

Remarks on Dirichlet problems with sublinear growth at infinity

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ABSTRACT. *We present some existence and multiplicity results for positive solutions to the Dirichlet problem associated with*

$$\Delta u + \lambda a(x)g(u) = 0,$$

under suitable conditions on the nonlinearity $g(u)$ and the weight function $a(x)$. The assumptions considered are related to classical theorems about positive solutions to a sublinear elliptic equation due to Brezis-Oswald and Brown-Hess.

Keywords: Boundary value problems, positive solutions, indefinite weight, bifurcation, multiplicity results.

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1. Introduction

This paper deals with the study of the Dirichlet problem

$$(\mathcal{D}) \quad \begin{cases} -\Delta u = \lambda a(x)g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

depending on the positive real parameter λ . The aim of this work is twofold. We present a partial survey and also give some new considerations about results of existence, uniqueness and multiplicity of positive solutions for the nonlinear eigenvalue problem (\mathcal{D}) under suitable conditions on the weight function $a(x)$ and the nonlinear term $g(u)$. By a *positive solution* of (\mathcal{D}) (or any other related equation considered in the paper) we mean a weak, strong or classical solution, depending on the properties of a , g and the domain Ω , such that $u(x) > 0$ for every $x \in \Omega$. For the moment, the definition given is deliberately broad in order to include different regularity conditions, which depend also on different approaches followed by the authors, that we are going to analyze. In the sequel we will introduce some specific conditions ensuring that the solutions we find are strong solutions which actually belong to $C_0^1(\bar{\Omega}) \cap C^{1,\theta}(\bar{\Omega})$ (for

each $\theta \in]0, 1[$). Furthermore, we refer to a *positive solution pair* for (\mathcal{D}) as a pair (λ, u) when we want to emphasize that $u(\cdot)$ is a positive solution of (\mathcal{D}) corresponding to the given parameter $\lambda > 0$. The set of all positive solutions pairs will be denoted by \mathcal{S} .

Starting with the Seventies, several authors have investigated the structure of the set of positive solutions assuming different features for the weight function and the nonlinearity. The case of the so-called *indefinite weight*, that is when $a(\cdot)$ changes its sign, has attracted much attention during the past decades. In this respect we recall the pioneering works of Manes and Micheletti [33], Hess and Kato [26], Brown and Lin [10] and López-Gómez [30] concerning the properties related to the principal eigenvalue. The research on the positive solutions for nonlinear indefinite weight problems has grown up at the end of the Eighties (see, for instance, [1, 2, 4, 5, 7, 9]) and is still a very active area of investigation. Concerning the nonlinearity, a great deal of results has been obtained when $g(u) = u^p$. In such a situation we usually refer to *superlinear* or *sublinear* problems according to the fact that $p > 1$ or $0 < p < 1$.

In this paper we focus our attention to the case in which g has a *linear growth at zero* and a *sublinear growth at infinity*. Problems where the nonlinearity presents such kind of growth naturally arise in the study of the steady states for reaction diffusion equations occurring in various mathematical models from population genetics or ecology. In this framework, typical assumptions require that the following limits

$$g_0 := \lim_{s \rightarrow 0^+} \frac{g(s)}{s}, \quad g_\infty := \lim_{s \rightarrow +\infty} \frac{g(s)}{s}$$

exist and are finite. The conditions of linear growth at zero and sublinear growth at infinity are then expressed by

$$(H_g) \quad g_0 > 0 = g_\infty.$$

In the present article, we will also consider more general behaviors for g at infinity.

The outline of this paper is as follows. In Section 2 we focus our attention on two main results of positive solutions for the sublinear elliptic Dirichlet problem, namely the theorem by Brezis and Oswald [8] and the one by Brown and Hess [9]. In particular, we compare such theorems in the case of problem (\mathcal{D}) . In Section 3 we restrict ourselves to the study of the one-dimensional ODE and, for the particular case of a constant weight, we present some bifurcation diagrams for the positive solutions by means of the analysis of the associated time-maps. This is a very classical approach which has been exploited by many authors (e.g. [11, 13, 29, 42, 44, 45]). In this context we show how some time-mapping estimates achieved by Opial (cf. [39]) turn out to be useful for our problem. In particular, this analysis suggests that we can get the existence

of positive solutions for every λ large if the condition (H_g) is replaced by the more general one

$$g_0 > 0 = \liminf_{s \rightarrow +\infty} \frac{\int_0^s g(\xi) d\xi}{s^2} .$$

The latter assumption is required in Section 4, in order to prove a result about existence of positive solutions for the general problem (\mathcal{D}) . Actually, we prove much more. Indeed, in the frame of Rabinowitz’s global bifurcation theorem we provide the existence of a bifurcation branch of positive solution pairs (λ, u) which is unbounded both in the u and the λ components (see Theorem 4.1 and Theorem 4.5). Our proof is inspired by some arguments developed in previous works by Hess and Kato [26], Coelho, Corsato, Obersnel and Omari [12] and, furthermore, Obersnel and Omari [36]. We also produce a counterexample (see Proposition 4.6) which shows that our assumptions are, in some sense, optimal. Section 5 is devoted to the comparison of the different uniqueness assumptions considered by Brezis-Oswald and Brown-Hess, by means of the analysis of an ODE equation with an indefinite weight. We give evidence (via numerical simulations) of the possibility of multiplicity results even if the map $s \rightarrow g(s)/s$ is decreasing on the positive real line. Finally, in Section 6 we briefly discuss how to extend our main results to a general linear second order strongly uniformly elliptic operator.

2. Remarks on Brezis-Oswald and Brown-Hess theorems

Let us begin by fixing some notations. Throughout the paper we denote by $\mathbb{R}^+ := [0, +\infty)$ the set of non negative real numbers and by $\mathbb{R}_0^+ :=]0, +\infty)$ the set of positive reals. Moreover, for $N \geq 1$ we suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain (i.e. open and connected) with sufficiently regular boundary. Specific conditions on $\partial\Omega$ will be given in Section 4.

Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying

$$(g_*) \quad g(0) = 0, \quad g(s) > 0 \quad \text{for } s > 0.$$

Let also $w \in L^\infty(\Omega) \setminus \{0\}$. In this setting we consider the following Dirichlet problem

$$\begin{cases} -\Delta u = w(x)g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

In [9], Brown and Hess presented a theorem on the existence and uniqueness of classical positive solutions for problem (1), assuming among other conditions that g and w are smooth functions and g is concave with a sublinear growth at infinity. The approach followed by the authors to prove the existence of nontrivial solutions is based on the use of the fixed point index in the framework of positive operators.

Another relevant contribution to sublinear elliptic equations was given by Brezis and Oswald in [8]. Their result applies to a more general Dirichlet problem of the form:

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

where $f : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is such that

- i) $s \mapsto f(x, s)$ is continuous on \mathbb{R}^+ for a.e. $x \in \Omega$;
- ii) $x \mapsto f(x, s)$ belongs to $L^\infty(\Omega)$ for every $s \geq 0$;
- iii) there is a constant $C > 0$ such that $f(x, s) \leq C(s+1)$ for a.e. $x \in \Omega$ and for every $s \geq 0$;
- iv) for each $\delta > 0$ there is a constant $C_\delta > 0$ such that $f(x, s) \geq -C_\delta s$ for a.e. $x \in \Omega$ and for every $s \in [0, \delta]$.

The approach in [8] is variational and the (weak) solutions u of (2) belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$. Consequently, by regularity theory, $u \in W^{2,p}(\Omega)$ for every $p < \infty$. In more recent years some extensions of [8] have been obtained, on the one hand considering some general second order elliptic operator instead of the Laplacian [6], and, on the other hand, relaxing the assumptions for the uniqueness of the solutions [27].

In our setting concerning problem (1), a consequence of the main results in [8] reads as follows.

THEOREM 2.1. *Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) and let $w \in L^\infty(\Omega) \setminus \{0\}$. Let us suppose that g_0 and g_∞ are finite and also*

$$\lambda_1(-\Delta - w(x)g_0) < 0 < \lambda_1(-\Delta - w(x)g_\infty). \quad (3)$$

Then there exists at least one positive solution $u(\cdot)$ to (1) with $u \in C_0^1(\overline{\Omega})$. Moreover, if $w(x) > 0$ for a.e. $x \in \Omega$ and $s \mapsto g(s)/s$ is decreasing on \mathbb{R}_0^+ , then the positive solution is unique and condition (3) is necessary, too.

Proof. We are going to apply [8, Theorem 2] with the position

$$f(x, s) := w(x)g(s).$$

First of all, conditions i) and ii) are obviously satisfied. Using the fact that $g(s)/s$ is continuous and positive on \mathbb{R}_0^+ with finite limits at zero and at infinity, we can find a positive constant $K := \sup_{s>0} \{g(s)/s\} < \infty$ such that $|f(x, s)| \leq \|w\|_\infty K s$, for all $s \geq 0$ and for a.e. $x \in \Omega$. In this way the growth conditions iii) and iv) are satisfied, too.

In order to complete the verification of the assumptions in [8, Theorem 2] we have also to check that

$$\lambda_1(-\Delta - a_0(x)) < 0 < \lambda_1(-\Delta - a_\infty(x)),$$

where

$$a_0(x) := \lim_{s \rightarrow 0^+} \frac{f(x, s)}{s}, \quad a_\infty(x) := \lim_{s \rightarrow +\infty} \frac{f(x, s)}{s}.$$

This clearly follows from (3). At this point, [8, Theorem 2] applies and ensures the existence of a nontrivial (weak) nonnegative solution $u(\cdot)$ to problem (1). By elliptic regularity theory, such a solution belongs to $C_0^1(\bar{\Omega})$ and, moreover, is strictly positive on Ω with negative outward derivative on $\partial\Omega$ (cf. also [8, Lemma 1]). About the uniqueness of the positive solution, we just observe that the conditions $w(x) > 0$ on Ω and $g(s)/s$ decreasing on \mathbb{R}_0^+ , imply that the map $s \mapsto f(x, s)/s$ is decreasing on \mathbb{R}_0^+ . Therefore, as a result of [8, Theorem 1], the conclusion follows. \square

Now we can make a first analysis of the nonlinear eigenvalue problem:

$$(\mathcal{D}) \quad \begin{cases} -\Delta u = \lambda a(x)g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$. For the weight function $a(x)$ we suppose, as in [26], that

$$(a_*) \quad a \in C(\bar{\Omega}) \text{ and there exists } x_0 \in \Omega \text{ such that } a(x_0) > 0.$$

Similar results can be obtained for

$$(a_{**}) \quad a \in L^\infty(\Omega) \text{ with } |\Omega^+| > 0$$

where $\Omega^+ := \{x \in \Omega : a(x) > 0\}$.

A direct application of Theorem 2.1 to the Dirichlet problem (\mathcal{D}) leads to the next results.

COROLLARY 2.2. *Let a satisfy (a_*) and let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) and (H_g) . Then, there exists $\Lambda^* > 0$ such that problem (\mathcal{D}) has a positive solution for each $\lambda > \Lambda^*$.*

Proof. We start by observing that the second inequality in (3) is trivially satisfied as it refers to the positivity of the first eigenvalue of $-\Delta$ with the Dirichlet boundary conditions. Therefore, we have only to check, for $\lambda > 0$ sufficiently large, the negativity of the first eigenvalue μ_1 of the problem

$$-\Delta u - \lambda g_0 a(x)u = \mu u, \quad u|_{\partial\Omega} = 0. \tag{4}$$

To this aim we recall some basic facts from the weighted eigenvalue problem

$$-\Delta u = \nu a(x)u, \quad u|_{\partial\Omega} = 0. \quad (5)$$

Under assumption (a_*) , according to [10, 14, 26, 33], there exists a sequence of real eigenvalues

$$0 < \nu_1 < \nu_2 \leq \nu_3 \leq \dots$$

to problem (5), with $\nu_n \rightarrow \infty$. Moreover, the principal eigenvalue ν_1 is simple with an associated positive eigenfunction (see, for instance, [14, Proposition 1.11 (c) and Theorem 1.13]). In such a situation, we can prove the thesis by taking

$$\Lambda^* := \nu_1/g_0. \quad (6)$$

Indeed, let us fix $\lambda > \Lambda^*$ and check that the principal eigenvalue μ_1 of (4) is negative. Let ϕ be the corresponding positive eigenfunction, so that ϕ satisfies

$$-\Delta\phi(x) - \nu a(x)\phi(x) = \mu_1\phi(x) := h(x), \quad \phi|_{\partial\Omega} = 0, \quad \phi(x) > 0 \text{ for } x \in \Omega,$$

with $\nu := \lambda g_0 > \Lambda^* g_0 = \nu_1$. If, by contradiction, $\mu_1 \geq 0$, then $h \geq 0$ and we enter in the setting of [26, Proposition 3] which, in turns, implies that $h = 0$ and $\nu = \nu_1$. The last equality clearly contradicts our choice of λ . Hence, $\mu_1 < 0$ and also the first inequality in (3) is satisfied. By Theorem 2.1 we are done.

The details of our proof are given only for completeness since the fact that $\mu_1 < 0$ for $\lambda > \nu_1/g_0$ is already contained in [25, Statement (1.15)], while the existence of $(\nu_1/g_0, 0)$ as a bifurcation point for positive solutions, is a main result in [26]. \square

We observe that Corollary 2.2 is basically a subcase of a general result by Brown and Hess (see for instance [9, Theorem 3 (ii) and Theorem 4]). Actually, in [9] the authors obtain a result of existence and uniqueness of positive classical solutions *if and only if* $\lambda > \nu_1/g_0$, provided that a and g are smooth functions with $g''(s) < 0$ for all $s > 0$. However, in absence of concavity type condition, we cannot guarantee (in general) the uniqueness of the positive solution (see Figure 1 (a)) or the fact that positive solutions exist *only if* $\lambda > \nu_1/g_0$ (see Figure 1 (b)). Even more complex situations may arise (see Figure 1 (c)).

COROLLARY 2.3. *Let a satisfy (a_*) and let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) and (H_g) .*

- *If $a(x) > 0$ for a.e. $x \in \Omega$ and $s \mapsto g(s)/s$ is decreasing on \mathbb{R}_0^+ then problem (\mathcal{D}) has a positive solution if and only if $\lambda > \nu_1/g_0$ and such a positive solution is unique [8].*
- *If $a(x)$ changes sign and, moreover, $g(s)$ is smooth on \mathbb{R}_0^+ with $g''(s) < 0$ for all $s > 0$, then problem (\mathcal{D}) has a positive solution if and only if $\lambda > \nu_1/g_0$ and such a positive solution is unique [9].*

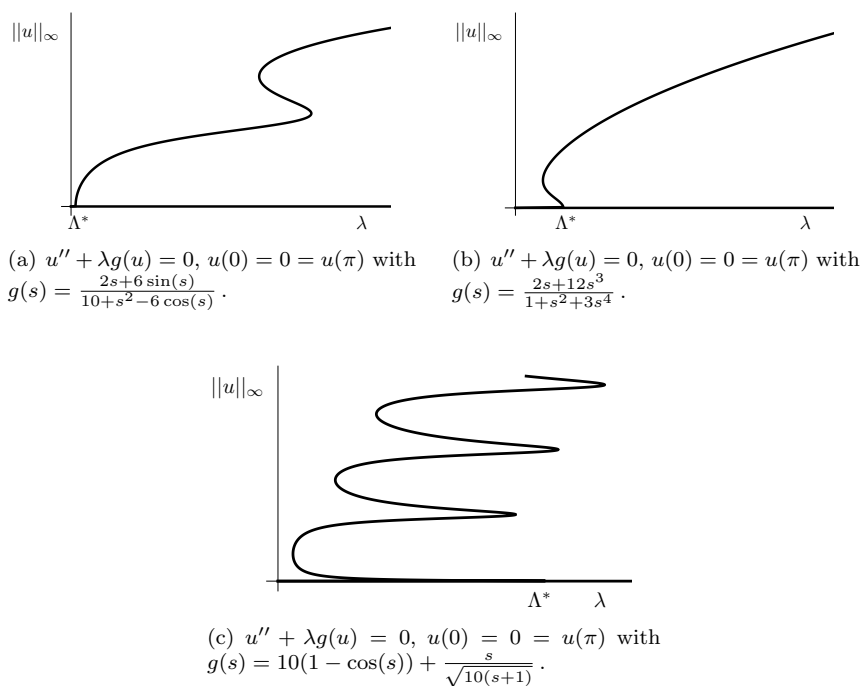


Figure 1: Bifurcation diagrams for one-dimensional Dirichlet (two-point boundary) problems.

Proof. The first part of the statement follows from Theorem 2.1, with the condition $\lambda > \nu_1/g_0$ obtained in the same manner as (6) in the proof of Corollary 2.2. The second part of the statement is precisely [9, Theorem 4]. \square

Note that if $g(s)$ is any strictly concave function satisfying (g_*) , then the map $s \mapsto g(s)/s$ is decreasing on \mathbb{R}_0^+ . The converse does not hold, a simple example is given by $g(s) = s/(1 + s^2)$. In this respect, a natural question is whether the result of uniqueness under the monotonicity request on $g(s)/s$ is still true also if the weight coefficient changes its sign. In general, the answer is negative even in the one-dimensional case, as it can be seen by the numerical analysis of some non-autonomous ODEs. A function with these features is presented in the next example. A more detailed discussion will be delivered in Section 5.

EXAMPLE 2.4. *Let*

$$g(s) := Ase^{(-Bs^2)} + \frac{s}{1 + |s|}, \quad A, B > 0. \tag{7}$$

Observe that $g_0 = A + 1$, $g_\infty = 0$ and $g(s)/s$ is strictly decreasing but not concave on \mathbb{R}_0^+ . There exist indefinite weights $a \in C([-T, T])$ such that for some values of $\lambda > 0$ the problem

$$\begin{cases} u'' + \lambda a(x)g(u) = 0, \\ u(-T) = 0 = u(T), \end{cases} \quad (8)$$

has multiple positive solutions.

If we restrict ourselves to the *autonomous case*, i.e. the case of a constant weight $a(x) \equiv 1$, problem (D) reduces to the following one

$$\begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (9)$$

where $\lambda > 0$. As already observed in [8, page 56], the next result holds.

COROLLARY 2.5. *Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) and (H_g) . Then problem (9) has a positive solution if*

$$\lambda > \lambda_1^* := \frac{\lambda_1(-\Delta)}{g_0}. \quad (10)$$

Moreover, if the map $s \mapsto g(s)/s$ is decreasing on \mathbb{R}_0^+ such positive solution is unique and (10) is also a necessary condition.

3. Phase-plane analysis in one dimension

In the one-dimensional case $N = 1$ we take as a domain an open interval $\Omega :=]a, b[$ and reduce problem (9) to the two-point boundary value problem

$$\begin{cases} u'' + \lambda g(u) = 0, \\ u(a) = 0 = u(b), \end{cases} \quad (11)$$

with $\lambda > 0$. As usual in this case, we indicate by $x = t$ the independent variable.

The set of positive solutions pairs is given by

$$\mathcal{S} = \{(\lambda, u) \in \mathbb{R}_0^+ \times C_0^1([a, b]) : u(\cdot) \text{ is a positive solution of (11)}\}.$$

Without loss of generality (due to the autonomous nature of system (11)) we also set

$$L := b - a$$

and observe that problem (11) is equivalent to

$$\begin{cases} u'' + \lambda g(u) = 0, \\ u(-L/2) = 0 = u(L/2). \end{cases}$$

In such a simplified setting, we can provide an interpretation of Corollary 2.5 in terms of time-mappings associated with the planar autonomous system

$$u' = y, \quad y' = -g(u), \tag{12}$$

which is equivalent to the scalar equation

$$u'' + g(u) = 0. \tag{13}$$

Since up to now we have assumed $g(s)$ to be defined only for $s \geq 0$, for convenience we take an odd extension of g on \mathbb{R} in order to have the solutions $(u(\cdot), y(\cdot))$ of (12) globally defined in the plane.

System (12) is conservative with energy

$$E(u, y) := \frac{1}{2}y^2 + G(u),$$

where

$$G(s) := \int_0^s g(\xi) d\xi.$$

Observe that the map $G : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies $G(0) = 0$ and is strictly increasing. For every $c > 0$, the solution $(u(\cdot), y(\cdot))$ of (12) satisfying the initial condition $(u(0), y(0)) = (c, 0)$ is unique, periodic and defined on the whole real line. We denote such a solution with (u_c, y_c) only when we want to stress its dependence on the parameter c .

The corresponding orbit/trajectory is given by the energy level set

$$\frac{1}{2}y^2 + G(u) = G(c).$$

From this relation we obtain that $u'(t) = \sqrt{2(G(c) - G(u(t)))}$ for all t in the maximal interval $[\alpha, 0]$ where both $u(t) \geq 0$ and $u'(t) = y(t) \geq 0$. Integrating $u'(t)/\sqrt{2(G(c) - G(u(t)))}$ on $[\alpha, 0]$ we can determine α as

$$\int_0^c \frac{ds}{\sqrt{2(G(c) - G(s))}} = \alpha.$$

This suggest to introduce the time-mapping formula

$$T(c) := 2 \int_0^c \frac{ds}{\sqrt{2(G(c) - G(s))}}. \tag{14}$$

In other words, $T(c)$ is the distance of two consecutive zeros of the solution $u(\cdot)$ of (13), where $u(t) \geq 0$ for all $t \in \mathbb{R}$ and $\|u\|_\infty = \max_{t \in \mathbb{R}} u(t) = c$. The time-mapping

$$\mathbb{R}_0^+ \ni c \mapsto T(c) \in \mathbb{R}_0^+$$

is a continuous function. By a rescaling in the time variable, it is straightforward to check what follows.

PROPOSITION 3.1. *Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) and let $R > 0$ be a fixed constant. For each $c > 0$, let us define*

$$v_{c,R}(t) := u_c\left(\frac{T(c)}{R}\left(t - \frac{a+b}{2}\right)\right).$$

Then, $v_{c,R}(t)$ is a solution of the equation

$$v'' + \left(\frac{T(c)}{R}\right)^2 g(v) = 0$$

with

$$v\left(\frac{a+b}{2}\right) = c, \quad v'\left(\frac{a+b}{2}\right) = 0$$

and, moreover, the following cases occur:

- $v_{c,R}(t) > 0 \forall t \in [a, b]$ if and only if $R > L$,
- $v_{c,R}(t) > 0 \forall t \in]a, b[$ with $v_{c,R}(a) = 0 = v_{c,R}(b)$ if and only if $R = L$,
- $v_{c,R}(t)$ vanishes in $]a, b[$ if and only if $R < L$.

Considering the second instance in the above proposition, we get immediately what follows.

PROPOSITION 3.2. *Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) . Then, problem (11) has a positive solution $u(\cdot)$ for some $\lambda > 0$ if and only if*

$$\lambda = \tau(c) := \left(\frac{T(c)}{L}\right)^2, \quad \text{for } c = \|u_c\|_\infty. \quad (15)$$

Moreover, the set \mathcal{S} of positive solution pairs is the Cartesian graph of a continuous curve

$$c \mapsto (\tau(c), v_c(\cdot)),$$

where

$$v_c(t) := u_c\left(\frac{T(c)}{L}\left(t - \frac{a+b}{2}\right)\right).$$

Proposition 3.2 permits to study the global bifurcation branches for positive solutions of (11) by analyzing the behavior of the time-mapping $T(\cdot)$. This approach has been already widely exploited by many authors under several different conditions on the nonlinearity (see, for instance, the classical works [29, 42, 45]). The behavior of $T(c)$ as $c \rightarrow 0^+$ or $c \rightarrow +\infty$, as well as other qualitative properties, like monotonicity, has been analyzed by Opial in [39]. In particular, according to [39], if the limits g_0 and g_∞ exist, then

$$\lim_{c \rightarrow 0^+} T(c) = \frac{\pi}{\sqrt{g_0}} \quad \text{and} \quad \lim_{c \rightarrow +\infty} T(c) = \frac{\pi}{\sqrt{g_\infty}}.$$

Moreover, $T(\cdot)$ is increasing (respectively, decreasing) on \mathbb{R}_0^+ provided that the map $s \mapsto g(s)/s$ is decreasing (respectively, increasing) on \mathbb{R}_0^+ .

If both g_0 and g_∞ are positive real numbers, then, by Proposition 3.2 we can recover a bifurcation result of Ambrosetti and Hess [3, Theorem A (iii)]. In fact, in this case, the set \mathcal{S} turns out to be a Cartesian graph joining the bifurcation point $(\pi/L)^2/g_0$ from the trivial solution to the bifurcation point $(\pi/L)^2/g_\infty$ from infinity.

On the other hand, from (H_g) we obtain

$$\lim_{c \rightarrow 0^+} T(c) = \frac{\pi}{\sqrt{g_0}} \quad \text{and} \quad \lim_{c \rightarrow +\infty} T(c) = +\infty.$$

Moreover, under the assumptions of Corollary 2.5, the map

$$\mathbb{R}_0^+ \ni c \mapsto \tau(c) \in \mathbb{R}_0^+$$

is monotone with

$$\inf \tau = \left(\frac{\pi}{L}\right)^2 / g_0 = \lambda_1^* \quad \text{and} \quad \sup \tau = +\infty.$$

From this point of view, one could say that Opial's monotonicity condition for the time-mapping is a dynamical interpretation of the uniqueness condition of Brezis-Oswald.

The inversion of $\tau(\cdot)$ complements Corollary 2.5 with a global bifurcation result in the sense that it ensures also the continuity of the map

$$] \lambda_1^*, +\infty) \ni \lambda \mapsto u_\lambda(\cdot),$$

where u_λ is the unique positive solution of (9) for a given λ (compare with [25, 26]).

The time-mapping approach based on Proposition 3.2 suggests the possibility of improving condition (H_g) . More precisely, if we are looking for positive solution pairs (λ, u) of (11) for all λ in an unbounded interval, we can replace the hypothesis $g_\infty = 0$ with appropriate assumptions which yet

ensure that $\sup \tau = +\infty$. For example, if we are interested in proving that $T(+\infty) = +\infty$, it will be sufficient to suppose that $G(s)/s^2 \rightarrow 0$ as $s \rightarrow +\infty$ (cf. [39, Théorème 11]), which is a more general condition than $g_\infty = 0$. With this purpose, we introduce the following constants

$$G_\infty := \liminf_{s \rightarrow +\infty} \frac{2G(s)}{s^2}, \quad G^\infty := \limsup_{s \rightarrow +\infty} \frac{2G(s)}{s^2}.$$

By the generalized L'Hôpital's rule, we know that

$$\liminf_{s \rightarrow +\infty} \frac{g(s)}{s} \leq G_\infty \leq G^\infty \leq \limsup_{s \rightarrow +\infty} \frac{g(s)}{s}.$$

Using [39, Corollaire 11], we find that

$$G_\infty = 0 \implies \limsup_{c \rightarrow +\infty} \tau(c) = +\infty. \quad (16)$$

Moreover, from [39, Théorème 16], we also know that

$$\limsup_{c \rightarrow +\infty} \tau(c) = +\infty \implies \liminf_{s \rightarrow +\infty} \frac{g(s)}{s} = 0.$$

In this setting, we obtain the following result which improves Corollary 2.5 in the one-dimensional case.

PROPOSITION 3.3. *Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) and suppose that the following hypothesis*

$$(H_G) \quad g_0 > 0 = G_\infty$$

holds. Then, the set \mathcal{S} of positive solutions pairs (λ, u) to problem (11) is a continuous curve which bifurcates from $(\lambda_1^, 0)$ and such that for each $\lambda > \lambda_1^*$ there exists at least one positive solution $u(\cdot)$ of (11) with $(\lambda, u) \in \mathcal{S}$. Furthermore, if $G^\infty > 0$, then, for each*

$$\lambda > \eta_* := \lambda_1(-\Delta)/G^\infty$$

there is an unbounded set of positive solutions $u(\cdot)$ of (11) with $(\lambda, u) \in \mathcal{S}$.

Proof. From (16) we know that assumption (H_G) implies

$$\lim_{c \rightarrow 0^+} \tau(c) = \lambda_1^* \quad \text{and} \quad \limsup_{c \rightarrow +\infty} \tau(c) = +\infty.$$

Thus, the continuity of the map $\tau(\cdot)$ on \mathbb{R}_0^+ implies that the range of τ contains the interval $] \lambda_1^*, +\infty)$. Then the first part of the claim follows from Proposition 3.2. On the other hand, since $G(\cdot)$ is monotone increasing, if we also

suppose that $G^\infty > 0$, then necessarily $G(s) \rightarrow +\infty$ as $s \rightarrow +\infty$. In this manner, we enter in the setting of [39, Corollaire 12] and so we have

$$\liminf_{c \rightarrow +\infty} T(c) \leq \pi/\sqrt{G^\infty}.$$

Hence

$$\liminf_{c \rightarrow +\infty} \tau(c) \leq \left(\frac{\pi}{L}\right)^2 / G^\infty = \eta_*.$$

We conclude that for each $\lambda \in]\eta_*, +\infty)$ the equation $\tau(c) = \lambda$ has infinitely many solutions. In fact,

$$\liminf_{c \rightarrow +\infty} \tau(c) < \lambda < \limsup_{c \rightarrow +\infty} \tau(c)$$

and, by the intermediate value theorem, there is a sequence $c_n \rightarrow +\infty$ of solutions of the equation $\tau(c) = \lambda$. To each such a solution $c_n > 0$ there corresponds a unique positive solution $u_n(\cdot)$ of (11) with $\|u\|_\infty = c_n$. Then also the second part of the claim follows from Proposition 3.2. \square

The consequence about the existence of infinitely many positive solutions is not related to the condition $g_0 > 0$ as it involves only the behavior of the time-mapping at infinity. In particular, infinitely many solutions can occur also when $G_\infty > 0$ as one can see in [17, 32, 35, 36, 38]. In this context, the following result can be given for problem (11) using Opial’s estimates, where we use the convection $1/0^+ = +\infty$ and $1/\infty = 0$.

PROPOSITION 3.4. *Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) and suppose also that*

$$0 \leq G_\infty < G^\infty \leq +\infty. \tag{17}$$

Then, for each

$$\lambda \in \left] \left(\frac{\pi}{L}\right)^2 / G^\infty, \left(\frac{\pi}{L}\right)^2 / G_\infty \right[$$

there is an unbounded set of positive solutions $u(\cdot)$ of (11).

Proof. We define

$$\eta_* := \left(\frac{\pi}{L}\right)^2 / G^\infty \quad \text{and} \quad \eta^* := \left(\frac{\pi}{L}\right)^2 / G_\infty.$$

As in the preceding proof, we also note that $G(s) \rightarrow +\infty$ as $s \rightarrow +\infty$. From [39, Corollaire 12] we find

$$\liminf_{c \rightarrow +\infty} T(c) \leq \pi/\sqrt{G^\infty} < \pi/\sqrt{G_\infty} \leq \liminf_{c \rightarrow +\infty} T(c).$$

Hence

$$\liminf_{c \rightarrow +\infty} \tau(c) \leq \eta_* < \eta^* \leq \limsup_{c \rightarrow +\infty} \tau(c).$$

By the intermediate value theorem, for each $\lambda \in]\eta_*, \eta^*[$ there is a sequence $c_n \rightarrow +\infty$ of solutions of the equation $\tau(c) = \lambda$. Now we conclude with the same argument as above. For an alternative proof see also [17, Theorem 3] (where the supplementary condition $g(s) \rightarrow +\infty$ as $s \rightarrow +\infty$ is assumed, due to the fact that the more general perturbed equation $-u'' = g(u) + h(x)$ is therein considered) or [35, Theorem 4]. \square

The next example provides a class of nonlinearities consistent with Proposition 3.4.

EXAMPLE 3.5. Let k, θ, A, B be given constants with $k, A > 0$, $\theta \in [0, 2\pi[$ and

$$|B| < \frac{2A}{\sqrt{k^2 + 4}}. \quad (18)$$

Define, for every $s \geq 0$,

$$G(s) := As^2 + Bs^2 \cos(k \log(1 + s) + \theta) \quad \text{and} \quad g(s) := G'(s).$$

Then $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a C^∞ -function and, by (18), one can easily check that (g_*) holds. Moreover

$$G_\infty = 2(A - B) < 2(A + B) = G^\infty \quad \text{and} \quad g_0 = 2(A + B \cos \theta) > 0.$$

By virtue of this result, given any pair of positive constants $\alpha < \beta$ we can easily find a function $G(\cdot)$ within the class introduced in Example 3.5 such that $G_\infty = \alpha$ and $G^\infty = \beta$. Indeed, it is sufficient to take $A = \alpha + \beta$, $B = \beta - \alpha$ and choose $k > 0$ sufficiently small. In this case, as a consequence of Proposition 3.4, Proposition 3.2 and the fact that $\lim_{c \rightarrow 0^+} \tau(c) = \lambda_1^*$, we can describe some features of the set of positive solution pairs associated with

$$\begin{cases} u'' + \lambda G'(u) = 0, \\ u(a) = 0 = u(b). \end{cases} \quad (19)$$

More in detail, this set is a Cartesian graph bounded in the λ -component and unbounded in the u -component, that bifurcates from $(\lambda_1^*, 0)$ and oscillates infinitely many times between the following values

$$\lambda_1^* = \eta_* = \left(\frac{\pi}{b-a} \right)^2 \frac{1}{2(A+B)} \quad \text{and} \quad \eta^* = \left(\frac{\pi}{b-a} \right)^2 \frac{1}{2(A-B)}.$$

This situation can be represented in Figure 2 for a particular function $G(\cdot)$ of this class.

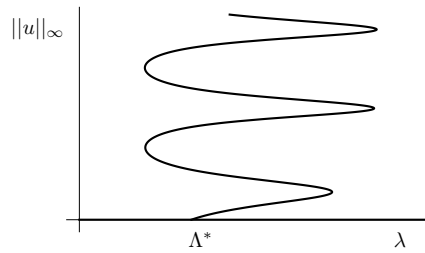


Figure 2: Bifurcation diagram in logarithmic scale for problem (19) with $G(s) = 2s^2 + s^2 \cos(2 \log(1 + s))$.

To give a general overview, we also discuss the other bifurcation diagrams which we can obtain in the dual situation, when the conditions at zero and at infinity are interchanged. To this aim, we introduce the constants

$$G_0 := \liminf_{s \rightarrow 0^+} \frac{2G(s)}{s^2}, \quad G^\infty := \limsup_{s \rightarrow 0^+} \frac{2G(s)}{s^2}$$

as well as

$$\rho_* := \left(\frac{\pi}{L}\right)^2 / G_0 \quad \text{and} \quad \rho^* := \left(\frac{\pi}{L}\right)^2 / G^\infty.$$

With these positions, we state the following result where we summarize all the possible combinations involving the lower and upper limits for G . The proof is omitted as it can be derived from Proposition 3.2 by some analogous arguments to those exposed above.

PROPOSITION 3.6. *Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) and let \mathcal{S} be the set of positive solutions pairs for (11). Then the following statements hold.*

- *If $G^0 > G_\infty$, then for each $\lambda \in]\rho_*, \eta^*[$ there exists at least one positive solution $u(\cdot)$ of (11) with $(\lambda, u) \in \mathcal{S}$.*
- *If $G_0 < G^\infty$, then for each $\lambda \in]\eta_*, \rho^*[$ there exists at least one positive solution $u(\cdot)$ of (11) with $(\lambda, u) \in \mathcal{S}$.*
- *If $G_0 < G^0$, then for each $\lambda \in]\rho_*, \rho^*[$ there is a sequence of positive solutions $u(\cdot)$ of (11) which converges uniformly to zero.*
- *If $G_\infty < G^\infty$, then for each $\lambda \in]\eta_*, \eta^*[$ there is a sequence of positive solutions $u_{\lambda,n}(\cdot)$ of (11) with $\|u_{\lambda,n}\|_\infty \rightarrow +\infty$ for $n \rightarrow \infty$.*

Suitably modifying the function in Example 3.5, we can find a class of functions such that $G_0 < G^0$.

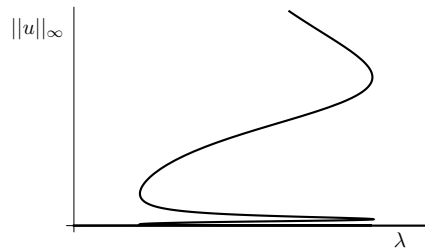


Figure 3: Bifurcation diagram in logarithmic scale for problem (19) with $G(s) = 2s^2 + s^2 \cos(2 \log(\frac{s}{1+s}))$.

EXAMPLE 3.7. Let k, θ, A, B be given constants with $k, A > 0$, $\theta \in [0, 2\pi[$ and B as in (18). Define, for every $s > 0$,

$$G(s) := As^2 + Bs^2 \cos(k \log(\frac{s}{s+1}) + \theta), \quad g(s) := G'(s) \text{ and } g(0) = 0.$$

Then $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a C^∞ -function satisfying (g_*) and such that

$$G_0 = 2(A - B) < 2(A + B) = G^0 \text{ and } g_\infty = 2(A + B \cos \theta) > 0.$$

In Figure 3 is represented a bifurcation diagram related to a function $G(\cdot)$ with nonlinear features described above. Moreover, we notice that it is not difficult to combine Example 3.5 and Example 3.7 in order to produce a class of functions such that both $G^0 > G_0$ and $G^\infty > G_\infty$ are valid.

To conclude, the analysis performed in this section by the use of the time-mapping shows some possibilities to improve Corollary 2.2 (at least to the one-dimensional case). With this in mind, now we come back to the study of the original Dirichlet problem (\mathcal{D}) to achieve a more general result.

4. Revisiting the sublinear case

In this section we consider a bounded domain $\Omega \subset \mathbb{R}^N$ with boundary of class $C^{1,1}$. Let X be the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ with its standard norm. We denote by

$$P_X := \{u \in X : u(x) \geq 0, \forall x \in \Omega\}$$

the positive cone in X .

This section is also developed under the following technical condition on the weight function.

- (a_#) Suppose that there exist an open set $\Omega_1 \subset \Omega$ and a constant $\eta > 0$ such that $a(x) \geq \eta$ for a.e. $x \in \Omega_1$.

Condition $(a_{\#})$ is always satisfied when (a_*) holds. Although it is slightly more restrictive than (a_{**}) , nevertheless it is a key hypothesis for the study of indefinite problems also considered by several authors. For a discussion about this topic we refer to [31, Ch. 9].

Our goal is still the generalization of Proposition 3.3 to problem (\mathcal{D}) . In view of the presence of the parameter λ in the equation, it seems natural to enter in a bifurcation setting, in order to obtain both the existence of solutions for each λ in a certain range and the existence of a continuum of solution pairs with the desired properties. With this respect, the following result holds.

THEOREM 4.1. *Let $a \in L^\infty(\Omega)$ satisfy $(a_{\#})$ and let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) and such that g_0 is finite. Then the following conclusions hold:*

- I) *If $g_0 > 0$, there exists an unbounded continuum $\mathcal{C} \subset \mathbb{R}_0^+ \times X$ containing $(\Lambda^*, 0)$ and such that $\mathcal{C} \setminus \{(\Lambda^*, 0)\}$ is made of positive solution pairs (λ, u) to problem (\mathcal{D}) .*
- II) *If $g_0 > 0$ and, moreover, $G_\infty = 0$, then for each $\lambda > \Lambda^*$ there is at least one positive solution $u(\cdot)$ with $(\lambda, u) \in \mathcal{C}$.*
- III) *If $g_0 > 0$ and also $G^\infty > G_\infty = 0$, then there is M^* such that for each $\lambda > M^*$ there is an unbounded set of positive solutions.*

Proof of I). The first part closely follows the schemes proposed in [26, Theorem 2] and [12, Theorem 2.2] which involve the global bifurcation theorem of Rabinowitz [40, Theorem 1.3]. In [26] the theory was developed for a continuous weight function, but it can be suitably adapted to cover the case in $(a_{\#})$.

First of all, we extend $g(s)$ by oddness to the whole real line (such extension will be still denoted by g). We fix a constant $p > N$ and consider the Nemytskii operator $F : X \rightarrow L^p(\Omega)$ associated with $f(x, u) := a(x)g(u)$, namely

$$F : X \rightarrow L^\infty(\Omega) \hookrightarrow L^p(\Omega), \quad u(\cdot) \mapsto f(\cdot, u(\cdot)).$$

For each $v(\cdot) \in L^p(\Omega)$ the Dirichlet problem

$$\begin{cases} -\Delta u = v(x) \\ u \in W_0^{1,p}(\Omega) \end{cases} \tag{20}$$

has a unique solution which belongs to $W^{2,p}(\Omega)$. Since $p > N$, this latter space is compactly embedded in $C^{1,\beta}(\bar{\Omega})$ for $0 \leq \beta < 1 - (N/p)$. We denote the inverse of the Laplacian operator by \mathfrak{L}^{-1} , which associates to each $v(\cdot) \in L^p(\Omega)$ the solution $u(\cdot) \in X$ of (20), via the following compositions

$$\mathfrak{L}^{-1} : L^p(\Omega) \rightarrow W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \hookrightarrow X.$$

In this way, problem (D) can be settled like a fixed point problem in the space X , as follows

$$u = \lambda Ku, \tag{21}$$

where $K : X \rightarrow X$ is the completely continuous operator defined as

$$K(u) := \mathfrak{L}^{-1}F(u).$$

Following [31] we define $C^{1,1^-}(\bar{\Omega}) := \bigcap_{0 < \theta < 1} C^{1,\theta}(\bar{\Omega})$. Observe that a solution of (21) belongs to $C_0^1(\bar{\Omega}) \cap C^{1,1^-}(\bar{\Omega})$ and is twice classically differentiable almost everywhere in Ω (see [22] and also [31, Theorem 5.8] to justify our assertions on the regularity results).

The existence of a finite g_0 allows to express the nonlinearity f as

$$f(x, s) = g_0 a(x)s + a(x)\gamma(s), \quad \text{with } \gamma(s)/s \rightarrow 0 \text{ as } s \rightarrow 0.$$

Therefore K admits a linearization at $u = 0$ of the form $\mathfrak{L}^{-1}A$, where A is the multiplication operator induced by the function $g_0 a(\cdot)$. We denote by \mathscr{W} the closure in $\mathbb{R} \times X$ of the set of nontrivial solution pairs (λ, u) of (21).

Let Λ^* be defined as in (6). According to Hess and Kato [26], as already observed in the proof of Corollary 2.2, the point $(\Lambda^*, 0)$ is a bifurcation point of the nonlinear problem (21). An application in this setting of Rabinowitz's global bifurcation theorem [40, 41] ensures that the set \mathscr{W} contains a maximal subcontinuum \mathscr{F} such that $\mathscr{F} \ni (\Lambda^*, 0)$ and \mathscr{F} is either unbounded or contains a point $(\hat{\lambda}, 0)$ where $\hat{\lambda}$ is a characteristic value of $\mathfrak{L}^{-1}A$ with $\hat{\lambda} \neq \Lambda^*$. On account of the fact that $(0, 0)$ is not a bifurcation point, \mathscr{F} is connected and $\Lambda^* > 0$, we firstly observe that

$$\mathscr{F} \subset \mathbb{R}_0^+ \times X.$$

We are going to prove that \mathscr{F} contains an unbounded sub-continuum \mathscr{C} starting from $(\Lambda^*, 0)$, satisfying $\mathscr{C} \setminus (\mathbb{R} \times \{0\}) \subset \mathbb{R}_0^+ \times \text{int}P_X$, which does not contain any point $(\hat{\lambda}, 0)$ with $\hat{\lambda} \neq \Lambda^*$. To this aim, we will show that

$$\mathscr{F} \setminus (\mathbb{R} \times \{0\}) \subset (\mathbb{R}_0^+ \times \text{int}P_X) \cup (\mathbb{R}_0^+ \times -\text{int}P_X) \tag{22}$$

and

$$\mathscr{F} \cap (\mathbb{R} \times \{0\}) = \{(\Lambda^*, 0)\}. \tag{23}$$

In fact, condition (23) implies that the second alternative of Rabinowitz bifurcation theorem does not occur and therefore \mathscr{F} is unbounded. Then, the continuum we are looking for can be defined as

$$\mathscr{C} := \{(\lambda, |u|) : (\lambda, u) \in \mathscr{F}\} \subset (\mathbb{R}_0^+ \times \text{int}P_X) \cup \{(\Lambda^*, 0)\}. \tag{24}$$

In this manner, assertion I) follows because it is obvious that \mathscr{C} is a closed connected unbounded set of solution pairs to (21) which contains $(\Lambda^*, 0)$ and moreover for each $(\lambda, u) \in \mathscr{C} \setminus \{(\Lambda^*, 0)\}$ we have $\lambda > 0$ and $u > 0$.

Our task is now to check conditions in (22) and (23). To do this we divide the proof into some steps.

Step 1. There is a neighborhood U of $(\Lambda^, 0)$ such that*

$$U \cap \mathcal{F} \subset (\mathbb{R}_0^+ \times \text{int}P_X) \cup (\mathbb{R}_0^+ \times -\text{int}P_X) \cup \{(\Lambda^*, 0)\}. \tag{25}$$

Indeed, if by contradiction there is no neighborhood U of $(\Lambda^*, 0)$ as above, then one could find a sequence (λ_n, u_n) of solutions to (21) with $\lambda_n \rightarrow \Lambda^*$ and $u_n \notin -\text{int}P_X \cup \text{int}P_X$, such that $0 < \|u_n\| \rightarrow 0$. Normalizing, we have

$$v_n = \lambda_n \frac{K(\|u_n\|v_n)}{\|u_n\|}, \quad \text{where } v_n := \frac{u_n}{\|u_n\|}.$$

By compactness, we can assume that $v_n \rightarrow v$ (up to a subsequence). Moreover,

$$v \notin -\text{int}P_X \cup \text{int}P_X.$$

Using the linearization of K at zero we obtain

$$v = \Lambda^* \mathcal{L}^{-1}Av, \quad \text{with } \|v\| = 1.$$

This means that v is an eigenfunction corresponding to the positive principal eigenvalue Λ^* and therefore

$$v \in -\text{int}P_X \cup \text{int}P_X.$$

A contradiction is thus achieved. For what follows, it is useful to note that $U \cap (\mathcal{F} \setminus (\mathbb{R} \times \{0\}))$ is nonempty.

Step 2. It holds that

$$\mathcal{F} \cap (\mathbb{R}_0^+ \times (-\partial P_X \cup \partial P_X)) = \{(\Lambda^*, 0)\}. \tag{26}$$

Suppose that $(\zeta, u_0) \in \mathcal{F} \cap (\mathbb{R}_0^+ \times (-\partial P_X \cup \partial P_X))$. The odd extension of $g(s)$ implies that also the operator K is odd. Therefore, when u is a solution of (21), $-u$ is a solution, too. Accordingly, without loss of generality, we can suppose that $(\zeta, u_0) \in \mathcal{F} \cap (\mathbb{R}_0^+ \times \partial P_X)$.

We claim that $u_0 = 0$. If, by contradiction, $u_0 \neq 0$, then $u_0(\cdot)$ is a nontrivial nonnegative solution to the problem

$$-\Delta u = \zeta a(x)g(u), \quad u|_{\partial\Omega} = 0,$$

which is equivalent to

$$-\Delta u + cu = (c + \zeta a(x)\phi(u))u, \quad u|_{\partial\Omega} = 0,$$

where we have introduced the auxiliary continuous function

$$\phi(s) := \begin{cases} g(s)/s & \text{for } s \neq 0, \\ g_0 & \text{for } s = 0. \end{cases} \tag{27}$$

Now, if we take

$$c \geq \zeta \|a\|_\infty \sup_{0 \leq s \leq \|u\|_\infty} \phi(s),$$

we obtain that

$$-\Delta u_0(x) + cu_0(x) \geq 0, \quad u_0|_{\partial\Omega} = 0,$$

with $u_0(x) \geq 0$ for all $x \in \Omega$ and $u_0 \not\equiv 0$. By the strong maximum principle $u_0 \in \text{int}P_X$ follows and this leads to a contradiction.

Since $u_0 = 0$, now we have $(\zeta, 0) \in \mathcal{F} \cap (\mathbb{R}_0^+ \times \partial P_X)$. So that, there exists a sequence (λ_n, u_n) of solutions to (21) with $\lambda_n \rightarrow \zeta > 0$ and $u_n \in \text{int}P_X$ such that $0 < \|u_n\| \rightarrow 0$. Normalizing as in Step 1 and passing up to a subsequence for $v_n := u_n/\|u_n\|$, we obtain

$$v = \zeta \mathfrak{L}^{-1}Av, \quad \text{with } \|v\| = 1 \text{ and } v > 0.$$

This means that v is a positive eigenfunction associated with the eigenvalue $\zeta > 0$. Therefore $\zeta = \Lambda^*$, as there is a unique positive eigenvalue having a positive eigenfunction.

Step 3. It holds that

$$\mathcal{F} \subset (\mathbb{R}_0^+ \times \text{int}P_X) \cup (\mathbb{R}_0^+ \times -\text{int}P_X) \cup \{(\Lambda^*, 0)\}. \tag{28}$$

Indeed, let us consider the set

$$\mathcal{F}' := \{(\lambda, u) \in \mathcal{F} : \lambda > 0, \pm u \in \text{int}P_X\} \cup \{(\Lambda^*, 0)\}.$$

By Step 1, the set \mathcal{F}' is open relatively to \mathcal{F} . We claim that \mathcal{F}' is closed in \mathcal{F} . To do this, we consider a sequence $(\lambda_n, u_n) \rightarrow (\zeta, u)$ with $\zeta > 0$ and $u_n \in \text{int}P_X \cup -\text{int}P_X$. If $u \in \text{int}P_X \cup -\text{int}P_X$, we are done. Otherwise, if $u \in -\partial P_X \cup \partial P_X$, from Step 2 we have $(\zeta, u) = (\Lambda^*, 0)$. The claim is thus proved. The connectedness of \mathcal{F} implies that $\mathcal{F}' = \mathcal{F}$ and (28) is verified.

Finally, the proof of I) is concluded because (22) and (23) directly follow from (28). □

Proof of II). Having already produced the continuum \mathcal{C} , we will prove that it is unbounded in the λ -component if $G_\infty = 0$. To this aim, we introduce the projection

$$p_1 : \mathbb{R} \times X \rightarrow \mathbb{R}, \quad (\lambda, u) \mapsto \lambda$$

and we show that $p_1(\mathcal{C}) \supset [\Lambda^*, +\infty)$.

Suppose, by contradiction, that the inclusion does not hold. So that there exists $\hat{\lambda} > \Lambda^*$ such that $\lambda < \hat{\lambda}$ for each $(\lambda, u) \in \mathcal{C}$.

Let a_1, b_1 be such that $\Omega \subset]a_1, b_1[\times \mathbb{R}^{N-1}$. As already observed in Section 3, the hypothesis $G_\infty = 0$ implies that $T^\infty = +\infty$, where

$$T^\infty := \limsup_{c \rightarrow +\infty} T(c). \tag{29}$$

Let us fix a constant $R > b_1 - a_1$ and let $d > 0$ be such that

$$T(d)^2 > R^2 \hat{\lambda} \|a\|_\infty. \tag{30}$$

According to Proposition 3.1 the function $v_{d,R}(t)$ is a solution of

$$v'' + \left(\frac{T(d)}{R}\right)^2 g(v) = 0$$

such that $v_{d,R}(t) > 0$ for all $t \in [a_1, b_1]$. Finally, from $v_{d,R}$ we define a function on \mathbb{R}^N as

$$\beta(x) := v_{d,R}(x_1), \quad \forall x = (x_1, \dots, x_N) \in \overline{\Omega}.$$

By construction, for each $\lambda \in]0, \hat{\lambda}[$, the function $\beta(x)$ is an upper solution which is not a solution for problem (\mathcal{D}) . Indeed, there exists a constant $\rho > 0$ such that

$$-\Delta\beta(x) \geq \hat{\lambda} \|a\|_\infty g(\beta(x)) + \rho, \quad \forall x \in \Omega \tag{31}$$

and, moreover,

$$\inf_{x \in \overline{\Omega}} \beta(x) = \eta > 0. \tag{32}$$

Now, we claim that

$$u(x) < \beta(x), \forall x \in \Omega, \tag{33}$$

for each positive solution $u(\cdot)$ such that $(\lambda, u) \in \mathcal{C}$. To prove this inequality we follow an argument close to the one in [36, Step 4] (for another possible proof, but involving a locally Lipschitz condition, we refer to [20, Theorem 2.2]).

Let us consider the set

$$\mathcal{C}' := \{(\lambda, u) \in \mathcal{C} : u(x) < \beta(x), \forall x \in \Omega\},$$

which is nonempty and open relatively to \mathcal{C} . In order to prove (33) we will show that \mathcal{C}' is also closed relatively to \mathcal{C} , so then we can conclude by the connectedness of \mathcal{C} .

Let $U(x) \leq \beta(x)$, for all $x \in \Omega$, be a solution of (\mathcal{D}) for some λ such that $(\lambda, U) \in \mathcal{C}$. We notice that $U(x) < \beta(x), \forall x \in \partial\Omega$. We are going to prove that $U(x) < \beta(x), \forall x \in \Omega$. Let us fix $\varepsilon > 0$ such that

$$4\varepsilon \hat{\lambda} \|a\|_\infty < \rho.$$

By the uniform continuity of $g(s)$ on the interval $[0, \|\beta\|_\infty]$, there exists $\delta > 0$ such that $|g(s') - g(s'')| < \varepsilon$ for each $s', s'' \in [0, \|\beta\|_\infty]$ with $|s' - s''| < \delta$. If there exists a point $x_0 \in \Omega$ such that $U(x_0) = \beta(x_0)$, then we can take a (small) open ball $B(x_0, r) \subset \Omega$ such that $|U(x) - U(x_0)| < \delta$ and $|\beta(x) - \beta(x_0)| < \delta$ for all $x \in B[x_0, r]$. As a consequence, we have

$$|g(\beta(x)) - g(U(x))| < 2\varepsilon, \quad \forall x \in B[x_0, r].$$

A comparison between (31) and $-\Delta U(x) = \lambda a(x)g(U(x))$ for a.e. $x \in B(x_0, r)$ (for $0 < \lambda < \hat{\lambda}$) shows that the function $W(x) := \beta(x) - U(x)$ satisfies $-\Delta W(x) \geq \rho/2$ for a.e. $x \in B(x_0, r)$ with $W \geq 0$ on $\partial B(x_0, r)$ and $W(x_0) = 0$. This contradicts the strong maximum principle on the ball $B(x_0, r)$ (cf. [24, Lemma 3.2 (interior form)]). Therefore we conclude that \mathcal{C}' is closed relatively to \mathcal{C} .

Therefore, from (33) we have that

$$\mathcal{C} \subset]0, \hat{\lambda}[\times]0, \beta(\cdot)].$$

Hence, \mathcal{C} is bounded in the product space and this contradicts the alternatives of Rabinowitz's global bifurcation theorem. Assertion II) is thus proved. \square

Proof of III). For the latter assertion, concerning the case $G^\infty > G_\infty = 0$, we rely to [36, Theorem 2.2] applied to the problem

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (34)$$

where

$$f(x, s) := \begin{cases} \lambda a(x)g(s) & \text{if } s \geq 0, \\ 0 & \text{if } s < 0. \end{cases}$$

With this respect, we observe that $f(x, s) \geq \lambda \eta g(s)$ for every $s \geq 0$ and a.e. $x \in \Omega_1$ and, moreover, $f(x, s) \leq h(s) := \lambda \|a\|_\infty g(s)$ for every $s \geq 0$ and a.e. $x \in \Omega$. By our special form of $f(x, s)$ (which, in particular, implies $f(x, 0) \equiv 0$), one can see that the assumptions (h_3) and $g(s) \rightarrow \infty$ as $s \rightarrow +\infty$ required in [36, Theorem 2.2] can be ignored. The condition $G_\infty = 0$ implies $\liminf_{s \rightarrow +\infty} (\int_0^s h(\xi) d\xi) / s^2 = 0$ as in (h_5) of [36, Theorem 2.2] and thus the existence of a sequence of upper solutions β_n tending to infinity uniformly in $\bar{\Omega}$ is guaranteed. On the other hand, given $\rho_N = N^N / (N-1)^{(N-1)}$ for $N \geq 2$ otherwise $\rho_1 = 1$ and let $R > 0$ be the radius of the largest ball contained in Ω_1 , according to [36, Remark 1] if

$$\lambda > M^* := \frac{\rho_N}{\eta G^\infty} \left(\frac{\pi}{2R} \right)^2,$$

then there exists a sequence of lower solutions α_n with $\max(\alpha_n) = \max_{\Omega_1}(\alpha_n)$ tending to infinity. The rest of the proof is similar to [36, Theorem 2.2]. It leads to the existence of an unbounded sequence of solutions u_n for (34) and the strong maximum principle (cf. [24, Lemma 3.2 (global form)]) guarantees that $u_n(x) > 0$ for all $x \in \Omega$. \square

The construction of an upper solution using conditions on the lower limit at infinity of $G(s)/s^2$ has been already exploited in [18, 36, 38].

One could argue that functions satisfying (H_G) and not (H_g) seem really artificial. Our opinion is that such kind of functions may look slightly unusual but not too weird. One can easily provide examples of functions in the class (g_*) and satisfying

$$0 = \liminf_{s \rightarrow +\infty} \frac{2G(s)}{s^2} < \limsup_{s \rightarrow +\infty} \frac{2G(s)}{s^2}. \tag{35}$$

This can be done in different manners. For example, by selecting an increasing sequence of positive reals $(a_n)_n$ such that

$$\lim_{n \rightarrow +\infty} n^{-2}a_{2n} = \ell \in]0, +\infty] \quad \text{and} \quad \lim_{n \rightarrow +\infty} n^{-2}a_{2n+1} = 0.$$

Then $G(s)$ can be constructed as a smooth function satisfying $G(0) = G'(0) = 0 < G''(0)$, $G'(s) > 0$ for all $s > 0$ and such that its graph interpolates the points (n, a_n) . This procedure, even if it permits to define functions satisfying our requests, still may look somehow artificial. For this reason, we show below how to define in an analytical manner suitable maps satisfying (g_*) and (H_G) by the use of elementary functions. Such nonlinearities are obtained by a modification of the ones considered in Example 3.5, as follows.

EXAMPLE 4.2. Let $\rho, \theta, A, k_1, k_2, p, q$, be positive constants, with $\theta \in [0, 2\pi]$, $A \geq e$, and $0 < q < 1 - p < 1$. Define, for every $s \geq 0$,

$$G(s) := \rho s^2 \left(1 + \cos(k_1 \log^p(A + s) + \theta) + k_2 \log^{-q}(A + s) \right).$$

If

$$k_1 p + k_2 q < 2k_2,$$

then $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, defined as $g(s) := G'(s)$, is a C^∞ -function satisfying (g_*) . Moreover,

$$G_\infty = 0 < 4\rho = G^\infty \quad \text{and} \quad g_0 \in]0, +\infty[.$$

Indeed, to check that $g(s) > 0$ for all $s > 0$, we just observe that

$$G'(s) \geq \rho s \left(2k_2 \xi^{-q} - k_1 p \xi^{p-1} - k_2 q \xi^{-q-1} \right), \quad \text{for } \xi := \log(A + s) > 1.$$

By the choice of the coefficients, we see that the term in parenthesis is strictly positive. All the other verifications are straightforward.

Under our assumptions, it is natural to ask whether there are further properties of the Rabinowitz's bifurcation continuum \mathcal{C} . Indeed, the following result holds.

PROPOSITION 4.3. *Let $a \in L^\infty(\Omega)$ satisfy $(a_\#)$ and let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) and such that g_0 is finite. If $g_0 > 0$, then the continuum \mathcal{C} defined in (24) is unbounded in the u -component.*

Proof. Let \mathcal{C} be the continuum obtained in I) of Theorem 4.1 and defined in (24). Suppose, by contradiction, that there exists $M > 0$ such that

$$\|u\| \leq M \text{ for all } (\lambda, u) \in \mathcal{C} \subset \mathbb{R}_0^+ \times X. \quad (36)$$

This, in turn, implies that $0 < u(x) \leq M$ for all $x \in \Omega$. Then, as a consequence of (g_*) and $g_0 > 0$, we find that $g(u(x)) \geq C_M u(x)$ for all $x \in \bar{\Omega}$, for

$$C_M := \inf_{0 < s \leq M} \frac{g(s)}{s} > 0.$$

In other words, for ϕ defined as in (27), we have that $\phi(u(x)) \geq C_M$ for every $(\lambda, u) \in \mathcal{C}$ and problem (\mathcal{D}) can be written as

$$\begin{cases} -\Delta u = \lambda a(x)\phi(u)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (37)$$

Now, let $z \in \Omega_1$ and $r > 0$ be such that the open ball $B = B(z, r)$ satisfies $\bar{B} \subset \Omega^+$ and, moreover, let $\rho_1 > 0$ be the first (positive) eigenvalue of the eigenvalue problem with positive weight

$$-\Delta u = \rho a(x)u, \quad u|_{\partial B} = 0.$$

We denote by ψ the associated positive eigenfunction with $\max_B \psi(x) = 1$.

We fix a constant $\hat{\lambda} > \rho_1/C_M$ such that there exists a (positive) solution \hat{u} of (37) with $(\hat{\lambda}, \hat{u}) \in \mathcal{C}$. We know that such a pair always exists because \mathcal{C} is unbounded in the product space and we are assuming (36). Let $v(x) = \vartheta\psi(x)$ (with $\vartheta > 0$) be the maximal eigenfunction of

$$-\Delta u = \rho_1 a(x)u, \quad u|_{\partial B} = 0$$

such that $v(x) \leq \hat{u}(x)$, $\forall x \in B$. By definition, we have $0 = v(x) < \hat{u}(x)$ on ∂B and $v(x_0) = \hat{u}(x_0)$ for some $x_0 \in B$. The function $W(x) := \hat{u}(x) - v(x)$ satisfies $-\Delta W(x) > 0$ for a.e. $x \in B$ with $W(x) > 0$ on ∂B and $\min_B W(x) = W(x_0) = 0$, thus contradicting the maximum principle. Therefore, our assertion is proved. \square

As a consequence of Theorem 4.1 and Proposition 4.3, we can say that Proposition 3.3 for the one-dimensional case is now extended to any sufficiently regular domain in \mathbb{R}^N . In particular, also Corollary 2.5 extends as follows (where the constant $\lambda_1^* := \lambda_1(-\Delta)/g_0$ is the one defined in (10)).

COROLLARY 4.4. *Let $a \in L^\infty(\Omega)$ satisfy $(a_\#)$ and let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) and (H_G) . Then there exists a continuum \mathcal{C} containing $(\lambda_1^*, 0)$ and such that $\mathcal{C} \setminus \{(\lambda_1^*, 0)\}$ is made of positive solution pairs (λ, u) to problem (9). The continuum \mathcal{C} is unbounded both in the u -component and the λ -component.*

Moreover, if the map $s \mapsto g(s)/s$ is decreasing on \mathbb{R}_0^+ since the conditions (H_g) and (H_G) are equivalent, then the set of positive solution pairs \mathcal{S} coincides with $\mathcal{C} \setminus \{(\lambda_1^, 0)\}$ and is the graph of a continuous map $]\lambda_1^*, +\infty[\ni \lambda \mapsto u_\lambda \in \text{int}P_X$.*

From the proof of Theorem 4.1 it is also clear that a more general version of Theorem 4.1 can be given as follows.

THEOREM 4.5. *Let $a \in L^\infty(\Omega)$ satisfy $(a_\#)$ and let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) and such that g_0 is finite. Then the following conclusions hold:*

- *If $g_0 > 0$, there exists an unbounded continuum $\mathcal{C} \subset \mathbb{R}_0^+ \times X$ containing $(\Lambda^*, 0)$ and such that $\mathcal{C} \setminus \{(\Lambda^*, 0)\}$ is made of positive solution pairs (λ, u) to problem (D). The continuum \mathcal{C} is always unbounded in the u -component.*
- *If $g_0 > 0$ and, moreover, $T^\infty = +\infty$, then the continuum \mathcal{C} is also unbounded in the λ -component and, therefore, for each $\lambda > \Lambda^*$ there is at least one positive solution $u(\cdot)$ with $(\lambda, u) \in \mathcal{C}$.*

The method of producing bounds for a PDEs using the ODE $u'' + g(u) = 0$ has been also considered in [36] and [28]. Sufficient conditions for validity of the time-mapping hypothesis have been presented in previous papers (see, for instance [19]).

Theorem 4.5 is useful to produce other existence results where explicit hypotheses on $g(s)$ or $G(s)$ at infinity can be employed in order to obtain $T^\infty = +\infty$. From [15], one could require that g is such that

$$\liminf_{s \rightarrow +\infty} \frac{g(s)}{s} = 0, \quad sg'(s) \leq Mg(s) \quad \text{for } s > d, \tag{38}$$

for some positive constant M . This hypothesis, according to Omari and Ye [37], is said to be a “desultorily sublinear condition”. For the PDE setting, it has been recently used for the Neumann problem in [43]. Condition (38) is independent on $G_\infty = 0$ as shown in an example of [15].

Finally, we notice that the assumption $\liminf_{s \rightarrow +\infty} g(s)/s = 0$ alone is not enough to guarantee the existence of positive solutions to problem (D) for $\lambda \geq \Lambda^* = \nu_1/k$. Indeed, we are able to provide a counterexample at least for a constant weight and in one-dimension case. Namely the following results holds.

PROPOSITION 4.6. *Let $\Omega \subset \mathbb{R}$ be a bounded open interval of length $|\Omega| = L$. For each positive constant k , there exists a continuous function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying (g_*) , with*

$$g_0 = k \quad \text{and} \quad \liminf_{s \rightarrow +\infty} \frac{g(s)}{s} = 0, \quad (39)$$

such that there is no positive solution pair for (11) when $\lambda \geq \lambda_1^ = (\frac{\pi}{L})^2/k$. The function g can be defined so that*

$$\lim_{s \rightarrow +\infty} \frac{2G(s)}{s^2} = \limsup_{s \rightarrow +\infty} \frac{g(s)}{s} = K, \quad (40)$$

for any prescribed value $K \in]k, +\infty]$.

Proof. Our example is inspired by some analogous considerations in [16, 34], however the proof here is completely different. We adopt a time-mapping technique as in Section 3. We discuss in detail the situation when K is a real number. The case $K = +\infty$ is can be treated in the same way with simple modifications.

We start by giving the general structure of the example. Let k, K be two given constants with $0 < k < K$. We consider a strictly increasing continuous function $q_1 : \mathbb{R}^+ \rightarrow \mathbb{R}_0^+$ with $q_1(0) = k$ and $q_1(+\infty) = K$. Then, let T_1 be the time-mapping associated with the autonomous scalar equation

$$u'' + g_1(u) = 0, \quad \text{for } g_1(s) := s q_1(s).$$

As usual, we set

$$G_1(s) := \int_0^s g_1(\xi) d\xi.$$

By the properties recalled in Section 3 we know that $T_1 : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is a strictly decreasing function with

$$\lim_{c \rightarrow 0^+} T_1(c) = \frac{\pi}{\sqrt{k}} \quad \text{and} \quad \lim_{c \rightarrow +\infty} T_1(c) = \frac{\pi}{\sqrt{K}}.$$

Next, we consider a strictly monotone increasing function $g_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $g_2(+\infty) = +\infty$ and such that

$$\lim_{s \rightarrow +\infty} \frac{g_2(s)}{s} = 0.$$

By the properties of $g_1(\cdot)$ and $g_2(\cdot)$ and since $g_1(s)/s \rightarrow K > 0$ as $s \rightarrow +\infty$, there exists a constant $d > 0$ such that

$$0 < g_2(s) < g_1(s), \quad \forall s \geq d.$$

Let $\varepsilon > 0$ be a fixed constant such that

$$3\varepsilon < \frac{\pi}{\sqrt{k}} - \frac{\pi}{\sqrt{K}} \tag{41}$$

and, subsequently, let us fix a constant $\theta \in]0, 1[$ such that

$$\sqrt{\theta} \geq \left(\frac{\pi}{\sqrt{K}} + \varepsilon \right) / \left(\frac{\pi}{\sqrt{k}} - \varepsilon \right). \tag{42}$$

At this moment, we can determine a constant $d^* \geq d$ such that

$$g_1(s) > \frac{1}{1 - \theta}, \tag{43a}$$

$$T_1(s) < \frac{\pi}{\sqrt{K}} + \varepsilon, \tag{43b}$$

$$\sqrt{8/g_2(s)} < \varepsilon, \tag{43c}$$

hold for all $s \geq d^*$.

Finally, we take two sequences $(d_n)_n$ and $(r_n)_n$ of positive real numbers with $d_n \nearrow +\infty$ and $r_n \searrow 0^+$ and $d_1 - r_1 > d^* + 2$.

We also define $I_n := [d_n - r_n, d_n + r_n]$. The function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ of our example will be defined as

$$g(s) := g_1(s) - \phi(s),$$

where $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function with

$$\begin{aligned} \phi(s) &= 0, \quad \forall s \in \mathbb{R}^+ \setminus \left(\bigcup_{n=1}^{\infty} I_n \right); \\ \max_{s \in I_n} \phi(s) &= \phi(d_n) := g_1(d_n) - g_2(d_n). \end{aligned}$$

If we denote by,

$$\Phi(s) := \int_0^s \phi(\xi) d\xi,$$

we also impose

$$\Phi(+\infty) \leq 1.$$

We notice that

$$g_1(s) \geq g(s) \geq g_2(s), \quad \forall s \geq d^* \geq d.$$

Moreover, $g(s) = g_1(s)$ for all $0 \leq s \leq d^* + 2$ and $g(d_n) = g_2(d_n)$. By definition of g , we have also that $G(s) = G_1(s) - \Phi(s)$. Hence (39) and (40) follow immediately.

If we denote by T the time-mapping associated with $u'' + g(u) = 0$, from the definition of g it is easy to check that

$$\lim_{c \rightarrow +\infty} T(c) = \lim_{c \rightarrow +\infty} T_1(c) = \frac{\pi}{\sqrt{K}}.$$

However, we want to prove more. Indeed, we claim that

$$T(c) < \frac{\pi}{\sqrt{k}} = \frac{\pi}{\sqrt{g_0}}, \quad \forall c > 0. \tag{44}$$

By construction, we have that $T(c) = T_1(c) < \pi/\sqrt{k}$, for all $c \in]0, d^* + 2]$. So, we consider now $c > d^* + 2$ and prove that $T(c) < \pi/\sqrt{k}$.

In fact, recalling the time-mapping formula given in (14) and using the fact that $c - 1 > d^*$, we have

$$\begin{aligned} T(c) &= 2 \int_0^{c-1} \frac{ds}{\sqrt{2(G(c) - G(s))}} + 2 \int_{c-1}^c \frac{ds}{\sqrt{2(G(c) - G(s))}} \\ &= 2 \int_0^{c-1} \frac{ds}{\sqrt{2(G_1(c) - G_1(s) - (\Phi(c) - \Phi(s)))}} + 2 \int_{c-1}^c \frac{ds}{\sqrt{2(\int_{\xi}^c g(\xi) d\xi)}} \\ &\leq 2 \int_0^{c-1} \frac{ds}{\sqrt{2(G_1(c) - G_1(s) - 1)}} + 2 \int_{c-1}^c \frac{ds}{\sqrt{2(\int_{\xi}^c g_2(\xi) d\xi)}} \\ &\leq \underbrace{\frac{2}{\sqrt{\theta}} \int_0^{c-1} \frac{ds}{\sqrt{2(G_1(c) - G_1(s))}}}_{G_1(c) - G_1(s) - 1 \geq \theta(G_1(c) - G_1(s)), \text{ by condition (43a)}} + 2 \int_{c-1}^c \frac{ds}{\sqrt{2(\int_{\xi}^c g_2(c-1) d\xi)}} \\ &< \frac{2}{\sqrt{\theta}} \int_0^c \frac{ds}{\sqrt{2(G_1(c) - G_1(s))}} + \sqrt{\frac{2}{g_2(c-1)}} \int_{c-1}^c \frac{ds}{\sqrt{c-s}} \\ &= \frac{T_1(c)}{\sqrt{\theta}} + \sqrt{\frac{8}{g_2(c-1)}} < \underbrace{\frac{1}{\sqrt{\theta}} \left(\frac{\pi}{\sqrt{K}} + \varepsilon \right)}_{\text{by (43b)}} + \underbrace{\varepsilon}_{\text{by (43c)}} \\ &\leq \underbrace{\frac{\pi}{\sqrt{k}} - \varepsilon + \varepsilon}_{\text{by (42)}} = \frac{\pi}{\sqrt{k}}. \end{aligned}$$

We have thus verified (44), so that by Proposition 3.2 we know that a positive solution to (11) can exist only for $\lambda < \lambda_1^*$. In other words, with our choice of the function g , there is no positive solution pair for problem (11) when $\lambda \geq \lambda_1^*$. \square

Following the instructions given in the proof, it is easy now to provide a concrete function g .

EXAMPLE 4.7. *As a model example, let us consider the following functions:*

$$q_1(s) = \begin{cases} k + \frac{2(K-k)}{\pi} \arctan(s) & \text{for } K < +\infty, \\ k + s \arctan(s) & \text{for } K = +\infty, \end{cases}$$

and

$$g_2(s) = \sqrt{s}.$$

The parameters involved in the construction can be explicitly computed once k and K are given.

For instance, let us take $k = 1$ and $K = 25$. In this case, we can choose $d = 1$. Next, we fix $\varepsilon = \pi/4$ and $\theta = 9/25$, in order to satisfy (41) and (42). With such a choice of the constants, simple computations show that $d^* = 170$ is more than adequate to have all the three conditions in (43) fulfilled. At this point, we take, for any positive integer n ,

$$d_n = 180 + n \quad \text{and} \quad r_n = \frac{2^{-n}}{25d_n}.$$

We define the function $\phi(s)$ as a piecewise linear function, namely

$$\phi(s) = \begin{cases} g_1(d_n) - g_2(d_n) - \frac{g_1(d_n) - g_2(d_n)}{r_n} |s - d_n| & \text{for } s \in I_n, \\ 0 & \text{for } s \notin I_n. \end{cases}$$

As a last step, we observe that

$$\int_0^{+\infty} \phi(\xi) d\xi = \sum_{n=1}^{\infty} r_n \phi(d_n) < \sum_{n=1}^{\infty} r_n g_1(d_n) < \sum_{n=1}^{\infty} K r_n d_n = \sum_{n=1}^{\infty} 2^{-n} = 1.$$

Therefore, all the required conditions are satisfied.

REMARK 4.8. *The function g , whose existence is asserted in Proposition 4.6, can be more than continuous. Indeed, it can be smooth as we like (it is just a matter of choosing q_1, g_2 and ϕ smooth functions). In particular, in Example 4.7 we can easily modify the choice of ϕ , taking a piecewise polynomial function instead of a piecewise linear function. Hence, when K is finite and g is $C^1(\mathbb{R}^+)$, we have $g(s)/s$ bounded but $\sup_{s>0} g'(s) = +\infty$. In this way our example shows that the second condition in (38) cannot be avoided.*

5. Concavity of $g(s)$ versus monotonicity of $g(s)/s$

In Corollary 2.3 we have recalled the uniqueness results to problem (\mathcal{D}) due to Brezis-Oswald [8] and the ones of Brown-Hess [9]. As already observed, when the weight function is positive, the hypothesis of Brezis-Oswald, concerning the monotonicity of $g(s)/s$, is more general than the requirement of Brown-Hess about the concavity of $g(s)$. On the other hand, the monotonicity of $g(s)/s$ is not enough to guarantee the uniqueness of positive solutions for an indefinite weight. Here we present an illustrative example in this direction, with the aid of some numerical computations.

Our example deals with the one-dimensional case

$$u'' + \lambda a(t)g(u) = 0, \quad (45)$$

where $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined by

$$g(s) := 10se^{-3s^2} + \frac{s}{|s| + 1} \quad (46)$$

and $a : \mathbb{R} \rightarrow \mathbb{R}$ is such that

$$a(t) := (1 - |t|)^5 \cos\left(\frac{9\pi}{2}|t|^{1.2}\right). \quad (47)$$

It is straightforward to check that g satisfies (g_*) and (H_g) . Moreover, the map $s \mapsto g(s)/s$ is strictly decreasing on \mathbb{R}_0^+ ; however, the function g is not concave. According to Section 3, the time-mapping associated with the autonomous system (12) is strictly increasing (see Figure 4) and therefore problem (11) has at most one positive solution for each $\lambda > 0$ and, in fact, there exists a unique positive solution if and only if $\lambda > \lambda_1^*$. This is precisely what Brezis and Oswald theorem asserts when applied to (11).

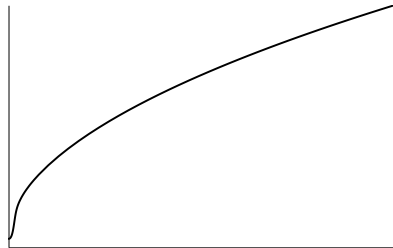


Figure 4: Time-mapping diagram for $u'' + g(u) = 0$ where the function g is defined as in (46).

We show now the effect of an indefinite weight on the number of positive solutions. The function $a(t)$ that we have selected for our simulations (see Figure 5) has been chosen just to give more evidence to the presence of multiple (positive) solutions. Multiple solutions can be obtained also for different sign-changing weights.

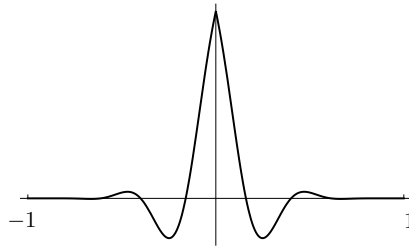


Figure 5: Graph of the function $a(t)$ as defined in (47) in the interval $\Omega =]-1, 1[$.

In our case, we give numerical evidence of at least five positive solutions for the Dirichlet problem associated with (45) on the domain $\Omega =]-1, 1[$. We start our analysis, for a fixed value of $\lambda = 80$, by shooting solutions from $t = -1$ with initial slope between $r_0 = 0.38$ and $r_1 = 10$. More in detail, for each $r \in [r_0, r_1]$, let $(u(\cdot, r), y(\cdot, r))$ be the solution of

$$\begin{cases} u' = y \\ y' = -\lambda a(t)g(u) \end{cases}$$

satisfying the initial condition $(u(-1, r), y(-1, r)) = (0, r)$. Then, in the phase-plane $(u, y) = (u, u')$, we consider the arc

$$\Gamma := \{(u(1, r), y(1, r)) : r \in [r_0, r_1]\}$$

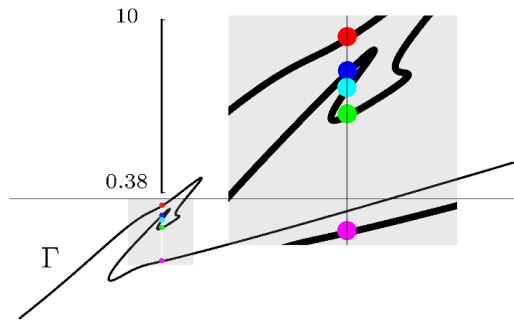
and look for points $p \in \Gamma \cap \{(0, y) : y < 0\}$. In the discretization of the interval $[r_0, r_1]$ we have taken a non-uniform distribution of nodes. The features of the function

$$f(t, u) = \lambda a(t)g(u), \quad t \in [-1, 1]$$

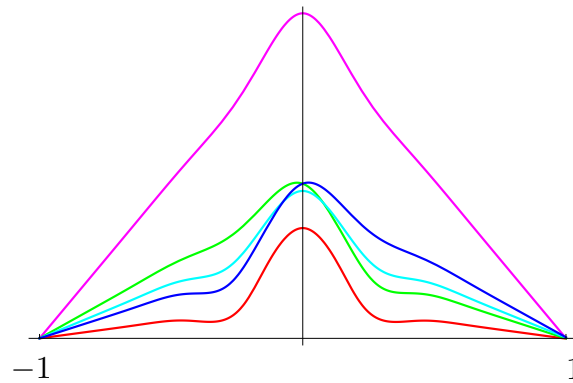
have required an increased number of nodes in the subinterval $[0.391, 0.393]$, in order to obtain a more accurate evaluation of the intersection points. The resulting curve Γ is shown in Figure 6 where we have also put in evidence the five intersection points.

For each intersection point $p = (0, \rho) \in \Gamma \cap \{(0, y) : y < 0\}$, we then solve the initial value problem

$$u'' + \lambda a(t)g(u) = 0, \quad u(-1) = 0, \quad u'(-1) = -\rho.$$

Figure 6: The curve Γ in the phase-plane.

The symmetry of the weight function (i.e. $a(-t) = a(t)$) guarantees that the solution $u(\cdot, -\rho)$ is a positive solution of the Dirichlet problem associated with (45) on $] -1, 1[$. The corresponding *five solutions* are represented in Figure 7. Notice that, three of these solutions are even functions, while the other two (called u_1 and u_2) are symmetric each other, that is $u_2(-t) = u_1(t)$.

Figure 7: The five positive solutions of (\mathcal{D}) .

Our example may have some interest also with respect to the result of Gidas, Ni and Nirenberg [21] on the symmetry of positive solutions. Notice that [21, Theorem 1'] does not apply because the function $[0, 1] \ni \xi \mapsto f(\xi, u)$ is not decreasing.

Another point of view in order to distinguish between symmetric and asym-

metric solutions is to consider the intersections between the curves

$$\begin{aligned} \Gamma^+ &:= \{(u(0, r), y(0, r)) : r \in [r_0, r_1]\}, \\ \Gamma^- &:= \{(u(0, r), -y(0, r)) : r \in [r_0, r_1]\}. \end{aligned}$$

The curve Γ^- can be equivalently described as the locus of the points at the time $t = 0$, shooting back from the negative y -axis with slope $r \in [-r_1, -r_0]$ at the time $t = 1$. In this way the set of intersection points $p \in \Gamma^+ \cap \{(x, 0) : x > 0\} = \Gamma^- \cap \{(x, 0) : x > 0\}$ are in bijection with the even positive solutions, while the set of intersection points $q \in \Gamma^+ \cap \Gamma^- \setminus \{(x, 0) : x > 0\}$ correspond to the positive solutions symmetric to each other but not even. This point of view is illustrated in Figure 8.

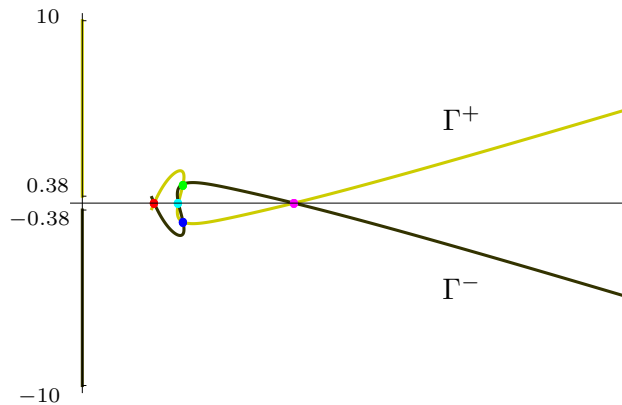


Figure 8: The curves Γ^+ and Γ^- in the phase-plane.

6. Final remarks

Our main results in Section 4 (namely Theorem 4.1 and Theorem 4.5) concern the existence of unbounded connected branches of positive solution pairs with regard to a nonlinear Dirichlet problem for the Laplace differential operator. Here we wish to sketch how to obtain the same kind of results in the case of a more general linear differential operator of the second order. Therefore we consider the problem

$$\begin{cases} \mathfrak{L}u = \lambda a(x)g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{48}$$

where \mathfrak{L} is a linear operator of the form

$$\mathfrak{L} := - \sum_{j,k=1}^N \alpha_{jk}(x) D_j D_k + \sum_{j=1}^N \alpha_j(x) D_j + \alpha_0(x).$$

In order to obtain the statement I) in Theorem 4.1 for this operator, we suppose that

$$\alpha_{jk} = \alpha_{kj} \in C(\overline{\Omega}) \quad \text{and} \quad \alpha_j, \alpha_0 \in L^\infty(\Omega), \quad \text{with} \quad \alpha_0 \geq 0.$$

Moreover, we also assume that \mathfrak{L} is strictly elliptic in Ω , indeed there exists a constant $\kappa > 0$ such that $\sum_{j,k} \alpha_{jk}(x) \xi_j \xi_k \geq \kappa \|\xi\|^2$ for all $x \in \Omega$ and $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$. Taking into account these assumptions and following [26], we can reproduce the same proof.

In order to obtain the statement II) in Theorem 4.1 we have to prove the existence of an upper solution β satisfying a condition analogous to (31). To this purpose, we first give the following lemma which is presented in a general form so that it can be applied in principle also in other contexts. We note also that our lemma presents some overlapping with a preceding result by Grossinho and Omari in [23, Lemma 2.1]. For the sequel, we recall the notation T^∞ introduced in (29), where $T(\cdot)$ is the time-mapping associated with the second order ODE $u'' + g(u) = 0$ (see Section 3).

LEMMA 6.1. *Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (g_*) and such that $T^\infty = +\infty$. Let $I := [t_0, t_1]$ and $B, M > 0$ be fixed real constants. Then for every measurable function $b : I \rightarrow \mathbb{R}$ with $|b(t)| \leq B$ for a.e. $t \in I$ and for every constant $K > 0$, there exists $k > K$, such that any solution $u(\cdot)$ of the initial value problem*

$$\begin{cases} u'' + b(t)u' + Mg(u) = 0, \\ u(t_0) = k, \quad u'(t_0) = 0, \end{cases} \quad (49)$$

is such that $u(t) > 0$ for all $t \in I$ and $u'(t) < 0$ for all $t \in]t_0, t_1]$.

Proof. Let $u(\cdot) : J \rightarrow \mathbb{R}^+$ be a solution of (49) defined on a right maximal interval of existence contained in I . For a.e. $t \in J$ we have that

$$\frac{d}{dt} \left(u'(t) e^{\mathcal{B}(t)} \right) + M e^{\mathcal{B}(t)} g(u(t)) = 0, \quad (50)$$

where we have set $\mathcal{B}(t) := \int_{t_0}^t b(\xi) d\xi$. Integrating on $[t_0, t]$, for $t \in J$ with $t > t_0$, it follows that

$$u'(t) = -M \int_{t_0}^t e^{\int_t^s b(\xi) d\xi} g(u(s)) ds$$

holds. This proves that $u'(t) < 0$ for all $t > t_0$ with $t \in J$.

We claim now that $J = I$ and $u(t) > 0$ for all $t \in I$. Suppose, by contradiction, that there exist a function $b : I \rightarrow \mathbb{R}$ satisfying $|b(t)| \leq B$ and a first point $t^* \in J$ such that $u(t^*) = 0$. We multiply equation (50) by $u'(t)e^{\mathcal{B}(t)}$ and so we obtain the relation

$$\frac{1}{2} \frac{d}{dt} \left(u'(t)e^{\mathcal{B}(t)} \right)^2 + Me^{2\mathcal{B}(t)} \frac{d}{dt} G(u(t)) = 0, \quad \forall t \in [t_0, t^*], \quad (51)$$

where, as usual, $G(s) := \int_0^s g(\xi) d\xi$. Notice that $\frac{d}{dt} G(u(t)) = g(u(t))u'(t) < 0$ for all $t \in]t_0, t^*[$. Integrating equation (51) on $[t_0, t] \subset [t_0, t^*[$ and after simple manipulations, we obtain

$$\begin{aligned} |u'(t)|^2 &= 2M \int_{t_0}^t e^{2 \int_t^s b(\xi) d\xi} \frac{d}{ds} (-G(u(s))) ds \\ &\leq 2Me^{2B|I|} (G(u(t_0)) - G(u(t))) = 2Me^{2B|I|} (G(k) - G(u(t))). \end{aligned}$$

Then, recalling that $u'(t) < 0$ on $]t_0, t^*[$, it follows that

$$-u'(t) \leq e^{B|I|} M^{\frac{1}{2}} \sqrt{2(G(k) - G(u(t)))}, \quad \forall t \in]t_0, t^*[.$$

From the previous inequality we have

$$\int_{u(t)}^{u(t_0)=k} \frac{ds}{\sqrt{2(G(k) - G(u(s)))}} \leq e^{B|I|} M^{\frac{1}{2}} (t - t_0), \quad \forall t \in]t_0, t^*[$$

and then, letting $t \rightarrow t^*$, we find

$$\frac{T(k)}{2} = \int_0^k \frac{ds}{\sqrt{2(G(k) - G(u(s)))}} \leq e^{B|I|} M^{\frac{1}{2}} (t^* - t_0) \leq M^{\frac{1}{2}} e^{B|I|} |I|.$$

Thus, using the fact that $\limsup_{c \rightarrow +\infty} T(c) = +\infty$, a contradiction is achieved. As a consequence, we conclude that $J = I$ and, moreover, $u(t) > 0$ for all $t \in I$. □

Now we use the preceding result to give an upper solution β as in the proof of II) in Theorem 4.1. By the use of the same notation, let a_1 and b_1 be such that $\Omega \subset]a_1, b_1[\times \mathbb{R}^{N-1}$. We proceed, by introducing the following constants:

$$M_0 > \hat{\lambda} \|a\|_\infty$$

and

$$M := \frac{M_0}{\kappa}, \quad b := \sup_{x \in \Omega} \left| \frac{\alpha_1(x)}{\alpha_{11}(x)} \right| \leq \frac{\|\alpha_1\|_\infty}{\kappa}.$$

Then, according to Lemma 6.1, let $u(\cdot) \in C^2([a_1, b_1])$ be such that

$$\begin{aligned} u''(t) - bu'(t) + Mg(u(t)) &= 0, \quad \forall t \in [a_1, b_1], \\ u(t) &> 0, \quad \forall t \in [a_1, b_1] \\ u'(t) &< 0, \quad \forall t \in]a_1, b_1]. \end{aligned} \tag{52}$$

We define

$$\beta(x) := u(x_1), \quad \forall x = (x_1, \dots, x_N) \in \bar{\Omega}.$$

By the positivity of $u(\cdot)$ on $[a_1, b_1]$ we have that (32) holds for a suitable constant η .

The choice of $\beta(x)$ implies that

$$\begin{aligned} \mathfrak{L}\beta(x) &= - \sum_{j,k=1}^N \alpha_{jk}(x) D_j D_k \beta(x) + \sum_{j=1}^N \alpha_j(x) D_j \beta(x) + \alpha_0(x) \beta(x) \\ &= -\alpha_{11}(x) u''(x_1) + \alpha_1(x) u'(x_1) + \alpha_0(x) u(x_1) \\ &\geq \alpha_{11}(x) \left(-u''(x_1) + \underbrace{\frac{\alpha_1(x)}{\alpha_{11}(x)} u'(x_1)}_{\text{using } u' < 0} \right) \\ &\geq \alpha_{11}(x) (-u''(x_1) + bu'(x_1)) \\ &= \alpha_{11}(x) Mg(u(x_1)) \geq \kappa Mg(u(x_1)) = M_0 g(u(x_1)) \\ &= \hat{\lambda} \|a\|_\infty g(u(x_1)) + (M - \hat{\lambda} \|a\|_\infty) g(u(x_1)) \\ &\geq \hat{\lambda} \|a\|_\infty g(\beta(x)) + \rho, \end{aligned}$$

where ρ is a suitable positive constant such that $(M - \hat{\lambda} \|a\|_\infty) g(u(t)) \geq \rho$ for all $t \in [a_1, b_1]$. Thus (31) is proved for \mathfrak{L} instead of $-\Delta$ and the rest of the proof of II) follows in the same manner. In conclusion, Theorem 4.1-I)-II) and Theorem 4.5 hold also for problem (48).

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