

The Concept of Separable Connectedness: Applications to General Utility Theory

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SUMMARY. - *We say that a topological set X is separably connected if for any two points $x, y \in X$ there exists a connected and separable subset $C(x, y) \subseteq X$ to which both x and y belong. This concept generalizes path-connectedness. With this concept we have improved some results on general utility theory: For instance, in 1987 Monteiro gave conditions (dealing with real-valued, continuous, order-preserving functions) on path-connected spaces in order to get continuous utility representations of continuous total preorders defined on the set. We have recently proved (in an article by Candeal, Hervés and Induráin, published in the *Journal of Mathematical Economics*, 1998) that Monteiro's results also work for the more general case of separably connected spaces. Then we study the particular situation of separable connectedness on spaces endowed with some extra structure, e.g. metric spaces.*

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1. Introduction

Looking for general results concerning the representability of continuous total orders by means of continuous order-preserving real-valued functions, a suitable starting point is a key result by Monteiro [1987], which proves that any countably bounded continuous total preorder defined on a path-connected space admits a continuous order-preserving representation. The technique used to prove that result consists in fixing a countable order dense subset on which we can easily find an order-preserving real-valued representation (for such subset), and then try to extend with continuity such order-preserving function to the whole set. The method used by Monteiro to find that crucial order-dense subset consists in matching countable boundedness and path-connectedness.

Going further, in Candeal et al. [1998] it was observed that Monteiro's techniques could be improved through the new concept of *separable connectedness*, more general than path-connectedness. Examples on which that improvement appeared clearly (i.e. separably connected but not path-connected totally preordered spaces) were then considered, paying an special attention to the particular case of metric spaces.

In the present work we intend to present as a survey such development, so introducing this key new concept of *separable connectedness* and analyzing some of its possibilities, at least in the case of looking for continuous utility functions on totally preordered topological spaces.

2. Definitions and previous results

Let X be a nonempty set. Let \succsim be a complete preorder (i.e.: a reflexive, transitive and complete binary relation) on X . (If \succsim is also antisymmetric, it is said to be a *total order*). We denote $x \prec y$ instead of $\neg(y \succsim x)$. Also $x \sim y$ will stand for $x \succsim y, y \succsim x$ ($x, y \in X$).

The preorder \succsim is said to be *representable* by a *utility function* if there exists a real-valued order-preserving function (*utility function*) $u : X \rightarrow \mathbb{R}$. Thus $x \succsim y \iff u(x) \leq u(y)$ ($x, y \in X$).

If X is endowed with a topology τ , the preorder \preceq is said to be *continuously representable* if there exists a utility function u that is continuous with respect to the topology τ on X and the usual topology on the real line \mathbb{R} . The preorder \preceq is said to be τ -continuous if the sets $U(x) = \{y \in X, x \preceq y\}$ and $L(x) = \{y \in X, y \preceq x\}$ are τ -closed, for every $x \in X$. It is said to be *countably bounded* if there exists a countable subset $D \subseteq X$ such that for every $x \in X$ there exist $a, b \in D$ with $a \preceq x \preceq b$. (D is said to *bound* \preceq).

On a totally preordered space (X, \preceq) one can always consider a natural topology, called *order topology*, whose subbasis is given by the subsets $\{z \in X, x \prec z\}$ and $\{z \in X, z \prec x\}$ ($x \in X$).

A topological space (X, τ) is said to be *second countable* if its topology admits a countable basis of τ -open sets, and *separable* if there exists a countable subset $D \subseteq X$ that meets every nonempty τ -open subset of X . A topological space (X, τ) is said to be *connected* if there is no partition of X into two disjoint nonempty τ -open subsets. Also, it is said to be *path-connected* if for every $a, b \in X$ there exists a continuous map (a *path*) $f_{a,b} : [0, 1] \rightarrow X$ such that $f(0) = a, f(1) = b$. Note that every path-connected space is connected and every convex set in a linear topological space is path-connected. In the particular case of a metric space X on which we shall consider the metric topology δ , second countability and separability are equivalent conditions (see Theorem 5.6 on Dugundji [1966], p. 187).

Finally, (X, τ) it is said to be *separably connected* (see Candeal et al. [1998]) if for every $a, b \in X$ there exists a connected and separable subset $C(a, b) \subseteq X$ such that $a, b \in C(a, b)$.

REMARK 2.1.

- (i) Obviously a connected and separable topological space is separably connected.
- (ii) The range of any path (continuous) is connected and separable. Therefore any path-connected space is separably connected.
- (iii) A separably connected set X is, in particular, connected because once we have fixed an element $a \in X$ we have that $X = \bigcup_{x \in X} C(a, x)$.

- (iv) The lexicographic square $[0, 1] \times [0, 1]$ endowed with the order topology that comes from the lexicographic ordering on the plane \mathbb{R}^2 is an example of a connected space that fails to be separably connected.
- (v) Not every separably connected space is path-connected. As an example, let $X = \{0\} \times [-1, 1] \cup \{(x, \sin(\frac{1}{x})) : x \in (0, 1]\}$, endowed with the Euclidean topology as a set of the plane \mathbb{R}^2 . It is well known that this set X is connected but not path-connected. And it is obviously separable (because \mathbb{R}^2 is metric and separable, the separability being hereditary on metric spaces).
- (vi) The Cartesian product of two separably connected spaces is also separably connected.
- (vii) The above fact (Remark 2.1 vi) can be used to provide a new example of separably connected but not path-connected space, in this case with the extra property of being non-separable: Consider, for instance, $\ell_\infty = \{(x_n)_{n \in \mathbb{N}} : \sup_{n \in \mathbb{N}} |x_n| < +\infty\}$ with its usual norm $\|\cdot\|_\infty$ and the function $\phi : \ell_\infty \rightarrow \mathbb{R}$ that maps $x = (x_n)_{n \in \mathbb{N}} \in \ell_\infty$ to $\Phi(x) = \sum_{n=1}^{\infty} (\frac{x_n}{2^{n-1}})$. Obviously Φ is linear. It is also continuous because $|\phi(x)| \leq \|x\|_\infty$ ($x \in \ell_\infty$). Denote by $\bar{0}$ the null element in ℓ_∞ . Let $\ell_\infty^+ = \{(x_n)_{n \in \mathbb{N}} \in \ell_\infty : x_n \geq 0 \ (n \in \mathbb{N})\}$. Finally, take

$$X = \{(x, t) \in \ell_\infty^+ \times \mathbb{R} : x \neq \bar{0}, t = \sin(\frac{1}{\phi(x)})\} \cup \{(\bar{0}, t) : t \in [-1, 1]\},$$

endowed with the product topology. (For further details see Candeal et al. [1998]).

- (viii) A sharp result (see Remark 3.5 (iv) in Candeal et al. [1998]) shows that if (X, \preceq) is a totally ordered set endowed with the order topology θ , then (X, θ) is separably connected if and only if it is path-connected.
- (ix) Balbás et al. [1998] posed the following problem: Either construct a metric space that be connected but not separably connected, or else prove that on metric spaces the connectedness

condition implies the separable connectedness condition. Actually, Balbás et al. conjectured that every connected metric space is separably connected. If such conjecture were true, then, as we shall see later, connected metric spaces would have quite good properties concerning the representability of continuous total preorders. However quite recently Aron and Maestre [1999] and, independently, Simon [1999] have shown that the conjecture is false.

(x) Some other neighbour topological conditions on a set X , as for instance

(i) compact connectedness (i.e.: for any two points $x, y \in X$ there exists a compact and connected subset $D_{\{x,y\}} \subseteq X$ to which they belong,

(ii) locally separable connectedness (i.e. : each point $x \in X$ has a base of separably connected neighbourhoods),

were introduced in Verdejo [1997]. (See also Balbás et al. [1998]).

3. Applications to general utility theory

Let X be a nonempty set. Following Herden [1991], a topology τ on X is said to be *useful* if every continuous total preorder \succsim on X can be represented by a τ -continuous utility function. It is said to be *strongly useful* (Candea et al. [1998]) if for every subset $Y \subseteq X$ and every complete preorder continuous on Y with respect to the topology that τ induces (on Y), there exists a continuous utility function from Y to \mathbb{R} representing such preorder.

This concept is more restrictive than the concept of just *useful topology*: consider an uncountable set X , fix an element $a \in X$ and define the topology τ by declaring that a subset $A \subseteq X$ is τ -open if either it is empty or else a belongs to A . It is not difficult to see that this topology is useful but not strongly useful. (See Candea et al. [1998]). In the particular case of metric spaces endowed with the metric topology δ , however, the following equivalences hold true:

THEOREM 3.1. *Let (X, δ) be a metric space. Then the following are equivalent:*

- i) The metric topology δ is strongly useful,*
- ii) the metric topology δ is useful,*
- iii) the metric topology δ is separable,*
- iv) the metric topology δ is second countable.*

A key step in the proof of this fact is the implication *ii) \implies iii)* that comes from an important result due to Estévez and Hervés [1995] that states that on any non-separable metric space (X, δ) it is possible to construct a δ -continuous total preorder that has no utility representation. For other implications, it is well-known in utility theory that second countable topologies are all useful. (see Eilenberg [1941] or Debreu [1964]). Moreover if X is second countable, then the restriction of δ to any subset $Y \subset X$ is also second countable. (See Dugundji [1966], theorem 7.2 on p. 176 and theorem 5.6 on p. 187).

Yi [1993] introduced the *extension property* for a topological space (X, τ) , by declaring that (X, τ) has the extension property if every complete continuous preorder defined on a closed subset $Y \subseteq X$ has a continuous extension to the whole space X .

Matching several results obtained by Monteiro [1987] and Yi [1993] the following extension of Theorem 3.1 has been obtained, for the particular case of *path-connected* metric spaces, in Candeal et al. [1998]:

THEOREM 3.2. *Let (X, δ) be a path-connected metric space. Then the following are equivalent:*

- i) The metric topology δ is strongly useful,*
- ii) the metric topology δ is useful,*
- iii) the metric topology δ is separable,*
- iv) the metric topology δ is second countable,*
- v) the metric topology δ has the extension property,*

vi) with respect to the metric topology δ , every δ -continuous complete preorder on X is countably bounded.

One of the keys to prove this Theorem 3.2 is the following result, first stated in Monteiro [1987]: *Every continuous complete preorder defined on a path-connected space is representable by a continuous utility function if and only if it is countably bounded.*

A second key is the following result, stated by Yi [1993]:

Let (X, τ) be a path-connected topological space. Suppose that X contains a closed subset S with uncountably many points such that the restriction of τ to S is the discrete topology on X . Then the topology τ on X fails to have the extension property.

Both key results used path-connectedness as a sufficient condition to get suitable *connected and separable* subsets of X that allow to conclude the proof of, respectively, each statement: For example, a preparatory result in Monteiro [1987] states that if \preceq is a complete and *countably bounded* preorder defined on a path-connected topological space (X, τ) then there exists a *countably bounded and separable* subset $F \subseteq X$ that bounds \preceq .

They were indeed these last two key results the ones which gave us (see Candeal et al. [1998]) the opportunity of introducing the new concept of *separably connectedness*. We noticed that it was possible to replace path-connectedness by the more general condition of separably connectedness both in Monteiro's and Yi's key results. Thus, we got:

THEOREM 3.3.

- i) *Let (X, τ) be a separably connected topological space. Suppose that X contains a closed subset S with uncountably many points such that the restriction of τ to S is the discrete topology on X . Then the topology τ on X fails to have the extension property.*
- ii) *Let (X, δ) be a separably connected metric space. Then δ has the extension property if and only if δ is separable.*
- iii) *Let (X, τ) be a separably connected topological space and let \preceq be a complete and countably bounded preorder defined on X . Then there exists a countably bounded and separable subset $F \subseteq X$ that bounds \preceq .*

- iv) Let (X, δ) be a separably connected metric space. Then δ is useful if and only if every δ -continuous complete preorder defined on X is countably bounded.*

As a strong final result, the article Candeal et al. [1998] concludes proving that we can substitute path-connected by separably connected in the statement of Theorem 3.2, so we have:

THEOREM 3.4. *Let (X, δ) be a separably connected metric space. Then the following are equivalent:*

- i) The metric topology δ is strongly useful,*
- ii) the metric topology δ is useful,*
- iii) the metric topology δ is separable,*
- iv) the metric topology δ is second countable,*
- v) the metric topology δ has the extension property,*
- vi) with respect to the metric topology δ , every δ -continuous complete preorder on X is countably bounded.*

REMARK 3.5.

- (i) Notice that the equivalences given on Theorem 3.4 are no longer valid for general metric spaces. For instance, let X be an uncountable set, and let $S \subset X$ be a subset such that both S and $X \setminus S$ are uncountable. Consider on X the trivial metric d , such that $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$ ($x, y \in X$). The metric topology δ is the discrete topology, so any complete preorder defined on S , or X , is plainly continuous. To extend a preorder from S to X just declare that any element in S is less preferred than any other element not belonging to S , and take a well ordering on $X \setminus S$ via the set-theoretical axiom of well-ordering. With such approach we conclude that δ has the extension property. Moreover, it is obvious that δ is not separable.*

- (ii) *Once again, let us insist in this fact: The key in Monteiro's [1987] result is to obtain (from the property of countable boundedness) a connected separable subset which bounds the complete preorder, and then to use Eilenberg's [1941] result (continuous complete preorders on a connected and separable topological space always admit a continuous representation) to get the existence of a utility representation. The proof in Monteiro [1987] needs the path-connectedness of X . However, let us assume, for instance, that X is a Cartesian product $X = X_1 \times X_2$ on which we define the complete preorder given by the lexicographic product of a complete preorder \preceq_1 on X_1 and a complete preorder \preceq_2 on X_2 . In a situation in which X_1 is connected and separable, but non path-connected, and X_2 is a nonseparable convex space, it follows that X is neither path-connected nor separable, so that Monteiro's result cannot be applied here. However, for any two points of X there exists a connected and separable subset of X to which those two points belong. (In other words: X is separably connected). So, it is then still possible to use Theorem 3.4 here.*
- (iii) *If the conjecture that appears in Balbás et al. [1998] (that we mentioned in Remark 2.1 (ix)) were true, then Theorem 3.4 would keep its validity for just connected metric spaces instead of separably connected metric spaces. But as we have said, the conjecture has been proved to be false. (See Aron and Maestre [1999] and Simon [1999]).*
- (iv) *In Aron and Maestre [1999] the authors claim that their example is generic, in the sense that it can be proved that every non-separable Banach space contains a connected but not separably connected subset as regards the norm topology.*
- (v) *Further studies on the concept of separable connectedness have been recently made by Verdejo [1997] and Estévez et al. [1999]. Of particular interest is the study of a correspondence R from a set A to a set X , defined and understood as a subset of the Cartesian product of A and X . If a pair (a, x) belongs to R it is usual to denote it by aRx ($a \in A$, $x \in X$). The correspondence R is said to admit a numerical representation if there ex-*

ist two real-valued functions $u : A \rightarrow \mathbb{R}$ and $v : X \rightarrow \mathbb{R}$ such that $aRx \iff u(a) < v(x)$ ($a \in A$, $x \in X$). Estévez et al. [1993, 1995] considered this type of correspondences, obtaining, in the setting of path-connected topological spaces, some necessary and sufficient conditions for the existence of continuous numerical representations. The studies made in Verdejo [1997] and Estévez et al. [1999] have also generalized those results (first obtained under path-connectedness) to the more general setting of separably connected topological spaces.

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