C|x + \eta n(x) - x_\epsilon|^\beta \epsilon^{-\theta(1+\beta)} \|\psi\|_{H^{-1/2}(\partial\Omega)} \\ &\leq C\epsilon^{\beta - \theta(1+\beta)} \|\psi\|_{H^{-1/2}(\partial\Omega)} \end{aligned}$$

and, also,

$$|(\gamma_1 - \gamma_0)(x + \eta n(x)) - (\gamma_1 - \gamma_0)(x_\epsilon)| \leq C\epsilon^\alpha$$

so that

$$\int_{D_\epsilon^{1,\theta}} (\gamma_0 - \gamma_1) \nabla u_\epsilon \nabla_x N = 2\epsilon \int_{\Sigma_\epsilon^\theta} (\gamma_0 - \gamma_1) \nabla u_\epsilon^i \nabla_x N + o(\epsilon),$$

where we set

$$u_\epsilon^i = u_\epsilon|_{D_\epsilon^1}, \quad u_\epsilon^e = u_\epsilon|_{\Omega \setminus \overline{D}_\epsilon^1}.$$

We now use the transmission conditions

$$u_\epsilon^i = u_\epsilon^e,$$
$$\gamma_1 \frac{\partial u_\epsilon^i}{\partial n} = \gamma_0 \frac{\partial u_\epsilon^e}{\partial n},$$

that are satisfied on $\partial D_\epsilon^{1,\theta}$ pointwise, in order to obtain, finally,

$$\begin{aligned} &\int_{D_\epsilon^{1,\theta}} (\gamma_0 - \gamma_1) \nabla u_\epsilon^i \nabla N \\ &\qquad = 2\epsilon \int_{\Sigma_\epsilon^\theta} (\gamma_0 - \gamma_1) \left\{ \frac{\gamma_0}{\gamma_1} \frac{\partial u_\epsilon^e}{\partial n} \frac{\partial N}{\partial n_x} + \frac{\partial u_\epsilon^e}{\partial \tau} \frac{\partial N}{\partial \tau_x} \right\} + o(\epsilon). \end{aligned}$$

Assume that

$$\|\nabla u_\epsilon^e - \nabla u_0^e\|_{L^\infty(\Sigma_\epsilon^\theta)} \leq C\epsilon^{\theta_1} \|\psi\|_{H^{-1/2}(\partial\Omega)} \tag{14}$$

for some $\theta_1 > 0$. Then

$$\begin{aligned} &\int_{D_\epsilon^{1,\theta}} (\gamma_0 - \gamma_1)(x) \nabla u_\epsilon^i(x) \nabla N(x, y) dx \\ &\qquad = 2\epsilon \int_{\Sigma_\epsilon^\theta} (\gamma_0 - \gamma_1) \left\{ \frac{\gamma_0}{\gamma_1} \frac{\partial u_0^e}{\partial n_1} \frac{\partial N}{\partial n_1} + \frac{\partial u_0^e}{\partial \tau_1} \frac{\partial N}{\partial \tau_1} \right\} d\sigma_x^\epsilon + o(\epsilon) \\ &\qquad = 2\epsilon \int_{\Sigma_1} (\gamma_0 - \gamma_1) \mathbb{M}^1(x) \nabla u_0(x) \nabla_x N(x, y) d\sigma_x + o(\epsilon), \end{aligned}$$

which concludes the proof.

So, we are left with the proof of (14). Let $2\epsilon < d < \frac{1}{2K}$ and let

$$\Omega_d^\epsilon = \left\{x \in \Omega \,:\, dist(x, \partial(\Omega \setminus D_\epsilon^1)) > d\right\}.$$

Since $u_\epsilon - u_0$ is solution of

$$\mathrm{div}(\gamma_0 \nabla(u_\epsilon - u_0)) = 0 \text{ in } \Omega \setminus D_\epsilon^1,$$

the regularity assumption on γ_0 implies that $u_\varepsilon - u_0 \in H^2_{loc}(\Omega \setminus D_\epsilon^1)$ (see [4, Theorem 2.1, Chapter 2]).

Consider $\Phi_\epsilon^k = \frac{\partial}{\partial x_k}(u_\epsilon - u_0)$ for $k = 1, 2$.

The function Φ_ϵ^k satisfies in $\Omega \setminus D_\epsilon^1$

$$\mathrm{div}\left(\gamma_0 \nabla \Phi_\epsilon^k\right) = -\mathrm{div}\left(\frac{\partial \gamma_0}{\partial x_k} \nabla(u_\epsilon - u_0)\right) =: F.$$

By Caccioppoli inequality and by Lemma 8 we have that

$$\begin{aligned}
\|\nabla \Phi_\epsilon^k\|^2_{L^2(\Omega^\epsilon_{d/2})} &\leq \frac{C}{d^2}\|\Phi_\epsilon^k\|^2_{L^2(\Omega^\epsilon_{d/4})} + \|F\|^2_{H^{-1}(\Omega^\epsilon_{d/4})} \\
&\leq C\left(\frac{1}{d^2}\|\Phi_\epsilon^k\|^2_{L^2(\Omega^\epsilon_{d/4})} + \|\nabla(u_\epsilon - u_0)\|^2_{L^2(\Omega)}\right) \\
&\leq C\left(\frac{1}{d^2} + 1\right)\|\nabla(u_\epsilon - u_0)\|^2_{L^2(\Omega)} \\
&\leq C\left(\frac{1}{d^2} + 1\right)\epsilon\|\psi\|^2_{H^{-1/2}(\partial\Omega)}.
\end{aligned}$$

Hence

$$\|\Phi_\epsilon^k\|_{H^1(\Omega^\epsilon_{d/2})} \leq C\sqrt{\epsilon} d^{-1}\|\psi\|_{H^{-1/2}(\partial\Omega)}.$$

Applying Lemma 3.3 to Φ_ϵ^k gives

$$\|\nabla \Phi_\epsilon^k\|_{L^{2+\eta}(\Omega_d^\epsilon)} \leq C\left(\|F\|_{H^{-1,2+\eta}(\Omega^\epsilon_{d/2})} + d^{\frac{2}{2+\eta}-1}\|\nabla \Phi_\epsilon^k\|^2_{L^2(\Omega^\epsilon_{d/2})}\right).$$

Now, by

$$\|F\|_{H^{-1,2+\eta}(\Omega^\epsilon_{d/2})} \leq C\|\nabla(u_\epsilon - u_0)\|_{L^{2+\eta}(\Omega^\epsilon_{d/2})}$$

and by the interior regularity estimates and Sobolev Immersion Theorem

$$\begin{aligned}
\|\nabla(u_\epsilon - u_0)\|_{L^{2+\eta}(\Omega^\epsilon_{d/2})} &\leq C\|u_\epsilon - u_0\|_{H^2(\Omega^\epsilon_{d/2})} \\
&\leq C\|u_\epsilon - u_0\|_{H^1(\Omega^\epsilon_{d/4})} \leq \|u_\epsilon - u_0\|_{H^1(\Omega)}
\end{aligned}$$

and hence

$$\begin{aligned}\|\nabla\Phi_\epsilon^k\|_{L^{2+\eta}(\Omega_d^\epsilon)} &\leq C\left(\|u_\epsilon-u_0\|_{H^1(\Omega)}+d^{\frac{2}{2+\eta}-1}\|\nabla\Phi_\epsilon^k\|^2_{L^2(\Omega^\epsilon_{d/2})}\right)\\ &\leq C\left(1+d^{\frac{2}{2+\eta}-2}\right)\sqrt{\epsilon}\,\|\psi\|_{H^{-1/2}(\partial\Omega)}.\end{aligned}$$

Finally, since $\frac{2}{2+\eta}-2<0$ and $d<1$, from last inequality, we derive

$$\|\nabla\Phi_\epsilon^k\|_{L^{2+\eta}(\Omega_d^\epsilon)}\leq Cd^{\frac{2}{2+\eta}-2}\sqrt{\epsilon}\,\|\psi\|_{H^{-1/2}(\partial\Omega)}.$$

On the other hand, applying Lemma 3.3 to $u_\epsilon-u_0$ we have

$$\|\Phi_\epsilon^k\|_{L^{2+\eta}(\Omega_d^\epsilon)}\leq Cd^{\frac{2}{2+\eta}-1}\sqrt{\epsilon}\|\psi\|_{H^{-1/2}(\partial\Omega)}.$$

By Sobolev Imbedding Theorem we than have

$$\left\|\frac{\partial}{\partial x_k}(u_\epsilon-u_0)\right\|_{L^\infty(\Omega_d^\epsilon)}\leq Cd^{\frac{2}{2+\eta}-2}\sqrt{\epsilon}\,\|\psi\|_{H^{-1/2}(\partial\Omega)}. \tag{15}$$

Now let $y\in\Sigma'_\epsilon$ and y_d be the closest point to y in Ω_d^ϵ. By (11) we have

$$|\nabla u_\epsilon(y)-\nabla u_\epsilon(y_d)|\leq C\frac{d^\beta}{\epsilon^{\theta(\beta+1)}}\|\psi\|_{H^{-1/2}(\partial\Omega)}. \tag{16}$$

Hence, by (15) and (16) we have

$$|\nabla(u_\epsilon^e-u_0)(y)|\leq C\left(d^\beta\epsilon^{-\theta(\beta+1)}+d^{-2+\frac{2}{2+\eta}}\sqrt{\epsilon}\right)\|\psi\|_{H^{-1/2}(\partial\Omega)}.$$

Choosing $\theta<\frac{\beta}{2(2-\frac{2}{2+\eta})(\beta+1)}$ we get

$$|\nabla(u_\epsilon^e-u_0)(y)|\leq C\epsilon^{\theta_1}$$

with $\theta_1>0$. □

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Authors' addresses:

Elena Beretta
Dipartimento di Matematica
Politecnico di Milano
via Bonardi, 9
20133 Milano, Italia
E-mail: elena.beretta@polimi.it

Elisa Francini
Dipartimento di Matematica e Informatica "U. Dini"
Università di Firenze
Viale Morgagni, 67A
50134 Firenze, Italia
E-mail: elisa.francini@unifi.it

Received April 4, 2016
Revised May 1, 2016
Accepted May 3, 2016