A CLASS OF LINEAR OPERATORS IN PERIODIC FUNCTION SPACES INCLUDING DIFFERENCE-DIFFERENTIAL OPERATORS (*)

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SOMMARIO. - *Si fa uno studio degli operatori lineari definiti negli spazi di Banach* C^n_T *delle funzioni T-periodiche e di classe* C^n, $u:\mathbf{R}\to\mathbf{C}$, $n \geqslant 0$, *per i quali la composizione con gli operatori di traslazione* $u \to u(.+\tau)$, $\tau \in \mathbf{R}$, *è commutativa. Si trovano gli autovalori e si dà una rappresentazione del tipo* $Lu = \int_0^T u(x+.)\,dG(x)$ *per mezzo di funzioni a variazione limitata. I risultati teorici sono applicati ad operatori definiti da equazioni differenziali alle differenze.*

SUMMARY. - *This is a study of linear operators for which composition with shift operators* $u \to u(.+\tau)$, $\tau \in \mathbf{R}$, *on Banach spaces* C^n_T *of T-periodic functions* $u:\mathbf{R}\to\mathbf{C}$, $n \geqslant 0$, *is commutative. Eigenvalues are found and representations of the type* $Lu = \int_0^T u(x+.)\,dG(x)$ *by functions of bounded variation are given. The abstract results are applied to operators given by difference-differential equations.*

1. Introduction.

For $T \in \mathbf{R}^+$, the space

$$C^n_T := \{u \in C^n(\mathbf{R},\mathbf{C}) \mid u^{(i)}(t+T) = u^{(i)}(t)\ \forall\ t \in \mathbf{R},\ i = 0,\dots,n\}$$

with the norm $\|u\|_{C^n_T} := \sum_{k=0}^{n} \max_t |u^{(k)}(t)|$ is a Banach space.

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Besides consider the space

$$L^2_T := \{u : \mathbf{R} \to \mathbf{C} \mid u(t+T) = u(t)\ \forall t \in \mathbf{R} \text{ and } u \in L^2([0,T],\mathbf{C})\}$$

which, equipped with the usual inner product $(u,v) := \int_0^T u(t)\overline{v(t)}dt$, is a Hilbert space of equivalence classes.

For every $n \geqslant 0$ the space C^n_T is continuously embeddable in the space L^2_T and, moreover, for every $m \geqslant n$ the embedding operator $I : C^m_T \to C^n_T$ is completely continuous.

For each real number τ let us define the following shift operator acting on T-periodic functions

$$S_\tau u(t) := u(t+\tau).$$

It is easy to see that S_τ is an isometry (isomorphism which preserves the norm) on the spaces C^n_T for every $n \geqslant 0$. Since $S_\tau \circ S_{\tau'} = S_{\tau+\tau'}$ for every τ, $\tau' \in \mathbf{R}$, the set $\mathcal{S}$ of all the shifts turns out to be a group.

Next consider the space $\mathcal{L}^{n,m} := \mathcal{L}(C^n_T, C^m_T)$ of the continuous linear operators mapping C^n_T into C^m_T and define an action of the group $\mathcal{S}$ onto $\mathcal{L}^{n,m}$ as follows

$$S_\tau x L := S_{-\tau} \circ L \circ S_\tau \qquad \forall L \in \mathcal{L}^{n,m} \text{ and } \forall \tau \in \mathbf{R}.$$

Therefore the set

$$\mathcal{LS}^{n,m} := \mathcal{LS}(C^n_T, C^m_T) := \{L \in \mathcal{L}^{n,m} \mid S_\tau x L = L \forall \tau \in \mathbf{R}\}$$

is a closed linear subspace of $\mathcal{L}^{n,m}$, and hence it is a Banach space.

Observe that the operators of $\mathcal{LS}^{n,m}$ are characterized by the property that they commute with the shifts, i.e.

$$L \circ S_\tau = S_\tau \circ L \qquad \forall \tau \in \mathbf{R}.$$

The spaces $\mathcal{LS}^{n,m}$ include various kinds of operators given, for example, by linear difference-differential equations (DDE's) with constant coefficients or by integral equations such as $u(t) = \int_0^T K(t-x)\,u(x)\,dx + f(t)$ with a T-periodic convolution kernel.

We give an integral representation theorem for the operators of $\mathcal{LS}^{0,n}$ and hence, in particular, for the T-periodic solutions of DDE's. Many representation theorems are known for initial value problems to DDE's (see for example L.E.El'sgol'ts-S. B. Norkin [3] and J. Hale [4]) and for ordinary differential equations in Banach spaces with many kinds of lateral conditions (see for example C. S. Hönig [5]). On the contrary similar results do not seem to exist for T-periodic solutions to DDE's.

We also obtain again some known results on the spectral theory and on the solvability of DDE's with constant coefficients (see for example L.E.El'sgol'ts-S. B. Norkin [2] and S. Invernizzi - F. Zanolin [6]). Furthermore the representation theorems provide a straightforward estimate of the rate of uniform convergence for the Fourier expansion of the solutions.

This research was suggested by a paper of A. Bellen [1] in which he studies an iterative monotone method for the numerical solution of nonlinear delay differential equations of the type

$$u^{(n)}(t) = f(t, u(t), u(t-\tau)) \qquad n = 1, 2,$$

in spaces of T-periodic functions. An iteration requires the solution of a linear difference-differential equation.

Moreover a maximum principle and the knowledge of upper and lower solutions are needed. The results of this paper are fully used in M. Zennaro [7], where some maximum principles are proved, and in A. Bellen - M. Zennaro [2], where a method for finding upper and lower solutions is given.

2. The spaces $\mathcal{LS}^{n,m}$. Eigenvalues and eigenspaces.

Define for every $k \in \mathbf{Z}$ the function $e_k(t) := exp\left(\frac{2k\pi i t}{T}\right)$. The set $E := \{e_k\}_{k \in \mathbf{Z}}$ is an orthogonal system in L^2_T, and is a fundamental set in C^n_T, since $span\, E$ is dense in C^n_T, for every $n \geqslant 0$.

The spaces $\mathcal{LS}^{n,m}$ can be characterized as follows.

THEOREM 2.1 - For every $L \in \mathcal{L}^{n,m}$ the statements

(i) - $L \in \mathcal{LS}^{n,m}$;

(ii) - For every $k \in \mathbf{Z}$ there exists $\lambda_k \in \mathbf{C}$ such that $Le_k = \lambda_k e_k$;

are equivalent.

Proof. Let (i) be true. Since $S_\tau e_k = e_k(\tau)\, e_k$ for every $\tau \in \mathbf{R}$, we have that $S_\tau \circ Le_k = L \circ S_\tau e_k = e_k(\tau)\, Le_k$ and then $Le_k(t+\tau) = e_k(\tau)\, Le_k(t)$ for every t, τ. For $t = 0$ we have $Le_k(\tau) = Le_k(0)\, e_k(\tau)$ for every $\tau \in \mathbf{R}$. Therefore (ii) is proved, with $\lambda_k = Le_k(0)$.

Conversely, assume (ii) to be true. It follows that if p is a trigonometric polynomial, i.e. $p = \sum_{i=-s}^{s} a_i e_i$, then

$$L \circ S_\tau p = \sum_i a_i L \circ S_\tau e_i = \sum_i a_i e_i(\tau) Le_i = \sum_i a_i e_i(\tau) \lambda_i e_i =$$

$$= S_\tau(\sum_i a_i \lambda_i e_i) = S_\tau \circ Lp.$$

Since $span\, E$ is dense in C^n_T and L is continuous, (i) holds, too. ■

Throughout the paper we shall mark the dependence of the numbers λ_k on L by λ^L_k.

It is easy to prove the following three corollaries.

COROLLARY 2.2 - *Let* $L, M \in \mathcal{L}\mathcal{S}^{n,m}$; then we have $\lambda^{L+M}_k = \lambda^L_k + \lambda^M_k$ for every $k \in \mathbf{Z}$.

COROLLARY 2.3 - Let $L \in \mathcal{L}\mathcal{S}^{n,m}$; if there exists $L^{-1} \in \mathcal{L}^{m,n}$, then $L^{-1} \in \mathcal{L}\mathcal{S}^{m,n}$ and $\lambda^{L^{-1}}_k = (\lambda^L_k)^{-1}$ for every $k \in \mathbf{Z}$.

COROLLARY 2.4 - Let $L \in \mathcal{L}\mathcal{S}^{n,m}$ and $M \in \mathcal{L}\mathcal{S}^{m,p}$; then $M \circ L \in \mathcal{L}\mathcal{S}^{n,p}$ and $\lambda^{M \circ L}_k = \lambda^M_k \cdot \lambda^L_k$ for every $k \in \mathbf{Z}$.

When the continuous operator L, acting from C^n_T into C^m_T, will be regarded as a continuous operator acting from C^q_T *into* C^p_T, we shall still denote it by L.

THEOREM 2.5 - Let L belong both to $\mathcal{L}\mathcal{S}^{n,m}$ and $\mathcal{L}\mathcal{S}^{q,p}$ and let M belong both to $\mathcal{L}\mathcal{S}^{m,p}$ and $\mathcal{L}\mathcal{S}^{n,q}$; then they commute, i.e. $L \circ M = M \circ L$.

Proof. By Corollary 2.4 we have that $L \circ M$ and $M \circ L$ belong to $\mathcal{L}\mathcal{S}^{n,p}$ and $\lambda^{M \circ L}_k = \lambda^{L \circ M}_k = \lambda^L_k \cdot \lambda^M_k$ for every $k \in \mathbf{Z}$. Therefore, if p is a trigonometric polynomial, i.e. $p = \sum_{i=-s}^{s} a_i e_i$, *then*

$$M \circ Lp = \sum_i a_i M \circ Le_i = \sum_i a_i \lambda_i^{M \circ L} e_i = \sum_i a_i \lambda_i^{L \circ M} e_i = \sum_i a_i L \circ Me_i = L \circ Mp.$$

Since $span\, E$ is dense in C^n_T and $L \circ M$, $M \circ L$ are continuous, it follows that $L \circ M = M \circ L$. ■

THEOREM 2.6 - $L \in \mathcal{L}\mathcal{S}^{n,m}$ implies $L \in \mathcal{L}\mathcal{S}^{n+k,m+k}$ for every $k \geqslant 1$.

Proof. For $n = 0$ and $m \geqslant 0$ this is a consequence of the representation theorems for the spaces $\mathcal{L}\mathcal{S}^{0,m}$ given in Section 3. Infact we shall see that for every $L \in \mathcal{L}\mathcal{S}^{0,m}$ there exists a function G, $[G] \in S^m_T$ (see Theorem 3.6), such that $Lf(t) = \int_0^T f(x+t)\, dG(x)$ for every $f \in C^0_T$. Now, if $f \in C^k_T$, it is easily seen that the following equalities hold for $i = 1, \dots, k$

$$(Lf)^{(i)}(t) = \int_0^T f^{(i)}(x+t)\, dG(x) = Lf^{(i)}(t)$$

and hence, since $f^{(k)}$ is continuous, we have that $Lf \in C^{m+k}_T$.

Moreover let $||L||_{0,m}$ be the norm of L as operator from C^0_T into C^m_T; then

$$||L||_{C^{m+k}_T} = \sum_{i=0}^{k-1} ||(Lf)^{(i)}||_\infty + \sum_{i=k}^{m+k} ||(Lf)^{(i)}||_\infty = \sum_{i=0}^{k-1} ||Lf^{(i)}||_\infty +$$

$$+ \sum_{i=0}^{m} ||(Lf^{(k)})^{(i)}||_\infty \leqslant \sum_{i=0}^{k-1} ||L||_{0,m} ||f^{(i)}||_\infty + ||Lf^{(k)}||_{C^m_T} \leqslant$$

$$\leqslant ||L||_{0,m} \sum_{i=0}^{k} ||f^{(i)}||_\infty = ||L||_{0,m} ||f||_{C^k_T}$$

and therefore L is continuous also from C^k_T into C^{m+k}_T.

Assume the theorem true for $n-1$ and $m \geqslant 0$. It is easy to see that the operator $J: u \to u' - u$ belongs to $\mathcal{LS}^{p,p-1}$ for every $p \geqslant 1$ and that there exists $J^{-1} \in \mathcal{LS}^{p-1,p}$. By Corollary 2.4 we have $L \circ J^{-1} \in \mathcal{LS}^{n-1,m}$, since $J^{-1} \in \mathcal{LS}^{n-1,n}$.

By the inductive hypothesis $L \circ J^{-1} \in \mathcal{LS}^{n-1+k,m+k}$ for every $k \geqslant 1$; since $J \in \mathcal{LS}^{n+k,n-1+k}$, we have that $L = L \circ J^{-1} \circ J \in \mathcal{LS}^{n+k,m+k}$. So the proof is complete. ■

For every $L \in \mathcal{LS}^{n,m}$ let us call *eigenvalue* of L each complex number λ such that $Lu = \lambda u$ for some $u \in C^n_T$, $u \neq 0$.

For every $\lambda \in \mathbf{C}$ and $L \in \mathcal{LS}^{n,m}$ define the set

$$K_{L,\lambda} := \{k \in \mathbf{Z} \mid \lambda^L_k = \lambda\}$$

which, obviously, may be empty. Besides, define

$$E_{L,\lambda} := \{e_k\}_{k \in K_{L,\lambda}}$$

and denote the sets $E_{L,0}$ and $K_{L,0}$ by E_L and K_L respectively.

Let $N_{L,\lambda}$ be the linear manifold of the functions $u \in C^n_T$ such that $Lu = \lambda u$. Note that $N_{L,0} = \mathit{kern}\, L$ and $N_{L,\lambda} = \{0\}$ if and only if λ is not an eigenvalue of L.

LEMMA 2.7 - If $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \varnothing$, then we have that $C^n_T = \overline{\mathit{span}\, E_1} \oplus \overline{\mathit{span}\, E_2}$ for every $n \geqslant 0$.

The proof is standard and is omitted for the sake of brevity.

Now we are able to prove the following theorem concerning the structure of the linear manifold $N_{L,\lambda}$.

THEOREM 2.8 - Let $L \in \mathcal{LS}^{n,m}$ and let λ be a complex number; then we have $N_{L,\lambda} = \overline{\mathit{span}\, E_{L,\lambda}}$, where $\mathit{span}\, \varnothing = \{0\}$.

Proof. Since L is continuous, we have $\overline{span\, E_{L,\lambda}}^{C^n_T} \subseteq N_{L,\lambda}$. Conversely, let $u \in N_{L,\lambda}$; then, by Lemma 2.7, we have that $u = v + w$, where $v \in \overline{span\, E_{L,\lambda}}^{C^n_T}$ and $w \in \overline{span\, (E - E_{L,\lambda})}^{C^n_T}$.

Consider the operator J defined in the proof of Theorem 2.6; by Theorem 2.5 we have $J^{-1} \circ L = L \circ J^{-1}$ and then $L(J^{-1}u) = J^{-1}(Lu) = \lambda J^{-1} u$. Since $J^{-1} u \in C^{n+1}_T$, its Fourier expansion $\sum_k a_k e_k$ converges uniformly with all its derivatives up to the $n-th$, i.e. in C^n_T, to $J^{-1} u$. Therefore, since L is continuous,

$$L(J^{-1}u) - \lambda J^{-1} u = \sum_k a_k (\lambda^L_k - \lambda)\, e_k = 0$$

and we have $\lambda = \lambda^L_k$ for every $a_k \neq 0$, i.e. $J^{-1} u \in \overline{span\, E_{L,\lambda}}^{C^n_T}$. On the other hand J^{-1} is an isomorphism of C^n_T onto C^{n+1}_T and J^{-1} maps $span\, E_{L,\lambda}$ into itself. Thus we have

$$J^{-1} v \in \overline{span\, E_{L,\lambda}}^{C^{n+1}_T} \subset \overline{span\, E_{L,\lambda}}^{C^n_T}$$

and

$$J^{-1} w \in \overline{span\, (E - E_{L,\lambda})}^{C^{n+1}_T} \subset \overline{span\, (E - E_{L,\lambda})}^{C^n_T}.$$

Hence, by Lemma 2.7, $J^{-1} w = 0$, i.e. $w = 0$ and $u = v$. ■

This theorem yields, as a corollary, the following result on the set of the eigenvalues of L.

COROLLARY 2.9 - If $L \in \mathcal{L}\mathcal{S}^{n,m}$, its eigenvalues are exactly $\{\lambda^L_k\}_{k \in \mathbf{Z}}$.

THEOREM 2.10 - Let $L \in \mathcal{L}\mathcal{S}^{n,m}$; then $L = 0$ if and only if $\lambda^L_k = 0$ for every $k \in \mathbf{Z}$.

Proof. If $L = 0$, it obviously follows that $\lambda^L_k = 0$ for every $k \in \mathbf{Z}$. Conversely, if $\lambda^L_k = 0$ for every $k \in \mathbf{Z}$, we have $Lp = 0$ for every $p \in span\, E$ and therefore, since $span\, E$ is dense in C^n_T and L is continuous, it follows that $L = 0$. ■

The following theorem can be proved by the same arguments of Theorem 2.8.

THEOREM 2.11 - Let $L \in \mathcal{L}\mathcal{S}^{n,m}$ and let $R(L)$ be the range of L; then $\overline{R(L)} = \overline{span\, (E - E_L)}$.

Since $\overline{span\, (E - E_L)}^{C^m_T} = C^m_T \cap (\overline{span\, E_L}^{L^2_T})^{\perp}$, we have immediately the following corollary.

COROLLARY 2.12 - Let $L \in \mathcal{L}\mathcal{S}^{n,m}$ and let $R(L)$ be closed; then the equation $Lu = f$ has a solution in C^n_T if and only if $(f, e_k) = 0$ for every $k \in K_L$.

3. Representation theorems for the spaces $\mathcal{LS}^{0,n}$.

First consider the case $\mathcal{LS}^{0,0}$.

LEMMA 3.1 - The space $\mathcal{LS}^{0,0}$ is isometrically isomorphic to C_T^{0*}, the dual space of C_T^0.

Proof. Indeed one can prove by direct arguments that the operator

$$K : \mathcal{LS}^{0,0} \to C_T^{0*} \qquad \text{such that} \quad K(L) := P_0^* \circ L,$$

where P_0^* is the evaluation functional defined by $P_0^* u := u(0)$, is linear and preserves the norm.

On the other hand there exists the inverse

$$K^{-1} : C_T^{0*} \to \mathcal{LS}^{0,0}$$

defined as follows:

for every $F^* \in C_T^{0*}$ and $u \in C_T^0$ $\quad K^{-1}(F^*)\, u(t) := F^* \circ S_t u.$ ■

Let us consider the space $BV_0([0,T],\mathbf{C})$ of the complex functions G defined in $[0,T]$ which are of bounded variation and are such that $G(x+0) = G(x)$ for every $x \in (0,T)$ and $G(0) = 0$ (see C. S. Hönig [5]).

$$\text{Let } \Phi(x) := \begin{cases} 0 & \text{if } x = 0 \text{ and } x = T \\ 1 & \text{if } 0 < x < T \end{cases}$$

$$S_T^0 := \frac{BV_0([0,T],\mathbf{C})}{span\{\Phi\}}.$$

The following lemma is a trivial consequence of the Riesz theorem.

LEMMA 3.2 - The space C_T^{0*} is isometrically isomorphic to the space S_T^0, and for every $F^* \in C_T^{0*}$ we have that

$$F^* u = \int_0^T u(x)\, dG(x) \qquad \text{for every } u \in C_T^0,$$

where $[G]$ is the element of S_T^0 corresponding to the linear functional F^* in the isometry.

Combining the results of Lemmata 3.1 - 3.2, we easily obtain the representation theorem for the operators of the space $\mathcal{LS}^{0,0}$.

THEOREM 3.3 - There exists an isometry $\mathcal{G}_0$ between the space $\mathcal{LS}^{0,0}$ and the space S_T^0, and for every $L \in \mathcal{LS}^{0,0}$ we have that

$$Lu(t) = \int_0^T u(x+t)\, dG(x)$$

for every $u \in C^0_T$ and for every real number t, where $[G] = \mathcal{G}_0(L)$.

Each function $G \in \mathcal{G}_0(L)$ will be called *representative function* of the operator L.

In the space S^0_T the norm is given by $\| [G] \|_0 := \inf_{\lambda \in \mathbf{C}} V(G+\lambda\Phi)$, where $V(G+\lambda\Phi)$ is the variation of $G+\lambda\Phi$.

Using Corollary 2.9 and Theorem 3.3 we can derive a result on the representation of all the eigenvalues of the operators which belong to the space $\mathcal{LS}^{0,0}$.

COROLLARY 3.4 - All the eigenvalues of $L \in \mathcal{LS}^{0,0}$ are given by

$$\lambda^L_k = \int_0^T e_k(x)\, dG(x), \text{ where } [G] = \mathcal{G}_0(L).$$

Proof. $Le_k(t) = \int_0^T e_k(x+t)\, dG(x) = \int_0^T e_k(t)\, e_k(x)\, dG(x) =$

$$= \Big(\int_0^T e_k(x)\, dG(x)\Big)\, e_k(t). \quad \blacksquare$$

Now consider $n \geqslant 1$ and observe that $L \in \mathcal{LS}^{0,n}$ implies $L \in \mathcal{LS}^{0,m}$ for every $m \leqslant n$.

Let us denote by S^n_T the subspace of S^0_T of the classes of the representative functions of the operators $L \in \mathcal{LS}^{0,n}$ for $n \geqslant 1$. In order to characterize the classes of S^n_T, we begin with the case $n = 1$.

THEOREM 3.5 - The space S^1_T is made up as follows:

$$S^1_T = \{ [G] \in S^0_T \mid G(x) = \int_0^x F(\xi)\, d\xi$$

for some F such that $F - F(0) \in BV_0([0,T], \mathbf{C})$ and $F(0) = F(T) \}$.

Proof. If such an F exists, we have immediately that $[G] \in S^1_T$. Conversely let us consider $L \in \mathcal{LS}^{0,1}$. The derivative operator D belongs to $\mathcal{LS}^{1,0}$ and then $D \circ L \in \mathcal{LS}^{0,0}$. According to Theorem 3.3, let $[G] = \mathcal{G}_0(L)$ and $[H] = \mathcal{G}_0(D \circ L)$.

Let $g(t) \equiv 1$; it follows that

$$D \circ Lg(t) = \frac{d}{dt} \int_0^T dG(x) = 0$$

and also

$$D \circ Lg(t) = \int_0^T dH(x) = H(T) - H(0)$$

and then $H(0) = H(T)$.

Now consider the function

$$K(x) := \int_0^x H(\xi)\, d\xi$$

and let $f \in C_T^0$; it follows that

$$\int_0^t D \circ Lf(\xi) d\xi = \int_0^t \left(\frac{d}{d\xi} \int_0^T f(x+\xi)\, dG(x) \right) d\xi =$$

$$= \int_0^T f(x+t)\, dG(x) - \int_0^T f(x)\, dG(x)$$

and also

$$\int_0^t D \circ Lf(\xi) d\xi = \int_0^t \left(\int_0^T f(x+\xi)\, dH(x) \right) d\xi =$$

$$= \int_0^T \left(\int_0^t f(x+\xi)\, d\xi \right) dH(x) = H(T) \int_0^t f(T+\xi)\, d\xi - H(0) \int_0^t f(\xi)\, d\xi -$$

$$- \int_0^T H(x) \left(\frac{d}{dx} \int_0^t f(x+\xi)\, d\xi \right) dx =$$

$$= [H(T) - H(0)] \int_0^t f(\xi)\, d\xi - \int_0^T H(x)\, [f(x+t) - f(x)]\, dx =$$

$$= - \int_0^T f(x+t)\, dK(x) + \int_0^T f(x)\, dK(x)$$

and then

$$\int_0^T f(x+t)\, d(G+K)(x) = \int_0^T f(x)\, d(G+K)(x) \qquad \forall\, t \in \mathbf{R}.$$

By integrating we obtain

$$\int_0^T \left(\int_0^T f(x+t)\, d(G+K)(x) \right) dt = \int_0^T \left(\int_0^T f(x+t)\, dt \right) d(G+K)(x) =$$

$$= T\bar{f}\int_0^T d(G+K)\,(x), \text{ where } \bar{f} := \frac{1}{T}\int_0^T f(\xi)\,d\xi \text{ is the mean of } f.$$

Moreover

$$\int_0^T \Big(\int_0^T f(x+t)\,d(G+K)\,(x)\Big)\,dt = T\int_0^T f(x)\,d(G+K)\,(x)$$

and therefore we can conclude that

$$\int_0^T f(x+t)\,d(G+K)\,(x) = \bar{f}\int_0^T d(G+K)\,(x).$$

If we put

$$\rho := \frac{1}{T}\int_0^T d(G+K)\,(x),$$

we have that $[G(x)] = [\rho x - K(x)]$ in S^0_T and hence we can suppose that $G(x) = \rho x - K(x)$.

Thus the function $F(x) := \rho - H(x)$ is such that $G(x) = \int_0^x F(\xi)\,d\xi$, $F - F(0) \in BV_0([0,T],\mathbf{C})$ and $F(0) = F(T)$.

The uniqueness of F is trivial. ■

The function F will be called *main derivative* of the representative function G.

Besides, we shall denote by $\mathcal{G}_1$ the restriction of $\mathcal{G}_0$ to $\mathcal{LS}^{0,1}$ and renorm the space S^1_T by $\|[G]\|_1 := \|[G]\|_0 + V(F)$. It is easy to verify that $\mathcal{G}_1$ is an isometry between $\mathcal{LS}^{0,1}$ and S^1_T.

For $n \geqslant 2$ the following characterization holds for S^n_T.

THEOREM 3.6 - The space S^n_T is made up as follows:

$$S^n_T = \{[G] \in S^0_T \mid G \in C^{n-1}([0,T],\mathbf{C}),\ [G^{(n-1)} - G^{(n-1)}(0)] \in S^1_T$$
and $G^{(k)}(0) = G^{(k)}(T)$ for $k = 1,\dots,n-1\}$.

Proof. Let $L \in \mathcal{LS}^{0,2}$; then $D\circ L \in \mathcal{LS}^{0,1}$. Let $[G] = \mathcal{G}_0(L)$ and $[H] = \mathcal{G}_0(D\circ L)$; we have that $[H] \in S^1_T$, so that H may be supposed to be continuous. By the same arguments of Theorem 3.5 we obtain $H(0) = H(T)$ and we can suppose $G(x) = \rho x - K(x)$, where we have put

$$K(x) := \int_0^x H(\xi)\, d\xi \quad \text{and} \quad \rho := \frac{1}{T}\int_0^T d(G+K)(x).$$

It follows that $G \in C^1([0,T], \mathbf{C})$ and that $G'(x) = \rho - H(x)$, so that $G'(0) = G'(T)$ and $[G' - G'(0)] \in S^1_T$.

Conversely it is easy to see that a function G which fulfils these properties defines an operator $L \in \mathcal{LS}^{0,2}$.

For $n \geqslant 3$ the proof can be easily carried out by induction. ■

Like before, $\mathcal{G}_n$ will denote the restriction of $\mathcal{G}_0$ to the subspace $\mathcal{LS}^{0,n}$. The space S^n_T will be renormed by

$$\| [G] \|_n := \| [G] \|_0 + V(G') + \ldots + V(G^{(n-1)}) + V(F),$$

where, according to the definitions, F is the main derivative of the function $G^{(n-1)} - G^{(n-1)}(0)$. $\mathcal{G}_n$ is an isometry between $\mathcal{LS}^{0,n}$ and S^n_T. Remark that, if $[G] \in S^n_T$ with $n \geqslant 2$, the main derivative of G is exactly the ordinary derivative G'.

4. Smooth operators.

It is interesting to consider the subspace of $\mathcal{LS}^{0,n}$ which consists of those operators which have a representative function with certain smoothness properties.

We shall say that $L \in \mathcal{LS}^{0,n}$ is *smooth* if it has a representative function which is of class C^n, and we shall denote by $\mathcal{SLS}^{0,n}$ the set of such operators.

For the operators of $\mathcal{SLS}^{0,n}$ the corresponding subspace of S^n_T is

$$SS^n_T := \{ [G] \in S^n_T \mid [G] \cap C^n([0,T], \mathbf{C}) \neq \varnothing \}.$$

We have obviously that $\mathcal{LS}^{0,n} \subset \mathcal{SLS}^{0,m}$ for every $m < n$.

THEOREM 4.1 - SS^n_T is closed in S^n_T for every $n \geqslant 0$.

Proof. If $n = 0$, it is a consequence of the fact that the convergence in BV_0 implies the uniform convergence; thus $BV_0 \cap C^0$ is closed in BV_0 and then SS^0_T is closed in S^0_T.

If $n = 1$, assume $\{[G_k]\} \to [G]$ in S^1_T and $[G_k] \in SS^1_T$. We can suppose $G_k \in C^1, G \in C^0, G_k \to G$ uniformly and $G'_k - G'_k(0) \in BV_0 \cap C^0$. Let H_k and H be respectively the main derivatives of G_k and G; it is easy to see that $H_k(x) = G'_k(x) + \rho_k[1 - \Theta(x)]$, where $\rho_k := G'_k(T) - G'_k(0)$ and

$$\Theta(x) := \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } 0 < x \leqslant T. \end{cases}$$

Since $H_k - H_k(0) \to H - H(0)$ in BV_0 and $G'_k(T) = H_k(0)$, we have that $G'_k - G'_k(0) - \rho_k\Theta \to H - H(0)$. On the other hand Θ is not a continuous function, and we recall that $BV_0 \cap C^0$ is closed in BV_0. Therefore $G'_k - G'_k(0) \to F \in BV_0 \cap C^0$ in BV_0 (and then uniformly, too) and $\rho_k \to \rho \in \mathbf{C}$. Thus, by integrating, we have that $G_k(x) - G'_k(0)\, x \to \int_0^x F(\xi)\, d\xi$ for every $x \in [0, T]$.

Moreover, from the convergence of $G_k(x)$ to $G(x)$, it follows that $G'_k(0) \to \eta \in \mathbf{C}$ and so we have $G(x) = \eta x + \int_0^x F(\xi)\, d\xi$, i.e. $G \in C^1([0,T], \mathbf{C})$, i.e. $[G] \in SS^1_T$.

If $n \geqslant 2$, the proof is carried out by induction. Assume that the theorem holds for $n-1$. Consider a sequence $\{[G_k]\}$ converging to $[G]$ in S^n_T such that $[G_k] \in SS^n_T$. We can suppose that $G_k \in C^n$, $G \in C^{n-1}$ and $G'_k - G'_k(0) \in C^{n-1}$. Since $n \geqslant 2$, by Theorem 3.6 and by the definition of the norm in the space S^n_T, we can conclude that $[G'_k - G'_k(0)] \in SS^{n-1}_T$, $[G' - G'(0)] \in S^{n-1}_T$ and

$$\{[G'_k - G'_k(0)]\} \to [G' - G'(0)]$$

in S^{n-1}_T; then, by the inductive hypothesis, $[G' - G'(0)] \in SS^{n-1}_T$, i.e. $G' \in C^{n-1}$ and so $G \in C^n$, i.e. $[G] \in SS^n_T$. ■

By virtue of the isometry $\mathcal{G}_n$, the subspace $\mathcal{SLS}^{0,n}$ is closed in $\mathcal{LS}^{0,n}$.

For $L \in \mathcal{SLS}^{0,n}$ with $n \geqslant 1$, the representation given by Theorem 3.3 takes the particular form

$$Lu(t) = \int_0^T \Gamma_L(x)\, u(x+t)\, dx \qquad \forall\, u \in C^0_T,$$

where Γ_L is uniquely determined in $C^{n-1}([0,T], \mathbf{C})$.

Moreover $\Gamma_L^{(n-1)} - \Gamma_L^{(n-1)}(0) \in BV_0 \cap C^0$ and, for $n \geqslant 2$, $\Gamma_L^{(k)}(0) = \Gamma_L^{(k)}(T)$ for $k = 0, 1, \ldots, n-2$. This is a trivial consequence of Theorems 3.5-3.6 and of the smoothness of L.

The function Γ_L will be called *associated kernel* to the smooth operator L.

Conversely it is easily seen that every function Γ which fulfils these properties defines an operator $L \in \mathcal{SLS}^{0,n}$.

THEOREM 4.2 - Let $\{L_k\}$ be a sequence of $\mathcal{SLS}^{0,n} (n \geqslant 1)$ which converges to L in $\mathcal{SLS}^{0,n}$; let Γ_k and Γ be respectively the associated kernels to L_k and L. Then $\{\Gamma_k\}$ converges uniformly to Γ with all the derivatives up to the $(n-1)$ – th.

Proof. Let $[G_k] = \mathcal{G}_n(L_k)$ and $[G] = \mathcal{G}_n(L)$. We can suppose $G_k, G \in C^n$ and hence we have $\Gamma_k^{(m)} = G_k^{(m+1)}$ and $\Gamma^{(m)} = G^{(m+1)}$ for $m = 0,1,\dots,n-1$. Since $\{[G_k]\} \to [G]$ in SS^n_T, we have that $V(\Gamma_k^{(m)} - \Gamma^{(m)}) \to 0$ for $m = 0,1,\dots,n-2$ and that

$$V(\Gamma_k^{(n-1)} + \rho_k[1-\Theta] - \Gamma^{(n-1)} - \rho[1-\Theta]) \to 0,$$

where $\rho_k := \Gamma_k^{(n-1)}(T) - \Gamma_k^{(n-1)}(0)$ and $\rho := \Gamma^{(n-1)}(T) - \Gamma^{(n-1)}(0)$ (see the proof of Theorem 4.1, case $n = 1$). It follows that $\Gamma_k^{(m)} - \Gamma_k^{(m)}(0) \to$ $\to \Gamma^{(m)} - \Gamma^{(m)}(0)$ uniformly for $m = 0,1,\dots,n-2$ and that $\Gamma_k^{(n-1)} -$ $- \Gamma_k^{(n-1)}(0) - \rho_k\Theta \to \Gamma^{(n-1)} - \Gamma^{(n-1)}(0) - \rho\Theta$ uniformly.

Since $\Theta \notin C^0$, while $\Gamma_k^{(n-1)}$ and $\Gamma^{(n-1)}$ are continuous, we have that $\Gamma_k^{(n-1)} - \Gamma_k^{(n-1)}(0) \to \Gamma^{(n-1)} - \Gamma^{(n-1)}(0)$ uniformly, too. We can suppose that $G_k \to G$ uniformly and then, since also $\Gamma_k - \Gamma_k(0) \to \Gamma - \Gamma(0)$ uniformly, it follows that

$$\int_0^x [\Gamma_k(\xi) - \Gamma_k(0)]\,d\xi \to \int_0^x [\Gamma(\xi) - \Gamma(0)]\,d\xi$$

uniformly, i.e. $G_k(x) - \Gamma_k(0)\,x \to G(x) - \Gamma(0)\,x$ uniformly, and hence $\Gamma_k(0) \to \Gamma(0)$ and $\Gamma_k \to \Gamma$ uniformly.

In this way, by $n-1$ passages, we can show that $\Gamma_k^{(m)}(0) \to \Gamma^{(m)}(0)$ and that $\Gamma_k^{(m)} \to \Gamma^{(m)}$ uniformly for $m = 1,\dots,n-1$, too. ■

5. Linear difference-differential equations with constant coefficients.

In this last section we apply the foregoing theory to linear difference-differential equations with constant coefficients such as

$$\text{(DDE)} \qquad u^{(n)}(t) + \sum_{k=0}^{n-1} \sum_{j=1}^{m_k} a_{kj}\, u^{(k)}(t + \tau_{kj}) = f(t)$$

where $n \geqslant 1$, $f \in C^0_T$, $a_{kj} \in \mathbf{C}$, $\tau_{kj} \in \mathbf{R}$. We look for a solution in the space C^n_T.

The equation DDE is of the form $Nu = f$, where $N \in \mathcal{LS}^{n,0}$. If D^k denotes the k – th derivative operator, we have that

$$N = D^n + \sum_{k=0}^{n-1} \sum_{j=1}^{m_k} a_{kj} D^k \circ S_{\tau_{kj}}.$$

LEMMA 5.1 - The operator $D^n - \omega I$ (which belongs to $\mathcal{LS}^{n,0}$) is invertible for every $\omega \in \mathbf{C}$ such that $\omega \neq (\frac{2k\pi i}{T})^n$ for every $k \in \mathbf{Z}$, and the inverse operator $J_{n,\omega}$ is smooth.

Proof. Let $\omega \neq (\frac{2k\pi i}{T})^n$ for every $k \in \mathbf{Z}$. By Theorem 2.8 it follows that $kern(D^n - \omega I) = \{0\}$.

Consider the following equation with boundary conditions:

$$\begin{cases} y^{(n)}(x) - (-1)^n \omega y(x) = 0 \\ y^{(k)}(0) = y^{(k)}(T) \quad \text{for } k = 0, 1, \dots, n-2 \\ y^{(n-1)}(T) - y^{(n-1)}(0) = (-1)^{n-1} \end{cases}$$

It is easy to see that this equation has a unique solution $\Gamma \in C^\infty(\mathbf{R}, \mathbf{C})$. Since Γ, restricted to $[0, T]$, fulfils the properties of the associated kernels, it defines an operator, say $J_{n,\omega}$, which belongs to $\mathcal{SLS}^{0,n}$.

Let $f \in C^0_T$; then we have

$$(D^n - \omega I) \circ J_{n,\omega} f(t) = \frac{d^n}{dt^n} \int_0^T \Gamma(x) f(x+t)\, dx - \omega \int_0^T \Gamma(x) f(x+t)\, dx =$$

$$= -\omega \int_0^T \Gamma(x) f(x+t)\, dx + (-1)^{n-1} [\Gamma^{(n-1)}(T) -$$

$$- \Gamma^{(n-1)}(0)] f(t) + (-1)^n \int_0^T \Gamma^{(n)}(x) f(x+t)\, dx =$$

$$= (-1)^{n-1} [\Gamma^{(n-1)}(T) - \Gamma^{(n-1)}(0)] f(t) -$$

$$- (-1)^n \int_0^T [\Gamma(x) - (-1)^n \omega \Gamma^{(n)}(x)] f(x+t)\, dx = f(t).$$

Hence $(D^n - \omega I) \circ J_{n,\omega} = I$ (the identity operator) and therefore we can conclude that $D^n - \omega I$ is invertible and that its inverse $J_{n,\omega}$ is smooth. ■

LEMMA 5.2 - Let $L \in \mathcal{LS}^{n,m}$, $M \in \mathcal{LS}^{n,m+1}$ and let there exist the inverse $L^{-1} \in \mathcal{LS}^{m,n}$. Then the following properties hold:

(i) - $kern(L+M)$ is a finite dimension subspace of C^n_T;

(ii) - $R(L)$ is closed in C^m_T;

(iii) - $L+M$ is invertible if and only if $kern\,(L+M) = \{0\}$;

(iv) - If $m = 0$ and L^{-1} is smooth, then also $(L+M)^{-1}$ is smooth, if it exists.

Proof. The properties (i), (ii), (iii) easily follow from the equality $L+M = L\circ(I+L^{-1}\circ M)$ and the complete continuity of M as operator from C^n_T into C^m_T.

In order to prove (iv), assume $m = 0$, L^{-1} to be smooth and $L+M$ to be invertible. Since

$$(L+M)^{-1} = (I+L^{-1}\circ M)^{-1}\circ L^{-1} =$$
$$= (I+L^{-1}\circ M - L^{-1}\circ M)\circ(I+L^{-1}\circ M)^{-1}\circ L^{-1} =$$
$$= L^{-1} - L^{-1}\circ M\circ(I+L^{-1}\circ M)^{-1}\circ L^{-1},$$

$L^{-1} \in \mathcal{S}\mathcal{L}\mathcal{S}^{0,n}$ and $L^{-1}\circ M\circ(I+L^{-1}\circ M)^{-1}\circ L^{-1} \in \mathcal{L}\mathcal{S}^{0,n+1} \subset \mathcal{S}\mathcal{L}\mathcal{S}^{0,n}$, we have that $(L+M)^{-1} \in \mathcal{S}\mathcal{L}\mathcal{S}^{0,n}$. ■

If we put $L := D^n - \omega I$ and $M := \omega I + \sum_{k=0}^{n-1}\sum_{j=1}^{m_k} a_{kj} D^k \circ S_{\tau_{kj}}$, $\omega \neq (\frac{2k\pi i}{T})^n$, equation DDE takes the form $(L+M)\,u = f$; the hypotheses of Lemma 5.2 are fulfilled for $m = 0$ and therefore properties (i), (ii), (iii), (iv) hold.

In particular, by (i) and (ii), we have that $dim\ kern\,N = d < \infty$ and that $R(N)$ is closed in C^0_T. Moreover, by (iv) and Lemma 5.1, the operator $N^{-1} = (L+M)^{-1}$ is smooth, if it exists. In this case the solution of DDE has the following form:

$$u(t) = \int_0^T \Gamma(x)\, f(x+t)\, dx$$

where Γ is the associated kernel to N^{-1}, which will be called *resolvent kernel* of DDE.

Now consider the following complex function of complex variable:

$$\varphi(z) := z^n + \sum_{k=0}^{n-1}\sum_{j=1}^{m_k} a_{kj}\, z^k\, exp(\tau_{kj} z)$$

which is called, according to L.E.El'sgol'ts - S. B. Norkin [3], the *characteristic quasipolynomial* of DDE.

Observe that the eigenvalues of N are $\lambda^N_k := \varphi(\frac{2k\pi i}{T})$, $k \in \mathbf{Z}$, and then the set $B := \left\{ exp\,(\frac{2k\pi i t}{T}) \,\middle|\, \varphi\,(\frac{2k\pi i}{T}) = 0 \right\}$ of complex exponential functions is a basis of $kern\, N$.

By Corollary 2.12 and Lemma 5.2, equation DDE has a solution $u \in C^n_T$ if and only if the function f is L^2_T-orthogonal to B.

Finally, by Lemma 5.2-(iii), we have that N is invertible if and only if $\varphi\,(\frac{2k\pi i}{T}) \neq 0$ for every $k \in \mathbf{Z}$.

If N is invertible, by Corollary 2.3, we have that $\lambda_k^{N^{-1}} = [\varphi(\frac{2k\pi i}{T})]^{-1}$ and then, by Corollary 3.4,

$$\frac{1}{T}\int_0^T \Gamma(x)\, exp(-\frac{2k\pi i x}{T})\, dx = \frac{1}{T}\lambda_{-k}^{N^{-1}} = [T\varphi(-\frac{2k\pi i}{T})]^{-1}.$$

So the Fourier expansion of the resolvent kernel Γ of DDE is

$$\sum_{k\in\mathbf{Z}} [T\varphi\,(-\frac{2k\pi i}{T})]^{-1}\, exp\,(\frac{2k\pi i x}{T}).$$

Since $\Gamma^{(n-1)}$ is continuous and of bounded variation in $[0,T]$, the Fourier expansion converges uniformly to Γ with all its derivatives up to the $(n-1)$-th in every closed subinterval $[\alpha,\beta] \subset [0,T)$. Since for $n \geqslant 2$ we have $\Gamma^{(k)}(0) = \Gamma^{(k)}(T)$ for $k = 0, 1, \dots, n-2$, the first $n-2$ derivatives of the expansion converge uniformly in $[0,T]$.

In any case the solution u of equation DDE has the form:

$$u(t) = \int_0^T (\sum_{k\in\mathbf{Z}} [T\varphi\,(-\frac{2k\pi i}{T})]^{-1}\, exp(\frac{2k\pi i x}{T}))\, f(x+t)\, dx =$$

$$= \sum_{k\in\mathbf{Z}} [T\varphi(-\frac{2k\pi i}{T})]^{-1} (\int_t^{t+T} exp(\frac{2k\pi i x}{T})\, f(x)\, dx)\, exp(-\frac{2k\pi i t}{T}) =$$

$$= \sum_{k\in\mathbf{Z}} [T\varphi(\frac{2k\pi i}{T})]^{-1} (\int_0^T exp(-\frac{2k\pi i x}{T})\, f(x)\, dx)\, exp(\frac{2k\pi i t}{T}),$$

which is nothing but the Fourier expansion of u.

By truncating the expansion to $2m+1$ terms, we have the approximation

$$u_m(t) = \int_0^T \Gamma_m(x)\, f(x+t)\, dx,$$

where $\Gamma_m(x) := \sum_{k=-m}^{m} [T\varphi\,(-\frac{2k\pi i}{T})]^{-1}\, exp\,(\frac{2k\pi i x}{T})$, which converges uniformly to u as m tends to ∞.

In order to give a bound to the error $\| u - u_m \|_\infty$, note that

$$\int_0^T \Gamma(x)\, p_m(x+t)\, dx = \int_0^T \Gamma_m(x)\, p_m(x+t)\, dx$$

for every trigonometric polynomial $p_m = \sum_{k=-m}^{m} a_k e_k$, and hence

$$u(t) - u_m(t) = \int_0^T [\Gamma(x) - \Gamma_m(x)]\, [f(x+t) - p_m(x+t)]\, dx.$$

We can choose p_m equal to p^*_m, the best L^2_T-approximation to f by trigonometric polynomials of degree $\leqslant m$, and denote by $e_m(f)$ the error $\| f - p^*_m \|_{L^2_T}$.

By the Cauchy-Schwartz inequality we have

$$\| u - u_m \|_\infty \leqslant \| \Gamma - \Gamma_m \|_{L^2_T} e_m(f) \leqslant \| \Gamma - \Gamma_m \|_{L^2_T} \| f - q^*_m \|_{L^2_T} \leqslant$$

$$\leqslant \sqrt{T} E_m(f) \| \Gamma - \Gamma_m \|_{L^2_T}$$

where q^*_m is the best uniform approximation to f by trigonometric polynomials of degree $\leqslant m$, and $E_m(f)$ is the error $\| f - q^*_m \|_\infty$.

We want to estimate $\| \Gamma - \Gamma_m \|_{L^2_T}$.

To this aim we consider the function $\Phi(y) := \dfrac{|\varphi(iy)|^2}{y^{2n}}$ which is defined and continuous in $\mathbf{R} - \{0\}$. We have immediately that $\lim_{y \to +\infty} \Phi(y) = \lim_{y \to -\infty} \Phi(y) = 1$; therefore, since $\varphi\left(\dfrac{2k\pi i}{T}\right) \neq 0$ for every $k \in \mathbf{Z}$, there exists $\sigma > 0$ such that $\left|\varphi\left(\dfrac{2k\pi i}{T}\right)\right|^2 > \sigma \left(\dfrac{2k\pi}{T}\right)^{2n}$ for every $k \in \mathbf{Z}$.

Hence

$$\| \Gamma - \Gamma_m \|_{L^2_T} = \Big(\sum_{k=-\infty}^{-m-1} \frac{1}{T} \left| \varphi\left(-\frac{2k\pi i}{T}\right) \right|^{-2} +$$

$$+ \sum_{k=m+1}^{+\infty} \frac{1}{T} \left| \varphi\left(-\frac{2k\pi i}{T}\right) \right|^{-2} \Big)^{1/2} \leqslant \frac{2T^n}{(2\pi)^n \sqrt{\sigma T}} \Big(\sum_{k=m+1}^{+\infty} k^{-2n} \Big)^{1/2}.$$

Since $k^{-2n} \leqslant \xi^{-2n}$ for every $\xi \in [k-1, k]$, we obtain

$$\sum_{k=m+1}^{+\infty} k^{-2n} \leqslant \int_m^{+\infty} \xi^{-2n}\, d\xi = [(2n-1)\, m^{2n-1}]^{-1}.$$

We can conclude that there exists a constant $c > 0$,

$$c := \frac{2T^n}{(2\pi)^n \sqrt{(2n-1)\,\sigma}},$$

depending on the operator N such that

$$\| u - u_m \|_\infty \leqslant c\, E_m(f)\, m^{-(n-1/2)}.$$

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