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On Approximation of Linear Second Order Elliptic Partial Differential Equations with Analytic Coefficients

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Summary. - *The linear second-order elliptic differential equation with real-valued coefficients that are entire functions on* $\Im^2$ *and whose coefficient* $c(x,y) \leq 0$ *on the disk* $D : x^2 + y^2 \leq 1$ *is given by*

$$\Delta^2 v + a(x,y)v_x + b(x,y)v_y + c(x,y)v = 0, \quad (x,y)\epsilon E^2.$$

The ideas of Bernstein and Saff have been applied by McCoy [9, 10] to study the singularities of certain second-order elliptic equations with singular coefficients. These results contains calculations of order and type of entire function potentials in terms of best polynomial approximation errors. Here some inequalities concerning order and type for the given equation have been obtained.

1. Introduction

The linear second order elliptic partial differential equation be given in normal form

$$L(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + a(x,y)\frac{\partial v}{\partial x} + b(x,y)\frac{\partial v}{\partial y} + c(x,y)v = 0 \qquad (1)$$

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with real analytic coefficients that are entire functions on $\Im^2$ and whose coefficient $c(x, y) \leq 0$ on the disk $D : x^2 + y^2 \leq 1$ are considered here. There are so many applications of the singularities of solutions to linear elliptic partial differential equations in several areas of mathematical physics [5, 7], for example, in potential scattering. Using function theoretic methods, R.P. Gilbert and D.L. Colton [8] determined necessary and sufficient conditions concerning the location of singularities of regular solution v in terms of corresponding information for a unique associated analytic function f on one complex-variable.

Using the standard procedure [1, 7, 8] the functions $a(x, y), b(x, y)$ and $c(x, y)$ analytically continue as $a(z, z^*), b(z, z^*)$ and $c(z, z^*)$ by change to the hyper - complex coordinates $z = x + iy, z^* = x - iy$ for $(x, y)\epsilon\Im^2$, reducing equation (1) to a complex valued hyperbolic equation

$$\begin{aligned} L(U) &= \frac{\partial^2 U}{\partial z \partial z^*} + A(z, z^*)\frac{\partial U}{\partial z^*} + B(z, z^*)\frac{\partial U}{\partial z} + C(z, z^*)U \\ &= 0, \qquad (2) \\ U(z, z^*) &= v\left[(z + z^*)/2, (z - z^*)/2i\right] \\ A(z, z^*) &= \left[a(z, z^*) + ib(z, z^*)\right]/4 \\ B(z, z^*) &= \left[a(z, z^*) - ib(z, z^*)\right]/4 \\ C(z, z^*) &= c(z, z^*)/4 \end{aligned}$$

and

$$V(z, z^*) = U(z, z^*) \exp\left\{\int_0^{z^*} A(z, \zeta)d\zeta - h(z)\right\}$$

for an arbitrary entire function h gives the Bergman canonical form of equation (2) [1, 8],

$$\begin{aligned} \beta(V) &= \frac{\partial^* V}{\partial v \partial z^*} + D(z, z^*)\frac{\partial V}{\partial z^*} + F(z, z^*)V = 0, \qquad (3) \\ F &= A_z + AB - C \\ D &= h' - \int_0^{z^*} A_\zeta d\zeta + B. \qquad (4) \end{aligned}$$

In view of [8] we have that regular V has a local representation $V = w[f]$ about the origin that is defined from a unique $w-$associate analytic function $f = f(z)$ by the integral operator $w[f]$,

$$\begin{aligned} v(z, z^*) &= w[f(\sigma)] = \int_L E(z, z^*, t) f(\sigma) d\mu(t), \\ \sigma &= z(1-t^2)/2, d\mu(t) = dt/(1-t^2)^{1/2} \end{aligned}$$

where L is the contour $t = e^{i\theta}$ from -1 to $+1$. The Bergman $E-$function follows

$$\begin{aligned} E(z, z^*, t) &= 1 + \sum_{n=1}^{\infty} t^{2n} z^n \int_0^{z^*} P^{(2n)}(z, \zeta) d\zeta, P^{(2)} \\ &= -2F, \qquad (5) \\ (2n+1)P^{(2n+2)} &= -2 \left[P_z^{(2n)} + DP^{(2n)} + F \int_0^{z^*} P^{(2n)} d\zeta \right], \quad (6) \end{aligned}$$

$n = 1, 2, \cdots$. The principal branch of the function element $V(z, z^*)$ continues analytically from its initial domain of definition by contour deformation to a (larger) domain of associated as given in the "Envelope Method" [5, 6]. Using this method, Gilbert and Colton [8, Theorem 1] show that the (principal branch) of $V(z, \overline{z})$ is singular at $z = \alpha$ if and only if, the w-associate f is singular at $z = \alpha/2$.

Now first we define real valued rational functions of type (n, v) as

$$r_{n,v}(z) p_n(z)/q_v(z), n, v = 0, 1, \cdots,$$

the ratio of real-valued relatively prime polynomials of degree n and v. It is clear that functions $r_{n,0}$ are simply the polynomials $p_n(z)$. Corresponding to $w-$associates the multi-valued function elements are

$$\varphi_{n,v}(z, z^*) = w[p_n(\sigma)/q_v(\sigma)],$$

and

$$\phi_n(z, z^*) = w[p_n(\sigma)]$$

whose principal branches are selected to approximate $V(z, z^*)$.

Further we define (mini-max) best approximation error using Chebyshev norms

$$\begin{aligned} e_{n,v}(f) &= \inf\{||f - r_{n,v}|| : r_{n,v} \epsilon R_{n,v}\}, \\ ||f - r_{n,v}|| &= \sup\{|f(z) - r_{n,v}(z)| : z \epsilon D\}, \\ E_{n,v}(V) &= \inf\{||V - \varphi_{n,v}|| : \varphi_{n,v} \epsilon R_{n,v}\}, \\ ||V - \varphi_{n,v}|| &= \sup\left\{|V(z,z^*) - \varphi_{n,v}(z,z^*)| : (z,z^*) \epsilon D^2\right\}, \end{aligned}$$

$n, v = 0, 1, \cdots, D^2 = D \times D$ and the errors in the best $''$polynomial$''$ approximates

$$e_n(f) = e_{n,0}(f), E_n(V) = E_{n,0}(V), n = 0, 1, \cdots$$

where

$$D_\delta \{z \epsilon C : |z| \leq \delta\} \quad with \quad D_1 \equiv D$$

and

$$\begin{aligned} R_{n,v} &= \{\varphi_{n,v} : \varphi_{n,v} = w[r_{n,v}], r_{n,v} \epsilon R_{n,v}\} \\ P_n &= \{\phi_n : \phi_n \epsilon R_{n,0}\}. \end{aligned}$$

The singularities of $V(z,\overline{z}) : \beta(V) = 0$. The study of the singularities of V and U reveals equivalence because $V(z,z^*)$ is singular at $(z_0, z_0^*) \epsilon \Im^2$ if and only if, $U(z,z^*)$ is singular at (z_0, z_0^*). Furthermore, $z^* = \overline{z}$ if and only if. $(x,y) \epsilon E^2$ so the singularities of v may be studied by noting those of $V(z,\overline{z})$. So we shall recognize those entire function element V whose analytic continuations from their initial domains of definition have no singularities located at finite distances from the origin. It follow via a function-theoretic extension of the Bernstein theorem so that we select a polydisk as the initial domain of definition.

The basis for this analysis is the Bergman and Gilbert Integral Operator Method [1, 3, 5, 6] which extends the classical theorems of S.N. Bernstein [2, 13] and E.B. Saff [12] from analytic function theory. Those classical results analyze the polar singularities of analytic f via approximation methods in the same way that the Hadamard and Mandelbrojt theorems [4] analyze the polar singularities of f via its Taylor's coefficients.

The Hadamard and Mandelborjt coefficient theorems have been extended to solutions of various classes the partial differential equations [3, 5, 6] via the Integral Operator Method. The ideas of Bernstein and Saff have been applied [9-11] along with these methods to study the singularities of certain second order elliptic partial differential equations with singular coefficients. Those results also contain calculations of order and type of entire function potentials in terms of best polynomial approximation errors. Here in this paper we shall obtain some inequalities concerning order and type of entire function element $V(z,\overline{z})$ in terms of approximation errors defined above. Our results and approach are different from those of P.A. McCoy [9-11].

Now we define the growth parameters such as order, lower order, type and lower type of entire function element $V(z,\overline{z})$ as

$$\begin{matrix}\rho(V)\\ \lambda(V)\end{matrix} = \lim_{\delta\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\delta,V)}{\log\delta} \tag{7}$$

$$\begin{matrix}T(V)\\ t(V)\end{matrix} = \lim_{\delta\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\delta,V)}{\delta^{\rho(V)}}. \tag{8}$$

Similarly the order ρ, lower order λ, type T and lower type t of the associated entire function f are defined as

$$\begin{matrix}\rho(f)\\ \lambda(f)\end{matrix} = \lim_{\delta\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m(\delta,f)}{\log\delta} \tag{9}$$

$$\begin{matrix}T(f)\\ t(f)\end{matrix} = \lim_{\delta\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m(\delta,f)}{\delta^{\rho(f)}} \tag{10}$$

where

$$\begin{aligned} M(\delta,V) &= \sup\left\{|V(z,\overline{z})| : (z,\overline{z})\epsilon D^2\right\} \\ m(\delta,f) &= \sup\left\{|f(z)| : z\epsilon D_\delta\right\}. \end{aligned}$$

2. Auxiliary Results

LEMMA 2.1. *Let $V(z,z^*)$ be a regular solution of $\beta(V)=0$ on the polydisk D^2. If the function element $V(z,\overline{z})$ has an analytic continuation as an entire function solution then there exists a sequence of mini-max polynomials $\{p_n^*\}$ such that*

$$||f(z)-p_n^*(z)|| \le K(\beta)m(\beta,f)(5/4\beta)^n, \beta > 5/4, n \ge 0. \tag{11}$$

Proof. For $V(z,z^*)$ a regular solution of $\beta(V)=0$ on the polydisk D^2, we assume that the function element $V(z,\overline{z})$ has analytic continuation as an entire function.

Let p_n^* be the mini-max polynomial for $e_n(f)$ and $\varphi_n^* = w[p_n^*]$. The entire function $f-p_n^*$ expands on $[-1+1]$ in a series of Chebyshev polynomials

$$T_n(z) = \frac{1}{2}\sum_{k=0}^{[n/2]} \frac{n}{n-k}\left(\begin{array}{c} n-k \\ k \end{array}\right)(2z)^{n-2k}, n = 0,1,2,\cdots$$

and continue analytically as

$$f(z) - p_n^*(z) = 2\sum_{k=n+1}^{\infty} \alpha_k T_k(z), \alpha_k = \alpha_k(f)$$

to the ellips

$$\overline{E}_\beta \equiv \{z\epsilon C : |z-1| + |z+1| < 2\beta\}, \beta > 4.$$

The Chebyshev coefficients $\alpha_k = \alpha_k(f)$ defined as contour integrals of f over the boundary $\partial\overline{E}_\beta$ are defined as

$$|\alpha_k| \leq \overline{M}(\beta,f)\beta^{-k}, k = 0,1,2,\cdots,$$

where

$$\overline{M}(\beta,f) = \sup\left\{|f(z)|, z\epsilon\overline{E}_\beta\right\}.$$

We have

$$\begin{aligned} e_n(f) = \|f(z) - p_n^*(z)\| &\leq \sup\left\{\left|2\sum_{k=n+1}^{\infty}\alpha_k T_k(z)\right|\right\} \\ &\leq 2\sum_{k=n+1}^{\infty}|\alpha_k|\sup|T_k(z)| \\ &\leq \frac{2\overline{M}(\beta,f)}{(\beta-1)}\sum_{k=n+1}^{\infty}(5/4\beta)^k \\ &\leq \frac{2\overline{M}(\beta,f)}{(\beta-1)}(2/\beta)^n, \beta > 2. \end{aligned}$$

The right hand inequality can be improved as

$$\sum_{k=n+1}^{\infty}(5/4\beta)^k < (5/4\beta)^n \quad for \ \ \beta > (5/2).$$

Hence we have

$$\|f(z) - p_n^*(z)\| \leq \frac{2\overline{M}(\beta, f)}{(\beta - 1)}(5/4\beta)^n, n \geq 0, \beta > 5/2. \tag{12}$$

Now define $D_1 = \{z : |z| \leq \beta\}$ then we have $\overline{E}_\beta \subset D_1$. Hence

$$\overline{M}(\beta, f) \leq m(\beta, f). \tag{13}$$

Using (13) in (12) we get the required result. □

Lemma 2.2. *Let $V(z, z^*)$ be a regular solution of $\beta(V) = 0$ on the polydisk D^2. Then the function element $V(z, \overline{z})$ has an analytic continuation as an entire function solution if, and only if,*

$$\lim_{n\to\infty}[E_n(V)]^{1/n} = 0. \tag{14}$$

Proof. Let $V(z, \overline{z})$ has an analytic continuation as an entire function. By the application of the maximum principle and $w-$operator with [8, Theorem 1], the equation

$$V(z, \overline{z}) = w[f(\sigma)]$$

the function

$$V(z, \overline{z}) - \varphi_n(z, \overline{z}) = w[f(\sigma) - p_n(\sigma)], n = 0, 1, \cdots$$

for each $p_n \epsilon P_n$ and $\varphi_n = w[P_n]$. As the function $V - \varphi_n$ are regular on D^2, the contour L is homogeneous to $L_0 = t = e^{i\theta} : \theta$ decrease from π to 0 and $|\sigma| \leq 1$ if $(z, t)\epsilon D \times L_0$. We have

$$\begin{aligned} |V(z, z^*) - \varphi_n(z, z^*) \ &\leq \ \int_{L_0} |E(z, z^*, t)\|f(\sigma) - p_n(\sigma)|d|\mu|(t) \\ &\leq \ C(E)\|f - p_n\|, n = 0, 1, 2, \cdots, \end{aligned}$$

where

$$C(E) = \sup\left\{\int_{L_0} |E(z, z^*, t)|d|\mu|(t) : |z|, |z^*| \leq 1\right\}$$

on D^2. The constant $C(E)$ is finite and $E(z, z^*, t)$ is continuous on $D^2 \times L_0$, so we have

$$|||V - \varphi_n||| \leq C(E)\|f - p_n\|$$

and

$$E_n(V) \leq C(E)e_n(f), n = 0, 1, 2, \cdots . \tag{15}$$

Combining (11) with (15) we get

$$E_n(V) \leq C(E)K(\beta)m(\beta, f)(5/4\beta)^n \beta > 5/4, n \geq 0 \tag{16}$$

or

$$\lim_{n\to}[E_n(V)]^{1/n} \leq 5/4\beta for \beta > 5/4,$$

it gives the equation (2.4) as $\beta \to \infty$.

Now for only if part, let $V(z, z^*)$ be regular on D^2 and assume that the Bernstein limit equation (14) is satisfied. The function V satisfies the Goursat data [1, 8],

$$V(z, 0) = g(z) = \int_{L_0} f(\sigma)d\mu(t), V(0, z^*) = g(0), z\epsilon D.$$

The analytic function g is singular at $z = 2\alpha$ if, and only if, f is singular at $z = \alpha$ [8]. To prove that $V(z, \overline{z})$ is an entire function we have to show same for $g(z)$. We observe the identities

$$g(z) - p_n(z) = V(z, 0) - \varphi_n(z, 0), z\epsilon D$$

$p_n \epsilon P_n$, and

$$\|g - p_n\| \leq \|V - \varphi_n\| \leq |||V - \varphi_n||| \, . \tag{17}$$

In view of classical Benrstein theorem, we have if

$$\varepsilon_n(g) = \inf \{\|g - p_n\| : p_n \epsilon P_n\}$$

and

$$[\varepsilon_n(g)]^{1/n} \to 0 \; as \;\; n \to, \quad then$$

$g = g(z)$ is entire. So using equation (17), we get the estimate

$$[\varepsilon_n(g)]^{1/n} \leq [E_n(V)]^{1/n}, n = 1, 2, \cdots .$$

Hence the proof is completed in view of our assumption. □

Lemma 2.3. *Let $V(z,\overline{z})$ has an analytic continuation as an entire function solution of $\beta(V)=0$ with $w-$associate f. Then the order, type, lower order and lower type of an entire function element $V(z,\overline{z})$ are less than or equal to the order, type, lower order and lower type of associate f respectively.*

Proof. Since function element $V(z,\overline{z})$ has analytic continuation as an entire function, by [8, Theorem 1], the same holds true for the $w-$associate $f=f(z)$ and we have

$$V(z,\overline{z})=b_2[f(\sigma)].$$

The non-negativity and normalization of the measure leads directly to the bound of above equality

$$M(\delta,V)\leq m(\delta,f),\delta>0. \tag{18}$$

Using the definition(7), (8), (9) and (10) with (18) the proof is completed. In view of above result, we get

$$\rho(V)\leq\rho(f),\lambda(V)\leq\lambda(f),T(V)\leq T(f),t(V)\leq t(f).$$

□

3. Main Results

Now we shall prove our main theorems:

Theorem 3.1. *The entire function element $V(z,\overline{z})$ is of finite order $\rho(V)$ if and only if*

$$\rho(f)\geq\limsup_{n\to\infty}\frac{n\log n}{-\log E_n(V)}. \tag{19}$$

Proof. Let

$$\limsup_{n\to\infty}\frac{n\log n}{-\log E_n(V)}=\mu.$$

First let $0<\mu<\infty$. Then for arbitrary $\varepsilon>0$, we have

$$E_n(V)>n^{-n/(\mu-\varepsilon)} \tag{20}$$

for a sequence of values of $\delta \to \infty$.

Using Lemma 2.1 with (13), we have

$$\begin{aligned}\log m(\beta, f) &\geq \log \overline{M}(\beta, f) \\ &\geq \log E_n(V) + \log(\beta - 1) - \log 2 + n\log(4\beta/5) \\ &\quad - \log C(E) + n\log(4\beta/5). \qquad (21)\end{aligned}$$

For the best approximate right hand side should be maximum. So we define the sequence $\beta_k = e[n_k]^{1/(\mu-\varepsilon)}, k = 1, 2, \cdots$. For $\beta_k \leq \beta \leq \beta_{k+1}, k > k_0$ we have from (20),

$$\begin{aligned}\log m(\beta, f) &> \frac{-n_k \log n_k}{(\mu - \varepsilon)} + \log(\beta_k - 1) - \log 2 \\ &\quad - \log C(E) + n_k \log(4\beta_k/5) \\ &= n_k \log\left(\frac{4e}{5}\right) + \log(\beta_k - 1) - \log 2 - \log C(E) \\ &= \left(\frac{\beta_{k+1}}{e}\right) \log\left(\frac{4e}{5}\right) + \log(\beta_k - 1) - O(1) \\ &> \left(\frac{\beta_{k+1}}{e}\right)^{\mu-\varepsilon} \log\left(\frac{4e}{4}\right)(1 + o(1)).\end{aligned}$$

Hence for $k > k_0$,

$$\begin{aligned}\log\log m(\beta, f) &> (\mu - \varepsilon)\log(\beta_{k+1}) + O(1) \\ &> (\mu - \varepsilon)\log\beta + O(1).\end{aligned}$$

or

$$\frac{\log\log m(\beta, f)}{\log\beta} > (\mu - \varepsilon) + o(1)$$

or

$$\rho(f) = \limsup_{\beta\to\infty} \frac{\log\log m(\beta, f)}{\log\beta} \geq \mu.$$

This inequality holds for $\mu = 0$ and if $\mu = \infty$ then $\rho(f) = \infty$. To prove the only if part let

$$\limsup_{n\to\infty} \frac{n\log n}{-\log E_n(V)} = \rho(V) \qquad (22)$$

is finite and positive. From (17) we have that

$$\varepsilon_n(g) \leq E_n(V) \tag{23}$$

From (22) for $\varepsilon > 0$ and $N = N(\varepsilon)$ we get

$$E_n(V) \leq 1/n^{n/(\rho(V)+\varepsilon)}, n \geq N(\varepsilon). \tag{24}$$

Combining (23) with (24), gives

$$\varepsilon_n^{1/n}(g) \leq 1/n^{n/(\rho(V)+\varepsilon)}, n \geq N(\varepsilon)$$

and $\lim_{n\to\infty} \varepsilon_n^{1/n}(g) = 0$. Therefore g continues analytically as an entire function of finite order by classical Bernstein theorem. The same is true for the $V(z,\overline{z})$. Hence the proof is completed. □

Theorem 3.2. *The entire function element $V(z,\overline{z})$ is of lower order $\lambda(V)$ if and only if*

$$\lambda(f) \geq \max_{\{n_k\}} \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{-\log E_{n_k}(V)}$$

Proof. For any sequence $\{n_k\}$, let us assume that

$$\liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{-\log E_{n_k}(V)} = \mu^*(n_k) = \mu^*.$$

Let $\mu^* > 0$. Then for $0 < \varepsilon < \mu^*$, we have

$$E_{n_k}(V) > [n_{k-1}]^{-n_k/(\mu^*-\varepsilon)}, k > k_0(\varepsilon). \tag{25}$$

Now define the sequence $\beta_k = e[n_{k-1}]^{1/(\mu^*-\varepsilon)}, k = 1, 2, \cdots$. For $\beta_k \leq \beta \leq \beta_{k+1}, k > k_0$, we have from Lemma 2.1 with (13), (25 and after a simple calculation as in Theorem 3.1,

$$\lambda(f) = \liminf_{\beta\to\infty} \frac{\log\log m(\beta, f)}{\log \beta} \geq \mu^*.$$

The inequality obviously holds if $\mu^* = 0$. Since $\{n_k\}$ was any increasing sequence, we obtain

$$\lambda(f) \geq \max \mu^*(n_k) = \max_{\{n_k\}} \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{-\log E_{n_k}(V)}.$$

The only if part can be proved in a similar manner as in Theorem 3.1. Hence the proof is completed. □

Theorem 3.3. *Let $V(z, z^*)$ be a regular solution of $\beta(V) = 0$ on the polydisk D^2. Then the function element $V(z, \overline{z})$ has an analytic continuation as an entire function of order $\rho(V)$ and type $T(V)$ if and only if*

$$\frac{T(f)}{M} \geq \limsup_{n\to\infty} \frac{n}{e\rho(f)}[E_n(V)]^{\rho(f)/n}, \tag{26}$$

where $M = (4/5)^{\rho(f)}$.

Proof. Let the entire function element $V(z, \overline{z})$ be of finite order $\rho(V)$ and type $T(V)$. Then the associate f is of order $\rho(f)$ and type $T(f)$ such that $\rho(V) \leq \rho(f)$ and $T(V) \leq T(f)$. Let $T(f) < \infty$. For arbitrary $\varepsilon > 0$, we have by (10)

$$\log m(\beta, f) < (T(f) + \varepsilon)\beta^{\rho(f)}, \beta > \beta_0(\varepsilon).$$

From (17), for any n and all $\beta > \beta_0$,

$$E_n(V) < K(\beta)\left(\frac{5}{4\beta}\right)^n \exp[(T(f) + \varepsilon)\beta^{\rho(f)}].$$

The right and side should be minimum for

$$\beta = \left\{\frac{(n+1)}{\rho(f)(T(f) + \varepsilon)}\right\}^{1/\rho(f)}.$$

The value of β is compatible with the condition $\beta > \beta_0$ for large values on n. Hence we have

$$E_n(V) \leq (5/4)^n K(\beta)\left\{\frac{e\rho(f)(T(f) + \varepsilon)}{n+1}\right\}^{(n+1)/\rho(f)}.$$

Hence

$$\limsup_{n\to\infty} n[E_n(V)]^{\rho(f)/n} \leq (5/4)^{\rho(f)} e\rho T(f).$$

This inequality holds if $T(f) = \infty$.

To prove sufficiency, let $V(z, z^*)$ be regular in polydisk D^2 for which (26) holds. Then we can see that

$$\lim_{n\to\infty} [E_n(V)]^{1/n} = 0.$$

From Lemma 2.2, we have

$$\lim_{n\to\infty}[\varepsilon_n(g)]^{1/n} \leq \lim_{n\to\infty}[E_n(V)]^{1/n},$$

which implies that

$$\lim_{n\to\infty}[\varepsilon_n(g)]^{1/n} = 0.$$

Using classical Bernstein theorem we see that $g(z)$ is entire function. By [13] $V(z,\overline{z})$ also admits as an entire function. By (26) we also have, for a given $\varepsilon > 0$,

$$\frac{n}{e\rho(f)}[E_n(V)]^{\rho(f)/n} < \left(\frac{5}{4}\right)^{\rho(f)}(T(f)+\varepsilon), \qquad n > n_0.$$

Hence

$$\limsup_{n\to\infty}\frac{n\log n}{-\log E_n(V)} \leq \rho(f).$$

Also we have $\rho(V) \leq \rho(f), T(V) \leq T(f)$, it follows that $V(z,\overline{z})$ is of finite order $\rho(V)$. That $V(z,\overline{z})$ is of type $T(V)$ follows from the necessary part. Hence the proof is completed. □

THEOREM 3.4. *Let $V(z,z^*)$ be a regular solution of $\beta(V) = 0$ on the polydisk D^2. Then the function element $V(z,\overline{z})$ has an analytic continuation as an entire function of order $\rho(V)$ and lower type $t(V)$ if*

$$\frac{t(f)}{M} \geq \liminf_{n\to\infty}\frac{n}{e\rho(f)}[E_n(V)]^{\rho(f)/n}.$$

Proof. Let

$$\liminf_{n\to\infty}\frac{n}{e\rho(f)}[E_n(V)]^{\rho(f)/n} = \eta, o < \eta < \infty.$$

For arbitrary $\varepsilon, 0 < \varepsilon < \eta$ and all sufficiently large $n > n_0 = n_0(\varepsilon)$, we have

$$E_n(V) > \left[\frac{e\rho(f)(\eta-\varepsilon)}{n}\right]^{n/\rho(f)}.$$

From (17) we have for $\beta > 5/4$

$$\begin{aligned} m(\beta, f) &\geq \frac{1}{C(E)K(\beta)} \left(\frac{4\beta}{5}\right)^5 E_n(V) \\ &> \frac{1}{C(E)K(\beta)} \left(\frac{4\beta}{5}\right)^5 \left[\frac{e\rho(f)(\eta-\varepsilon)}{n}\right]^{n/\rho(f)}, \qquad n > n_0. \end{aligned}$$

Hence

$$\begin{aligned} \log m(\beta, f) &> n\log(4\beta/5) + \frac{n}{\rho(f)} \log\{e\rho(f)(\eta-\varepsilon)/n\} \\ &\quad - \log C(E) - \log K(\beta). \end{aligned} \tag{27}$$

Let

$$n = \rho(f)(\eta-\varepsilon)\left(\frac{4\beta}{5}\right)^{\rho(f)}.$$

Then for large values of $\beta, n > n_0$, we have

$$\log m(\beta, f) > (\eta-\varepsilon)\left(\frac{4\beta}{5}\right)^{\rho(f)} - \log C(E) - \log K(\beta).$$

or

$$\lim_{n\to\infty} \frac{\log m(\beta, f)}{\beta^{\rho(f)}} > (\eta-\varepsilon)\left(\frac{4}{5}\right)^{\rho(f)} - o(1).$$

or

$$t(f) \geq \eta \left(\frac{4}{5}\right)^{\rho(f)}$$

or

$$\liminf_{n\to\infty} \frac{n}{e\rho(f)}[E_n(V)]^{\rho(f)/n} \leq \left(\frac{5}{4}\right)^{\rho(f)} t(f).$$

If $\eta = \infty$ then $t(f)$ is infinite. This inequality also holds if $\eta = 0$.

This completes the proof of Theorem 3.4. □

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