

Low Frequency Electromagnetic Scattering. The Impedance Problem for a Sphere

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SUMMARY. - We consider the low-frequency scattering problem of a plane electromagnetic wave by a small sphere, on the boundary of which an impedance condition is satisfied. The impedance boundary condition was introduced by Leontovich (1948) and it accounts for situations where the obstacle is not perfectly conducting but the exterior field will not penetrate deeply into the scatterer. It provides a method to simulate the material properties of the surface of highly absorbing coating layers. For the near electromagnetic field we obtain the low-frequency coefficients of the zeroth and the first order while in the far field we derive the leading non-vanishing terms for the scattering amplitude, the scattering and the absorption cross-sections.

1. Introduction

The problem of scattering of an electromagnetic plane wave by a resistive object, whose characteristic dimension is much less than the wavelength of the incident wave has taken less attention than the

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corresponding problem for an acoustic plane or spherical wave. In this work we develop the low-frequency technique for a scatterer, where Leontovich boundary condition holds on its surface [6]. While no general results for this impedance boundary condition are known we present a new method for obtaining solutions in the special case of a spherical obstacle. The problem of low frequency scattering, both for the sphere and the ellipsoid, appears in the work of Dassios and Kleinman [2], where the zeroth order approximations are obtained and the new results in the present paper are the relative first order low-frequency approximations as well as the exact form of the absorption cross-section.

We attack the problem from the point of view of Helmholtz decomposition theorem [3], in which the vector field solutions are expressed as a sum of a longitudinal and a transverse part. The same separation holds for the low-frequency approximations [4], which are reduced to the solution of an exterior boundary value problem for the vector Laplace equation [6].

The longitudinal part of the solution may be represented as the gradient of a scalar potential and consequently all the techniques, applied for the scalar Laplace equation are at once available to provide result. The second part, namely the transverse component, is obtained from a vector potential by taking the curl. This field may always be derived from scalar fields and the gauge requirement that the field has zero divergence leads again to the scalar Laplace equation [3].

The scalar Laplace equation with boundary conditions on a sphere is simply representable if we separate it in spherical polar coordinates. The form of the solution for the exterior scattering problem is a spherical harmonic of the n -th order times r^{-n-1} [1].

It is shown that in the case of plane wave incidence both the near and far field approximations are expressible in terms of a finite number of multiples [2]. This is not true in the case of spherical wave incidence, where the near field is constructed from infinitely many multiple terms. This difference is due to the fact that in the near field every plane wave component generates an appreciable interaction with the scatterer.

The formulation of the problem is presented in Section 2. Sec-

tion 3 involves the decomposition of the scattering problem into low-frequency approximations [4]. The reduction of the initial wave problem to a set of exterior problems for the approximation coefficients and the relative explicit results are partially provided in Section 4. The low-frequency approximation for the cross-sections furnished in Section 4 shows that the absorption cross-section is by two orders of magnitude larger than the scattering cross-section.

2. Statement of the problem

Consider a sphere S_R with radius R , centered at the origin, which lies in a homogeneous isotropic medium V with magnetic permeability μ , electric primitivity ε and conductivity σ . An incident plane-polarized electromagnetic wave $(\mathbf{E}^i, \mathbf{H}^i)$ propagates in the direction $\hat{\mathbf{k}}$ and has the following form

$$\mathbf{E}^i(\mathbf{r}) = \hat{\mathbf{a}}e^{ik\hat{\mathbf{k}}\cdot\mathbf{r}}, \quad \mathbf{H}^i(\mathbf{r}) = \frac{1}{Z}\hat{\mathbf{b}}e^{ik\hat{\mathbf{k}}\cdot\mathbf{r}} \quad (1)$$

where the polarization vectors $\hat{\mathbf{a}}, \hat{\mathbf{b}}$ and the direction $\hat{\mathbf{k}}$ form a right-handed orthogonal system $\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \hat{\mathbf{k}}, \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = 0$. Here

$$k = \sqrt{\varepsilon\mu\omega^2 + i\mu\sigma\omega}, \quad \text{Im}k \geq 0$$

is the wave number with angular frequency ω . The harmonic time-dependent factor $e^{-i\omega t}$ is suppressed throughout this work.

The impedance scattering problem in the exterior V^e of the sphere S_R is formulated as follows. Find the total field

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^i(\mathbf{r}) + \mathbf{E}^s(\mathbf{r}), \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}^i(\mathbf{r}) + \mathbf{H}^s(\mathbf{r}), \quad \mathbf{r} \in V^e \quad (2)$$

which solves the Maxwell's equations

$$\nabla \times \mathbf{E} = ikZ\mathbf{H}, \quad \nabla \times \mathbf{H} = -\frac{1}{Z}ik\mathbf{E} \quad (3)$$

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{H} = 0$$

and satisfies the Leontovich or impedance boundary condition

$$\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{E}) = -Z_s Z (\hat{\mathbf{n}} \times \mathbf{H}), \quad (4)$$

while the scattered fields $(\mathbf{E}^s, \mathbf{H}^s)$ satisfy the Silver-Müller radiation conditions

$$\lim_{r \rightarrow \infty} \left(\mathbf{r} \times \left\{ \begin{array}{c} \nabla \times \mathbf{E}^s \\ \nabla \times \mathbf{H}^s \end{array} \right\} + ikr \left\{ \begin{array}{c} \mathbf{E}^s \\ \mathbf{H}^s \end{array} \right\} \right) = \mathbf{0} \quad (5)$$

uniformly over directions.

The dimensionless parameter Z_s denotes the surface impedance of the obstacle relative to the characteristic impedance

$$Z = \frac{\mu\omega}{\sqrt{\varepsilon\mu\omega^2 + i\mu\sigma\omega}}$$

of the medium and may vary on S_R . Here $\hat{\mathbf{n}}$ is the outward unit normal to the sphere.

The impedance problem will reduce to the perfect conductor as the surface impedance approaches zero.

The solutions \mathbf{E} , \mathbf{H} of the above scattering problem enjoy the following integral representation (Stratton-Chu) formula

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \mathbf{E}^i(\mathbf{r}) + \frac{ik}{4\pi} \int_{S_R} [ikZ G(\mathbf{r}, \mathbf{r}') (\hat{\mathbf{n}}' \times \mathbf{H}(\mathbf{r}')) \\ &\quad + \nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') (\hat{\mathbf{n}}' \cdot \mathbf{E}(\mathbf{r})) - (\nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') \times (\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{r}')))] \mathbf{d}\mathbf{s}(\mathbf{r}') \\ \mathbf{H}(\mathbf{r}) &= \mathbf{H}^i(\mathbf{r}) + \frac{ik}{4\pi} \int_{S_R} [-ik \frac{1}{Z} G(\mathbf{r}, \mathbf{r}') (\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{r}')) \\ &\quad + \nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') (\hat{\mathbf{n}}' \cdot \mathbf{H}(\mathbf{r})) - (\nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') \times (\hat{\mathbf{n}}' \times \mathbf{H}(\mathbf{r}')))] \mathbf{d}\mathbf{s}(\mathbf{r}') \end{aligned} \quad (6)$$

where $G(\mathbf{r}, \mathbf{r}')$ is the fundamental solution of the Helmholtz equation in V^e .

Taking into consideration the impedance boundary conditions, we obtain the following integral representations for the solutions of the scattering problem

$$\begin{aligned}
 \mathbf{E}(\mathbf{r}) &= \mathbf{E}^i(\mathbf{r}) + \frac{ik}{4\pi} \int_{S_R} [-ik \frac{1}{Z_s} G(\mathbf{r}, \mathbf{r}') (\hat{\mathbf{n}}' \times (\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{r}')))] \\
 &+ \nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') (\hat{\mathbf{n}}' \cdot \mathbf{E}(\mathbf{r})) - (\nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') \times (\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{r}')))] \mathbf{d}\mathbf{s}(\mathbf{r}'), \\
 \mathbf{H}(\mathbf{r}) &= \mathbf{H}^i(\mathbf{r}) + \frac{ik}{4\pi} \int_{S_R} [-ik Z_s G(\mathbf{r}, \mathbf{r}') (\hat{\mathbf{n}}' \times (\hat{\mathbf{n}}' \times \mathbf{H}(\mathbf{r}')))] \\
 &+ \nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') (\hat{\mathbf{n}}' \cdot \mathbf{H}(\mathbf{r})) - (\nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') \times (\hat{\mathbf{n}}' \times \mathbf{H}(\mathbf{r}')))] \mathbf{d}\mathbf{s}(\mathbf{r}').
 \end{aligned} \tag{7}$$

The electric scattering amplitude $\mathbf{g}_e(\hat{\mathbf{r}})$ is given by the surface integral

$$\begin{aligned}
 \mathbf{g}_e(\hat{\mathbf{r}}) &= -\frac{ik^3}{4\pi} \hat{\mathbf{r}} \times \\
 &\left[\hat{\mathbf{r}} \times \int_{S_R} [-Z\hat{\mathbf{r}}' \cdot (\hat{\mathbf{n}}' \times \mathbf{H}(\mathbf{r}')) + \hat{\mathbf{n}}' \cdot \mathbf{E}(\mathbf{r}')] \mathbf{r}' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \mathbf{d}\mathbf{s}(\mathbf{r}') \right] \\
 &- \frac{ik^3}{4\pi} \hat{\mathbf{r}} \times \int_{S_R} [\hat{\mathbf{r}}' \cdot (\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{r}')) + Z\hat{\mathbf{n}}' \cdot \mathbf{H}(\mathbf{r}')] \mathbf{r}' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \mathbf{d}\mathbf{s}(\mathbf{r}')
 \end{aligned} \tag{8}$$

and the magnetic scattering amplitude, with simplification from the boundary condition, is given by

$$\mathbf{g}_m(\hat{\mathbf{r}}) = \frac{1}{Z} \hat{\mathbf{r}} \times \mathbf{g}_e(\hat{\mathbf{r}}).$$

The scattering amplitudes contain all the obstacle information and it is easily seen that is tangential in nature.

We define the differential scattering cross-section or radar cross-section to be the power, scattered in the direction $\hat{\mathbf{r}}$, relative to the incident power flux in the direction of incidence and given explicitly by

$$\sigma(\hat{\mathbf{r}}) = \frac{4\pi}{k^2} |\mathbf{g}_e(\hat{\mathbf{r}})|^2.$$

The scattering cross-section is defined as the value of $\sigma(\hat{\mathbf{r}})$ averaged over all directions, i.e.

$$\sigma_s(\hat{\mathbf{r}}) = \frac{1}{k^2} \int_{S^2} |\mathbf{g}_e(\hat{\mathbf{r}})|^2 \mathbf{d}\mathbf{s}(\hat{\mathbf{r}}). \tag{9}$$

A similar expression is used to define the absorption cross-section as

$$\sigma_a(\hat{\mathbf{r}}) = \operatorname{Re} \int_{S_R} \frac{1}{Z_s} \left(|\mathbf{E}(\hat{\mathbf{r}}')|^2 - |\hat{\mathbf{n}}' \cdot \mathbf{E}(\hat{\mathbf{r}}')|^2 \right) \mathbf{d}\mathbf{s}(\hat{\mathbf{r}}'). \quad (10)$$

The extinction cross-section which measures, in units of area, the total energy that the scatterer removes from the incident wave either by scattering in all directions, or by absorption is given by

$$\sigma_e(\hat{\mathbf{r}}) = \sigma_s(\hat{\mathbf{r}}) + \sigma_a(\hat{\mathbf{r}}).$$

3. The low-frequency expansion

When the characteristic dimension of the scatterer is much smaller than the wavelength of the incident wave, all the fields are analytic functions of the wavenumber [2]. In other words, the assumption that $kr \ll 1$ implies that the total fields can be expanded in power series with respect to k and have the following form

$$\begin{aligned} \mathbf{E}^i(\mathbf{r}) &= \hat{\mathbf{a}} \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} (\hat{\mathbf{k}} \cdot \mathbf{r})^n, & \mathbf{H}^i(\mathbf{r}) &= \frac{1}{Z} \hat{\mathbf{b}} \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} (\hat{\mathbf{k}} \cdot \mathbf{r})^n, \\ \mathbf{E}(\mathbf{r}) &= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{E}_n(\mathbf{r}), & \mathbf{H}(\mathbf{r}) &= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{H}_n(\mathbf{r}). \end{aligned} \quad (11)$$

Substituting the expansions into the vector Helmholtz equation, adding the boundary condition and equating like powers of ik we arrive at the following sequence of potential problems for the low-frequency coefficients

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E}_n(\mathbf{r}) &= -n(n-1)\mathbf{E}_{n-2}(\mathbf{r}), \\ \nabla \times \nabla \times \mathbf{H}_n(\mathbf{r}) &= -n(n-1)\mathbf{H}_{n-2}(\mathbf{r}) \\ \nabla \times \mathbf{E}_n(\mathbf{r}) &= nZ\mathbf{H}_{n-1}(\mathbf{r}), \\ \nabla \times \mathbf{H}_n(\mathbf{r}) &= -\frac{1}{Z}n\mathbf{E}_{n-1}(\mathbf{r}) \\ \nabla \cdot \mathbf{E}_n(\mathbf{r}) &= 0, \\ \nabla \cdot \mathbf{H}_n(\mathbf{r}) &= 0 \end{aligned} \quad (12)$$

for $\mathbf{r} \in V^e$ and the following boundary conditions

$$\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{E}_n(\mathbf{r})) = -Z_s Z (\hat{\mathbf{n}} \times \mathbf{H}_n(\mathbf{r})), \quad \mathbf{r} \in S_R \quad (13)$$

In the order to establish the behaviour of the approximations $\mathbf{E}_n, \mathbf{H}_n$ in the far field we replace 11 into the integral representation 7. This give us that

$$\begin{aligned} \mathbf{E}_n(\mathbf{r}) &= \hat{\mathbf{a}}(\hat{\mathbf{k}} \cdot \mathbf{r})^n \\ &- \frac{1}{4\pi Z_s} \int_{S_R} \hat{\mathbf{n}}' \times (\hat{\mathbf{n}}' \times \sum_{j=1}^{n-1} \binom{n-1}{j} |\mathbf{r} - \mathbf{r}'|^{j-1} \mathbf{E}_{n-j-1}(\mathbf{r}')) \mathbf{d}\mathbf{s}(\mathbf{r}') \\ &- \frac{1}{4\pi} \int_{S_R} (\hat{\mathbf{n}}' \cdot \sum_{j=3}^n \binom{n}{j} (j-1) |\mathbf{r} - \mathbf{r}'|^{j-3} \mathbf{E}_{n-j}(\mathbf{r}')) \cdot \mathbf{r}' \mathbf{d}\mathbf{s}(\mathbf{r}') \\ &+ \frac{1}{4\pi} \int_{S_R} (\mathbf{r}' \times (\hat{\mathbf{n}}' \times \sum_{j=3}^n \binom{n}{j} (j-1) |\mathbf{r} - \mathbf{r}'|^{j-3} \mathbf{E}_{n-j}(\mathbf{r}'))) \mathbf{d}\mathbf{s}(\mathbf{r}') \\ &+ \mathbf{0} \left(\frac{1}{r} \right), \quad r \rightarrow \infty; \end{aligned} \quad (14)$$

$$\begin{aligned} \mathbf{H}_n(\mathbf{r}) &= \frac{1}{Z} \hat{\mathbf{b}}(\hat{\mathbf{k}} \cdot \mathbf{r})^n \\ &- \frac{Z_s}{4\pi} \int_{S_R} \hat{\mathbf{n}}' \times (\hat{\mathbf{n}}' \times \sum_{j=1}^{n-1} \binom{n-1}{j} |\mathbf{r} - \mathbf{r}'|^{j-1} \mathbf{H}_{n-j-1}(\mathbf{r}')) \mathbf{d}\mathbf{s}(\mathbf{r}') \\ &- \frac{1}{4\pi} \int_{S_R} (\hat{\mathbf{n}}' \cdot \sum_{j=3}^n \binom{n}{j} (j-1) |\mathbf{r} - \mathbf{r}'|^{j-3} \mathbf{H}_{n-j}(\mathbf{r}')) \mathbf{r}' \mathbf{d}\mathbf{s}(\mathbf{r}') \\ &+ \frac{1}{4\pi} \int_{S_R} (\mathbf{r}' \times (\hat{\mathbf{n}}' \times \sum_{j=3}^n \binom{n}{j} (j-1) |\mathbf{r} - \mathbf{r}'|^{j-3} \mathbf{H}_{n-j}(\mathbf{r}'))) \mathbf{d}\mathbf{s}(\mathbf{r}') \\ &+ \mathbf{0} \left(\frac{1}{r} \right), \quad r \rightarrow \infty \end{aligned} \quad (15)$$

for every $n = 0, 1, 2, \dots$

The electric scattering amplitude is expressible in terms of the coefficients in the low-frequency expansions by substituting the expansions 11 into the expression for $\mathbf{g}_e(\hat{\mathbf{r}})$ given in Section 2, expanding the exponential that appears in the integrand and collecting like powers of ik . The corresponding expression for the electric scattering amplitude that results is

$$\begin{aligned}
\mathbf{g}_e(\hat{\mathbf{r}}) &= \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{(ik)^{n+3}}{n!} \hat{\mathbf{r}} \times \\
&\times \left[\hat{\mathbf{r}} \times \int_{S_R} [-Z\hat{\mathbf{r}} \cdot (\hat{\mathbf{n}}' \times \sum_{j=0}^n \binom{n}{j} (-1)^j (\hat{\mathbf{r}} \cdot \mathbf{r}')^j \mathbf{H}_{n-j}(\mathbf{r}')) \right. \\
&+ \hat{\mathbf{n}}' \cdot \sum_{j=0}^n \binom{n}{j} (-1)^j (\hat{\mathbf{r}} \cdot \mathbf{r}')^j \mathbf{E}_{n-j}(\mathbf{r}')] \mathbf{r}' \mathbf{d}\mathbf{s}(\mathbf{r}') \Big] \\
&+ \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{(ik)^{n+3}}{n!} \hat{\mathbf{r}} \times \\
&\int_{S_R} [\hat{\mathbf{r}} \cdot (\hat{\mathbf{n}}' \times \sum_{j=0}^n \binom{n}{j} (-1)^j (\hat{\mathbf{r}} \cdot \mathbf{r}')^j \mathbf{E}_{n-j}(\mathbf{r}')) \\
&+ Z\hat{\mathbf{n}}' \cdot \sum_{j=0}^n \binom{n}{j} (-1)^j (\hat{\mathbf{r}} \cdot \mathbf{r}')^j \mathbf{H}_{n-j}(\mathbf{r}')] \mathbf{r}' \mathbf{d}\mathbf{s}(\mathbf{r}').
\end{aligned} \tag{16}$$

4. Near and far field approximations

For the zeroth order approximations $\mathbf{E}_0, \mathbf{H}_0$ we have that they are harmonic vector functions, which assumes the asymptotic form

$$\mathbf{E}_0(\mathbf{r}) = \hat{\mathbf{a}} + \mathbf{0} \left(\frac{1}{r} \right), \quad \mathbf{H}_0(\mathbf{r}) = \frac{1}{Z} \hat{\mathbf{b}} + \mathbf{0} \left(\frac{1}{r} \right), \quad r \rightarrow \infty \tag{17}$$

solve the differential equations

$$\begin{aligned}
\Delta \left\{ \begin{array}{l} \mathbf{E}_0(\mathbf{r}) \\ \mathbf{H}_0(\mathbf{r}) \end{array} \right\} &= \mathbf{0}, \\
\nabla \times \left\{ \begin{array}{l} \mathbf{E}_0(\mathbf{r}) \\ \mathbf{H}_0(\mathbf{r}) \end{array} \right\} &= \mathbf{0}, \quad \text{in } V^e \\
\nabla \cdot \left\{ \begin{array}{l} \mathbf{E}_0(\mathbf{r}) \\ \mathbf{H}_0(\mathbf{r}) \end{array} \right\} &= 0,
\end{aligned} \tag{18}$$

and the boundary condition

$$\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{E}_0(\mathbf{r})) = -Z_s Z (\hat{\mathbf{n}} \times \mathbf{H}_0(\mathbf{r})), \quad \mathbf{r} \in S_R. \tag{19}$$

From the Helmholtz' theorem [3] we have that any vector field, which is finite, uniform, continuous and vanishes at infinity, may be

expressed as a sum of a gradient of a scalar potential and a curl of a zero-divergence vector potential. We shall call these two parts the irrotational (longitudinal) and the solenoidal (transverse) part, respectively.

For $\mathbf{E}_0, \mathbf{H}_0$ we need to arrange things so that the fields have zero curl and zero divergence. This condition gives us that they have only longitudinal part, which we shall obtain as a gradient of scalar potentials.

Thus, for the zeroth order coefficients we have

$$\mathbf{E}_0(\mathbf{r}) = \hat{\mathbf{a}} + \nabla\Psi_0(\mathbf{r}), \quad \mathbf{H}_0(\mathbf{r}) = \frac{1}{Z}\hat{\mathbf{b}} + \nabla\Phi_0(\mathbf{r}), \quad \mathbf{r} \in V^e. \quad (20)$$

The condition for zero divergence yields, that the potentials Ψ_0, Φ_0 are solutions of the scalar Laplace equation.

Let us introduce a spherical polar coordinate system with its origin at the centre of the sphere S_R and unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$. Then we shall consider that the polarization vectors $\hat{\mathbf{a}}, \hat{\mathbf{b}}$ and the direction $\hat{\mathbf{k}}$ have the following coordinate components

$$\begin{aligned} \hat{\mathbf{a}} &= (\sin\theta \cos\varphi, \cos\theta \cos\varphi, -\sin\varphi), \\ \hat{\mathbf{b}} &= (\sin\theta \sin\varphi, \cos\theta \sin\varphi, \cos\varphi), \\ \hat{\mathbf{k}} &= (\cos\theta, -\sin\theta, 0). \end{aligned} \quad (21)$$

just as the Cartesian unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. Substituting the expression for the gradient ∇ in spherical polar coordinates into the boundary condition 19, we obtain the relations

$$\frac{1}{R}\Psi_{0,\theta} + a_2 + \frac{ZZ_s}{R\sin\theta}\Phi_{0,\varphi} + Z_s b_3 = 0, \quad (22)$$

$$\frac{1}{R}\Psi_{0,\varphi} + a_3 \sin\theta - \frac{ZZ_s}{R}\Phi_{0,\theta} \sin\theta - Z_s b_2 \sin\theta = 0,$$

where we denote by $a_i, b_i; i = 1, 2, 3$ the three coordinate components of the polarization vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$. After differentiating the first equation with respect to φ and the second one – to θ and eliminating the function Ψ_0 we obtain the equation

$$\Phi_{0,\theta\theta} + \frac{\cos\theta}{\sin\theta}\Phi_{0,\theta} - \frac{1}{\sin^2\theta}\Phi_{0,\varphi\varphi} - \frac{2R}{Z}\sin\theta \sin\varphi = 0. \quad (23)$$

If we separate the radial and angular parts in Φ_0 the equation (23) becomes the associated Legendre equation from which we find that

$$\Phi_0 = -\frac{R^3}{Zr^2}P_1^1(\cos\theta)\sin\varphi, \quad (24)$$

with P_1^1 be the associated Legendre function.

The same approach, but differentiating the first equation in 22 with respect to θ , the second - to φ and eliminating now the function Φ_0 , we use to find that

$$\Psi_0 = -\frac{R^3}{r^2}P_1^1(\cos\theta)\cos\varphi \quad (25)$$

Finally the zeroth order approximations \mathbf{E}_0 , \mathbf{H}_0 are obtained from Ψ_0 and Φ_0 by taking the gradient.

The first order low-frequency coefficients \mathbf{E}_1 , \mathbf{H}_1 are again solutions of the vector Laplace equation and have zero divergence. From the relations 12 we obtain that

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E}_1(\mathbf{r}) &= \mathbf{0}, & \nabla \times \nabla \times \mathbf{H}_1(\mathbf{r}) &= \mathbf{0}, \\ \nabla \times \mathbf{E}_1(\mathbf{r}) &= Z\mathbf{H}_0(\mathbf{r}), & \nabla \times \mathbf{H}_1(\mathbf{r}) &= -\frac{1}{Z}\mathbf{E}_0(\mathbf{r}), & \mathbf{r} \in V^e \\ \nabla \cdot \mathbf{E}_1(\mathbf{r}) &= 0, & \nabla \cdot \mathbf{H}_1(\mathbf{r}) &= 0. \end{aligned} \quad (26)$$

Using the aforementioned breaking of the vector solutions into a longitudinal and a transverse part we write \mathbf{E}_1 , \mathbf{H}_1 as

$$\begin{aligned} \mathbf{E}_1(\mathbf{r}) &= \hat{\mathbf{a}}(\mathbf{r} \cdot \hat{\mathbf{k}}) + \nabla \times (r\Psi_1^1(\mathbf{r})\hat{\mathbf{r}}) + \nabla\Psi_1^2(\mathbf{r}), \\ \mathbf{H}_1(\mathbf{r}) &= \frac{1}{Z}\hat{\mathbf{b}}(\mathbf{r} \cdot \hat{\mathbf{k}}) + \nabla \times (r\Phi_1^1(\mathbf{r})\hat{\mathbf{r}}) + \nabla\Phi_1^2(\mathbf{r}) \end{aligned} \quad (27)$$

where the scalars Ψ_1^1 , Ψ_1^2 , Φ_1^1 and Φ_1^2 are solutions of the Laplace equation in the exterior of the sphere in the form

$$r^{-n-1}P_n^m(\cos\theta)\sin\varphi \quad \text{or} \quad r^{-n-1}P_n^m(\cos\theta)\cos\varphi.$$

First, we need to arrange things so that

$$\begin{aligned}
 \nabla \times \mathbf{E}_1(\mathbf{r}) &= \nabla \times \hat{\mathbf{a}}(\mathbf{r} \cdot \hat{\mathbf{k}}) + \nabla \times \nabla \times (r\Psi_1^1(\mathbf{r})\hat{\mathbf{r}}) \\
 &= \hat{\mathbf{b}} + \nabla \left(\frac{\partial}{\partial r} (r\Psi_1^1(\mathbf{r})) \right) = Z\mathbf{H}_0(\mathbf{r}) \\
 &= Z \left[\frac{1}{Z}\hat{\mathbf{b}} + \nabla \left(-\frac{R^3}{Zr^2}P_1^1(\cos\theta)\sin\varphi \right) \right], \\
 \nabla \times \mathbf{H}_1(\mathbf{r}) &= \nabla \times \frac{1}{Z}\hat{\mathbf{b}}(\mathbf{r} \cdot \hat{\mathbf{k}}) + \nabla \times \nabla \times (r\Phi_1^1(\mathbf{r})\hat{\mathbf{r}}) \\
 &= -\frac{1}{Z}\hat{\mathbf{a}} + \nabla \left(\frac{\partial}{\partial r} (r\Phi_1^1(\mathbf{r})) \right) = -\frac{1}{Z}\mathbf{E}_0(\mathbf{r}) \\
 &= -\frac{1}{Z} \left[\hat{\mathbf{a}} + \nabla \left(-\frac{R^3}{r^2}P_1^1(\cos\theta)\cos\varphi \right) \right].
 \end{aligned} \tag{28}$$

This, give us the relations

$$\begin{aligned}
 \frac{\partial}{\partial r} (r\Psi_1^1(\mathbf{r})) &= -\frac{R^3}{r^2}P_1^1(\cos\theta)\sin\varphi, \\
 \frac{\partial}{\partial r} (r\Phi_1^1(\mathbf{r})) &= \frac{R^3}{Zr^2}P_1^1(\cos\theta)\cos\varphi.
 \end{aligned} \tag{29}$$

Now, from the Leontovich boundary condition, applying the same technics used to obtain the zeroth order coefficients, we find the functions Ψ_1^1 , Ψ_1^2 , Φ_1^1 and Φ_1^2 in the following form

$$\begin{aligned}
 \Psi_1^1(\hat{\mathbf{r}}) &= \frac{R^3}{r^2}P_1^1(\cos\theta)\sin\varphi, \\
 \Psi_1^2(\hat{\mathbf{r}}) &= -\frac{R^4}{6} \left[\frac{R}{r^3}P_2^1(\cos\theta) + \frac{9Z_s}{r^2}P_1^1(\cos\theta) \right] \cos\varphi, \\
 \Phi_1^1(\hat{\mathbf{r}}) &= -\frac{R^3}{Zr^2}P_1^1(\cos\theta)\cos\varphi, \\
 \Phi_1^2(\hat{\mathbf{r}}) &= -\frac{R^4}{6Z} \left[\frac{R}{r^3}P_2^1(\cos\theta) + \frac{9}{Z_s r^2}P_1^1(\cos\theta) \right] \sin\varphi.
 \end{aligned} \tag{30}$$

Finally, the first order approximations are

$$\begin{aligned}
 \mathbf{E}_1(\hat{\mathbf{r}}) &= \hat{\mathbf{a}}(\mathbf{r} \cdot \hat{\mathbf{k}}) + R^3 \nabla \times \left(\frac{1}{r}P_1^1(\cos\theta)\sin\varphi.\hat{\mathbf{r}} \right) \\
 &\quad - \frac{R^4}{6} \nabla \left[\left(\frac{R}{r^3}P_2^1(\cos\theta) + \frac{9Z_s}{r^2}P_1^1(\cos\theta) \right) \cos\varphi \right], \\
 \mathbf{H}_1(\hat{\mathbf{r}}) &= \frac{1}{Z}\hat{\mathbf{b}}(\mathbf{r} \cdot \hat{\mathbf{k}}) - \frac{R^3}{Z} \nabla \times \left(\frac{1}{r}P_1^1(\cos\theta)\cos\varphi.\hat{\mathbf{r}} \right) \\
 &\quad - \frac{R^4}{6Z} \nabla \left[\left(\frac{R}{r^3}P_2^1(\cos\theta) + \frac{9}{Z_s r^2}P_1^1(\cos\theta) \right) \sin\varphi \right].
 \end{aligned} \tag{31}$$

The explicit forms of the low-frequency approximations \mathbf{E}_0 , $\mathbf{E}_1, \mathbf{H}_0$ and \mathbf{H}_1 allows the calculation of the electric scattering amplitude up to the order k^4 through the expression

$$\begin{aligned}
\mathbf{g}_e(\hat{\mathbf{r}}) = & \\
& -\frac{ik^3}{4\pi} \left(\hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \int_{S_R} [-Z\hat{\mathbf{r}} \cdot (\hat{\mathbf{r}}' \times \mathbf{H}_0(\mathbf{r}')) + \mathbf{r}' \cdot \mathbf{E}_0(\mathbf{r}')] \mathbf{r}' \mathbf{d}\mathbf{s}(\mathbf{r}') \right] \right. \\
& \left. + \hat{\mathbf{r}} \times \int_{S_R} [\hat{\mathbf{r}} \cdot (\hat{\mathbf{r}}' \times \mathbf{E}_0(\mathbf{r}')) + Z\hat{\mathbf{r}} \cdot \mathbf{H}_0(\mathbf{r}')] \mathbf{r}' \mathbf{d}\mathbf{s}(\mathbf{r}') \right) \\
& + \frac{k^4}{4\pi} \left(\hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \int_{S_R} [-Z\hat{\mathbf{r}} \cdot (\hat{\mathbf{r}}' \times (\mathbf{H}_1(\mathbf{r}') - (\hat{\mathbf{r}} \cdot \mathbf{r}') \mathbf{H}_0(\mathbf{r}')) \right. \right. \\
& \left. \left. + \mathbf{r}' \cdot (\mathbf{E}_1(\mathbf{r}') - (\hat{\mathbf{r}} \cdot \mathbf{r}') \mathbf{E}_0(\mathbf{r}')) \right] \mathbf{r}' \mathbf{d}\mathbf{s}(\mathbf{r}') \right) \\
& + \hat{\mathbf{r}} \times \int_{S_R} [\hat{\mathbf{r}} \cdot (\hat{\mathbf{r}}' \times (\mathbf{E}_1(\mathbf{r}') - (\hat{\mathbf{r}} \cdot \mathbf{r}') \mathbf{E}_0(\mathbf{r}')) \\
& \left. + Z\hat{\mathbf{r}} \cdot (\mathbf{H}_1(\mathbf{r}') - (\hat{\mathbf{r}} \cdot \mathbf{r}') \mathbf{H}_0(\mathbf{r}')) \right] \mathbf{r}' \mathbf{d}\mathbf{s}(\mathbf{r}') \right) + \hat{\mathbf{O}}(k^5). \tag{32}
\end{aligned}$$

With the use of the addition theorem for spherical harmonics over the unit sphere and after tedious calculation of a number of particular surface integrals we arrived at the expression

$$\begin{aligned}
\mathbf{g}_e(\hat{\mathbf{r}}) = & i(kR)^3 \begin{pmatrix} 0 \\ (1 + P_1(\cos \theta)) \cos \varphi \\ -(1 + P_1(\cos \theta)) \sin \varphi \end{pmatrix} \\
& + (kR)^4 \begin{pmatrix} 0 \\ -\left(\frac{Z_s P_1(\cos \theta)}{2} + \frac{8 + P_2(\cos \theta)}{6Z_s} \right) \cos \varphi \\ \left(\frac{3P_1(\cos \theta)}{2Z_s} + Z_s \frac{4 - P_2(\cos \theta)}{6} \right) \sin \varphi \end{pmatrix} + \hat{\mathbf{O}}(k^5), kR \rightarrow 0. \tag{33}
\end{aligned}$$

Substituting \mathbf{g}_e in 9 for the scattering cross-section we easily obtain that

$$\begin{aligned}
\sigma_s = & 4\pi R^2 \left(\frac{4}{3} (kR)^4 + \left(\frac{5}{24} + \frac{19}{60Z_s^2} + \frac{Z_s^2}{15} \right) (kR)^6 \right) \\
& + 0 \left((kR)^8 \right), \quad kR \rightarrow 0.
\end{aligned}$$

Now, from the formula 10, replacing the approximation coefficient for the total field we evaluate the absorption cross-section up to the order k^2

$$\sigma_a = 4\pi R^2 (kR)^2 \frac{15}{8} \left(\frac{1}{Z_s} + Z_s \right) + O(kR)^4, \quad kR \rightarrow 0.$$

We observe that the low-frequency approximation of σ_a is by two orders of magnitude higher than σ_s , which means that the absorption is much more prominent than scattering. Note that, for plane wave incidence, both the near and the far fields are expressed in terms of finite number of multiples. The reason for this behaviour is connected to the actual distance between the singularities of the incident and that of the scattered field. In our case this distance is infinite and only the first few multiples of the incident field interact with the scatterer. These multiples appear in the asymptotic form of the total field and the solution has as many multiples as those that survive at infinity.

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