ON A PROCEDURE FOR THE SIMULTANEOUS DETERMINATION OF ALL ZEROS OF A POLYNOMIAL (*)

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SOMMARIO. - *In questo lavoro si considera una procedura d'iterazione per la simultanea determinazione di tutti gli zeri di un polinomio nel caso che essi siano reali e distinti. Si ottengono condizioni sufficienti per la convergenza di una tale procedura.*

SUMMARY. - *We consider in this paper an iteration procedure for the simultaneous determination of all zeros of a polynomial in the case when they are real and distinct. The sufficient conditions for the convergence of that procedure are obtained.*

Let us consider the system of equations

(1) $$x_i = f_i(x_1, x_2, \dots, x_n) \quad (i = 1, 2, \dots, n)$$

where the f_i and their derivatives $\frac{\partial f_i}{\partial x_j}$ $(i, j = 1, 2, \dots, n)$ are real and continuous functions for $A_i \le x_i \le B_i$, where A_i and B_i are given numbers. Let $x'_1, x'_2, \dots, x'_n$ be an isolated solution of (1), with $A_i \le x'_i \le B_i$. As is known, the quoted isolated solution of the system (1) can be obtained by the iteration method. In order to obtain it we start with a given initial approximate solution $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}$, with $A_i \le x_i^{(0)} \le B_i$, and form the iteration sequences

(2) $$x_i^{(k+1)} = f_i(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}) \quad (i = 1, 2, \dots, n; k = 0, 1, 2, \dots) .$$

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If we set

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad f(X) = \begin{bmatrix} f_1(X) \\ f_2(X) \\ \vdots \\ f_n(X) \end{bmatrix},$$

then the system (1) and the iteration sequences (2) can be written as

$$(1') \qquad X = f(X)$$

and

$$(2') \qquad X^{(k+1)} = f(X^{(k)}) \quad (k = 0, 1, 2, \ldots).$$

For the derivative $f'(X)$ we now have

$$f'(X) = \left[\frac{\partial f_i}{\partial x_j}\right].$$

The domain of X, $f(X)$ and $f'(X)$ is $D\{A_i \le x_i \le B_i\}$, where A_i and B_i are given real numbers.

Here we use the norm notation

$$||X||_m = \max_i |x_i|.$$

With respect to domain D we shall introduce the norm

$$(3) \qquad ||f'(X)||_I = \max_{X \in D} ||f'(X)||_m,$$

where

$$(4) \qquad ||f'(X)||_m = \max_i \sum_{j=1}^{n} \left| \frac{\partial f_i(X)}{\partial x_j} \right|.$$

In our paper we shall use the following theorem related to the convergence of the iterative procedure (2′), that is (2), and which may be found in [1].

THEOREM 1. *Let the functions $f(X)$ and $f'(X)$ be continuous in the domain D, and, in D, let the inequality*

$$\|f'(X)\|_I \le q < 1 , \tag{5}$$

where q is a constant, hold true.

If the successive approximations

$$X^{(k+1)} = f(X^{(k)}) \quad (k = 0,1,2,\ldots) \tag{6}$$

lie in D, then the iteration process (6) *converges and the limiting vector*

$$X' = \lim_{k\to\infty} X^{(k)}$$

is the sole solution of system (1) *in domain D.*

COROLLARY 1. *The iteration process* (6) *converges if*

$$\sum_{j=1}^{n} \left| \frac{\partial f_i(X)}{\partial x_j} \right| \le q_i < 1 \quad (i = 1,2,\ldots,n) \tag{7}$$

when $X \in D$.

Obviously, from the system of inequalities (7) follows condition (5) of Theorem 1.

NOTE 1. For the approximation of $X^{(k)}$ the following estimate holds:

$$\|X' - X^{(k)}\|_m \le \frac{q^k}{1-q} \|X^{(1)} - X^{(0)}\|_m \quad (k = 1,2,\ldots) , \tag{8}$$

where $X^{(1)} = f(X^{(0)})$.

NOTE 1$'$. The convergence of the iterative process (2) (or (2$'$)) in the general case is linear.

Here a proof of Note 1$'$ will be given.

If the iterative process (2$'$) converges to the solution X' of the equation (1$'$), then

$$X' = f(X') . \tag{9}$$

Let $X^{(k)}$ be an approximate solution of equation (1′) obtained by the process (2′) for k large enough. Setting $X' = X^{(k)} + (X' - X^{(k)})$ and using Taylor's formula we obtain

$$f(X') = f(X^{(k)}) + f'(\bar{X})(X' - X^{(k)}) \tag{10}$$

whence $\bar{X} = X^{(k)} + \theta(X' - X^{(k)}), 0 < \theta < 1$.

Keeping in mind (2′) and (9), we obtain from (10)

$$X' - X^{(k+1)} = f'(\bar{X})(X' - X^{(k)}) ,$$

whence

$$||X' - X^{(k+1)}||_m \leq \max_{X \in D} ||f'(X)||_m ||X' - X^{(k)}||_m . \tag{11}$$

Because of (3), (4) and (5) we see that (11) is reduced to

$$||X' - X^{(k+1)}||_m \leq q||X' - X^{(k)}||_m . \tag{12}$$

From (12) we deduce that the convergence of the iterative process (2′) (or (2)) is linear, that is of the order 1.

1. Consider a real polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 , \quad (a_n \neq 0,\ n \geq 3) \tag{13}$$

with all zeros $x_1, x_2, \ldots, x_n$ real and distinct.

Let us set then

$$d = \min_{i \neq j} |x_i - x_j| \quad (i,j = 1,2,\ldots,n) .$$

For d one usually takes an estimate of the form

$$d \geq m \ (m > 0) ,$$

(see for example [2], also [3]).

When discussing a real polynomial whose all zeros x_i are real and distinct, which is our case, the lower bound for d can be also determined if all the zeros of the polynomial (13) are sufficiently separated using corresponding intervals.

Let

$$(14)\quad [b_i, d_i],\ (i = 1,2,\dots,n),\ b_1 < d_1 < b_2 < d_2 < \dots < b_n < d_n$$

denote intervals which contain the respective zeros x_i of the polynomial (13). Then

$$(d_i, b_{i+1}) \quad (i = 1,2,\dots,n-1)$$

are intervals in which the polynomial (13) has no zeros. It is obvious that

$$d \geq \min_i (b_{i+1} - d_i) \quad (i = 1,2,\dots,n-1)\ .$$

We shall write polynomial (13) in the form

$$a_n \prod_{j=1}^{n} (x - x_j) = P(x),\ \text{whence}\ \frac{1}{x - x_i} + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{x - x_j} = \frac{P'(x)}{P(x)}\ \text{and}$$

$$(15)\qquad \frac{1}{(x - x_i)^3} + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{(x - x_j)^3} = Q(x) \quad (i = 1,2,\dots,n)$$

where

$$(16)\quad Q(x) = \frac{P'''(x)P^2(x) - 3P''(x)P'(x)P(x) + 2(P'(x))^3}{2P^3(x)}\ .$$

We shall take respectively from each of the intervals (14) one point. Let these points be c_i $(i = 1,2,\dots,n)$, no one of them being a zero of (13). We then have from (15), for $x = c_i$,

$$(17)\qquad \frac{1}{(c_i - x_i)^3} + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{(c_i - x_j)^3} = Q(c_i) \quad (i = 1,2,\dots,n)\ .$$

We shall consider (17) as a system of n equations in n unknowns $x_1, x_2, \dots, x_n$ which can be written in the form

$$(18)\quad x_i = c_i - \left[Q(c_i) - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{(c_i - x_j)^3} \right]^{-\frac{1}{3}} \qquad (i = 1,2,\dots,n)\ .$$

The system (18) is of the form (1). We shall apply the method of iteration for its solution. Let $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}$ be initial approximate values of zeros of the polynomial (13), with $x_i^{(0)} \in [b_i, d_i]$, $(i = 1,2,\dots,n)$. Applying the iteration procedure (2) to the system (18) we obtain

$$(19) \qquad x_i^{(k+1)} = c_i - \left[Q(c_i) - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{(c_i - x_j^{(k)})^3} \right]^{-\frac{1}{3}},$$

$$(i = 1,2,\dots,n;\ k = 0,1,2,\dots) .$$

We shall demonstrate the following theorem, related to the convergence of iteration sequences (19).

THEOREM 2. *Let the polynomial* (13) *have all its zeros* $x_i (i = 1,2, \dots,n)$ *real and distinct contained in the intervals* (14), *respectively. If in every interval* (14) *we choose arbitrarily* c_i *and* $x_i^{(0)}$ *and if then all the approximations* $x_i^{(k+1)}$ $(k = 0,1,2,\dots)$ *obtained by the procedure* (19) *remain in the intervals* (14), *then the iteration sequences* (19) *converge respectively to the zeros* x_i *of the polynomial* (13), *when*

$$(20) \qquad \max_i (d_i - b_i) \leq \frac{d}{s},$$

where

$$d = \min_{i\neq j} |x_i - x_j| ,$$

s is a number greater than 2 for which

$$(21) \qquad \left[\left(\frac{s}{s-1} \right)^4 + \frac{\pi^4}{45} \right] \cdot \frac{1}{s^4} \leq q < 1$$

and q is a constant.

Proof. The system (18) is of the form (1). Denoting its right sides by f_i $(i = 1,2,\dots,n)$ respectively, we shall have

$$(22) \qquad \frac{\partial f_i}{\partial f_j} = \begin{cases} 0 \text{ for } j = i \\ -(c_i - x_i)^{-4} \left[Q(c_i) - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{(c_i - x_j)^3} \right]^{-\frac{4}{3}} & \text{for } j \neq i . \end{cases}$$

Substituting in the relation (22) the quantities $Q(c_i)$ by the corresponding values from (17) we shall obtain

$$(23) \qquad \frac{\partial f_i}{\partial f_j} = \begin{cases} 0 \text{ for } j = i \\ \frac{-(c_i - x_i)^4}{(c_i - x_j)^4} \text{ for } j \neq i \end{cases} .$$

By (23), the left sides in (7) reduce to

$$(24) \qquad \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{|c_i - x_i|^4}{|c_i - x_j|^4} \quad (i = 1, 2, \ldots, n) .$$

By hypothesis, the zeros $x_1, x_2, \ldots, x_n$ of (13) are real and distinct and the minimal distance between neighbouring zeros is $d > 0$. Let zeros $x_1, x_2, \ldots, x_n$ be in order

$$x_1 < x_2 < \ldots < x_n .$$

If $x_i, c_i, x_i^{(0)} \in [b_i, d_i]$, then, having in mind (20)

$$(25) \qquad |c_i - x_i| \le \max_i (d_i - b_i) \le \frac{d}{s}$$

and

$$(25') \qquad |x_i^{(0)} - x_i| \le \max_i (d_i - b_i) \le \frac{d}{s} \; (s > 2), \; (i = 1, 2, \ldots, n) .$$

If $c_i < x_i$, because of (25), we now have

$$(26) \qquad \begin{aligned} &|c_i - x_j| > (j - i) d \text{ for } j - i > 0, \\ &|c_i - x_j| > (i - j - 1) d \text{ for } i - j - 1 > 0 , \\ &|c_i - x_j| \ge \frac{s-1}{s} d \text{ for } i - j - 1 = 0 \; (i, j = 1, 2, \ldots, n) . \end{aligned}$$

If $c_i > x_i$, then, having in mind (25)

$$(27) \qquad \begin{aligned} &|c_i - x_j| > (j - i - 1) d \text{ for } j - i - 1 > 0, \\ &|c_i - x_j| \ge \frac{s-1}{s} d \text{ for } j - i - 1 = 0 , \\ &|c_i - x_j| > (i - j) d \text{ for } i - j > 0 \; (i, j = 1, 2, \ldots, n) . \end{aligned}$$

From (26) and (27) one obtains

$$(28)\quad \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{|c_i - x_j|^4} < \frac{1}{d^4}\left[\left(\frac{s}{s-1}\right)^4 + \right.$$
$$\left. + 2\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots + \frac{1}{\left(\frac{n-2}{2}\right)^4}\right)\right], \quad (i = 1, 2, \ldots, n)$$

for n even ≥ 4, and

$$(29)\quad \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{|c_i - x_j|^4} < \frac{1}{d^4}\left[\left(\frac{s}{s-1}\right)^4 + \right.$$
$$\left. + 2\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots + \frac{1}{\left(\frac{n-1}{2}\right)^4}\right)\right], \quad (i = 1, 2, \ldots, n)$$

for n odd ≥ 3.

Since

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = \frac{\pi^4}{90},$$

relations (28) and (29) reduce to

$$(30)\qquad \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{|c_i - x_j|^4} < \frac{1}{d^4}\left[\left(\frac{s}{s-1}\right)^4 + \frac{\pi^4}{45}\right].$$

Multiplying (30) by $|c_i - x_i|^4$, one obtains, keeping in mind (25)

$$(31)\quad \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{|c_i - x_i|^4}{|c_i - x_j|^4} < \left[\left(\frac{s}{s-1}\right)^4 + \frac{\pi^4}{45}\right] \cdot \frac{1}{s^4} \quad (i = 1, 2, \ldots, n).$$

Because of (21) we see that (31) reduces to

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{|c_i - x_i|^4}{|c_i - x_j|^4} \leq q < 1 \quad (i = 1, 2, \ldots, n),$$

which by (23) and (24) represents the conditions (7). The condition (20) implies that (7) is satisfied. In view of Corollary 1 and Theorem 1, the preceding fact proves Theorem 2.

NOTE 2. For the approximations of $x_i^{(k)}$ the following estimates hold

(32)
$$|x_i^{(k)} - x_i| \leq \frac{q^k}{1-q} \max_i |x_i^{(1)} - x_i^{(0)}| \quad (i = 1,2,\ldots,n;\ k = 1,2,\ldots) .$$

NOTE 2′. The convergence of the iterative process (19) is linear.

COROLLARY 2. *The iteration process* (19) *converges if*

(32′)
$$\max_i (d_i - b_i) \leq 0.4915563988\, d .$$

The estimate (32) follows from the estimate (8), while the conclusion in Note 2′ follows from the conclusion of Note 1′.

Set

(33)
$$F(s) = \left[\left(\frac{s}{s-1}\right)^4 + \frac{\pi^4}{45}\right] \cdot \frac{1}{s^4} .$$

For $s \in (0,1)$ we have $F(s) > 2$. For $s \in (1,\infty)$ the function $F(s)$ is continuous and monotonically decreasing, where $F(s) \to \infty$ when $s \to 1$ and $F(s) \to 0$ when $s \to \infty$. Thus there is only one value $s' \in (1,\infty)$ for which $F(s') = 1$, which means that s' is the positive root of the equation

(34)
$$\left[\left(\frac{s}{s-1}\right)^4 + \frac{\pi^4}{45}\right] \cdot \frac{1}{s^4} = 1 .$$

It is not hard to establish that for s' we have the bounds

$$2.034354556 < s' < 2.034354557 .$$

Besides that, for $s > s'$ one has

(35)
$$F(s) = q < F(s') < 1 .$$

In view of (33) the relation (35) represents the condition (21), which means that (20) holds for $s > s'$.

REMARK. We saw that the condition (20) is fulfilled for $s > s'$, where s' is the positive root of the equation (34). Therefore in (20) one may take $s = 2.034354557$. In that case (20) reduces to

$$\max_i (d_i - b_i) \leq 0.4915563988\, d, \tag{36}$$

so that Corollary 2 is proved.

For given $s > s'$, in view of (21), one has

$$q = \left[\left(\frac{s}{s-1} \right)^4 + \frac{\pi^4}{45} \right] \cdot \frac{1}{s^4} < 1, \tag{37}$$

and according to (20) one should have

$$\max_i (d_i - b_i) \leq \frac{d}{s}. \tag{38}$$

The constant q obtained in this way serves in (32) for the valuation of the approximations $x_i^{(k)}$.

From (37) we may deduce that q quickly decreases as s gets larger, which in view of (38) means that q decreases quickly when $\max_i (d_i - b_i)$ decreases.

For a given $q < 1$ one can find s from (37), and then from (38) determine how large $\max_i (d_i - b_i)$ should be. In the following table we give some cases:

q	s	$\max_i (d_i - b_i)$	q	s	$\max_i (d_i - b_i)$
0.5	2.246	$\leq 0.4452\, d$	0.001	7.900	$\leq 0.1265\, d$
0.1	2.940	$\leq 0.3401\, d$	0.0001	13.698	$\leq 0.0730\, d$
0.01	4.677	$\leq 0.2138\, d$	0.00001	24.390	$\leq 0.0410\, d$

In practice $\max_i (d_i - b_i)$ should be determined so that (36) holds. Then from (38) one takes $s = \frac{d}{\max_i (d_i - b_i)}$. With this value of s one determines the constant q from (37).

As mentioned, for the quantity d one usually gives the bound of the form $d \geq m$ ($m > 0$), so that the condition (20) will be satisfied for

$$\max_i (d_i - b_i) \leq \frac{m}{s}, \tag{39}$$

while the condition (36) will be satisfied when

$$\max_i (d_i - b_i) \leq 0.4915563988\, m. \tag{40}$$

In practice instead of d one most often operates with the quantity m.

In iterative procedures for simultaneous determination of all zeros x_i of a polynomial (when the zeros are all distinct) the initial values $x_i^{(0)}$ must be chosen so that

$$|x_i^{(0)} - x_i| \leq h \quad (i = 1, 2, \ldots, n).$$

To prove the convergence of these iterative procedures, the constant h, in the general case must satisfy the condition

$$h < \frac{d}{2}.$$

In some iterative procedures one requires that h be much smaller than $\frac{d}{2}$.

We mention here the procedure

(41)

$$x_i^{(k+1)} = x_i^{(k)} - \frac{P(x_i^{(k)})}{\prod_{\substack{j=1\\ j\neq i}}^{n} (x_i^{(k)} - x_j^{(k)})} \quad (i = 1, 2, \ldots, n;\ k = 0, 1, 2, \ldots)$$

which is used more than 20 years. The appearance of the procedure (41) is linked with the papers [4] and [5].

For the iterative procedure (41), K. Dočev [4] proved the following proposition:

If $d = \min_{i \neq j} |x_i - x_j|$ and

$$|x_i^{(0)} - x_i| < h \quad (i = 1, 2 \ldots n) \tag{41'}$$

where

$$h = d \cdot \frac{(1+q)^{1/(n-1)}}{2(1+q)^{1/(n-1)} - 1} \quad (0 < q < 1) , \tag{42}$$

then the iterative procedure (41) is convergent, where

$$|x_i^{(k)} - x_i| \le h q^{2^k - 1} \quad (i = 1, 2, \ldots, n;\ k = 0, 1, 2, \ldots) .$$

By the procedure (41), we deduce from (42) that the quantity h must be much smaller than $\frac{d}{2}$ and it decreases with the increase of the degree n of the polynomial. Thus for $n = 3$ we must have $h < 0.2265409197\ d$, for $n = 6$ we must have $h < 0.1146128658\ d$, and for $n = 10$ we must have $h < 0.0690099084\ d$. A similar situation arises with some interval methods for simultaneous determination of all zeros of a polynomial (see [6]).

Good aspects of the procedure (19) are in the following:

1) one can take wider intervals $[b_i, d_i]$ containing initial values $x_i^{(0)}$,
2) the widths of the intervals $[b_i, d_i]$, depending on (20), do not depend on the degree n of the polynomial,
3) a relatively small decrease of the intervals $[b_i, d_i]$ enables one to arrive at the approximations $x_i^{(k)}$ with a satisfying precision,
4) the calculation of the values $x_i^{(k+1)}$ are not complicated, since the quantities c_i and $Q(c_i)$ are not changed.

2. EXAMPLE. The procedure (19) will be applied to the Legendre polynomial of degree 6

$$P_6(x) = \frac{1}{16}(231 x^6 - 315 x^4 + 105 x^2 - 5) . \tag{43}$$

It is known that the Legendre polynomial of degree n

$$P_n(x) = \frac{1}{n! 2^n} \frac{d^n}{dx^n} (x^2 - 1)^n$$

has all zeros real and different and that they all lie in the interval $(-1, 1)$.

The zeros of the polynomial (43) are the zeros of the polynomial

$$P(x) = 231 x^6 - 315 x^4 + 105 x^2 - 5 \tag{44}$$

and they are symmetrically distributed with respect to the point 0. Since $P(-0.95) > 0$, $P(-0.92) < 0$; $P(-0.67) < 0$, $P(-0.64) > 0$; $P(-0.25) > 0, P(-0.21) < 0; P(0.21) < 0, P(0.25) > 0; P(0.64) > 0$, $P(0.67) < 0$; $P(0.92) < 0$, $P(0.95) > 0$; then in every interval $[b_i, d_i]$:

$$(45) \qquad [-0.95,-0.92],\ [-0.67,-0.64],\ [-0.25,-0.21]\ ,$$

$$[0.21,0.25],\ [0.64,0.67],\ [0.92,0.95]$$

lies a zero of the polynomial (44). Here

$$(46) \qquad \max_i (d_i - b_i) = 0.04$$

and

$$(47) \qquad \min_i (b_{i+1} - d_i) = 0.25 = m \leq d\ .$$

According to (47) we have

$$0.4915563988\ m = 0.1228890997$$

which in view of (46) means that

$$(48) \qquad \max_i (d_i - b_i) < 0.4915563988\ m\ .$$

Taking into account (47) we obtain from (48)

$$\max_i (d_i - b_i) < 0.4915563988\ d\ ,$$

so that also the condition (32′) is fulfilled. According to (39), we now have $s = \frac{m}{\max_i(d_i - b_i)} = 6.25$. For $s = 6.25$ from (37) we obtain

$$q < 0.002735\ .$$

If for quantities $c_i \in [b_i, d_i]$ and $x_i^{(0)} \in [b_i, d_i]$ we take for example:

$c_1 = -0.94\,, c_2 = -0.63\,, c_3 = -0.23\,, c_4 = 0.22\,, c_5 = 0.65\,, c_6 = 0.93\,,$

$$x_1^{(0)} = -0.93\,,\ x_2^{(0)} = -0.65\,,\ x_3^{(0)} = -0.24\,,$$
$$x_4^{(0)} = 0.24\,,\ x_5^{(0)} = 0.66\,,\ x_6^{(0)} = 0.94\ ,$$

then by applying the procedure (19), taking into account (16), we set approximations:

$$\begin{array}{lll}
x_1^{(1)} = -0.9324695197\,, & x_1^{(2)} = -0.9324695142\,, & x_1^{(3)} = -0.9324695142 \\
x_2^{(1)} = -0.6612096179\,, & x_2^{(2)} = -0.6612093865\,, & x_2^{(3)} = -0.6612093865 \\
x_3^{(1)} = -0.2386191882\,, & x_3^{(2)} = -0.2386191861\,, & x_3^{(3)} = -0.2386191861 \\
x_4^{(1)} = 0.2386191878\,, & x_4^{(2)} = 0.2386191861\,, & x_4^{(3)} = 0.2386191861 \\
x_5^{(1)} = 0.6612093679\,, & x_5^{(2)} = 0.6612093865\,, & x_5^{(3)} = 0.6612093865 \\
x_6^{(1)} = 0.9324695142\,, & x_6^{(2)} = 0.9324695142\,, & x_6^{(3)} = 0.9324695142
\end{array}$$

According to (32), in our case

$$|x_i^{(2)} - x_i| < 0.0000000841 \quad (i = 1, 2, \ldots, 6)\ .$$

For the application of the procedure (41), according to (41′) and (42) we must have $|x_i^{(0)} - x_i| < 0.1146128658\ m = 0.0286532165$, that is $\max_i (d_i - b_i) < 0.0286532165$, which is not fulfilled here. This means that in applying the procedure (41) the initial values $x_i^{(0)}$ must be taken from smaller intervals than the intervals (45).

When dealing with a complex or real polynomial $P(z)$ of degree n, whose all zeros $z_1, z_2, \ldots, z_n$ are distinct, then we have the following proposition:

If $\bar{z}_i$ is an approximate value of a zero z_i of a polynomial $P(z)$ and if

$$\left|\frac{nP(\bar{z}_i)}{P'(\bar{z}_i)}\right| < \frac{m}{2} \leq \frac{d}{2}\,, \tag{49}$$

then

$$|\bar{z}_i - z_i| < \frac{|P(\bar{z}_i)|}{|P'(\bar{z}_i)| - \frac{2n-2}{m}|P(\bar{z}_i)|} \tag{50}$$

independently of the way we obtained the approximate value $\bar{z}_i$. The proposition (50) may be found in [7].

In our case, for $\bar{z}_i = x_i^{(2)}$ $(i = 1,2,\ldots,6)$, the conditions (49) are fulfilled. According to (50) we have

$$|x_i^{(2)} - x_i| < 0.000000000036 \quad (i = 1,2,\ldots,6)\ .$$

Let us state at the end that there exist several procedures for the simultaneous determination of all zeros of a polynomial. Some of them may be found in [8] and [9].

References

[1] DEMIDOVICH B.P. and MARON I.A., *Computational mathematics*, translated from russian by George Yankovsky, Mir publishers Moscow, Third printing 1981.

[2] BEREZIN I.S. and ŽIDKOV N.P., *Metodi vičislenij*, tom II, p. 89, Moscow 1960 (in Russian).

[3] PREŠIĆ S.B., *On the minimal distance of the zeros of a polynomial*, Publ. Inst. Math. *38* (52) (1985), 35-38.

[4] DOČEV K., *A modified Newton method for the simultaneous approximation of all roots of the given algebraic equation* (in Bulgarian), Fiz.-Math.Spis. Bulgar, Akad. Nauk *5* (1962), 136-139.

[5] PREŠIĆ S.B., *Un procédé itératif pour la factorisation des polynomes*, C.R. Acad. Sci. Paris *262* (1966), 862-863.

[6] KJURKCHIEV N.V. and MARKOV S.M., *Two interval methods for algebraic equations with real roots*, PLISKA Studia matematica bulgarica, Vol.5 (1983), 118-131.

[7] SIMEUNOVIĆ D.M., *On some estimates of approximate values of zeros of polynomials* (to appear in ZAMM-Z. angew. Math. Mech. *70* (1990).

[8] EHRLICH L.W., *A modified Newton method for polynomials*, Comm. ACM *10* (1967), 107-108.

[9] NOUREIN A.W., *An improvement on two iteration methods for simultaneous determination of the zeros of a polynomial*, Internat. J. Comput. Math. *3* (1977), 241-252.