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## ABSTRACT<sup>1</sup>

In this paper we present a dynamic programming algorithm for pricing variable annuities with Guaranteed Minimum Withdrawal Benefits (GMWB) under a general Lévy processes framework. The GMWB gives the policyholder the right to make periodical withdrawals from her policy account even when the value of this account is exhausted. Typically, the total amount guaranteed for withdrawals coincides with her initial investment, providing then a protection against downside market risk. At each withdrawal date, the policyholder has to decide whether, and how much, to withdraw, or to surrender the contract. We show how different levels of rationality in the policyholder's withdrawal behaviour can be

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modelled. We perform a sensitivity analysis comparing the numerical results obtained for different contractual and market parameters, policyholder behaviours, and different types of Lévy processes.

KEYWORDS: Variable annuities, GMWB, Dynamic approach, Lévy processes, Policyholder's behaviour

# 1. Introduction

Variable annuities are very flexible life insurance contracts that package several types of options and guarantees, at policyholder discretion. Typically, a lump sum premium is paid at contract inception and is invested in one or more mutual funds chosen by the policyholder among a range of alternative opportunities. Then this initial investment sets up a reference portfolio ('policy account') and each option or guarantee is financed by periodical deductions from the policy account value.

Guarantees are commonly referred to as GMxBs (Guaranteed Minimum Benefit of type 'x'), where 'x' stands for accumulation (A), death (D), income (I) or withdrawal (W). In particular, GMABs and GMDBs provide guarantees in the accumulation phase, prior to retirement, although sometimes the GMDB is offered also after retirement. In a GMIB, that consists of a (possibly indexed, or participating) deferred life annuity, the guarantee usually concerns the annuitized amount or the annuitization rate. However, GMABs and GMDBs can be found also in other types of life insurance contracts such as unit-linked or participating policies, and GMIBs become, after conversion, traditional life annuities. The GMWB, instead, is undoubtedly the most interesting feature of variable annuities and is the most popular rider selected by variable annuity customers, see Ledlie et al. (2010). GMWBs are similar to an income drawdown, because they entitle the policyholder to make periodical withdrawals from her account, even when the account value is reduced to 0. Typically, this guarantee concerns the entire initial investment of the policyholder, that can be withdrawn within a given period of time. At the end of the withdrawal period, or at death, any remaining fund in the reference portfolio is paid back to the policyholder or to her estate. Recently, Guaranteed Lifelong Withdrawals (GLW), protecting the policyholder against the risk of underfunding due to high longevity, have been introduced in the market.

When a variable annuity contains a GMWB (or GLW) rider, there is an amount, fixed or time-dependent, that the policyholder is entitled to withdraw at some specified dates (typically, annually or semiannually). Withdrawals below this fixed amount are allowed, while withdrawals above this amount, if permitted, are subject to a penalty. Then, the prediction of the policyholder behaviour is a key-element in the valuation of such guarantee. In particular, under the so called 'static' (or 'passive') approach, it is assumed that the policyholder withdraws exactly the amount contractually specified (see Milevsky and Salisbury (2006)). The 'dynamic' approach assumes instead that the policyholder chooses the amounts to withdraw according to some optimal policy. In-between these two approaches there is the 'mixed' one, coined by Bacinello et al. (2011), that assumes a static behaviour with respect to the choice of the withdrawal amounts, but a dynamic one with respect to surrender decisions.

General information on variable annuity features can be found in Ledlie et al. (2010) and Abbey and Henshall (2007). The market for variable annuities has been steadily growing in the past 20 years. However, sales fell during the recent financial crisis and many companies offering these products had to eventually exit the business as a result of poor, or lack of, hedging of the guarantees attached.

In this paper we present a dynamic programming algorithm aimed at pricing a variable annuity with a GMWB under the dynamic approach. This algorithm is general enough

to allow for different levels of policyholder rationality in her withdrawal behaviour so that, in particular, the static and the mixed approach can be accommodated as special cases. Variants and extensions of the basic GMWB contract are easily dealt with. We overcome some well-known problems arising from assuming normality of the reference fund returns, as very often done in the literature, by putting ourselves in a general Lévy framework. This class of stochastic processes is flexible enough to allow for jumps and other desirable properties displayed by the empirical distribution of asset returns (such as fat tails and skewness) and is straightforward to implement. We present extensive numerical examples and compare the results obtained for different market and contractual parameters, policyholder behaviours, as well as for different types of Lévy processes.

The paper is structured as follows. In Section 2 we review the existing literature on GMWBs, focussing in particular on the dynamic approach. In Section 3 we describe the variable annuity contract and the discrete time framework adopted for the valuation. In Section 4 we briefly introduce and recall the main properties of Lévy processes. In Section 5 we develop the dynamic programming algorithm and in Section 6 we present the numerical results. Finally, Section 7 concludes the paper.

## 2. Review on the Literature on GMWBs

The pricing and hedging variable annuity contracts has attracted the interest of many academics and practitioners. This review focuses on GMWBs and GLWs and in no way claims to be exhaustive. We classify in Table 1 the papers of which we are aware according to the following features: type of benefit (GMWB, GLW), assumption on policyholder behaviour (static\* - to be explained below - mixed or dynamic), statistical assumption on fund return distribution and numerical tool used for the evaluation.

With the term *static\** we extend the static behaviour described in the introduction by including any fixed withdrawal or surrender strategy. More precisely, in this class we include deterministic strategies (such as the static discussed before), strategies based on the value of state variables (eg withdrawal or surrender behaviours based on the moneyness of the guarantees) that is, in the language of stochastic processes, adapted strategies, and also randomization of such strategies. As opposed to the mixed or dynamic behaviour, the *static\** approach is not the result of an optimization process. We point out that the *static\** approach, frequently adopted by practitioners in the analysis of products and e.g. in profit testing exercises, is appealing, somewhat intuitive and straightforward to implement even under sophisticated assumptions on the evolution of the state (market, mortality, ...) variables. However, it is undoubtedly hard to anticipate correctly the policyholder behaviour, e.g. to specify her policy as a function of the moneyness of the guarantees (see Knoller et al. (2011), Kent and Morgan (2008) and Abbey and Henshall (2007) for the different factors influencing lapse rates). On the other hand, the dynamic approach overcomes this subjective side by taking a worst case scenario from the insurer's point of view, but is subject to the curse of dimensionality and hence very often forces to adopt a very simple setup. Looking at Table 1, one can see that few papers go beyond the assumption of normality for the fund returns. Notable exceptions are Chen et al. (2008), where the jump diffusion model of Merton is considered, and Bélanger et al. (2009) and Forsyth and Vetzal (2012), where regime switching type processes are used. These

Paper	Rider	PH behaviour	Fund process	Tool
Boyle et al. (2005)	GMWB	static*	GBM	MC
Milevsky and Salisbury (2006)	GMWB	static/dynamic	GBM	PDE
Bauer et al. (2008)	GMWB	static*	GBM	MC
Chen and Forsyth (2008)	GMWB	dynamic	GBM	PDE
Dai et al. (2008)	GMWB	dynamic	GBM	PDE
Chen et al. (2008)	GMWB	dynamic	GBM/Merton	PDE
Shah and Bertsimas (2008)	GLW	static	SIR+SV	quasi analytic/MC
Peng et al. (2012)	GMWB	static	SIR	analytical/MC
Bélanger et al. (2009)	GMWB	dynamic	RS	PDE
Piscopo (2009)	GLW	static	GBM	MC
Bacinello et al. (2011, 2012)	GMWB/GLW	static/mixed	SIR+SV+SM	(LS)MC
Kling et al. (2011)	GLW	static*/mixed	SV	(LS)MC
Piscopo and Haberman (2011)	GLW	static	GBM+SM	MC
Moenig and Bauer (2011)	GMWB	dynamic	GBM	DP
Steinorth and Mitchell (2012)	GLW	dynamic	GBM	DP
Holz et al. (2012)	GMWB/GLW	static*/mixed	GBM	DP/MC
Forsyth and Vetzal (2012)	GLW	dynamic	RS	PDE

Table 1: GMWB=Guaranteed Minimum Withdrawal Benefit, GLW=Guaranteed Lifelong Withdrawal; GBM=Geometric Brownian Motion, Merton=Merton Jump Diffusion model, RS=Regime Switching, SIR=Stochastic Interest Rates, SV=Stochastic Volatility, SM=Stochastic Mortality; PDE=Partial Differential Equations, DP=Dynamic Programming, MC=Monte Carlo LSMC=Least-Squares Monte Carlo

contributions use partial differential equations to solve the corresponding optimization problem.

In this paper, we show how an approach based on dynamic programming can accommodate any fund return distribution within the class of Lévy processes, allowing therefore a great variety of statistical features such as kurtosis and skewness.

### 3. The structure of the contract

#### *a. Description*

Although there are different ways in which a GMWB can be arranged within a variable annuity contract, in what follows we focus on a specific case, that is the most common in practice. At contract inception, the policyholder pays a lump-sum premium, that is invested in a well diversified mutual fund, hence a reference portfolio backing the variable annuity is set up in this way. The current value of this portfolio defines the first of two accounts which the policyholder is entitled to, called ‘personal account’. After that, the policyholder has the right to make periodical withdrawals, even if her personal account value is reduced to zero. Usually the total withdrawals guaranteed during the life of the contract amount to her whole initial investment. Then, in this case, the guarantee becomes effective if the reference portfolio is completely exhausted before the initial premium has been totally recouped. The second account, called ‘guarantee account’, keeps updated the total amount of money that the policyholder is still guaranteed for withdrawals. The cost of the guarantee is financed by periodical deductions from the personal account value (‘insurance fees’). The amount that the policyholder is entitled to withdraw at

each available date is usually subject to a withdrawal level, fixed or time-dependent, over which some penalty is applied. At maturity, the policyholder (or her estate) receives the maximum between the balance of the personal account and the guarantee account.

When the contract contains a surrender option, the policyholder is allowed to terminate the contract before maturity. In this case she receives a cash amount, called surrender value, usually equal to the balance of the personal account with a proportional penalty if it is higher than the specified withdrawal level. We notice that there is an important difference between surrender and withdrawal of all the guarantee account ('total withdrawal'), at least when the balance of the personal account is higher than that of the guarantee account. In case of surrender, in fact, the contract ceases to exist, while in case of total withdrawal the contract remains still in force if the personal account is not completely exhausted, and the insurance fee continues to be applied. Then, at maturity, the balance of the personal account is entirely paid back. Moreover, after total withdrawal the policyholder may still decide to surrender the contract before maturity if she values that the periodical application of the insurance fee prevails on the (lump sum) surrender penalty.

Finally, we observe that the introduction of mortality risk can be easily handled in the dynamic programming algorithm, as we will see in Sections 3d and 5. We refer, more in detail, to the case in which the contract expires before maturity (and before surrender) if the insured dies, with the payment of a lump sum benefit specified in the contract, typically equal to the balance of one of the two accounts or to the maximum between them.

### *b. Model and Valuation*

We now formalize what just described. Let  $A_t$  and  $W_t$  denote the time  $t$  guarantee account and personal account respectively, before any decision at  $t$  is made. Moreover, let  $S_t$  denote the unit price at time  $t$  of the reference fund,  $U$  the lump sum premium,  $T$  the maturity of the policy.

Assume that withdrawals are allowed only at times  $t_i$ ,  $i = 1, 2, \dots, N - 1$ , with  $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$ , where  $t_0$  denotes the inception of the contract.<sup>2</sup> The return on the fund over  $(t_i, t_{i+1})$ ,  $i = 0, \dots, N - 1$ , is then

$$R_{t_i} = \frac{S_{t_{i+1}} e^{q(t_{i+1}-t_i)}}{S_{t_i}} - 1,$$

where  $q$  is the dividend yield, assumed to be constant. Let  $\theta_{t_i}$  denote the decision made at time  $t_i$  by the policyholder. In the simplest case,  $\theta_{t_i}$  is just the amount withdrawn at  $t_i$ , but examples involving other types of decisions can be considered. For the moment we think of  $\theta_{t_i}$  as some element of a set of admissible decisions  $\Theta_{t_i}$ , which can depend on the current value of the state variables  $A_{t_i}$  and  $W_{t_i}$ .

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<sup>2</sup>If  $t_0$  coincides, instead, with the end of an accumulation period and  $U$  is the (possibly guaranteed) accumulation benefit, then  $t_0$  could be included in the set of possible withdrawal times.



The accounts evolve according to the following equations:

$$W_{t_{i+1}} = g_1(W_{t_i}, R_{t_i}, \theta_{t_i}), \quad (1)$$

$$A_{t_{i+1}} = g_2(W_{t_i}, A_{t_i}, \theta_{t_i}), \quad (2)$$

for some nonnegative functions  $g_1, g_2$ . Hence  $W_{t_{i+1}}$  is determined by the current personal account value, the fund return and the decision, while  $A_{t_{i+1}}$  depends on the current value of both accounts and the decision. The initial conditions are  $W_0 = A_0 = U$ .<sup>3</sup> The cash flow paid to the policyholder at  $t_i$ ,  $i = 1, \dots, N - 1$ , is denoted by  $C_{t_i}$  and depends on the current decision and state variables, written

$$C_{t_i} = g_3(W_{t_i}, A_{t_i}, \theta_{t_i}) \quad (3)$$

for a nonnegative function  $g_3$ . For  $i = N$ ,  $C_{t_N} = C_T$  is specified separately, typically as a function of  $W_T, A_T$ .<sup>4</sup>

Let  $\pi$  denote a possible sequence of (withdrawals) decisions, i.e.  $\pi = (\theta_{t_1}, \theta_{t_2}, \dots, \theta_{t_{N-1}})$  with  $\theta_{t_i} \in \Theta_{t_i}$ . The initial value of the cash flows generated from holding the GMWB variable annuity and adopting the sequence of decisions  $\pi$  is given by:

$$V_0^\pi = E \left[ \sum_{i=1}^N e^{-r t_i} C_{t_i} \right], \quad (4)$$

where  $E$  denotes the expectation taken under a suitable risk-adjusted measure and  $r$  is the (assumed constant) risk-free rate. Finally, the no-arbitrage value of the variable annuity is given by:

$$V_0 = \sup_{\pi} V_0^\pi, \quad (5)$$

where the supremum is taken over all sequences  $\pi = (\theta_{t_1}, \theta_{t_2}, \dots, \theta_{t_{N-1}})$  of withdrawal decisions satisfying the constraint  $\theta_{t_i} \in \Theta_{t_i}$  and where the personal and guarantee account satisfy (1)-(2). This approach assumes that the policyholder behaves rationally and acts so as to maximize the expected present value of all the cash flows generated by the GMWB variable annuity. This problem can be solved using the dynamic programming algorithm, as explained in Section 5. In the rest of this section, we exemplify the framework introduced by specifying the set of admissible decisions and the state equations.

### 1) DYNAMIC WITHDRAWALS

At any date  $t_i$ ,  $i = 1, \dots, N - 1$ , the policyholder can choose to withdraw any amount  $\theta_{t_i}$  up to the guarantee account  $A_{t_i}$ , that is the decision set is  $\Theta_{t_i} = [0, A_{t_i}]$ . The evolution of the accounts between  $t_i$  and  $t_{i+1}$ , equations (1)-(2), is now given by

$$W_{t_{i+1}} = \max \{W_{t_i} - \theta_{t_i}, 0\} (1 + R_{t_i})(1 - \varphi(t_{i+1} - t_i)), \quad (6)$$

$$A_{t_{i+1}} = A_{t_i} - \theta_{t_i}, \quad (7)$$

<sup>3</sup>For  $i = 0$ , we set conventionally  $\theta_0 = \theta_{t_0} = 0$ , so that  $W_{t_1}$  and  $A_{t_1}$  are determined only by the single premium  $U$  and the first period return  $R_{t_0}$ .

<sup>4</sup>Different type of contracts, or more general frameworks, can be represented with a similar scheme, by adding state variables and their state equations. The functions  $g_i$ ,  $i = 1, 2, 3$ , may depend on these additional variables as well.

where  $\varphi$  is the insurance fee, applied while the contract is still in force. Note that once  $W_t$  or  $A_t$  hits the value 0, they stay at this value thereafter. Hence withdrawals continue while the guarantee account is positive, even if the personal account is insufficient.

Although the policyholder can withdraw any amount  $0 \leq \theta_{t_i} \leq A_{t_i}$ , a proportional penalty at rate  $\kappa_1$  applies in case withdrawals exceed a withdrawal level  $G$ , which is typically equal to  $\frac{A_0}{N}$ .<sup>5</sup> The cash flow paid to the policyholder, equation (3), now becomes

$$C_{t_i} = \begin{cases} \theta_{t_i} & \text{if } 0 \leq \theta_{t_i} \leq G \\ G + (1 - \kappa_1)(\theta_{t_i} - G) & \text{if } \theta_{t_i} > G \end{cases} \quad (8)$$

$$= \theta_{t_i} - \kappa_1 \max\{\theta_{t_i} - G, 0\}.$$

Note that the penalty charge is applied on the portion of  $\theta_{t_i}$  exceeding  $G$ . At maturity, the policyholder receives the maximum between the remaining balance in both accounts, that is  $C_{t_N} = \max\{W_T, A_T\}$ .

## 2) STATIC WITHDRAWALS

The policyholder is constrained to withdraw the amount  $G$ , provided this is lower than the guarantee account, or the guarantee account otherwise. This behaviour is obtained by setting the set of decisions at  $t_i$  as the singleton  $\Theta_{t_i} = \{\min\{G, A_{t_i}\}\}$ . The accounts and cash-flow are still defined by (6), (7) and (8). More generally, the static\* approach defined in Section 2 corresponds to fixing a sequence  $\pi$  of withdrawal decisions, and the value of the contract is then  $V_0^\pi$ .

As shown in Milevsky and Salisbury (2006), under the static approach the variable annuity contract can be decomposed (in the case  $t_i = i$  and  $G = W_0/T$ ) into an immediate annuity with instalment  $G$  and maturity  $T$  and a Quanto-Asian put option corresponding to the guarantee of receiving at maturity the policyholder account net of the last instalment, if positive. More specifically, the pay-off of the put option is

$$W_0 \frac{1}{Y_T} \max\{1 - \bar{Y}_T, 0\},$$

where

$$Y_t = S_t^{-1}(1 - \varphi)^{-t}, \quad \bar{Y}_T = \frac{1}{T} \sum_{t=1}^T Y_t.$$

## 3) DYNAMIC+SURRENDER

We introduce the possibility of surrendering the GMWB before maturity. Without loss of generality, we assume that surrender can take place only at the discrete withdrawal

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<sup>5</sup>An alternative to (7) is to specify the guarantee account evolution including a reset provision in case of withdrawals above  $G$  (see for instance Milevsky and Salisbury (2006)):

$$A_{t_{i+1}} = \begin{cases} A_{t_i} - \theta_{t_i} & \text{if } 0 \leq \theta_{t_i} \leq G \\ \max\{\min\{A_{t_i} - \theta_{t_i}, W_{t_i} - \theta_{t_i}\}, 0\} & \text{if } \theta_{t_i} > G \end{cases}.$$

The extension to a time dependent withdrawal level is straightforward, while the inclusion of reset provisions on the guaranteed withdrawal amount requires an additional state variable.

dates  $t_i$ ,  $i = 1, 2, \dots, N - 1$ . Letting  $\theta_{t_i} = s$  denote the decision to surrender the contract at  $t_i$ , the set of possible choices for the policyholder is enlarged and becomes  $\Theta_{t_i} = [0, A_{t_i}] \cup \{s\}$ . The state equations and the cash flow now need to be redefined. The amount received by the policyholder in case of surrender is given by the personal account value  $W_{t_i}$ , net of a penalty  $\kappa_2$  applied as before to the portion of  $W_{t_i}$  exceeding  $G$  (or  $A_{t_i}$ , if less). We then modify definition (8) as follows:

$$C_{t_i} = \begin{cases} \theta_{t_i} & \text{if } 0 \leq \theta_{t_i} \leq G \\ \theta_{t_i}(1 - \kappa_1) + \kappa_1 G & \text{if } \theta_{t_i} > G \\ W_{t_i}(1 - \kappa_2) + \kappa_2 \min \{G, A_{t_i}, W_{t_i}\} & \text{if } \theta_{t_i} = s \end{cases}. \quad (9)$$

The state equations now just reflect the fact that, in case of surrender, the accounts are set to 0, that is

$$W_{t_{i+1}} = \begin{cases} \max \{W_{t_i} - \theta_{t_i}, 0\} (1 + R_{t_i})(1 - \varphi(t_{i+1} - t_i)) & \text{if } 0 \leq \theta_{t_i} \leq A_{t_i} \\ 0 & \text{if } \theta_{t_i} = s \end{cases}, \quad (10)$$

$$A_{t_{i+1}} = \begin{cases} A_{t_i} - \theta_{t_i} & \text{if } 0 \leq \theta_{t_i} \leq A_{t_i} \\ 0 & \text{if } \theta_{t_i} = s. \end{cases} \quad (11)$$

Note that, provided  $\kappa_2 \geq \kappa_1$ , surrender is never optimal if  $W_{t_i} \leq A_{t_i}$ .

#### 4) MIXED (STATIC+SURRENDER)

As in the static approach, the policyholder is behaving passively with respect to partial withdrawals, but can choose to surrender in a dynamic fashion. The decision set at  $t_i$  is now  $\Theta_{t_i} = \{\min\{A_{t_i}, G\}, s\}$ , while the cash flow and the state equations (9)-(11) are unchanged.

#### c. Fair Pricing and Comparison

Recall that the cost of the guarantee is charged to the policyholder through the application of the proportional insurance fee rate  $\varphi$  to the personal account. Hence the contract is fairly priced if and only if its initial value  $V_0$ , computed under any of the approaches introduced in 3b, coincides with the initial premium  $U$ . Then, the fair fee rate  $\varphi^*$  can be defined as a solution of the following equation:

$$V_0(\varphi) = U \quad (12)$$

where, with a slight abuse of notation, we explicitly indicate that  $V_0$  is a function of the proportional fee rate  $\varphi$ .

Denote now by  $V_0^{\text{dynamic}}$ ,  $V_0^{\text{static}}$ ,  $V_0^{\text{surrender}}$  and  $V_0^{\text{mixed}}$  the initial values of the contract under each of the assumptions in 3b, and by  $\varphi^{\text{dynamic}}$ ,  $\varphi^{\text{static}}$ ,  $\varphi^{\text{surrender}}$  and  $\varphi^{\text{mixed}}$  the corresponding fair fees. It is clear that

$$V_0^{\text{static}} \leq V_0^{\text{dynamic}} \leq V_0^{\text{surrender}}, \quad \varphi^{\text{static}} \leq \varphi^{\text{dynamic}} \leq \varphi^{\text{surrender}}.$$

The same inequalities hold when  $V_0^{\text{dynamic}}$  and  $\varphi^{\text{dynamic}}$  are replaced respectively by  $V_0^{\text{mixed}}$  and  $\varphi^{\text{mixed}}$ . No direct comparison seems possible between values and fees computed under the dynamic and mixed approach. The spread  $V_0^{\text{dynamic}} - V_0^{\text{static}}$  can be interpreted as the extra cost, in terms of single premium, required to add to the ‘static’ contract (the policyholder can only withdraw the amount  $G$  and receive the remaining personal account at maturity) the possibility to withdraw any amount up to the personal account. Similar interpretations apply to the difference  $V_0^{\text{surrender}} - V_0^{\text{mixed}}$  or to the corresponding spreads computed in terms of fair fees, while the differences  $V_0^{\text{mixed}} - V_0^{\text{static}}$  and  $V_0^{\text{surrender}} - V_0^{\text{dynamic}}$  can be seen as the extra cost required to add the surrender option to the ‘static’ and ‘dynamic’ contract respectively.

#### d. Mortality Risk

We assume now that, if the insured dies before maturity  $T$ , the contract prematurely expires with the payment of a death benefit. To simplify the treatment, we suppose that, in case of death between two withdrawal dates  $t_i$  and  $t_{i+1}$ , the death benefit, denoted by  $D_{t_{i+1}}$ , is paid at  $t_{i+1}$ , after which the values of the personal and guarantee account are set to 0.

If the contract does not include a GMDB, the death benefit is given by the personal account, i.e.  $D_{t_{i+1}} = W_{t_{i+1}}$ . If instead a GMDB is included, in case of death there is a guaranteed amount  $G_{t_{i+1}}^{\text{d}}$  so that the death benefit is  $\max\{G_{t_{i+1}}^{\text{d}}, W_{t_{i+1}}\}$ . The guaranteed amount can be computed, for instance, according to:<sup>6</sup>

- $G_{t_{i+1}}^{\text{d}} = A_{t_{i+1}}$  (return of premium net of withdrawals);
- $G_{t_{i+1}}^{\text{d}} = \max\{G_{t_i}^{\text{d}} - \theta_{t_i}, 0\}e^{r^{\text{d}}(t_{i+1}-t_i)}$  with  $G_0^{\text{d}} = A_0$  (roll-up of residual premium at some interest rate  $r^{\text{d}}$ ).

## 4. Lévy Processes Framework

In order to model the fund value, we start with a stochastic process  $(X_t)_{t \geq 0}$ , with  $X_0 = 0$ , defined on the basic probability space equipped with the risk neutral measure introduced in the previous section. We assume that  $X_t$  is a Lévy process, that is  $X_t$  has right-continuous with left limits paths,  $X_s - X_t$  is independent of  $(X_u)_{0 \leq u \leq t}$  and is distributed as  $X_{s-t}$ , for  $0 \leq t < s$ . For a comprehensive description of Lévy processes, their properties and applications we refer to Cont and Tankov (2004) and Schoutens (2003). Lévy processes are a combination of a linear drift, a Brownian motion, and a jump process. A Lévy process  $(X_t)$  is determined by its characteristic function

$$\Phi_t(u) := E[e^{iuX_t}] = [\Phi_1(u)]^t$$

and, in particular, all moments of  $X_t$  can be numerically recovered from the knowledge of  $\Phi_t$ , when they are not available in closed form. If  $\Phi_t$  is integrable, then  $X_t$  has density

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<sup>6</sup>Of course  $G_{t_{i+1}}^{\text{d}}$  is meaningless if  $\theta_{t_i} = s$ , being the contract no longer in force after surrender. Note moreover that the valuation of the GMDB guarantee with a roll-up of premium would require to treat  $G_t^{\text{d}}$  as a state variable.

given by:

$$f_t(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izx} \Phi_t(z) dz.$$

We model the reference portfolio value  $S_t$  as an exponential Lévy process:

$$S_t = S_0 e^{(r-q+d)t + X_t},$$

where  $q$  is the dividend yield and  $d = -\frac{1}{t} \ln \Phi_t(-i) = -\ln \Phi_1(-i)$  represents the adjustment so that  $(S_t e^{-(r-q)t})$  is a martingale under the risk-neutral measure.

In the numerical experiments we consider the following examples of Lévy processes, commonly used in finance applications, although we could in principle use any exponential Lévy model to represent the fund dynamics.

**(1) Geometric Brownian Motion (GBM)**

$$\Phi_t(u) = \exp\left(iu\mu t - \frac{1}{2}\sigma^2 u^2 t\right),$$

with  $\mu \in \mathbb{R}$ ,  $\sigma > 0$ .

**(2) Merton Jump Diffusion (MJD)**

$$\Phi_t(u) = \exp\left(iu\mu t - \frac{1}{2}\sigma^2 u^2 t + \lambda t \left(e^{ium - \frac{1}{2}c^2 u^2} - 1\right)\right),$$

with  $\mu, m \in \mathbb{R}$ ,  $\sigma, c, \lambda > 0$ .

**(3) Variance-Gamma (VG)**

$$\Phi_t(u) = \exp\left(-\frac{t}{\nu} \ln\left(1 - iu\mu\nu + \frac{1}{2}\sigma^2 u^2 \nu\right)\right),$$

with  $\mu \in \mathbb{R}$ ,  $\sigma, \nu > 0$ .

**(4) Carr, Geman, Madan, Yor (CGMY)**

$$\Phi_t(u) = \exp(c \Gamma(-y) t [(m - iu)^y - m^y + (g + iu)^y - g^y]),$$

with  $g, m \geq 0$ ,  $c > 0$ ,  $y < 2$  and  $\Gamma$  is the gamma function.

Example (1) is a pure diffusion, without jump component. A jump component in the equity returns is introduced by Merton (1976) through a compound Poisson process, leading to Example (2), where  $\lambda$  denotes the jump intensity and  $m$  and  $c$  are the mean and standard deviation of the log jump sizes, assumed to be normally distributed. This jump component produces a finite number of jumps within any finite time interval, i.e., the process exhibits finite activity, allowing to capture rare and large events such as market crashes or corporate defaults. However, market prices can also experiment very frequent jumps of different sizes within any finite time interval. This property is captured by infinite activity processes, e.g. by Example (3), which is a pure jump process with infinite

activity and paths of finite variation. The Variance-Gamma process was first introduced by Madan and Seneta (1990) and Madan and Milne (1991), and then extended by Madan et al. (1998). Here, in Example (3), we refer to this latter extension. In particular, the VG process can be seen as a Brownian motion with constant drift  $\mu$  and volatility  $\sigma$ ,<sup>7</sup> with a stochastic time change defined through a gamma process with unit mean rate and variance rate  $\nu$ . Alternatively, the VG can also be seen as the difference between two independent gamma processes with suitable parameters. This process has a lot of desirable properties consistent with empirical evidence; it allows, e.g., to control skewness and kurtosis of the return distribution and to correct some biases in option pricing implied by the Black and Scholes (1973) model. A further generalization of the VG process is the Carr et al. (2002) model (CGMY), given by Example (4), that allows for both a diffusion and a jump component. Moreover, it can be suitably parametrized in order to capture finite or infinite activity as well as finite or infinite variation.

## 5. Dynamic programming algorithm

The value of the GMWB is found by implementing the following standard dynamic programming algorithm for discrete stochastic control problems (see e.g. Bertsekas (2005) and Seierstad (2009)). As we act in a Markovian framework, for each  $t_i$ ,  $i = 1, \dots, N$ , and each value of the guarantee account  $A_{t_i}$  and personal account  $W_{t_i}$ , we denote the no-arbitrage value at date  $t_i$  of the variable annuity as  $V(t_i, A_{t_i}, W_{t_i})$ . The initial value of the GMWB, that is the solution of (5), is found by solving the Bellman recursive equation, which proceeds backward in time for  $i = N - 1, \dots, 1$ :

$$V(t_i, A_{t_i}, W_{t_i}) = \sup_{\theta \in \Theta_{t_i}} E [C_{t_i} + e^{-r(t_{i+1}-t_i)} V(t_{i+1}, A_{t_{i+1}}, W_{t_{i+1}}) | A_{t_i}, W_{t_i}],$$

$$V(t_N, A_{t_N}, W_{t_N}) = \max\{A_{t_N}, W_{t_N}\}.$$

Note that the equations for the cash-flow and the accounts are given by (6)-(8) (or (9)-(11)). The initial value of the contract is then found by computing

$$V_0 = E [e^{-r(t_1-t_0)} V(t_1, A_{t_1}, W_{t_1}) | A_{t_0} = W_{t_0} = U].$$

When mortality is taken into account (see Section 3d), the Bellman equation for the value function  $V$  (provided mortality and the financial variables are independent) becomes

$$V(t_i, A_{t_i}, W_{t_i}) =$$

$$= \sup_{\theta \in \Theta_{t_i}} \left\{ C_{t_i} + {}_{t_{i+1}-t_i}p_{x+t_i} E [e^{-r(t_{i+1}-t_i)} V(t_{i+1}, A_{t_{i+1}}, W_{t_{i+1}}) | A_{t_i}, W_{t_i}] \right.$$

$$\left. + {}_{t_{i+1}-t_i}q_{x+t_i} E [e^{-r(t_{i+1}-t_i)} D_{t_{i+1}} | A_{t_i}, W_{t_i}] \right\}, \quad i = 1, 2, \dots, N - 1,$$

$$V(t_N, A_{t_N}, W_{t_N}) = \max\{A_{t_N}, W_{t_N}\},$$

---

<sup>7</sup>The original model by Madan and Seneta (1990) was instead without drift.

where  $x$  is the age of the insured at time  $t_0 = 0$ ,  ${}_u p_y$  is the (risk-neutral) probability of surviving age  $y + u$  conditional on surviving age  $y$ , while  ${}_u q_y = 1 - {}_u p_y$  is the corresponding death probability. The initial value of the contract is then

$$V_0 = {}_{t_1} p_x E \left[ e^{-rt_1} V(t_1, A_{t_1}, W_{t_1}) \mid A_0 = W_0 = U \right] \\ + {}_{t_1} q_x E \left[ e^{-rt_1} D_{t_1} \mid A_0 = W_0 = U \right].$$

Note that, if there is no GMDB ( $D_{t_{i+1}} = W_{t_{i+1}}$ ) and  $\theta \neq s$ , the expected discounted payoff in case of death simplifies to

$$E \left[ e^{-r(t_{i+1}-t_i)} D_{t_{i+1}} \mid A_{t_i}, W_{t_i} \right] = \max\{W_{t_i} - \theta, 0\}(1 - \varphi(t_{i+1} - t_i)),$$

and is 0 for  $\theta = s$ .

The execution of the algorithm requires a discretization over the state variables  $W$  and  $A$  and interpolation of the value function over the resulting grid in order to compute the expectation (see for instance Judd (1998)). As the density of the 1 year log return can be straightforwardly computed through Fourier inversion, the expectation can be calculated via numerical integration.

#### a. Algorithm

We outline the algorithm employed to value a GMWB variable annuity under the dynamic+surrender approach. The valuation under the alternative approaches described in 3b requires minor and obvious modifications.

**Step 0.** For each  $i = 0, \dots, N$ , discretize the state space  $[0, A_0]$  for  $A_{t_i}$  and  $[0, \infty)$  for  $W_{t_i}$ :

$$\mathbb{A} = \{a_1, \dots, a_H\}, \quad 0 = a_1 < a_2 < \dots < a_H = A_0, \\ \mathbb{W} = \{w_1, \dots, w_K\}, \quad 0 = w_1 < w_2 < \dots < w_K.$$

**Step 1.** Start at  $t_N = T$  by setting  $V(t_N, a_h, w_k) = \max\{a_h, w_k\}$  for each  $(a_h, w_k) \in \mathbb{A} \times \mathbb{W}$ .

**Step 2.** Proceed backwards: for  $i = N - 1, \dots, 1$

**I** - interpolate the  $H \cdot K$  triplets  $(a_h, w_k, V(t_{i+1}, a_h, w_k))$ ,  $h = 1, \dots, H$  and  $k = 1, \dots, K$ , to construct the function  $\tilde{V}(t_{i+1}, a, w)$  for  $0 \leq a \leq A_0$  and  $w \geq 0$ ;

**II** - for each  $(a_h, w_k) \in \mathbb{A} \times \mathbb{W}$  compute

$$V(t_i, a_h, w_k) = \sup_{\theta \in \Theta_{t_i}} \left\{ C_{t_i} + 1_{\theta \neq s} e^{-r(t_{i+1}-t_i)} \int_{-\infty}^{\infty} \tilde{V}(t_{i+1}, \tilde{a}, \tilde{b}) f_1(z) dz \right\},$$

where

$$C_{t_i} = \begin{cases} \theta - \kappa_1 \max\{\theta - G, 0\} & \text{if } 0 \leq \theta \leq a_h \\ w_k(1 - \kappa_2) + k_2 \min\{G, a_h, w_k\} & \text{if } \theta = s \end{cases},$$

$$\tilde{a} = a_h - \theta,$$

$$\tilde{b} = \max\{w_k - \theta, 0\} e^{(r-q+d)(t_{i+1}-t_i)+z} (1 - \varphi(t_{i+1} - t_i)).$$

Computing the sup in II requires discretization of the control set  $\Theta_{t_i}$  to select the supremum.

**Step 3.** The value of the contract at inception is

$$V_0 = e^{-r t_1} \int_{-\infty}^{\infty} \tilde{V}(t_1, U, U e^{(r-q+d)t_1+z}(1-\varphi t_1)) f_1(z) dz.$$

Note that  $f_1$  and  $d$  have been introduced in Section 4 and can be computed before implementing the above algorithm. In particular, the density  $f_1$  is obtained through inversion of the characteristic function  $\Phi_1$  (see Bailey and Swartztrauber (1994)) and then the constant  $d$  can be calculated via numerical integration. Similarly, numerical integration (e.g. simple trapezoidal rule or Gauss quadrature) can be used to compute the integrals in Step 2.II and Step 3 of the algorithm.

## 6. Numerical results

We fit the four models introduced in Section 4 to option prices on the S&P 500 observed on 31 December 2012, using maturity specific interest rates and dividend yields. We consider both call and put options for maturities up to 2 years, and discard options too far in or out of the money. When not available in closed form, plain vanilla option prices in a Lévy framework can be computed easily using Fourier inversion techniques, see for instance Jackson et al. (2008/09). The fitting results in the parameter estimates are contained in Table 2, together with other key statistics.<sup>8</sup> The densities of the 1 year log return for the different estimated models are displayed in Figure 1.<sup>9</sup>

model	GBM	Merton	VG	CGMY
	$\sigma = 0.1361$	$\sigma = 0.1114$	$\sigma = 0.1301$	$c = 0.6817$
		$\lambda = 0.5282$	$\mu = -0.3150$	$g = 18.0293$
		$m = -0.1825$	$\nu = 0.1753$	$m = 57.6250$
		$c = 0.1094$		$y = 0.8000$
volatility (%)	13.61	21.58	18.53	15.59
skewness	0	2.1783	-0.7430	-0.3156
kurtosis	3	9.9050	3.9237	3.2743

Table 2: Parameters of the Lévy processes obtained by calibration to S&P 500 option prices

From Figure 1 and Table 2 one can observe that the calibration leads to notable differences among the four models here considered. Nevertheless, the simplest and the most sophisticated model, namely GBM and CGMY, are relatively close to each other, both in terms of moments and numerical results (see Tables 3-5). On the other hand, the Merton and VG models, although they differ in terms of skewness and kurtosis, produce

<sup>8</sup>Moments related to Lévy processes can be straightforwardly computed using cumulants, see Cont and Tankov (2004).

<sup>9</sup>In the CGMY model we fix  $y = 0.8$ , implying a finite variation, infinite activity process.



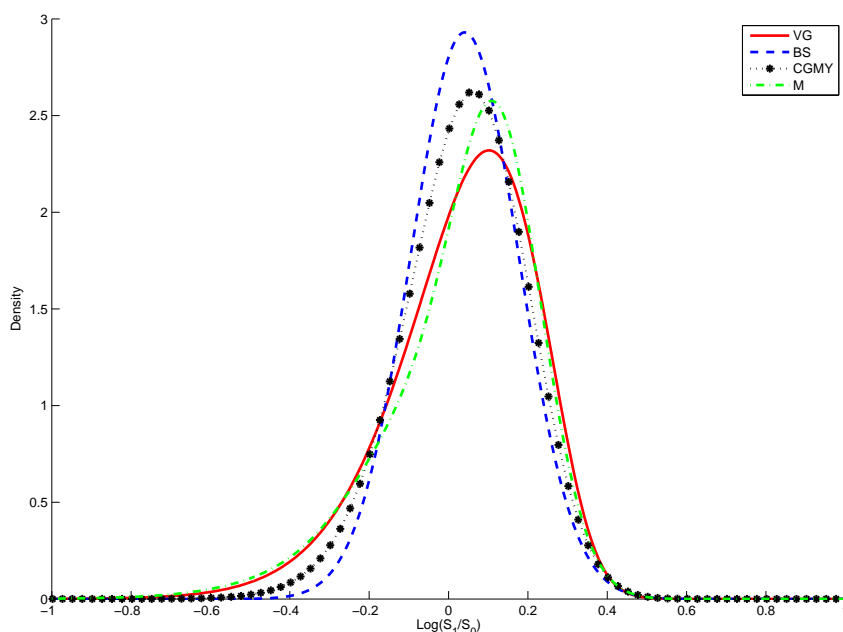


Figure 1: Density of 1 year log return

comparable results that are always much higher than those obtained with the GBM and CGMY. This can be imputed to the heavier tail displayed by the Merton and VG models, so that the guarantees implicit in the GMWB are underpriced by models that are not able to capture extreme movements in the fund process.

A comparative static analysis is performed for the contract value and fair fees for the different models and contract parameters, market interest rate and policyholder behaviour. If not otherwise mentioned, we use the CGMY model and the following parameter values as *benchmark case*:  $t_i = i$ ,  $T = 20$ ,  $r = 5\%$ ,  $\kappa := \kappa_1 = \kappa_2 = 5\%$ ,  $q = 0$ ,  $U = 100$ ,  $G = U/T$ . In Table 3 we report the fair fee rates  $\varphi^{\text{surrender}}$  and, in brackets,  $\varphi^{\text{dynamic}}$ , in basis points, for different levels of the market interest rate  $r$ . Similar results are reported in Table 4, for different maturities  $T$ , and in Table 5, for different values of the penalty rate  $\kappa$ .

The fair fee rates  $\varphi^{\text{surrender}}$  and  $\varphi^{\text{dynamic}}$  decrease with  $r$ , as expected. Note that the fee required to compensate the surrender option, given by the spread  $\varphi^{\text{surrender}} - \varphi^{\text{dynamic}}$ , decreases with  $r$  in each model. In particular, this spread ranges from 42-173 b.p when  $r = 3\%$  to 0-55 b.p. when  $r = 7\%$ . The surrender option is not priced by the GBM model for  $r \geq 5\%$  and by the CGMY model for  $r \geq 7\%$ , confirming the importance of the fund distribution tail.

Similar findings can be seen in Table 4, as contracts with a longer maturity, with or without the surrender feature, require a lower fee to be fair. As  $T$  increases, several effects on the contract value can be highlighted, and the overall impact is negative. Firstly, the insurance fee is applied over a longer period; secondly, the GMWB guarantee has a lower

$r$ (%)	3	4	5	6	7
GBM	115 (73)	61 (50)	38 (38)	29 (29)	22 (22)
Merton	323 (150)	251 (117)	193 (91)	144 (73)	104 (59)
VG	318 (145)	243 (112)	181 (86)	130 (68)	89 (54)
CGMY	197 (103)	128 (74)	79 (54)	45 (42)	35 (35)

Table 3:  $\varphi^{\text{surrender}} - \varphi^{\text{dynamic}}$  in b.p., for different risk-free interest rates.

$T$	10	15	20	25	30
GBM	72 (72)	49 (49)	38 (38)	31 (30)	26 (26)
Merton	215 (158)	200 (115)	193 (91)	189 (77)	186 (66)
VG	205 (150)	188 (109)	181 (86)	177 (72)	174 (62)
CGMY	107 (101)	85 (71)	79 (54)	75 (43)	72 (38)

Table 4:  $\varphi^{\text{surrender}} - \varphi^{\text{dynamic}}$  in b.p., for different contract maturities.

value since the market rate is higher than the minimum interest rate guaranteed on the personal account, which in our examples is 0% as  $G = U/T$ ; finally, the guarantee is offered over a longer period, and this instead has a positive impact on the contract value. Unlike Table 3, here the spread  $\varphi^{\text{surrender}} - \varphi^{\text{dynamic}}$  increases with the contract duration, in line with the fact that American options premiums increase with the time to maturity, even though, in our case, this higher value is recovered on average over a longer period.<sup>10</sup> An exception is the GBM model under which, for the combination of parameters under scrutiny (in particular  $r = 5\%$ ,  $\kappa = 5\%$ ), surrendering the contract is never optimal.

The penalties for non guaranteed withdrawals or surrender have an obvious depressing effect on the fair fee rates, as can be seen in Table 5. Recall in fact that, for fair contracts, there is a trade-off between  $\varphi$  and  $\kappa$ , since the cost of the GMWB guarantee is recouped through the fee and the penalty. If no penalty is applied the fair fee turns out to be extremely high, in particular if surrender is allowed, while for exceedingly high penalties all models show that surrendering the contract is never optimal. One is lead to think that, for these penalty levels, withdrawing amounts greater than  $G$  is always not optimal. Also the spread  $\varphi^{\text{surrender}} - \varphi^{\text{dynamic}}$  decreases with  $\kappa$ , as  $\varphi^{\text{surrender}}$  is more sensitive than  $\varphi^{\text{dynamic}}$  to changes of  $\kappa$ . Again, the GBM produces the lowest fair fees, and only for low

<sup>10</sup>Due to the surrender feature, this period can be shorter than the time to maturity.

$\kappa$ (%)	0	2.5	5	7.5	10
GBM	543 (183)	181 (72)	38 (38)	30 (30)	24 (24)
Merton	803 (262)	428 (147)	193 (91)	79 (72)	62 (62)
VG	821 (262)	429 (144)	181 (86)	70 (67)	58 (58)
CGMY	665 (216)	284 (103)	79 (54)	44 (44)	37 (37)

Table 5:  $\varphi^{\text{surrender}}$  ( $\varphi^{\text{dynamic}}$ ) in b.p., for different penalties.

penalties the surrender decision may be convenient.

Coming now to the differences among the valuation approaches described in Sections 3b, in Table 6 we report the results pertaining the fair fee rate obtained for different penalty levels. Note that these results have been constructed using the CGMY model. We observe that the possibility of acting dynamically, i.e. withdrawing amounts different

$\kappa$ (%)	0	2.5	5	7.5	10
$\varphi^{\text{static}}$	13	13	13	13	13
$\varphi^{\text{mixed}}$	51	13	13	13	13
$\varphi^{\text{dynamic}}$	216	103	54	44	37
$\varphi^{\text{surrender}}$	665	284	79	44	37

Table 6: Fair fees in b.p., benchmark case, for different penalties and policyholder behaviours.

from  $G$ , has always a significant impact on the cost of the guarantee, as measured by the difference  $\varphi^{\text{dynamic}} - \varphi^{\text{static}}$  in the case of ‘European’ contracts (without surrender) and by  $\varphi^{\text{surrender}} - \varphi^{\text{mixed}}$  in the case of ‘American’ contracts (including surrender). However, since for high penalties the surrender option does not require an additional fee ( $\varphi^{\text{mixed}} = \varphi^{\text{static}}$ ,  $\varphi^{\text{surrender}} = \varphi^{\text{dynamic}}$ ), this extra value seems to be mainly due to the possibility of withdrawing amounts lower than  $G$ , rather than higher. From another point of view, there seems to be a compound effect on fees when guarantees are combined. For instance, when  $\kappa = 2.5\%$ , adding the surrender option to a static contract has no cost (under the CGMY model), while adding the same option to a dynamic contract requires an extra cost of 181 b.p. .

In Table 7 we report the values  $V_0^{\text{static}}$  and  $V_0^{\text{dynamic}}$  for different penalty levels, assuming no fees are subtracted from the personal account. Then, the difference between these contract values and  $U = 100$  represents the total cost of the GMWB guarantee that should be charged as a lump sum at inception. The difference under the static approach (independent of the penalty level) is particularly important, while in the dynamic approach optimal decisions are driven also by the fees and hence would be different if a fee were applied. We do not report the corresponding American contract values (i.e. under

mixed and dynamic+surrender approaches) because they are equal to the corresponding European values. Indeed, if no fee is charged, on one hand one has nothing to gain in case of surrender because she does not receive more than the personal account that, net of partial withdrawals, grows on average at the market interest rate. On the other hand, she loses the GMWB guarantee, that is the possibility of withdrawing money in the future while the guarantee account is still positive even if the personal account is not. In addition, if  $\kappa > 0$  the portion of personal account exceeding  $G$  (or  $A_t$ , if less) is penalized, while there are no penalizations if one withdraws amounts not greater than  $G$  in the future and remains in the contract until maturity. From Table 7 we notice that, in the

$\kappa$ (%)	0	2.5	5	7.5	10
$V_0^{\text{static}}$	101.63	101.63	101.63	101.63	101.63
$V_0^{\text{dynamic}}$	107.44	106.90	106.14	105.45	104.83

Table 7: Comparison of contract values under different policyholder behaviours, benchmark case, for different penalties when  $\varphi = 0$ .

dynamic approach, the contract values are slightly decreasing with  $\kappa$ , while the difference between  $V_0^{\text{dynamic}}$  and  $V_0^{\text{static}}$  can be substantial, even when  $\kappa = 10\%$ . We argue that, when  $\kappa = 10\%$ , the difference  $V_0^{\text{dynamic}} - V_0^{\text{static}} = 3.2$  is mainly due to the possibility of withdrawing less than the guaranteed amount  $G$ , while, when  $\kappa = 0\%$ , the extra difference  $107.44 - 104.83 = 2.61$  is imputable to the possibility of withdrawing more than  $G$ . We observe that the cost of the guarantee in the static approach is moderate, due to the relatively high spread between the market interest rate and the minimum guaranteed rate rewarding the guarantee account over a period of 20 years. The value of the Quanto-Asian put (see Section 3b), that is the difference between the contract value  $V_0^{\text{static}} = 101.63$  and the expected present value of the annuity of guaranteed withdrawals  $G = 5$ , equal to 61.64, almost 40% of the single premium  $U$ , is instead substantial.

Let us now introduce mortality in the contract. We consider a male aged  $x = 55$  years at inception and derive his one-year survival probabilities  $p_{x+i}$ ,  $i = 0, 1, \dots, T - 1$ , from the PNML00 (Pensioners, males, Normal, lives) provided by the CMI (Continuous Mortality Investigation). Then we stress these probabilities replacing them with  $p_{x+i}^\alpha$ ,  $\alpha \geq 0$ , that is equivalent to multiplying by  $\alpha$  the mortality intensity. In Table 8 we report the results obtained with the CGMY model, in terms of fair fee rates, under the various approaches here considered for different values of the parameter  $\alpha$  and the contract maturity  $T$ , when the death benefit is given by the personal account value (hence no death guarantee). For comparison, we report also the values for  $\alpha = 0$ , corresponding to a contract without mortality. In Table 9 we report similar results, but only under the dynamic+surrender approach and for the case in which the contract embeds a GMDB given by the maximum between the values of personal and guarantee account.

From Tables 8 and 9 we do not see a great impact of mortality on the contract fair fees, and we also do not notice a significant increase when we introduce a GMDB guarantee. Moreover, the effect of mortality (higher as  $\alpha$  increases) is not clear: for European contracts the fair fee rate decreases with  $\alpha$ , while it increases for American contracts. Actually mortality may have an effect similar to the introduction of the surrender option.

$T$	10	15	20	25	30
$\varphi^{\text{static}}$					
$\alpha = 0.0$	49	24	13	8	5
$\alpha = 0.5$	48	23	12	7	4
$\alpha = 1.0$	47	22	11	6	4
$\alpha = 2.0$	45	20	9	5	2
$\varphi^{\text{mixed}}$					
$\alpha = 0.0$	49	24	13	8	5
$\alpha = 0.5$	48	23	12	7	4
$\alpha = 1.0$	47	22	11	6	4
$\alpha = 2.0$	45	20	9	5	2
$\varphi^{\text{dynamic}}$					
$\alpha = 0.0$	101	71	54	43	38
$\alpha = 0.5$	102	72	56	46	41
$\alpha = 1.0$	103	73	57	49	44
$\alpha = 2.0$	104	75	61	53	50
$\varphi^{\text{surrender}}$					
$\alpha = 0.0$	107	85	79	75	72
$\alpha = 0.5$	108	86	79	75	73
$\alpha = 1.0$	108	86	79	75	73
$\alpha = 2.0$	108	87	80	76	74

Table 8: Fair fee in b.p., benchmark case, for different contract maturities and mortality tables obtained by stressing the one-year survival probabilities using  $p_y^{(\alpha)} = p_y^\alpha$ .

Unlike surrender, on one hand the time of death cannot be chosen by the policyholder, but, on the other hand, contract termination by death is not penalized. Finally, we observe that, at least in the benchmark case, the transition from the static to the mixed approach has no effect.

## 7. Conclusions

In this paper we present a dynamic programming algorithm for the valuation of variable annuities with Guaranteed Minimum Withdrawal Benefits. A very crucial aspect underlying the valuation of such products is to predict how the policyholder behaves with respect to her withdrawal decisions. Our algorithm is general enough to encompass different levels of rationality in the policyholder behavior, so that it results particularly suitable to meet different purposes of an insurance company (e.g., for pricing purposes it is reasonable to assume an approach based on the worst case scenario, while for realistic risk-management valuations an intermediate approach seems to be more appropriate). Moreover, the algorithm can be easily extended in order to include other policyholder decisions in addition to those concerning her withdrawal behavior (e.g., switching between different reference portfolios, acquisition of new guarantees or cancellation of existing

$T$	10	15	20	25	30
$\alpha = 0.0$	107	85	79	75	72
$\alpha = 0.5$	108	86	79	76	73
$\alpha = 1.0$	109	87	80	77	74
$\alpha = 2.0$	110	89	82	78	76

Table 9:  $\varphi^{\text{surrender}}$  in b.p. for a GMDB contract, benchmark case, for different contract maturities and different mortality tables obtained by stressing the one-year survival probabilities using  $p_y^{(\alpha)} = p_y^\alpha$ .

ones) or other contract features such as reset provisions. Another important contribution of our paper with respect to the existing literature concerns the model assumptions governing the evolution of the reference portfolio. In this respect not only we go beyond the classical Black and Scholes (1973) model, but put ourselves in the general class of Lévy processes. In the numerical section we perform a sensitivity analysis choosing as examples four different types of Lévy processes. This analysis highlights the relevance of the specific assumption adopted in the valuation, i.e., the model risk, and in particular the fact that GMWB guarantees can be grossly underpriced by models that are not able to capture extreme movements in the fund process.

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