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PREFERENCE-BASED APPROACH TO RISK SHARING

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To my grandmothers

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1
Introduction

In this chapter we present the main motivations of the thesis together with a detailed explanation of the structure of the work.

1.1 Motivations

In this thesis we present several conditions for the existence of *optimal solutions* to the problem of *optimal risk sharing* by starting from the assessment of the individual preferences of the agents and by considering a topological context.

It is well known that optimal risk sharing is an argument that deserves both theoretical and practical interest. It typically appears when considering the classical *reinsurance problem* in insurance mathematics when the *original insurer* cedes a part of a risk X to a *reinsurer*, but now it is also widely used in a variety of financial and economical applications.

In general, the problem of finding optimal risk allocations of a given *risk* X belonging to some space $\mathcal{L}_+$ of nonnegative random variables on a common probability space consists in determining an allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X) = \{(Y_1, ..., Y_m) \mid X = \sum_{i=1}^m Y_i\}$ which is optimal according to some criterion. Therefore, there are m agents (agencies) and we want to determine a *Pareto optimal allocation* Y_i^* $(i = 1, ..., m)$. The set $\mathcal{A}(X)$ is called the *feasible set* corresponding to the risk X.

In particular, if each agent is endowed with an initial exposure X_i we shall denote by $\mathcal{S}$ the set of all *individually rational* feasible allocations for which each agent is at least as well as under the initial allocation $(X_1, .., X_m)$. In such a context, the risk sharing problem is traduced on finding *individually rational Pareto-optimal allocations* $(Y_1^*, ..., Y_m^*) \in \mathcal{S}$, (namely *optimal solutions*). This clearly implies the definition of an approach necessary to describe agents decision making behaviour.

Throughout the present work, we denote by $\precsim$ a preorder (i.e., a reflexive and transitive binary relation) on a set $\mathcal{S}$. A preorder is said to be *total* if for any two elements $Y,\ Z \in \mathcal{S}$ either $Y \precsim Z$ or $Z \precsim Y$.

In the literature, usually a *functional approach* is considered. By applying the universally accepted restriction according to which the preferences of the generic agent i only depend on its own share Y_i of the risk, if we refer to a so called *reward approach*, an *optimal solution* to the aforementioned problem is the solution to the following *multi-objective maximization problem* associated to m

assigned real-valued functions $U_1, ..., U_m$:

$$
\begin{aligned}
(1) \qquad & \sup \; (U_1(Y_1), U_2(Y_2), ..., U_m(Y_m)) \\
& \text{sub} \\
& (Y_1, .., Y_m) \in \mathcal{S}.
\end{aligned}
$$

A vector $(Y_1^*, ..., Y_m^*) \in \mathcal{S}$ is a solution to the previous problem provided that for no $(Y_1, .., Y_m) \in \mathcal{S}$ it holds that $U_i(Y_i) \geq U_i(Y_i^*)$ for all $i \in \{1, ..., m\}$ with at least one strict inequality.

The functions (functionals) U_i are axiomatically defined, say. By referring to a classical context of decision making under uncertainty, at least implicitly the consideration of the previous problem implies that U_i is the *utility function* of a *total preorder* $\precsim_i$ representing the preferences of agent i (i.e., for all $i \in \{1, ..., m\}$ and for all individual shares $Y_i,\ Z_i$, we have that $Y_i \precsim_i Z_i$ if and only if $U_i(Y_i) \leq U_i(Z_i)$). Therefore, we can say that every function U_i naturally induces a total preorder $\precsim_i$.

It should be noted that two main observations arise at this point:

1. The consideration of total preorders is very restrictive and a more realistic approach should require that nontotal preorders are also incorporated in the model;

2. In a more appropriate approach we should start from the individual (in the meantime not necessarily total) preorders $\precsim_i$.

It is well known that if a preoder $\precsim$ is not total, than it cannot be represented by a utility function in the classical sense, as described above. Indeed, the consideration of a (two-ways) utility functional automatically implies that the preorder is total, due to the fact that clearly the natural (pre)order $\leq$ on the real line $\mathbb{R}$ is total.

Based on these fundamental considerations, we shall consider the following definitions of Pareto-optimal allocation and optimal solution in a preference-based setting:

(2) An allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ is said to be *Pareto optimal* if for no other allocation $(Y_1, ..., Y_m) \in \mathcal{A}(X)$ it occurs that $Y_1^* \precsim_i Y_1, ..., Y_m^* \precsim_m Y_m$ with at least one index i such that $Y_i^* \prec_i Y_i$. If $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ belongs to the set $\mathcal{S}$ of all the feasible allocations for which each agent is at least as well as under the initial allocation $(X_1, .., X_m)$, then $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ is an *optimal solution*

We denote by $\prec_i$ the *strict part* (*asymmetric part*) of the individual preorder $\precsim_i$.

It should be noted that if every preorder $\precsim_i$ is total and it is represented by a utility function U_i, then problem (2) coincides with problem (1). This consideration motivates our approach as being more general and appropriate than the usual ones.

1.2 Structure of the thesis

The thesis is organized in two parts.

The first part is devoted to introduce fundamental notions on topological preordered spaces, existence of maximal elements, preferences and functionals on $\mathcal{L}_+$ spaces.

We start recalling the basic concepts concerning preorders and the existence of maximal elements on compact sets. Indeed, as we have said before, a topological context is assumed since L^p-spaces come naturally into consideration.

We start presenting the popular notions of *upper semicontinuity* of a preorder on a topological space and *upper semicontinuity* of a real-valued function. We recall that a preorder $\precsim$ on a set $\mathcal{S}$ is said to be *upper semicontinuous* if, for every $Y \in \mathcal{S}$, the upper section $i_{\precsim}(Y) = \{Z \in \mathcal{S} : Y \precsim Z\}$ is a closed subset of $\mathcal{S}$. In particular, in the case we consider a metric space, the previous definition is equivalent to *sequential upper semicontinuity.*

In order to prove the existence of maximal elements for not necessarily total preorders we present in a detailed way the so called “folk theorem", which guarantees the existence of a maximal element for every (not necessarily total) preorder on a compact set provided that the preorder is upper semicontinuous. Unless there are generalizations of this result that recently appeared in the literature, this remains a rather general result that fits many situations. Its proof is based on the well known *Zorn's Lemma.*

Then we briefly describe fundamental properties of preferences on $\mathcal{L}_+$ spaces. We start presenting *monotonicity conditions* with respect to stochastic orders; in particular, when *comonotone allocations* are considered, we can limit ourself to comonotone feasible allocations, provided that appropriate *monotonicity conditions* with respect to the *second stochastic order* $\precsim_{SSD}$ are imposed. This kind of arguments are based on the well known *improvement theorem.*

We also introduce some aspects on functionals on $\mathcal{L}_+$ in order to fully characterize distortion risk measures.

The second part is devoted on analyzing the problem of Optimal risk sharing in a preference based approach, that is, we shall study the preordered sets representing individual and coalition preference decision making behaviour among feasible allocations.

We start considering a "coalition" preference decision making behaviour, expressed by the *coalition preorder* $\precsim$ on $\mathcal{A}(X)$. The aforementioned restriction according to which the individual preorder $\precsim_i$ only depends on the individual share of agent i (for $i = 1, ..., m$) allows us to consider a *social preorder* $\precsim = \bigcap_{i=1}^m \precsim_i$. From the previous considerations it is clear that we shall assume that every individual preference $\precsim_i$ is actually defined on $\mathcal{A}(X)$. If we define $\mathcal{A}(X)|_i$ as follows: $\mathcal{A}(X)|_i = \{Y_i \in \mathcal{L}_+ : \exists\ (Y_1, .., Y_{i-1}, Y_{i+1}, .., Y_m)\ s.t.\ (Y_1, .., Y_m) \in \mathcal{A}(X)\}$ it is clear that $\precsim_i$ is defined on $\mathcal{A}(X)$ but it is restricted to elements on $\mathcal{A}(X)|_i$.

We prove that there exists an optimal solution in a general preference-based setting provided that every individual preorder $\precsim_i$ is *upper semicontinuous* and the feasible set $\mathcal{A}(X)$ is compact. We also consider, when possible, the set $\mathcal{S} = \{(Y_1, ..., Y_m) \in \mathcal{A}(X) : X_i \precsim_i Y_i, ..., X_m \precsim_i Y_m\}$, with $(X_1, ..., X_m) \in \mathcal{A}(X)$, i.e., we start from an initial allocation and therefore we consider the *individually rational case.* The consideration of the *weak** - *topology* is of help in order to guarantee the compactness of $\mathcal{S}$.

Other results in this direction concern the case of *translation invariant* individual preorders $\precsim_i$ (i.e., preorders for which $Y_i \precsim Z_i$ if and only if $Y_i + c \precsim Z_i + c$, for all $Y_i,\ Z_i \in \mathcal{L}_+$ and for every constant c). The case of individual preferences expressed by total preorders endowed with translation invariant utility functions is particularly favorable since it guarantees that determining Pareto optimal allocations is in fact equivalent to determining optimal solutions for every choice of the initial exposures.

The optimal risk sharing problem in a functional approach is frequently identified with the *sup-convolution problem* relative to the functions $U_1, ..., U_m$:

$$(3) \quad U_1 \square U_2 \square ... \square U_m(Y_1, ..., Y_m) = \sup \sum_{i=1}^{m} U_i(Y_i).$$

It is well known that problems (1) and (3) do not coincide, in general, but the solutions to problem (3) are always solutions to problem (1).

The reference to the general preference-based setting allows us to use the classical representations of nontotal preorders in order to take advantage of the above sup-convolution problem under upper semicontinuity of the functions U_i together with the assumption of compactness of the feasible set $\mathcal{A}(X)$. By the way, a relevant example in our framework of a (upper-semi)continuous functional is provided by the *Choquet integral*, when we consider the topology L^∞ of (essentially) bounded functions on a common probability space.

Indeed, the reader may recall that there are essentially two kinds of representation of a not necessarily total preorder $\precsim$ on a set $\mathcal{S}$:

1. The representation based on the existence of an *order-preserving function* U (i.e., a $\precsim$-increasing function U which preserves the strict part $\prec$ of $\precsim$);
2. The so called *multi-utility representation* of the preorder $\precsim$, according to which there exists a family $\mathcal{U}$ of $\precsim$-increasing functions such that, for all $Y, Z \in \mathcal{S}$, $Y \precsim Z$ is equivalent to $U(Y) \leq U(Z)$ for all $U \in \mathcal{U}$.

The aforementioned representations are basically different since the first one only provides, say, the essential information about the preorder $\precsim$ for the purpose of determining its maximal elements, while the second fully characterizes the preorder. We use these two notions in order to guarantee the existence of optimal solutions, and to this aim we appropriately refer to well known results in mathematical utility theory (for example, Rader's theorem). In particular, we guarantee the existence of upper-semicontinuous order -preserving functions U_i or else we assume the existence of a *finite multi-utility representation* $\mathcal{U}$ in order to determine the optimal solutions by means of a maximization of one single function, as in problem (3).

We also show, in a functional setting, that problems (1) and (3) coincide in case that the functions U_i are all *comonotone super-additive* and *positively homogeneous*, therefore in some sense completing a well known result from the literature according to which the two problems coincide in case that all the functions are translation invariant.

It is clear that a *risky approach* is perfectly symmetrical to ours, and it is based on the consideration of the *inf-convolution problem*.

As a natural extension to the sup-convolution problem, we study the problem of risk sharing in the presence of different risky outcomes. In such a context, the m agents make a choice among different risky outcomes by comparing the shares corresponding to every risky option. This problem will be referred to the existence of maximal elements for a not necessarily total *coalition prorder*. Under particular assumptions that guarantee the existence of the sup-convolution for every risky outcome, the coalition preorder is total and the related utility function is the associated sup-convolution.

This work ends with brief considerations about comonotonicity and risk sharing re-adapting the main propositions and theorems of the thesis to the case of comonotone allocations, together with well known results from the literature.

Part I

Background

2

Topological preordered spaces and the existence of maximal elements

In this chapter we present fundamental notions of real representation of topological preordered spaces and the existence of maximal elements

2.1 Introduction

In the following chapters we will study the problem of optimal risk sharing among two or more agents. As we have previously introduced, this problem basically consists in finding the conditions that allow a feasible allocation to be both Pareto optimal and individually rational (namely optimal solution).

As we will see, the problem related to the existence of optimal solutions can be expressed in terms of the existence of maximal elements for a not necessarily total *coalition preorder*.

In this chapter we briefly express the conditions for the existence of maximal elements for topological preordered spaces.

In the first section we are concerned with the real representation of preordered sets. Since we deal with not necessarily total preorders, we present the basic concepts of an *order-preserving function* and that of a *multi-utility representation* of a preorder. The elements will be denoted by small letters in the usual way, unless in the next chapter we shall consider (topological) vector spaces of real random variables, therefore indicated by capital letters.

The second section is devoted to the (semi)continuous real representation of topological preordered spaces and to the existence of maximal elements.

We start presenting the popular notions of *upper semicontinuity* of a preorder on a topological space and *upper semicontinuity* of a real-valued function. In particular, in the case we consider a metric space, the previous definitions are equivalent to *sequential upper semicontinuity*.

Then, we recall the basic properties of compact spaces in order to prove the so called *folk theorem*, according to which an upper semicontinuous preorder on a compact topological space admits a maximal element. In parallel we furnish a brief characterization of topology generated by maps, necessary to introduce theorems of functional analysis. We also furnish a separate proof of the folk theorem concerning the case when the preorder has a finite upper semicontinuous multi-utility representation on a compact space.

The interest of this latter case will become clear in the next chapter, when the existence of optimal solutions will be related to the existence of maximal elements for a unique *coalition* (or *social*) preorder.

2.2 Preference and Utility

In this paragraph and throughout the thesis, the symbol $\neg$ stands for "not".

Definition 2.2.1 (preorders)**.** A *preorder* $\precsim$ on a nonempty set $\mathcal{S}$ is a binary relation on $\mathcal{S}$ which is *reflexive* (i.e., $x \precsim x$ for all $x \in \mathcal{S}$) and *transitive* (i.e., $x \precsim y$ and $y \precsim z$ imply $x \precsim z$ for all $x, y, z \in \mathcal{S}$).

If in addition $\precsim$ is *antisymmetric* (i.e., $x \precsim y$ and $y \precsim x$ imply $x = y$ for all $x, y \in \mathcal{S}$), then we shall refer to $\precsim$ as an *order*.

If $\precsim$ is a preorder (order) on $\mathcal{S}$, then the related set $(\mathcal{S}, \precsim)$ will be referred to as a *preordered set* (respectively, an *ordered set*).

A preorder $\precsim$ is said to be *total* if $x \precsim y$ or $y \precsim x$ for all $x, y \in \mathcal{S}$.

Definition 2.2.2 (indifference, strict preference and incomparability)**.** Given a preorder $\precsim$ on a set $\mathcal{S}$, define, for every $x, y \in \mathcal{S}$, the binary relations $\sim$ (*indifference relation*), $\prec$ (*strict preference relation*) and $\bowtie$ (*incomparability relation*):

$$x \sim y \Leftrightarrow (x \precsim y) \text{ and } (y \precsim x), \tag{2.1}$$

$$x \prec y \Leftrightarrow (x \precsim y) \text{ and } \neg(y \precsim x), \tag{2.2}$$

$$x \bowtie y \Leftrightarrow \neg(x \precsim y) \text{ and } \neg(y \precsim x). \tag{2.3}$$

Remark 2.2.3. The strict part $\prec$ of any preorder $\precsim$ on a set $\mathcal{S}$ is *acyclic*, i.e. it satisfies the following property for all elements $x_0, ..., x_n \in \mathcal{S}$ and every positive integer $n > 1$:

$$(x_0 \prec x_1) \textit{ and } (x_2 \prec x_3) \textit{ and ... and } (x_{n-1} \prec x_n) \Rightarrow \neg(x_n \prec x_0).$$

Definition 2.2.4 (increasing function)**.** Given a preordered set $(\mathcal{S}, \precsim)$, a function $u : (\mathcal{S}, \precsim) \to (\mathbb{R}, \leq)$ is said to be a *increasing* on $(\mathcal{S}, \precsim)$ if, for all $x, y \in \mathcal{S}$,

$$x \precsim y \Rightarrow u(x) \leq u(y)$$

Definition 2.2.5 (lower and upper sections)**.** Given a preorder $\precsim$ on a set $\mathcal{S}$, for every $x \in \mathcal{S}$ we set the following subsets of $\mathcal{S}$:

$$l(x) = \{y \in \mathcal{S} \mid y \prec x\}, \;\; r(x) = \{z \in \mathcal{S} \mid x \prec z\},$$

$$d(x) = \{y \in \mathcal{S} \mid y \precsim x\}, \;\; i(x) = \{z \in \mathcal{S} \mid x \precsim z\}.$$

2.2.1 Order-preserving functions

Definition 2.2.6 (order-preserving function)**.** Given a preordered set $(\mathcal{S}, \precsim)$, a function $u : (\mathcal{S}, \precsim) \to (\mathbb{R}, \leq)$ is said to be an *order-preserving function* on $(\mathcal{S}, \precsim)$ if it is increasing on $(\mathcal{S}, \precsim)$ and, for all $x, y \in \mathcal{S}$,

$$x \prec y \Rightarrow u(x) < u(y).$$

Remark 2.2.7. It is clear that a function $u : (\mathcal{S}, \precsim) \to (\mathbb{R}, \leq)$ on a totally preordered set $(\mathcal{S}, \precsim)$ is order-preserving if and only if, for all $x, y \in \mathbb{S}$,

$$x \precsim y \Leftrightarrow u(x) \leq u(y). \tag{2.4}$$

In this case, u is said to be a *utility function* on $(\mathcal{S}, \precsim)$.

In economic literature, an order preserving function is often referred to as a *Richter-Peleg utility function* (see Richter[33] and Peleg[31]). This kind of representation furnishes, in some sense, an approximate description of a not necessary total preorder and does not allow to fully characterize the original preorder.

We now recall the central concept of multi-utility representation which has been recently introduced in order to deal with the non-total cases in an effective way.

2.2.2 Multi-utility representations

Definition 2.2.8 (multi-utility representation)**.** A *multi-utility representation* of a preorder $\precsim$ on a set $\mathcal{S}$ is a family $\mathcal{U}$ of functions $u : (\mathcal{S}, \precsim) \longrightarrow (\mathbb{R}, \leq)$ such that that for all $x, y \in \mathcal{S}$,

$$x \precsim y \Leftrightarrow [u(x) \leqslant u(y), \text{ for all } u \in \mathcal{U}]. \tag{2.5}$$

The multi-utility representation is in some sense the best kind of representation since it characterizes the preorder by means of a family of real-valued functions. Please notice that, if $\mathcal{U}$ is a multi-utility representation of a preorder $\precsim$ on a set $\mathcal{S}$, we have that, for $x, y \in \mathcal{S}$,

$$x \bowtie y \Leftrightarrow [\exists u_1,\ u_2 \in \mathcal{U} : (u_1(x) < u_1(y)) \ and \ (u_2(x) > u_2(y))].$$

The concept of a multi-utility representation was introduced by Ok[27] in the case of a finite representing family and then deeply investigated by Evren and Ok,[22] who studied the difficult problem of guaranteeing the existence of a continuous representation of this kind. Such a problem was also studied more recently by Bosi and Herden.[7] The case of finite multi-utility representations was carefully considered by Kaminski.[24]

2.3 Existence of Maximal elements

In the sequel, the *natural topology* on the real line will be denoted by τ_{nat}.

Definition 2.3.1 (topological proerdered space)**.** A triplet $(\mathcal{S}, \precsim, \tau)$ is said to be a *topological preordered space* if $(\mathcal{S}, \precsim)$ is a preordered set and $(\mathcal{S}, \tau)$ is a topological space.

Throughout the thesis, we shall frequently refer to the case when the topology $\tau = \tau_{||\cdot||}$ is induced by a *metric* $|| \cdot ||$ on $\mathcal{S}$.

2.3.1 Semicontinuity

Definition 2.3.2 (continuous and upper semicontinuous preorder)**.** A preorder $\precsim$ on a topological space $(\mathcal{S}, \tau)$ is said to be

1. *upper* (*lower*) *semicontinuous* if $i(x)$ ($d(x)$) is a closed subset of $\mathcal{S}$ for every $x \in \mathcal{S}$;

2. *continuous* if it is both lower and upper semicontinuous.

It is clear that a total preorder $\precsim$ on a topological space $(\mathcal{S}, \tau)$ is upper semicontinuous if and only if $l(x)$ is an open subset of $\mathcal{S}$ for every $x \in \mathcal{S}$.

Since in the sequel we will often deal with topological metric spaces, it is of interest to recall the well known result showing that closed sets are indeed closed as far as sequences are concerned.

Proposition 2.3.3. *Assume that $\mathcal{S}$ is a subset of a metric space $\mathcal{A}$. Then, the following are equivalent:*

1. $\mathcal{S}$ is closed;

2. if $\{x_n\}$ is a convergent sequence of elements in $\mathcal{S}$, then the limit $a = \lim_{n\to\infty} x_n$ always belongs to $\mathcal{S}$.

Remark 2.3.4. From the previous Proposition, it is clear that if a preorder $\precsim$ on a topological metric space $(\mathcal{S}, \tau)$ is *upper* (*lower*) *semicontinuous*, then $i(x)$ ($d(x)$) is a sequentially closed subset of $\mathcal{S}$ for every $x \in \mathcal{S}$.

Let us now recall the basic notion of an upper semicontinuous function.

Definition 2.3.5 (upper semicontinuous real-valued function)**.** A real-valued function u on an arbitrary topological space $(\mathcal{S}, \tau)$ is said to be *upper semicontinuous* if

$$u^{-1}(]-\infty, \alpha[) = \{x \in \mathcal{S} : u(x) < \alpha\}$$

is an open set for every $\alpha \in \mathbb{R}$. Analogously, a real-valued function u on an arbitrary topological space $(\mathcal{S}, \tau)$ is said to be *lower semicontinuous* if

$$u^{-1}(]\alpha, +\infty[) = \{x \in \mathcal{S} : \alpha < u(x)\}$$

is an open set for every $\alpha \in \mathbb{R}$.

In the case when we consider a metric space, the previous definition of an upper semicontinuous function is equivalent to the following definition of a sequentially upper semicontinuous function, that we also present for the sake of completeness.

Definition 2.3.6 (sequentially upper semicontinuous real-valued function)**.** A real-valued function u on an arbitrary topological space $(\mathcal{S}, \tau)$ is said to be *sequentially upper semicontinuous* if the following condition is verified :

for every point $x \in X$ and every sequence $\{x_n\}_{n \in \mathbb{N}}$ in $(\mathcal{S}, \tau)$ converging to x the equation $u(x) \geq \lim_{n\to\infty} \sup \; f(x_n) = \inf_k \sup_{n>k} f(x_n)$ holds.

It is important to notice that every upper semicontinuous preorder $\precsim$ on a topological space $(\mathcal{S}, \tau)$ has an upper semicontinuous multi-utility representation (see Ok,[27] Proposition 2).

This result was slightly improved by Bosi and Zuanon,[9] in particular providing that if a preorder $\precsim$ has an upper semicontinuous multi-utility representation, then the preorder is upper semicontinuous.

Proposition 2.3.7. *Let $\precsim$ be a preorder on a topological space $(\mathcal{S}, \tau)$. Then the following conditions are equivalent.*

1. *$\precsim$ has an upper semicontinuous multi-utility representation;*

2. *$\precsim$ is upper semicontinuous.*

Proof. (1) $\Rightarrow$ (2). Assume that $\precsim$ has an upper semicontinuous multi-utility representation. If for two elements $x, z \in \mathcal{S}$ we have that $z \in \mathcal{S} \backslash i(x)$, then there exists an upper semicontinuous increasing function $u : (\mathcal{S}, \tau) \to (\mathbb{R}, \leq, \tau_{nat})$ such that $u(z) < u(x)$ and therefore $u^{-1}(]-\infty, u(x)[)$ is an open decreasing subset of $\mathcal{S}$ containing z such that

$$u^{-1}(]-\infty, u(x)[) \cap i(x) = \emptyset$$

. Therefore $i(x)$ is closed for every $x \in \mathcal{S}$.

(2) $\Rightarrow$ (1) Assume that $\precsim$ is upper semicontinuous on $(\mathcal{S}, \tau)$ and denote by $\chi(A)$ the indicator function of any subset $A \subset \mathcal{S}$ (i.e., $\chi(x) = 1$ if $x \in A$, and $\chi(x) = 0$ if $x \notin A$). Then observe that

$$\{\chi(i_{\precsim}(x))\}|_{x \in \mathcal{S}}$$

is an upper semicontinuous multi-utility representation of $\precsim$. □

The problem concerning the existence of semicontinuous or continuous order-preserving functions was extensively treated in the literature since its relevance in mathematical economics and game theory (see, e.g., Bridges and Mehta[13]).

2.3.2 Compactness

In a topological space a fundamental property is the compactness.

Let us now recall the definition of a compact topological space and a well known characterization of compactness.

Definition 2.3.8. A topological space $(\mathcal{S}, \tau)$ is said to be *compact* if for every family $\{O_\alpha\}_{\alpha \in I}$ of open subsets of $\mathcal{S}$ such that

$$\mathcal{S} \subseteq \bigcup_{\alpha \in I} O_\alpha$$

there exists a finite subfamily $\{O_{\alpha_1}, ..., O_{\alpha_n}\}$ such that

$$\mathcal{S} \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}$$

In other terms, every *open cover* of $\mathcal{S}$ admits a finite subcover.

Definition 2.3.9. A collection $\{F_\alpha\}_{\alpha\in I}$ of subsets of a set $\mathcal{S}$ is said to have the *finite intersection property* if the intersection

$$\bigcap_{i=1}^{n} F_{\alpha_i}$$

of every subfamily $\{F_{\alpha_1}, ..., F_{\alpha_n}\}$ of $\{F_\alpha\}_{\alpha\in I}$ is nonempty.

In the following chapters we will frequently deal with subspace of compact sets, then we include the following theorem.

Theorem 2.3.10. *A closed subspace of a compact topological space* $(\mathcal{S}, \tau)$ *is compact.*

The following theorem is well known and appears in every book of general topology (see e.g. Engelking[20]).

Theorem 2.3.11. *A topological space* $(\mathcal{S}, \tau)$ *is compact if and only if every family* $\{F_\alpha\}_{\alpha\in I}$ *of closed subsets of* $\mathcal{S}$ *with the finite intersection property actually satisfies*

$$\bigcap_{\alpha\in I} F_\alpha \neq \emptyset$$

For the sake of completeness, we include the statement of the following famous theorem.

Theorem 2.3.12. *(Baire Maximum Value Theorem) Let* $\mathcal{S}$ *be a compact topological space and* u *an upper semicontinuous real function on* X*. Then,*

$$\{x \in \mathcal{S} : f(x) = \sup\ u(X)\}$$

is a nonempty compact subset of $\mathcal{S}$*.*

Definition 2.3.13 (maximal elements)**.** Given a preordered set $(\mathcal{S}, \precsim)$, an element $x^* \in \mathcal{S}$ is said to be a *maximal element* (for $\precsim$) if for no $x \in \mathcal{S}$ it occurs that $x^* \prec x$.

Since we will frequently deal with non-empty subsets of ordered set, we include the following theorem:

Theorem 2.3.14. *Every finite non-empty subset of a ordered set has maximal and minimal elements.*

We include the simple proof of the following proposition, since it concerns a situation that will frequently occur in the next chapter.

Proposition 2.3.15. *If* $(\mathcal{S}, \tau, \precsim)$ *is a compact topological preordered space and*

$$u : (\mathcal{S}, \tau, \precsim) \rightarrow (\mathbb{R}, \tau_{nat}, \leq)$$

is an upper semicontinuous order-preserving function for $\precsim$ *on* $\mathcal{S}$*, then there is a maximal element* $x^* \in \mathcal{S}$*.*

Proof. By contraposition, assume that there is no maximal element relative to $\precsim$ on $\mathcal{S}$. Then

$$\bigcup_{x \in \mathcal{S}} u^{-1}(]-\infty, u(x)[)$$

is an open cover of $\mathcal{S}$. Since $\mathcal{S}$ is a compact topological space, there exists a finite subset $\{x_1, ..., x_n\}$ of $\mathcal{S}$ such that

$$\bigcup_{i=1}^{n} u^{-1}(]-\infty, u(x_i)[)$$

is also an open cover of $\mathcal{S}$. Since it is clear that the family

$$\{u^{-1}(]-\infty, u(x_i)[)\}_{i=1,\ldots,n}$$

is linearly ordered by set inclusion, assume without loss of generality that

$$u^{-1}(]-\infty, u(x_1)[) \subset u^{-1}(]-\infty, u(x_2)[) \subset ... \subset u^{-1}(]-\infty, u(x_n)[)$$

. Then x_n should belong to some set

$$u^{-1}(]-\infty, u(x_{\bar{i}})[) \text{ } with \text{ } \bar{i} < n$$

and we arrive at the contradiction that

$$x_n \in u^{-1}(]-\infty, u(x_n)[)$$

. This consideration completes the proof. □

We now recall the classical Rader's Theorem (see Rader,[32] and Bosi and Zuanon[8]).

We start recalling the definitions of *second countable* and *separable* topological spaces.

Definition 2.3.16. A topological space $(\mathcal{S}, \tau)$ is said to be *second countable* if there exists a countable family $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ of open subsets of $\mathcal{S}$ such that every open set O is expressed as the union of a subfamily of $\mathcal{B}$. $\mathcal{B}$ is said to be a countable basis for the topology τ.

Definition 2.3.17. A topological space $(\mathcal{S}, \tau)$ is said to be *separable* if there exists a countable subset D of $\mathcal{S}$ such that $O \cap D \neq \emptyset$ for every nonempty open set O

Then, we can introduce the classical Rader's Theorem as follows:

Theorem 2.3.18. *Every upper semicontinuous total preorder on a second countable topological space has an upper semicontinuous utility representation*

Remark 2.3.19. Recall that a compact metric space $(\mathcal{S}, \tau_{||\cdot||})$ is *separable* and therefore second countable (see e.g, Engelking[20]).

Corollary 2.3.20. *If* $(\mathcal{S}, ||\cdot||)$ *is a compact metric space and* $\precsim$ *is an upper semicontinuous total preorder on* $(\mathcal{S}, ||\cdot||)$*, then there is an upper semicontinuous utility function* u *for* $\precsim$*, and* $x^* \in \mathcal{S}$ *is a maximal element for* $\precsim$ *if and only if*

$$u(x^*) = \sup\ u(\mathcal{S})$$

Proof. Since $\mathcal{S}$ is a compact metric space, it is separable and therefore second countable. Therefore, the upper semicontinuous total preorder $\precsim$ on $(\mathcal{S}, \tau_{||\cdot||})$ admits an upper semicontinuous utility function u by Rader's theorem. It is clear that there is a maximal element x^* for $\precsim$ by the above Proposition 2.3.15. Finally, the fact that $x^* \in \mathcal{S}$ is a maximal element for $\precsim$ if and only if $u(x^*) = \sup\ u(\mathcal{S})$ is an immediate consequence of the assumption according to which $\precsim$ is total. □

2.3.3 Topology generated by maps

We are now interested on including a brief characterization of *weak topology* (see Conway,[16] Rudin[35] and Pedersen[30]), that is necessary to understand some important applications of Functional Analysis Theorems. Until now we have seen that, given a topology (collection of open sets) we can decide whether a function is continuous. We can now reverse this argument. Consider a set X, a collection of topological spaces $(Y_i)_{i\in I}$ and a collection of maps $(f_i)_{i\in I}$ such that:

$$f_i : X \to Y_i \quad \forall i \in I$$

We would like to define a topology on X such that all $(f_i)_{i\in I}$ are continous, and we want to do this in the coarsest way, that is the topology with the fewest open sets. So we need to characterize the collection of open sets O_X. Obviously every preimage $f_i^{-1}(O_i)$ of every open set $O_i \in Y_i$ under any f_i must be included in O_X. Then, finite intersection of these open sets should be open and then the union of (possibly infinitely many) finite intersections should be open. The next Lemma states that this collection is a topology:

Lemma 2.3.21. *The collection of all unions of finite intersection of sets of the form $(f_i^{-1}(O_i))_{i\in I}$ where O_i is an open set in Y_i, is a topology. It is called the weak topology of X with respect to $(f_i)_{i\in I}$, denoted by $\sigma(X,(f_i)_{i\in I})$.*

Le now X be a topological vector space over some field K. Recall the following definitions:

Definition 2.3.22. A linear functional on X is a linear map $X \to K$.

Definition 2.3.23. The dual space of X, denoted by X^*, is the space of all linear functionals on X that are continuous with respect to the given topology.

Definition 2.3.24. The weak* topology on X is the coarsest topology (the topology with the fewest open sets) such that each element of X^* remains a continuous function.

2.3.4 Tychonov theorem

A particular class of weak topology is the product topology.

Let $X_{\alpha\in A}$ be topological spaces, where A is the index set. Then we have the following definitions:

Definition 2.3.25. The cartesian product of X_α nonempty sets is given by:

$$\prod_{\alpha\in A} X_\alpha = \{f : A \to \bigcup_{\alpha\in A} X_\alpha, f(\alpha) \in X_\alpha\}$$

that is, the sets of maps with domain A such that for each $\alpha \in A$ the map selects an element of X_α

If we consider now the projection maps:

$$p_\alpha : \prod_{\alpha\in A} X_\alpha \to X_\alpha$$

we can introduce the product topology as follows:

Definition 2.3.26. The product topology on $\prod_{\alpha\in A} X_\alpha$ is the coarest topology such that all projection maps p_α are continuous.

Then we can introduce the well known Tychonov theorem:

Theorem 2.3.27. *Let $\{X_\alpha : \alpha \in A\}$ be a family of compact topological spaces. Then the cartesian product $\prod_{\alpha\in A} X_\alpha$ is compact in the product topology.*

2.3.5 Banach Alaoglu theorem

The Tychonov Theorem is essential to proove a fundamental compactness result of Functional Analysis, the Banach-Alaoglu Theorem (see Alaoglu[2])

Theorem 2.3.28. *Let X be a normed space, then for any $r > 0$, the closed ball*

$$B_r = \{x^* \in X^* \ s.t. \ \|x^*\| \leq r\}$$

of its dual space is compact in the weak topology.*

This theorem will be applied, for example, to state the compactness in the weak topology $\sigma(L^\infty, L^1)$ of closed balls in L^∞.

We can now consider the classical and well-known Zorn's lemma (see Kuratowski[26] and Zorn[39]) indicating a condition for the existence of maximal elements for a preordered set.

2.3.6 Zorn's Lemma

Lemma 2.3.29. *(Zorn's Lemma) Suppose that a partially (pre)ordered set $\mathcal{S}$ has the property that every chain $\mathcal{C}$ (i.e. totally ordered subset) has an upper bound in $\mathcal{S}$ (i.e., there exists $k \in \mathcal{S}$ such that $x \precsim k$ for all $x \in \mathcal{C}$). Then the set S contains at least one maximal element.*

The Zorn's Lemma, in one of its equivalent form, implies the Tychonov Theorem in topology, to which it is also equivalent (see Kelley[25]).

In particular, we will apply the Zorn's Lemma in order to prove the "folk" theorem in the sequel, guaranteeing the existence of a maximal element for an upper semicontinuous preorder on a compact topological space. For reader's convenience, we present also its proof.

2.3.7 "Folk" theorem

Theorem 2.3.30. *Let $(\mathcal{S}, \tau)$ be a compact topological space, and $\precsim$ an upper semicontinuous preorder on $(\mathcal{S}, \tau)$. Then there exists a maximal element relative to $\precsim$.*

Proof. From the above Zorn's Lemma, the thesis follows if every chain $\mathcal{C}$ has an upper bound in $\mathcal{S}$. Let $\mathcal{C}$ be a chain in $\mathcal{S}$. Then, any element of

$$\bigcap_{z \in \mathcal{C}} i(z)$$

provides an upper bound for $\mathcal{C}$. Therefore we only need to prove that

$$\bigcap_{z \in \mathcal{C}} i(z) \neq \varnothing$$

. Since $i(z)$ is closed for every $z \in \mathcal{S}$, and $\mathcal{S}$ is compact, it suffices to show that the family

$$\{i(z) : z \in \mathcal{C}\}$$

has the finite intersection property (see Theorem 2.3.11). So, let us consider a finite subset $\widetilde{\mathcal{C}}$ of $\mathcal{C}$ and the associated finite collection

$$\{i(z) \mid z \in \widetilde{\mathcal{C}}\}$$

. Since

$$\bigcap_{z \in \widetilde{\mathcal{C}}} i(z) = i(\max \widetilde{\mathcal{C}})$$

is nonempty (see Theorem 2.3.14), the thesis follows. □

In the next chapter we shall frequently refer to existence of an upper semicontinuous multi-utility representation.

2.3.8 Upper semicontinuous multi-utility representations

Definition 2.3.31 (upper-semicontinuous multi-utility representation)**.** An *upper semicontinuous multi-utility representation* of a preorder $\precsim$ on a topological space $(\mathcal{S}, \tau)$ is a family $\mathcal{U}$ of upper semicontinuous functions $u : (\mathcal{S}, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ which is also a multi-utility representation of $\precsim$.

In the particular case when a finite upper semicontinuous multi-utility representation exists, it is possible to provide a proof of the folk theorem (i.e., Theorem 2.3.30) which doesn't use Zorn's lemma, but only lexicographic arguments, say.

The following theorem is an adaptation from Evren and Ok.[22]

Theorem 2.3.32. *Let $(\mathcal{S}, \tau)$ be a compact topological space, and let $\precsim$ be a preorder on $(\mathcal{S}, \tau)$ which admits a finite upper semicontinuous multi-utility representation $\mathcal{U} = \{u_1, ..., u_n\}$. Then there exists a maximal element relative to $\precsim$.*

Proof. Define $\mathcal{S}_1 = \text{argmax } \{u_1(x) : x \in \mathcal{S}\}$. Observe that, from Theorem 2.3.12, $\mathcal{S}_1$ is a compact subset of $\mathcal{S}$. Define subsequently, for $i = 1, ..., n$,

$$\mathcal{S}_i = \text{argmax } \{u_i(x) : x \in \bigcap_{h=1}^{i-1} \mathcal{S}_h\}$$

. Then $x^* \in \text{argmax } \{u_n(x) : x \in \mathcal{S}_n\}$ is a maximal element for $\precsim$. Indeed, for all $y \in \mathcal{S}$,

$$\begin{aligned} x^* \precsim y \quad &\Leftrightarrow \quad u_i(x^*) \leq u_i(y) \;\; \forall i \in \{1, ..., n\} \Rightarrow y \in \bigcap_{i=1}^{n} \mathcal{S}_i \\ &\Rightarrow \quad u_i(x^*) = u_i(y) \;\; \forall i \in \{1, ..., n\}, \end{aligned}$$

and therefore $x^* \prec y$ is false, implying that x^* is actually a maximal element for $\precsim$ on $\mathcal{S}$. □

In the following chapter we are going to introduce fundamental functional properties that are recurrent in the general risk sharing functional-approach.

3

Preferences on $\mathcal{L}_+$ spaces

In this chapter we restrict our attention to random variables from $\mathcal{L}_+$ spaces (i.e., a topological vector space of nonnegative random variables), analyzing some properties on preferences necessary to introduce the risk sharing problem formulation

3.1 Introduction

Since we often deal with (topological) metric spaces, it is necessary to introduce basic notions on normed vector spaces.

In particular, we restrict our attention to random variables belonging to topological vector spaces of nonnegative random variables. In this case, the concept of norm (sup-norm), clearly induces a metric.

We study the conditions that allow a preorder to be consistent with stochastic orders. Particularly relevant is the case preferences are consistent with second stochastic order dominance since this concept will be useful in the case we deal with comonotone allocations in the risk-sharing setting.

Then, we introduce some aspects on functionals on $\mathcal{L}_+$ in order to fully characterize distortion risk measures. Particularly relevant is the case of Choquet integral with respect to a convex probability distortion, since it is an example of translation invariant, comonotone superadditive, positively homoseneous and normalized functional.

The interest of this latter case will become clear in the next chapters, when the existence of optimal solutions will be related to the existence of solutions to the *sup convolution problem*.

3.2 Continuity with respect to the Norm Topology

Denote by $\mathbb{R}$ ($\mathbb{R}^+$) the set of all real numbers (respectively, the set of all nonnegative real numbers).

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a *probability space*, and denote by $\mathbb{1}_F$ the *indicator function* of any subset F of Ω.

Let $\mathcal{L}_+$ be a vector space of nonnegative *real random variables* on $(\Omega, \mathcal{F}, \mathcal{P})$. In particular, $\mathcal{L}_+$ could be specialized as the space L^∞_+ of nonnegative bounded random variables, or else the space L^1_+ (L^2_+) of *integrable* (respectively, *square integrable*) nonnegative random variables on $(\Omega, \mathcal{F}, \mathcal{P})$.

We recall the well known general definition of L^p-*space* $(1 \leq p < \infty)$.

Definition 3.2.1 (L^p-space)**.** The space $L^p(\Omega)$ consists of all measurable functions $X : \Omega \longrightarrow \mathbb{R}$ such that

$$\int \mid X \mid^p d\mathcal{P} = \mathbb{E} \mid X^p \mid < \infty.$$

The L_p-norm of $X \in L^p(\Omega)$ is defined by

$$\mid\mid X \mid\mid_{L^p} = \left(\int \mid X \mid^p d\mathcal{P} \right)^{\frac{1}{p}}.$$

Example 3.2.2. *A classical example of a L_2-norm continuous utility functional is*

$$U(X) = \mathbb{E}[X] - \alpha Var[X] \quad (\alpha > 0).$$

Remark 3.2.3. We just recall that this kind of functional is very popular in the literature, since the consideration of the variance can be explained with the incorporation of "transaction costs".

Definition 3.2.4 (sup-norm topology)**.** The *sup-norm* on the space $L^\infty_+(\Omega, \mathcal{F})$ of all bounded measurable functions on the measurable space $(\Omega, \mathcal{F})$ is defined as follows:

$$\mid\mid X \mid\mid = \sup_{\omega \in \Omega} \mid X(\omega) \mid.$$

The *norm topology* on $L^\infty_+(\Omega, \mathcal{F})$ is the topology corresponding to the norm above.

Remark 3.2.5. Since this topology focuses on situations involving extremal events, such as catastrophes, this may be called the "topology of fear". This topology is focused on extremals, and as a result is much more restrictive in defining "proximity". The function "sup" is continuous with respect to the sup-norm, but not continuous with respect to the standard "averaging" topology.

3.3 Preferences consistence with stochastic orders

We briefly discuss the conditions that allow a preference relation to be consistent with second stochastic order. The importance of second stochastic dominance consistency will be clear when we will face the problem of efficient risk sharing restricting our attention to comonotone allocations.

We start introducing basic definitions in order to fully characterize stochastic orders.

If for two random variables $X, Y \in \mathcal{L}_+$ we have that $X(\omega) \leq Y(\omega)$ for $\omega \in \Omega$ $\mathcal{P}$-almost surely, then we shall simply write $X \leq Y$. We have that $\leq$ is a preorder on $\mathcal{L}_+$.

Definition 3.3.1 (total preorder)**.** A preorder $\precsim$ on a vector space $\mathcal{L}_+$ is said to be *total* if $X \precsim Y$ or $Y \precsim X$ for all random variables $X, Y \in \mathcal{L}_+$.

Definition 3.3.2 (Monotonicity)**.** A preorder $\precsim$ on a vector space $\mathcal{L}_+$ is said to be *monotone* if $X \leq Y$ implies that $X \precsim Y$ for all random variables $X, Y \in \mathcal{L}_+$.

Definition 3.3.3 (translation invariance)**.** A preorder $\precsim$ on a vector space $\mathcal{L}_+$ of nonnegative random variables is said to be *translation invariant* if the following condition holds for every positive real number c (identified with the constant random variable equal to c), and all random variables $X, Y \in \mathcal{L}_+$,

$$X \precsim Y \Leftrightarrow X + c \precsim Y + c. \tag{3.1}$$

Denote by $S_X(t) = 1 - F_X(t) = \mathcal{P}(\{\omega \in \Omega : X(\omega) > t\})$ the *decumulative distribution function* of any random variable $X \in \mathcal{L}_+$.

Recall that two random variables X and Y are said to be equivalent, or equal in law, or equal in distribution, iff they have the same probability distribution function, i.e.,

$$X \overset{d}{=} Y \Leftrightarrow F_X(x) = F_Y(y) \ \forall \ x \in \mathbb{R}$$

The shorthand $X \sim U(0,1)$ is used to indicate that the random variable X has the standard uniform distribution with minimum 0 and maximum 1. A standard uniform random variable X has probability density function

$$f(x) = 1 \quad 0 < x < 1.$$

For the sake of convenience, a constant random variable equal to $c \in \mathbb{R}$ will be also be denoted by c.

Definition 3.3.4 (first order stochastic dominance)**.** Let X, Y be random variables $\in \mathcal{L}_+$ with distribution functions F_X and F_Y. Then, the following statements are equivalent:

- X is said to precede Y in first order stochastic dominance, denoted by $X \precsim_{st} Y$;
- $F_Y(t) \leq F_X(t)$ for all $t \in \mathbb{R}$;
- $E[f(X)] \leq E[f(Y)]$ for all increasing functions f.

Definition 3.3.5 (second order stochastic dominance)**.** Let X, Y be random variables $\in \mathcal{L}_+$ with distribution functions F_X and F_Y. Then, the following statements are equivalent:

- X is said to precede Y in second order stochastic dominance, denoted by $X \precsim_{SSD} Y$;
- $\int_{-\infty}^{x} F_Y(t)dt \leq \int_{-\infty}^{x} F_X(t)dt$ for all $x \in \mathbb{R}$;
- $E[f(X)] \leq E[f(Y)]$ for all increasing concave functions f.

Definition 3.3.6 (stop-loss order stochastic dominance)**.** Let X, Y be random variables $\in \mathcal{L}_+$. Then, the following statements are equivalent:

- X is said to precede Y in stop-loss order, denoted by $X \precsim_{sl} Y$;
- $E[(X-d)_+] = E[max\{X-d, 0\}] \leqslant E[(Y-d)_+] = E[max\{Y-d, 0\}]$, for all $d \in \mathbb{R}$;
- $E[f(X)] \leq E[f(Y)]$ for all increasing convex functions f.

Definition 3.3.7 (convex order stochastic dominance)**.** Let X, Y be random variables $\in \mathcal{L}_+$. Then, the following statements are equivalent:

- X is said to precede Y in convex order, denoted by $X \precsim_{CX} Y$;
- $X \precsim_{sl} Y$ and $E[X] = E[Y]$;

Remark 3.3.8. Note that convex order is equivalent to ordering with respect to second stochastic dominance with equal means (see Rothschild and Stiglitz[47]).

3.4 Comonotonic sets

The concept of comonotonicity is actually a robust tool in order to solve several research and applicative problems in capital allocation and risk sharing.

In this section we are going to briefly list fundamental implications of comonotonicity. We start from the definition of comonotone random variables.

Definition 3.4.1 (comonotone random variables)**.** Two random variables $X, Y \in \mathcal{L}_+$ are said to be *comonotone* if

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$$

for $\omega_1, \omega_2 \in \Omega$ $\mathcal{P}$-almost surely.

3.4.1 The Univariate case

We want now to introduce comonotonic properties of a set $X \subseteq \mathbb{R}^m$, in particular considering the correlation between the definition of comonotonicity and the total order structure of $\mathbb{R}$.

The following characterization of comonotonic random vectors are well known, see for istance Dhaene.[19]

Definition 3.4.2 (Comonotonic Set)**.** The set $X \subseteq \mathbb{R}^m$ is said to be comonotonic if it is $\leq$-totally ordered, i.e. if for any $(X_1, ..., X_m), (X'_1, ..., X'_m) \in X$, either $(X_1, ..., X_m) \leq (X'_1, ..., X'_m)$ or $(X'_1, ..., X'_m) \leq (X_1, ..., X_m)$.

Theorem 3.4.3. *The following statements are equivalent:*

1. *The random vector* $(X_1, ..., X_m)$ *is comonotonic;*
2. $F_{(X_1,....,X_m)}(x_1, ..., x_m) = min\{F_{X_1}(x_1), ..., F_{X_m}(x_m)\}$
3. $(X_1, ..., X_m) \stackrel{d}{=} \{F^{-1}_{X_1}(U), ..., F^{-1}_{X_m}(U)\}$ *where* $U \sim Unif[0, 1]$*;*
4. *There exists a random variable* Z *and nondecreasing functions* $f_i(i = 1, ..., m)$ *s.t.* $(X_1, ..., X_m) \stackrel{d}{=} (f_1(Z), ..., f_m(Z))$.

Note that only the total order structure of $\mathbb{R}$ is needed in order to define comonotonic random vectors. In fact, the definition of comonotonicity can be extended to sets of measurable functions with values in any totally preordered set $\mathcal{C}$ endowed with the σ-algebra induced by the total preorder $\leq_C$. Then, definition 3.4.1 becomes (see Chateauneuf[14]):

Definition 3.4.4. Two random variables $X, Y \in \mathcal{L}_+$ are said to be *comonotone* if and only if

$$(X(\omega_1) <_C X(\omega_2)) \Longrightarrow (Y(\omega_1) \leq_C Y(\omega_2))$$

for $\omega_1, \omega_2 \in \Omega$ $\mathcal{P}$-almost surely.

3.4.2 The Multivariate case

As an extension to the previous case we consider comonotonic properties of subsets of $(\mathbb{R}^m)^n$.

Definition 3.4.5 (Comonotonic Set)**.** The set $X \subseteq (\mathbb{R}^m)^n$ is said to be comonotonic if it is $\leq$-totally ordered, i.e. if for any $(X_1, ..., X_m), (X'_1, ..., X'_m) \in X$ with $X_i \subseteq \mathbb{R}^n, i = 1, ..m$, either $(X_1, ..., X_m) \leq (X'_1, ..., X'_m)$ or $(X'_1, ..., X'_m) \leq (X_1, ..., X_m)$.

Theorem 3.4.6. *The following statements are equivalent:*

1. *The random vector* $(X_1, ..., X_m)$ *is comonotonic;*
2. $F_{(X_1,....,X_m)}(x_1^1, ..., x_1^n, ..., x_m^1, ..., x_m^n) =$
   $min\{F_{X_1}(x_1^1, ..., x_1^n), ..., F_{X_m}(x_m^1, ..., x_m^n)\}$
   for all $(x_1^1, ..., x_1^n, ..., x_m^1, ..., x_m^n) \in (\mathbb{R}^m)^n$
3. *There exists a random variable* Z *and nondecreasing functions* $(f_i, ..., g_i), i = 1, ..., n$ *s.t.*
   $(X_1, ..., X_m) \overset{d}{=} ((f_1(Z), ..., f_n(Z)), ..., (g_1(Z), ..., g_n(Z)).$

.

3.5 Functionals on $\mathcal{L}_+$

In the following we are going to introduce fundamental proporties of functionals on $\mathcal{L}_+$ in order to introduce distortion risk measures restricting our attention the the well known mathematical propertis of the Choquet integral.

Definition 3.5.1 (basic properties of a real functional)**.** A functional U from $\mathcal{L}_+$ into $\mathbb{R}$ is said to be

1. *Monotone* if $U(X) \leq U(Y)$ for all $X, Y \in \mathcal{L}_+$ such that $X \leq Y$;

   the financial meaning of monotonicity is clear: the risk of a financial instrument with the payoff X is at least as much as another one with the payoff Y, if former incurs at least as much losses as the latter in every state of economy.
2. *Normalized* if $U(\mathbb{1}_\Omega) = 1$;
3. *Monotone with respect to First Order Stochastic Dominance* if $U(X) \leq U(Y)$ for all $X, Y \in \mathcal{L}_+$ such that $S_X(t) \leq S_Y(t)$ for all $t \in \mathbb{R}^+$;
4. *Positively Homogeneous* (i.e., $U(\gamma X) = \gamma U(X)$ for every $\gamma \in \mathbb{R}^+$ and $X \in \mathcal{L}_+$;

   From a financial perspective, positive homogeneity implies that a linear increase of the return by a positive factor leads to a linear increase in risk by the same factor.

5. *Translation Invariant* if $U(X+c) = U(X) + c$ for all $X \in \mathcal{L}_+$ and $c \in \mathbb{R}^+$;

   Translation invariance is motivated by the interpretation of $U(X)$ as a reserve requirement, i.e., the amount which should be raised in order to make X acceptable from the point of view of a supervising agency.

6. *Law invariant* if $U(X) = U(Y)$ for all $X, Y \in \mathcal{L}_+$ with distribution functions F_X and F_Y such that $F_X = F_Y$;

   this assumption is essential for a functional to be estimated from empirical data.

7. *Comonotone Additive* if $U(X+Y) = U(X) + U(Y)$ for all comonotone $X, Y \in \mathcal{L}_+$;

8. *Comonotone Subadditive* (*Superadditive*) if $U(X+Y) \leq U(X) + U(Y)$ (respectively, $U(X+Y) \geq U(X) + U(Y)$) for all comonotone $X, Y \in \mathcal{L}_+$;

9. *Sublinear* (*Superlinear*) if U is positively homogeneous and *subadditive* (*superadditive*), i.e., $U(X+Y) \leq U(X)+U(Y)$ $(U(X+Y) \geq U(X)+U(Y))$ for all $X, Y \in \mathcal{L}_+$);

   financial implications of this subadditivity is obviously related to diversification effect. Though Artzner et al. (see Artzner[3]) treat sub-additivity as a essential demand for constructing a risk measure in order for it to be coherent, empirical indications prescrives that subadditivity does not always hold in reality (see Föllmer et al.[23]) .

10. *Convex* if $U(\lambda X + (1-\lambda Y)) \leq \lambda U(X) + (1-\lambda)U(Y)$ for all $X, Y \in \mathcal{L}_+,\ 0 \leq \lambda \leq 1$;

    convexity explaines the diversification property relaxing the requirement that a risk measure must be more sensitive to aggregation of large risks.

It is clear that a comonotone additive functional is also translation invariant. Further, a functional U is translation invariant if and only if $U(X+c) = U(X)+c$ for all $X \in \mathcal{L}_+$ and $c \in \mathbb{R}$ (i.e., the nonnegativity of c can be removed, see e.g. Marinacci and Montrucchio[28]).

Remark 3.5.2. If U is comonotone superadditive, positively homogeneous and normalized functional, then $U(X+c) \geq U(X) + c$ for all $X \in \mathcal{L}_+$ and $c \in \mathbb{R}$.

The following proposition is found in Parker [29, Lemma 6] (see also Bosi and Zuanon [10, Lemma 3.2]). We present its proof here for reader's convenience.

We recall that a real-valued functional U on a metric space $(\mathcal{S}, ||\cdot||)$ said to be *uniformly continuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \mathcal{S}$, $|| x - y || < \delta$ implies $| U(x) - U(y) | < \varepsilon$. It is clear that uniform continuity implies continuity.

Proposition 3.5.3. *If U is a monotone, positively homogeneous and comonotone subadditive functional on $L^\infty_+(\Omega, \mathcal{F})$, then U is uniformly continuous with respect to the sup-norm topology on $L^\infty_+(\Omega, \mathcal{F})$.*

Proof. Consider any two measurable real-valued functions $X, Y \in L^\infty_+(\Omega, \mathcal{F})$ and let U be a functional with the indicated properties. Then we have that

$$\begin{aligned} U(X) - U(Y) &\leq U(|| X - Y || \mathbb{1}_\Omega + Y) - U(Y) \\ &\leq || X - Y || \, U(\mathbb{1}_\Omega) + U(Y) - U(Y) \\ &= || X - Y || \, U(\mathbb{1}_\Omega). \end{aligned}$$

Analogously, it can be shown that $U(Y) - U(X) \leq || X - Y || \, U(\mathbb{1}_\Omega)$. Hence, we have that $| U(X) - U(Y) | \leq || X - Y || \, U(\mathbb{1}_\Omega)$. This consideration completes the proof. □

Example 3.5.4 (Upper semicontinuous not continuous functional)**.** Let $\mathcal{G}$ be any family of probability distortions, and for the sake of convenience denote by $U_g(X)$ the Choquet integral of any random variable X with respect to the distorted probability $g \circ \mathcal{P}$. Then define a functional U on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ by imposing, for all $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$,

$$U(X) := \inf_{g \in \mathcal{G}} U_g(X).$$

We claim that U is upper semicontinuous on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ endowed with the norm topology. Indeed consider any $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ and $\alpha \in \mathbb{R}$ with $U(X) < \alpha$. Then there exists $g \in \mathcal{G}$ such that $U_g(X) < \alpha$ and, since U_g is continuous and therefore in particular upper semicontinuous by Proposition 3.5.3, we have that $U_g^{-1}([0,\alpha[)$ is an open set containing X such that $U_g(Z) < \alpha$ for all $Z \in U_g^{-1}([0,\alpha[)$.

3.5.1 Coherent risk measures

In Artzner et al (see Artzner[3]), coherent risk measure is defined through the following set of axioms:

Definition 3.5.5. A real-valued functional ρ on $\mathcal{L}$ is said to be a *coherent risk measure* if ρ is:

1. *monotone*;
2. *translation invariant*;
3. *sublinear*.

Coherent measures have the following general form:

$$\rho(X) = \sup_{Q \in \mathcal{Q}} E_Q[-X]$$

where $\mathcal{Q}$ is some class of probability measures on the state space Ω.

3.5.2 Distortion risk measures

In this subsection, we discuss distortion risk measures.

Distortion risk measures were introduced by Wang[37] and can be defined as the distorted expectation of any non-negative loss random variable X, so they are closely related to the distortion expectation theory. For instance, Tsanakas and Desli[36] fully describe how risk measures can be interpreted from several perspectives, including a clarifying explanation of the connection between distortion risk measures and distortion expectation theory.

There are two key elements to define a distortion risk measure: first, the associated distortion function; and, second, the concept of the Choquet Integral (see Choquet[15]).

For a complete literature review of distortion risk measures, see Denuit et al.[18] and Balbàs et al.[4]

Definition 3.5.6 (Choquet integral with respect to a probability distortion)**.** A *probability distortion* g is a real-valued, nondecreasing and nonnegative function $g : [0,1] \longrightarrow [0,1]$ such that $g(0) = 0$ and $g(1) = 1$.

A real-valued functional U on $L^{\infty}_{+}(\Omega, \mathcal{F}, \mathcal{P})$ is said to be the *Choquet integral* with respect to the *distorted probability* $\mu = g \circ \mathcal{P}$ if, for all $X \in L^{\infty}_{+}(\Omega, \mathcal{F}, \mathcal{P})$,

$$U(X) = \int X dg \circ \mathbb{P} = \int_0^{+\infty} g(S_X(t))dt. \tag{3.2}$$

In particular we have that the Choquet integral satisfies the following properties (see Denneberg ([17])):

1. *monotonicity*;
2. *positive homogeneity*;
3. *translation invariance*;
4. *comonotone additivity*;
5. In the generalized case, distortion risk measures are *not additive*;
6. distortion risk measures are *sub-additive* if and only if the distortion function g is concave;
7. For a non-decreasing distortion function g, the associated risk measure ρ is consistent with the first stochastic dominance. The proof is given in Hardy and Wirch[38]
8. For a non-decreasing concave distortion function g, the associated risk measure ρ is consistent with the second stochastic dominance. As a result, every coherent distortion risk measure is consistent with respect to second-order stochastic dominance.

9. For a strictly concave distortion function g, the associated risk measure ρ is strictly consistent with the second stochastic dominance.

From these considerations, it is clear that the Choquet integral with respect to a probability distortion is a coherent risk measure.

Many different distortions g have been proposed in the literature. Some well-known ones are presented below.

- The mathematical expectation $U_g(X) = E[X]$ is a distortion risk measure whose distortion function is the identity function

$$g(x) = x$$

  provided the mathematical expectation exists.

- The value at risk VaR has been adopted as a standard tool to assess the risk and to calculate capital requirements in the financial industry. Value-at-Risk at level α of a random variable X (which we often call loss), is defined as follows:

$$VaR_\alpha(X) = inf\{x|F_X(x) \geq \ \alpha\} = F_X^{-1}(\alpha),$$

  where F_X is the distribution function (cdf) of X and α is the confidence or the tolerance level $0 \leq \alpha \leq 1$.

  A disadvantage when using VaR in the financial context is that the capital requirements for adverse scenarios based on the measure can be underestimated.

  Another problem related to VaR is that it may fail the subadditivity property.

  In particular, VaR is a distortion risk measure whose distortion function g is represented by:

$$g(x) = \begin{cases} 0, \ if \ \ 0 \leq x < 1-\alpha; \\ 1, \ if \ \ 1-\alpha \leq x \leq 1. \end{cases}.$$

The distortion function is discontinuous in this case due to the jump at $x = 1 - \alpha$. This implies that VaR is not coherent. As a result, VaR does not represent a "well" behaved distortion function.

- Tail Value-at-Risk (TVaR) may be interpreted as the mathematical expectation beyond VaR, and is defined as:

$$TVaR_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_\lambda(X) d\lambda$$

The TVaR risk measure does not suffer the two disvantages discussed above for VaR and appear to be a more powerful measure for assessing the actual risks faced by companies and financial institutions.

In particular TVaR is a distortion risk measure whose distortion function g is represented by:

$$g(x) = \begin{cases} \frac{x}{1-\alpha}, \ if \ \ 0 \leq x < 1-\alpha; \\ 1, \ if \ \ 1-\alpha \leq x \leq 1. \end{cases} .$$

- The conditional value at risk CVaR is the conditional expectation of X subject to $X \geq VaR_\alpha(X)$, i.e.

$$CVaR_\alpha(X) = E[X | X \geq VaR_\alpha(X)]$$

In particular, CVaR is a distortion risk measure whose distortion function g is represented by:

$$g(x) = min(\frac{x}{1-\alpha}, 1) \quad x \in [0, 1]$$

Remark 3.5.7. CVaR is known as tail conditional expectation in Artzner et al.,[3] conditional tail expectation in Wirch and Hardy,[38] mean shortfall in Bertsimas et al.,[6] and expected shortfall in Acerbi et al.[1]

The use of distortion risk measurses in the risk sharing setting will be explained in the following chapter, in particular referring to the risk redistribution problem with distortion risk measures.

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Part II

Preference-based approach to risk sharing

4

Existence of individually rational Pareto optimal allocations

In this chapter we analyze the problem of optimal risk sharing in a preference based approach, that is, we shall study the preordered sets representing individual and coalition preference decision making behaviour among feasible allocations. In particular, we study the existence of feasible allocations that are both Pareto optimal and individually rational, namely optimal solutions.

4.1 Introduction

The problem of optimal risk sharing among two or more agents has been studied in several contexts, mostly with a risk functional approach, that is, representing the attitude towards risk of each agent by utility functionals.

In such a context, an optimal risk sharing problem can be formulated as follows: m agents with individual exposures X_i are interested in sharing an optimal re-allocation of their own risks X_i. Let $X = \sum_{i=1}^{m} X_i$ be the total exposure of the agents and let U_i be the preference-functional of the i-th agent. The risk sharing problem consists in finding an optimal allocation $Y_i, \{i = 1, .., m\}$ of (uncertain) shares of X such that $X = \sum_{i=1}^{m} Y_i$.

We will discuss in the following paragraphs the characterization of optimal allocation. We just want now to consider that the key elements in the functional risk sharing approach described above, are the preference functionals U_i. We find in the literature these functionals expressed in terms of expected utility, non expected utility, risk measures, in such a way that depends on evolving of theories of risks.

It started with the pioneering works of Borch[14] and Arrow[7] with applications to insurance and reinsurance problems, where the attitude towards risk of each agent is represented by von Neumann Morgenstern expected utility. This approach was at the base of several papers, for example Wilson[51] and Aase.[1] Further extensions devoped for various decision criteria that depend on the risk measure approach, in particular in terms of coherent or convex risk measures, see for example Young,[52] Kaluska,[39],[40] Barrieu and El Karoui,[9] Jouini et al.,[38] Ludkovski and Young,[44] Acciaio,[2] Bourgert and Rüschendorf.[16] Recently the optimal risk sharing problem was studied in Grechuk and Zabarankin[34] and Grechuk et al.[35] with general deviation measures, in such a context that involves not only risk preferences of individuals but also their reward-preferences. The risk-reward risk sharing approach is also at the base of the work of Carlier et al.[18]

Since the works of Borch, Arrow and Wilson, it is well known that efficient risk sharing is mutually related to the comonotonicity property. We will study in the next chapters the implication of Comonotonicity in risk sharing with the related connection to the literature.

In this chapter, we are interested in describing a preference based approach, that is, we shall study the preordered sets representing individual and coalition preference decision making behaviour.

We start introducing the problem formulation related to finding an optimal solution, that is, a feasible allocation that is both *Pareto optimal* and *individually rational*, we characterize Pareto optimal allocations describing the order-conditions that allow an individual to prefer an allocation to another, and then we define a *coalition preorder* representing the attitude of all the agents to prefer an allocation to another one.

Then we study the existence of optimal solutions illustrating that this problem can be related to the existence of maximal elements for the *coalition preorder* in $\mathcal{A}(X)$. Identically, we extend these considerations explaining the equivalence between optimal solutions and maximal elements for the *coalition preorder* in the set S of all the feasible allocations for which each agent is at least as well as under the initial exposure. We will synthesize this problem by finding conditions under which we can define an upper semicontinuous *coalition preorder* on the compact set S.

Then, we introduce the so called multi-objective maximization problem in order to produce sufficient conditions for the existence of an optimal solution.

It is important to notice that we are interested on describing the not-necessarily total ordered structure of preference relations in the context of risk sharing. In literature we found an extensive application of risk sharing problems starting from the assumptions that agents preferences are endowed with particular utility functions. This clearly implies that agents preferences are total (at least implicitly, even if the authors do not mention preferences at all).

In our context, the expression of preferences relations by using order preserving functions or a multi utility representation clearly implies that we are considering a not necessarily total order structure.

4.2 Problem formulation

Consider an uncertain payoff X and m agents endowed with their own initial exposures $(X_1, .., X_m)$, with $X = \sum_{i=1}^m X_i$. Agent i has preferences over her own risks which are expressed by a (not necessarily total) preorder $\precsim_i$ $(i = 1, ..., m)$.

Divide X into uncertain shares $Y_1, ..., Y_m$ in such a way that $X = \sum_{i=1}^m Y_i$, be the total exposure.

Definition 4.2.1 (feasible allocations)**.** For every risk X, denote by $\mathcal{A}(X)$ the set of all the *feasible allocations* of X, i.e. the set

$$\mathcal{A}(X) = \{(Y_1, ..., Y_m) \mid X = \sum_{i=1}^m Y_i\}. \tag{4.1}$$

It is clear that the set $\mathcal{A}(X)$ of all feasible allocations is a *convex* subset of the product space ${\mathcal{L}_+}^m$ (i.e., $\alpha Y + (1-\alpha)Z \in \mathcal{A}(X)$ for all $Y, Z \in \mathcal{A}(X)$ and $\alpha \in [0,1]$).

We now present the central concept of *Pareto optimal allocation.*

Definition 4.2.2 (Pareto optimal allocation)**.** An allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ is said to be *Pareto optimal* if for no other allocation $(Y_1, ..., Y_m) \in \mathcal{A}(X)$ it occurs that $Y_1^* \precsim_i Y_1, ..., Y_m^* \precsim_m Y_m$ with at least one index i such that $Y_1^* \prec_i Y_i$.

We omit the immediate proof of the following proposition.

Proposition 4.2.3. *Assume that the individual preorder $\precsim_i$ is total for every $i \in \{1, ..., m\}$. Then the following conditions are equivalent concerning an allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$:*

1. *$(Y_1^*, ..., Y_m^*)$ is Pareto optimal;*
2. *for every allocation $(Y_1, ..., Y_m) \in \mathcal{A}(X)$ such that $Y_i^* \precsim_i Y_i$ for $i \in \{1, ..., m\}$ it occurs that $Y_i^* \sim_i Y_i$ for $i \in \{1, ..., m\}$.*

The following definition is of basic importance.

Definition 4.2.4. An allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ is said to be *individually rational* if all agents are at least as well off under $(Y_1^*, ..., Y_m^*)$ as under the initial exposures X_i $(i \in \{1, ..., m\})$, so that $X_i \precsim_i Y_i^*$ for all $i \in \{1, ..., m\}$.

Definition 4.2.5. An allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ is said to be *optimal* if it is both *Pareto optimal* and *individually rational.*

To study the existence of Pareto optimal allocations and optimal solutions we will define a coalition preorder, expression of preferences aggregation of the individuals. We clearly characterize this concept in the following paragraph.

4.2.1 The coalition preorder

We are now interested in considering a "coalition" preference decision making behaviour, expressed by the *coalition preorder* $\precsim$ on $\mathcal{A}(X)$.

In particular, we say that a coalition of m agents prefers the allocation $Y = (Y_1, ..., Y_m)$ over $Z = (Z_1, ..., Z_m)$ if and only if every agents prefers her own share Y_i over Z_i. Then, we can define the *coalition preorder* $\precsim$ in this way:

$$(Z_1, ..., Z_m) \precsim (Y_1, ..., Y_m) \;\Leftrightarrow\; Z_i \precsim_i Y_i \;\forall i \in \{1, .., m\}. \tag{4.2}$$

Remark 4.2.6. As we already discussed in the introduction of the thesis, we apply the universally accepted restriction according to which the preferences of the generic agent i only depend on its own share Y_i of the risk. From this aforementioned restriction, the *coalition preorder* $\precsim$ can be defined as the intersection of the individual preorders, that is:

$$\precsim = \bigcap_{i=1}^{m} \precsim_i . \tag{4.3}$$

From the previous considerations it is clear that we shall assume that every individual preference $\precsim_i$ is actually defined on $\mathcal{A}(X)$.

If we define $\mathcal{A}(X)|_i$ as follows:

$$\mathcal{A}(X)|_i = \{Y_i \in \mathcal{L}_+ : \exists\ (Y_1, .., Y_{i-1}, Y_{i+1}, .., Y_m)\ s.t.\ (Y_1, .., Y_m) \in \mathcal{A}(X)\}$$

it is clear that $\precsim_i$ is defined on $\mathcal{A}(X)$ but it is restricted to elements on $\mathcal{A}(X)|_i$.

Remark 4.2.7. Observe that the preorder $\precsim$ is not necessarily total, even if $\precsim_i$ is total for every i. Indeed, for two feasible allocations $Y = (Y_1, ..., Y_m)$ and $Z = (Z_1, ..., Z_m)$ there may exist two indexes i, j with $Y_i \prec_i Z_i$ and $Z_j \prec_j Y_j$. This consideration justifies in full the material and technique presented in the previous chapter and in particular the considerations on the existence of maximal elements for not necessarily total preorders.

Remark 4.2.8. It should be noted that in the particular case when every individual preorder $\precsim_i$ is total and admits a utility representation U_i, for all $Y = (Y_1, ..., Y_m)$ and $Z = (Z_1, ..., Z_m)$ in $\mathcal{A}(X)$ it occurs that $(Y_1, ..., Y_m) \precsim (Z_1, ..., Z_m)$ if and only if $U_i(Y_i) = U_i(Y) \leq U_i(Z) \leq U_i(Z_i)$. Therefore, we have that $\mathcal{U} = \{U_1, ..., U_m\}$ is a finite multi-utility representation of the coalition preorder $\precsim$.

In the following section we are going to introduce fundamental concepts necessary to fully characterize the existence of Pareto optimal allocations and optimal solutions.

4.3 Existence of optimal solutions

In this section we are going to study the conditions that allow a feasible allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ to be both Pareto optimal and individually rational, restricting our attention on the coalition preorder defined in the previous section.

We start considering the characterization of Pareto optimal allocations, equivalent to the problem concerning the existence of maximal elements for the coalition preorder $\precsim$.

Then we will extend this equivalence in order to characterize optimal solutions and we introduce a so called multi-objective maximization problem in order to produce sufficient conditions for the existence of an optimal solution.

4.3.1 Characterization of Pareto optimal allocations

Let us now recall the definition of a *translation invariant* preorder.

Definition 4.3.1. A preorder $\precsim$ on a vector space $\mathcal{L}_+$ of nonnegative random variables is said to be *translation invariant* if the following condition holds for every positive real number c (identified with the constant random variable equal to c), and all random variables $X, Y \in \mathcal{L}_+$,

$$X \precsim Y \Leftrightarrow X + c \precsim Y + c. \tag{4.4}$$

Remark 4.3.2. It is easy to check that a preorder $\precsim$ is translation invariant if and only if actually the above condition (4.4) holds true for every constant random variable c.

Remark 4.3.3. It should be noted that, if $\precsim$ is a translation invariant total preorder on $\mathcal{L}$, then for all random variables $X, Y \in \mathcal{S}$, and every real number c,

$$X \prec Y \Leftrightarrow X + c \prec Y + c. \tag{4.5}$$

Indeed, in this case we have that $\neg(X+c \prec Y+c) \Leftrightarrow \neg((X+c \precsim Y+c)$ *and* $\neg(Y+c \precsim X+c)) \Leftrightarrow Y+c \precsim X+c \Leftrightarrow Y \precsim X \Leftrightarrow \neg(X \prec Y)$.

It is clear that a total preorder $\precsim$ on $\mathcal{L}$ is translation invariant provided that it admits a *translation invariant* utility function U (i.e., $U(X+c) = U(X) + c$ for all $X \in \mathcal{L}_+$ and $c \in \mathbb{R}$).

In the following proposition we present a simple but useful property exhibited by Pareto optimal allocations in case of translation invariant individual total preorders.

Proposition 4.3.4. *Assume that $\precsim_i$ is a translation invariant total preorder for all $i \in \{1, ..., m\}$ and consider any m-tuple of real numbers $(\pi_i, ..., \pi_m)$ such that $\sum_{i=1}^m \pi_i = 0$. Then the following conditions are equivalent.*

1. $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ *is Pareto optimal;*

2. $(Y_1^* + \pi_1, ..., Y_m^* + \pi_m) \in \mathcal{A}(X)$ *is Pareto optimal.*

Proof. (1) $\Rightarrow$ (2) Let $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ be Pareto optimal and consider the allocation $(Y_1, ..., Y_m) \in \mathcal{A}(X)$ such that $Y_i \precsim_i (Y_i^* + \pi_i)$ for $i \in \{1, .., m\}$. By translation invariance of the total preorder $\precsim_i$ we have that

$$Y_i \precsim_i (Y_i^* + \pi_i) \Leftrightarrow Y_i - \pi_i \precsim_i (Y_i^* + \pi_i - \pi_i) \Leftrightarrow Y_i - \pi_i \precsim_i Y_i^*$$

So

$$Y_i - \pi_i \precsim_i Y_i^* \quad i \in \{1, .., m\}$$

Note that the allocation $(Y_1 - \pi_1, ..., Y_m - \pi_m)$ with $\sum_{i=1}^m \pi_i = 0$ is in $\mathcal{A}(X)$ and is only "weakly" dominated by $(Y_1^*, ..., Y_m^*)$ that is

$$[(Y_i - \pi_i \precsim_i Y_i^*) \text{ and } \neg(Y_i^* \succ_i Y_i - \pi_i)] \quad i \in \{1, .., m\}$$

The preorder $\precsim_i$ is total, so

$$\neg(Y_i^* \succ_i Y_i - \pi_i) \Leftrightarrow Y_i^* \precsim_i Y_i - \pi_i \quad i \in \{1, .., m\}$$

So we have

$$[(Y_i - \pi_i \precsim_i Y_i^*) \wedge (Y_i^* \precsim_i Y_i - \pi_i) \quad i \in \{1, .., m\}$$

that is

$$Y_i - \pi_i \sim_i Y_i^* \quad i \in \{1, .., m\}$$

that is a condition of Pareto indifference because $(Y_1^*, ..., Y_m^*)$ is Pareto optimal; By translation invariance of the total preorder $\precsim_i$

$$Y_i - \pi_i \sim_i Y_i^* \Leftrightarrow Y_i - \pi_i + \pi_i \sim_i Y_i^* + \pi_i \Leftrightarrow Y_i \sim_i Y_i^* + \pi_i \quad i \in \{1, .., m\}$$

Hence, the allocation $(Y_1^* + \pi_1, ..., Y_m^* + \pi_m) \in \mathcal{A}(X)$ is Pareto optimal.
(2) $\Rightarrow$ (1). Analogous. □

Alternative Proof. (1) $\Rightarrow$ (2) By contraposition, consider $(Y_1^*+\pi_1, ..., Y_m^*+\pi_m) \in \mathcal{A}(X)$ which is not Pareto optimal. Then there exists $(Z_1^* + \pi_1, ..., Z_m^* + \pi_m) \in \mathcal{A}(X)$ such that

$$[Y_i^* + \pi_i \precsim_i Z_i^* + \pi_i] \ \forall i \ \wedge [\exists \ i \in \{1, ..., m\} s.t. Y_i^* + \pi_i \prec_i Z_i^* + \pi_i]$$

. By translation invariace of the total preorder $\precsim_i$ we have that

$$Y_i^* \precsim_i Z_i^* \ \forall i \ \wedge [\exists \ i \in \{1, ..., m\} s.t. Y_i^* \prec_i Z_i^*].$$

Hence $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ is not Pareto optimal.
(2) $\Rightarrow$ (1). Analogous. □

Corollary 4.3.5. *Assume that $\precsim_i$ is a translation invariant total preorder for all $i \in \{1, ..., m\}$ and let $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ be a Pareto optimal allocation. Then $(Y_1^* + Z_1, ..., Y_m^* + Z_m) \in \mathcal{A}(X)$ is also a Pareto optimal allocation provided that the following condition holds for some uniquely determined m-tuple of real numbers $(\pi_i, ..., \pi_m)$ such that*

$$Z_i \sim_i \pi_i \text{ for all } i \in \{1, ..., m\} \text{ and } \sum_{i=1}^{m} \pi_i = 0. \tag{4.6}$$

Proof. We have that $Y_i^*+Z_i \sim_i Y_i^*+\pi_i$, implying that also $(Y_1^*+Z_1, ..., Y_m^*+Z_m) \in \mathcal{A}(X)$ is Pareto Optimal from the above Proposition 4.3.4. □

The existence of optimal allocations is guaranteed when there are Pareto optimal allocations and $\precsim_i$ has a translation invariant utility U_i for all $i \in \{1, ..., m\}$. This fact is illustrated in the following easy proposition.

Proposition 4.3.6. *Assume that $\precsim_i$ is a translation invariant total preorder for all $i \in \{1, ..., m\}$ with a translation invariant utility function U_i. Let $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ be Pareto optimal. Then the following conditions are equivalent for every m-tuple of real numbers $(\pi_i, ..., \pi_m)$ such that $\sum_{i=1}^{m} \pi_i = 0$:*

1. $(Y_1^* + \pi_1, ..., Y_m^* + \pi_m) \in \mathcal{A}(X)$ *is optimal;*

2. $U_i(X_i) - U_i(Y_i^*) \leq \pi_i$.

Proof. Just consider that, under our assumptions, $(Y_1^* + \pi_1, ..., Y_m^* + \pi_m) \in \mathcal{A}(X)$ is optimal if and only if, for all $i \in \{1, ..., m\}$,

$$U_i(X_i) \leq U_i(Y_i^*) + \pi_i = U_i(Y_i^* + \pi_i).$$

□

Remark 4.3.7. In the case of individual total preorders with translation invariant utilities, the above Proposition 4.3.6 guarantees that determining Pareto optimal allocations is in fact equivalent to determining optimal solutions for every choice of the initial exposures.

As we have previously already introduced, the problem concerning the existence of Pareto optimal allocations can be related to the problem concerning the existence of maximal elements for the coalition preorder $\precsim$ defined in (4.3). The following proposition illustrates this possibility.

Proposition 4.3.8. *For every risk X and for every feasible allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ the following conditions are equivalent:*

(i) $(Y_1^*, ..., Y_m^*)$ *is Pareto optimal;*

(ii) $(Y_1^*, ..., Y_m^*)$ *is maximal for $\mathcal{A}(X)$ with respect to the coalition preorder $\precsim = \bigcap_{i=1}^m \precsim_i$.*

Proof. (ii) $\Rightarrow$ (i). By contraposition, consider a feasible allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ which is not Pareto optimal. Then there exists $(Y_1', ..., Y_m') \in \mathcal{A}(X)$ such that:

$$[Y_i^* \precsim_i Y_i' \ \forall\ i \in \{1, ..., m\}] \wedge [\exists\ \bar{i} \in \{1, ..., m\}\ s.t.\ Y_{\bar{i}}^* \prec_i Y_{\bar{i}}'].$$

Therefore,

$$[(Y_1^*, ..., Y_m^*) \precsim (Y_1', ..., Y_m')] \wedge [\neg((Y_1', ..., Y_m') \precsim (Y_1^*, ..., Y_m^*))]$$

clearly implies that

$$(Y_1^*, ..., Y_m^*) \prec (Y_1', ..., Y_m')$$

. Hence, $(Y_1^*, ..., Y_m^*)$ is not maximal for $\precsim$.

(i) $\Rightarrow$ (ii). By contraposition, consider $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ which is not maximal for $\precsim$. Then there exists $(Y_1', ..., Y_m') \in \mathcal{A}(X)$ such that

$$(Y_1^*, ..., Y_m^*) \prec (Y_1', ..., Y_m')$$

, and this is equivalent to require that

$$[(Y_1^*, ..., Y_m^*) \precsim (Y_1', ..., Y_m')] \;\wedge\; [\neg((Y_1', ..., Y_m') \precsim (Y_1^*, ..., Y_m^*))]$$

with

$$\neg[(Y_1', ..., Y_m') \precsim (Y_1^*, ..., Y_m^*)] = \bigcup_{i=1}^{m} [\neg(Y_i' \precsim Y_i^*)]$$

Hence, there exists $\bar{i} \in \{1, ..., m\}$ such that:

$$[Y_{\bar{i}}^* \precsim_{\bar{i}} Y_{\bar{i}}'] \;\wedge\; [\neg(Y_{\bar{i}}' \precsim_{\bar{i}} Y_{\bar{i}}^*)]$$

clearly implies that

$$Y_{\bar{i}}^* \prec_{\bar{i}} Y_{\bar{i}}'$$

. Therefore we have that

$$[Y_i^* \precsim_i Y_i' \;\forall\; i \in \{1, ..., m\}] \wedge [\exists\; \bar{i} \in \{1, ..., m\} \; s.t. \; Y_{\bar{i}}^* \prec_i Y_{\bar{i}}'].$$

This means that $(Y_1^*, ..., Y_m^*)$ is not Pareto optimal. □

Remark 4.3.9. Please notice that the previous proposition does not require any restrictive assumption on the preorders $\precsim_i$.

4.3.2 Characterization of optimal solutions

Until now we considered Pareto optimality of allocations in $\mathcal{A}(X)$ providing the equivalence between Pareto optimality and maximality with respect to the coalition preorder $\precsim$ on $\mathcal{A}(X)$.

We now extend the previous considerations in order to study the existence of optimal solutions, i.e., we study Pareto optimality in the set $\mathcal{S}$ of all the feasible allocations for which each agent is at least as well as under the initial allocation $(X_1, .., X_m)$ defined as follows:

$$\mathcal{S} = \{(Y_1, ..., Y_m) \in \mathcal{A}(X) \mid (X_1, .., X_m) \precsim (Y_1, ..., Y_m)\} \tag{4.7}$$

As a natural extension of Proposition 4.3.8, we state the equivalence between optimal solutions and maximal elements with respect to the coalition preorder $\precsim$ on $\mathcal{S}$.

Proposition 4.3.10. *For every risk X and for every feasible allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ the following conditions are equivalent:*

(i) $(Y_1^*, ..., Y_m^*)$ *is Pareto optimal and individually rational;*

(ii) $(Y_1^*, ..., Y_m^*)$ *is maximal for $\mathcal{S}$ with respect to the coalition preorder $\precsim = \bigcap_{i=1}^m \precsim_i$.*

Proof. (ii) $\Rightarrow$ (i). By contraposition, consider a feasible allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ which is not individually rational pareto optimal. This implies that $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ could be either pareto optimal (but not individually rational) or not pareto optimal (but individually rational).

If $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ is not pareto optimal (but individually rational), there exists $(Y_1', ..., Y_m') \in \mathcal{A}(X)$ such that:

$$[Y_i^* \precsim_i Y_i' \ \forall\ i \in \{1, ..., m\}] \wedge [\exists\ \bar{i} \in \{1, ..., m\}\ s.t.\ Y_{\bar{i}}^* \prec_i Y_{\bar{i}}'].$$

Since $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ is individually rational, then also $(Y_1', ..., Y_m') \in \mathcal{A}(X)$ is individually rational. Therefore,

$$(Y_1', ..., Y_m') \in \mathcal{S}$$

So, $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ could not be maximal for $\mathcal{S}$ with respect to the coalition preorder $\precsim = \bigcap_{i=1}^m \precsim_i$.

Consider now the case $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ be pareto optimal (but not individually rational). From Proposition 4.3.8 $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ pareto optimal implies that $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ is maximal for $\mathcal{A}(X)$ with respect to the coalition preorder $\precsim = \bigcap_{i=1}^m \precsim_i$. Since $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ is not individually rational, it can't be maximal for $\mathcal{S}$.

(i) $\Rightarrow$ (ii) analogous □

4.3.3 Existence of optimal solutions

Let now $\mathcal{L}_+$ be a *topological vector space*. This is the case of a vector space endowed with a topology which makes the vector operations continuous. Recall that a *normed space* is always a topological vector space when we consider the associated *norm topology*.

From Proposition 4.3.10, we can analyze the existence of maximal elements for the coalition preorder to study the existence of optimal solutions.

In particular, we are going to consider the conditions for the existence of $\precsim$-maximal elements for $\mathcal{S} = \{(Y_1, ..., Y_m) \in \mathcal{A}(X) \mid (X_1, .., X_m) \precsim (Y_1, ..., Y_m)\}$.

The following condition will be assumed:

A1: for every i and every $Z \in \mathcal{A}(X)$, $i_{\precsim_i}(Z) = \{Y \in \mathcal{A}(X) \mid Z \precsim_i Y\}$ is τ-closed (i.e., $\precsim_i$ is *upper semicontinuous* for every i).

The following theorem provides sufficient topological conditions for the existence of optimal solutions, by using the existence of maximal elements for the coalition preorder.

Theorem 4.3.11. *There exists a Pareto optimal and individually rational element* $(Y_1^*, ..., Y_m^*)$ *of* $\mathcal{S} = \{(Y_1, ..., Y_m) \in \mathcal{A}(X) \mid (X_1, .., X_m) \precsim (Y_1, ..., Y_m)\}$, *where* $\precsim = \bigcap_{i=1}^m \precsim_i$ *is the coalition preorder on* $\mathcal{S}$, *provided that* $\precsim_i$ *is an upper semicontinuous preorder for every* i *and the induced topology* $\tau_{\mathcal{S}}^m$ *on* $\mathcal{S}$ *is compact.*

Proof. From Proposition (4.3.10), it suffices to show that the coalition preorder $\precsim = \bigcap_{i=1}^{m} \precsim_i$ on $\mathcal{S}$ has a maximal element. Since $\precsim_i$ is an upper semicontinuous preorder for every i, we have that also the coalition preorder $\precsim$ on $\mathcal{S}$ is upper semicontinuous. Indeed, we have that

$$\begin{aligned} i_{\precsim}(Z) = \quad & i_{\precsim}((Z_1, ..., Z_m)) = \{(Y_1, ..., Y_m) \in \mathcal{S} : (Z_1, ..., Z_m) \precsim (Y_1, ..., Y_m)\} \\ = \quad & \bigcap_{i=1}^{m} i_{\precsim_i}(Z) \end{aligned}$$

is a closed subset of $\mathcal{S}$ for every $Z \in \mathcal{S}$. Therefore, from Theorem 2.3.30, $\precsim$ admits a maximal element. □

Remark 4.3.12. If $\mathcal{L}_+$ is a metric space and a set $\mathcal{A}(X)$ is compact, we have that $\mathcal{A}(X)$ is a compact metric space when we consider the *product metric.* Therefore $\mathcal{A}(X)$ is separable, or equivalently second countable (see e.g.Engelking[29]).

Remark 4.3.13. Under particular assumptions, it is possible to apply the Banach Alaoglu Theorem (see 2.3.28), to provide the compactness of $\mathcal{S}$, as stated in the following corollary.

Corollary 4.3.14. *The following condition will be assumed:*

A1: for every i and every $Z \in \mathcal{A}(X)$, $i_{\precsim_i}(Z) = \{Y \in \mathcal{A}(X) \mid Z \precsim_i Y\}$ is closed in the weak topology $\sigma(L^\infty, L^1)$.*

A2: $\mathcal{A}(X)$ is closed in the weak topology $\sigma(L^\infty, L^1)$.*

Then, $\mathcal{S}$ is compact.

Proof. The set

$$\mathcal{A}(X)|_i = \{Y_i \in \mathcal{L}_+ : \exists \ (Y_1, .., Y_{i-1}, Y_{i+1}, .., Y_m) \ s.t. \ (Y_1, .., Y_m) \in \mathcal{A}(X)\}$$

is a subset of the closed ball

$$B(0, \parallel X \parallel)$$

that is compact in the weak* topology $\sigma(L^\infty, L^1)$ by the Banach Alaoglu Theorem (see 2.3.28).

The Thichonof theorem guarantees also the compactness in the product topology. The set

$$\mathcal{S} = \{(Y_1, ..., Y_m) \in \mathcal{A}(X) \mid (X_1, .., X_m) \precsim (Y_1, ..., Y_m)\}$$

is clearly a subset of $\mathcal{A}(X)$. Since $\precsim_i$ is $\sigma(L^\infty, L^1)$ upper semicontinuous preorder for every i, then also $\precsim$ on $\mathcal{S}$ is $\sigma(L^\infty, L^1)$ upper semicontinuous preorder. Then, $\mathcal{S}$ is a closed subset of a compact set, then it is compact. □

Remark 4.3.15. It is clear that if we assume the compactness of $\mathcal{A}(X)$ instead of $\mathcal{S}$, Theorem 4.3.11 holds since $\mathcal{S}$ is a closed subset of the compact set $\mathcal{A}(X)$.

4.3.4 The multi-objective maximization problem

Let us now introduce the so called *multi-objective maximization problem* associated to m real-valued functions $U_1, ..., U_m$ (see e.g. Kaminski[41])).

Definition 4.3.16. A solution to the problem

$$\begin{aligned} &\sup \ (U_1(Y_1), U_2(Y_2), ..., U_m(Y_m)) \\ &\text{sub} \\ &(Y_1, .., Y_m) \in \mathcal{S} \end{aligned} \tag{4.8}$$

is $(Y_1^*, ..., Y_m^*)$ provided that one of the following equivalent conditions hold:

1. for all $(Y_1, .., Y_m) \in \mathcal{S}$, $U_i(Y_i) \geq U_i(Y_i^*)$ for all $i \in \{1, ..., m\}$ imply $U_i(Y_i) = U_i(Y_i^*)$ for all $i \in \{1, ..., m\}$;

2. for no $(Y_1, .., Y_m) \in \mathcal{S}$ it holds that $U_i(Y_i) \geq U_i(Y_i^*)$ for all $i \in \{1, ..., m\}$ with at least one strict inequality;

3. for all $(Y_1, .., Y_m) \in \mathcal{S}$, if $U_i(Y_i) > U_i(Y_i^*)$ for some $i \in \{1, ..., m\}$, then there exists some $j \in \{1, ..., m\}$ such that $U_j(Y_j) < U_j(Y_j^*)$.

In the following proposition we are going to use the previous concept in order to produce sufficient conditions for the existence of an optimal solution in the risk sharing setting.

Proposition 4.3.17. *Let U_i be an order-preserving function for the individual preorder $\precsim_i$ ($i \in \{1,...,m\}$) on $\mathcal{S}$. Then the following statements are valid:*

1. *If $(Y_1^*,...,Y_m^*)$ is a solution to the problem (4.8), then it is maximal for S with respect to the coalition preorder $\precsim = \bigcap_{i=1}^m \precsim_i$;*

2. *If $\precsim_i$ is a total preorder for all $i \in \{1,...,m\}$, then an optimal solution $(Y_1^*,...,Y_m^*)$ is a solution to the problem (4.8).*

Proof. We prove statement 1. by contraposition. Assume that $(Y_1^*,...,Y_m^*)$ is not maximal for S with respect to the coalition preorder $\precsim = \bigcap_{i=1}^m \precsim_i$. Then there exists $(Y_1',...,Y_m') \in \mathcal{S}$ such that

$$(Y_1^*,...,Y_m^*) \prec (Y_1',...,Y_m')$$

, and this is equivalent to require that

$$[(Y_1^*,...,Y_m^*) \precsim (Y_1',...,Y_m')] \ \wedge \ [\neg((Y_1',...,Y_m') \precsim (Y_1^*,...,Y_m^*))]$$

with

$$\neg[(Y_1',...,Y_m') \precsim (Y_1^*,...,Y_m^*)] = \bigcup_{i=1}^m [\neg(Y_i' \precsim Y_i^*)]$$

Hence, there exists $\bar{i} \in \{1,...,m\}$ such that:

$$[Y_{\bar{i}}^* \precsim_{\bar{i}} Y_{\bar{i}}'] \ \wedge \ [\neg(Y_{\bar{i}}' \precsim_{\bar{i}} Y_{\bar{i}}^*)]$$

clearly implies that

$$Y_{\bar{i}}^* \prec_{\bar{i}} Y_{\bar{i}}'$$

. Therefore we have that

$$[Y_i^* \precsim_i Y_i' \ \forall \ i \in \{1,...,m\}] \wedge [\exists \ \bar{i} \in \{1,...,m\} \ s.t. \ Y_{\bar{i}}^* \prec_i Y_{\bar{i}}'].$$

Therefore, since U_i is an order-preserving function for $\precsim_i$ for all $i \in \{1, ..., m\}$, it is clear that

$$[U_i(Y_i^*) \leq U_i(Y_i') \; \forall \; i \in \{1, ..., m\}] \wedge [\exists \; \bar{i} \in \{1, ..., m\} \; s.t. \; U_{\bar{i}}(Y_{\bar{i}}^*) < U_{\bar{i}}(Y_{\bar{i}}')]$$

contradicting the fact that $(Y_1^*, ..., Y_m^*)$ is a solution to the problem (4.8).
Statement 2. will be also proved by contraposition. Assume that $\precsim_i$ is a total preorder for all $i \in \{1, ..., m\}$ and that $(Y_1^*, ..., Y_m^*)$ is not a solution to the problem (4.8). Then

$$[U_i(Y_i^*) \leq U_i(Y_i') \; \forall \; i \in \{1, ..., m\}] \wedge [\exists \; \bar{i} \in \{1, ..., m\} \; s.t. \; U_{\bar{i}}(Y_{\bar{i}}^*) < U_{\bar{i}}(Y_{\bar{i}}')]$$

Since U_i is in this case a utility function for $\precsim_i$ for all $i \in \{1, ..., m\}$, we have that

$$[Y_i^* \precsim_i Y_i' \; \forall \; i \in \{1, ..., m\}] \wedge [\exists \; \bar{i} \in \{1, ..., m\} \; s.t. \; Y_{\bar{i}}^* \prec_i Y_{\bar{i}}'].$$

contradicting the fact that $(Y_1^*, ..., Y_m^*)$ is optimal. This consideration completes the proof. □

From Proposition 4.3.10 we know that if $(Y_1^*, ..., Y_m^*)$ is maximal for $\mathcal{S}$ with respect to the coalition preorder $\precsim = \bigcap_{i=1}^m \precsim_i$ then it is individually rational pareto optimal.

So we can readapt Proposition 4.3.17 in this way:

Proposition 4.3.18. *Let U_i be an order-preserving function for the individual preorder $\precsim_i$ ($i \in \{1, ..., m\}$) on $\mathcal{S}$. Then the following statements are valid:*

1. *If $(Y_1^*, ..., Y_m^*)$ is a solution to the problem (4.8), then it is an optimal solution;*

2. *If $\precsim_i$ is a total preorder for all $i \in \{1, ..., m\}$, then an optimal solution $(Y_1^*, ..., Y_m^*)$ is a solution to the problem (4.8).*

The following corollary concerning the case of total preorders and the corresponding utilities is immediate and we omit its proof.

Corollary 4.3.19. *Let $\precsim_i$ be a total preorder for all $i \in \{1, ..., m\}$ and let U_i be a utility function for $\precsim_i$ for all $i \in \{1, ..., m\}$. Then the following statements are equivalent:*

1. $(Y_1^*, ..., Y_m^*)$ *is a solution to the problem (4.8);*
2. $(Y_1^*, ..., Y_m^*)$ *is an optimal solution.*

From Proposition 4.3.6 we know that in the case of individual total preorders with translation invariant utilities, determining Pareto optimal allocations is in fact equivalent to determining optimal allocations for every choice of the initial exposures.

So, from Proposition 4.3.6 and Corollary 4.3.19, we can readapt Proposition 4.3.18 for the case of individual total preorders with translation invariant utilities.

We start modifying the multi-objective optimization problem restricted on $\mathcal{A}(X)$:

Definition 4.3.20.

$$\begin{aligned}
&\sup\ (U_1(Y_1), U_2(Y_2), ..., U_m(Y_m)) \\
&\text{sub} \\
&(Y_1, .., Y_m) \in \mathcal{A}(X)
\end{aligned} \tag{4.9}$$

Then we have the following Proposition:

Proposition 4.3.21. *Let $\precsim_i$ be a total preorders with translation invariant utility U_i, ($i \in \{1, ..., m\}$) on $\mathcal{A}(X)$. Then the following statements are equivalent:*

1. $(Y_1^*, ..., Y_m^*)$ *is a solution to the problem (4.9);*
2. $(Y_1^*, ..., Y_m^*)$ *is an optimal solution.*

As we will see in the following chapter, the problem of multi-objective maximization can be traduced to that of maximizing a single function, and under particular conditions, the two problems coincide.

5

The sup-convolution problem

In this chapter we introduce the sup-convolution problem, strictly related to the multi-objective maximization problem analyzed in the previous chapter. We characterize optimal solutions for both not necessarily total and total preorders. Then, we study the case of agents making a choice over different risky outcomes and then sharing the risks of the selected outcome.

5.1 Introduction

The *sup-convolution problem* is of help since it allows us to reduce the research of optimal solutions to the maximization of a single function. In particular, in the risk sharing context, the problem is to characterize optimal allocations of a risk X to the m agents under all of the feasible allocations of X, i.e. under all decompositions $(Y_1, ..., Y_m)$ such that $X = \sum_{i=1}^{m} Y_i$.

Since it is of interest the characterization of optimal solutions, we can formulate the *sup-convolution problem* restricting our attention on the set $\mathcal{S}$ of all the feasible allocations for which each agent is at least as well as under the initial exposure.

Definition 5.1.1 (sup-convolution problem)**.** The sup-convolution problem relative to the functions $U_1, ..., U_m$ on $\mathcal{S}$ is defined as follows

$$U_1 \square U_2 \square ... \square U_m(Y_1, ..., Y_m) = \sup\{\sum_{i=1}^{m} U_i(Y_i) | X = \sum_{i=1}^{m} Y_i.\} \tag{5.1}$$

The sup-convolution problem supports a rich literature, see for instance Harsanyi,[37] Wilson,[51] Rubinstein,[48] Borch,[14] Aase,[1] Filipovic and Kupper,[31] Burgert and Rüschendorf,[16] Barrieu and El Karoui,[9] Jouni,[38] Barrieu and Scandolo.[10]

In such a context explained by Definition 5.1.1, the risk sharing problem is related to the maximization of the "overall" utility of the m agents by some kind of exchange contracts. There are several cases in literature concerning the sup-convolution of utility functions in a risk sharing context, for example Filipovic and Kupper[31] and Jouini et al[38] studied the problem of sup-convolution in the case of monetary utility functions.

The sup-convolution problem is equivalently reduced to a *inf-convolution problem* in the case where agents preferences are expressed by risk-functions ρ_i, generally risk measures. In such a context, the inf-convolution problem represents the value of the optimal risk-allocation problem formulation, interpreting the problem of minimazing the total risk of a risk sharing contract.

Definition 5.1.2 (inf-convolution problem). The inf-convolution problem relative to the functions $\rho_1, ..., \rho_m$ on $\mathcal{S}$ is defined as follows

$$\rho_1 \Box \rho_2 \Box ... \Box \rho_m(Y_1, ..., Y_m) = \inf\{\sum_{i=1}^{m} \rho_i(Y_i) | X = \sum_{i=1}^{m} Y_i.\} \tag{5.2}$$

In literature we can find several cases concerning the problem of reducing a maximization of "overall" utility to a inf-convolution setting.

In particular, Barrieu and El Karoui[9] studied the problem of maximizing the aggregate expected utility of two agents having access to a financial market to reduce their risk. This problem is equivalently traduced in a more general framework involving convex risk measures and their inf-convolution.

Similarly, Burgert and Rüschendorf[16] studied the optimal risk allocation problem or equivalently the problem of risk sharing with m agents endowed with risk measures $\{\rho_1, .., \rho_m\}$, in particular convex risk measures and their inf-convolution. The problem of minimizing the total risk of a risk sharing contract can be considered as an optimistic attitude towards risk, typical for insurance and reinsurance contracts. As opposite, Burgert and Rüschendorf[16] considered also the case of a "cautious" risk attitude where the problem of optimal risk allocation is reduced to maximizing the total risk in the worst case. In other terms, from a regulatory point of view, the risk measures should be chosen by the traders (agents) in a most cautious way in order not to underestimate the whole risk.

In the following section we are going to study the sup-convolution problem to characterize optimal solutions for the risk-sharing setting, in particular considering the equivalence between solutions to the sup-convolution problem and the solutions of the multi-objective optimization problem defined in the previous chapter.

We start considering agents preferences expressed by not necessarily total preorders. Then we consider optimal solutions considering agents preferences represented by total preorders. In particular, as we have already justified in the previous chapter, the case of individual total preorders $\precsim_i$ with translation invariant utility functions U_i is particularly favorable since it characterize the equivalence between pareto optimal allocations and optimal solutions and guarantees that U_i is an upper semicontinuous utility function for an upper semicontinuous total preorder.

In the last section we study the problem of risk sharing in the presence of different risky outcomes. This problem is of interest for example in the case of building-projects selection exposed to catastrophic events. In such a context, the m agents select different risky outcomes and then share the potential risks of the selected project. This problem will be referred to the existence of maximal elements for a not necessarily total coalition preorder. Under particular assumptions that guarantee the existence of the sup-convolution for every risky outcome, the coalition preorder is total and the related utility function is the associated sup-convolution.

5.2 Characterization of optimal solutions

In this section we are going to characterize optimal solutions based on the solution of the sup-convolution problem.

We start considering agents preferences endowed with not necessarily total preorders and we exploit the equivalence between the sup- convolution problem and the multi-objective optimization problem of the previous chapter in the case when the individual preorders are expressed by order preserving functions.

As a natural extension of the latter considerations, we consider the case of individual preorders expressed by upper semicontinuous order preserving functions or a finite upper semicontinuous multi-utility representation.

Then we study the case of agents preferences endowed with total preorders, guaranteeing the existence of upper semicontinuous utility functions for the upper semicontinuous total preorders.

5.2.1 Optimal solutions for not necessarily total preorders

In this paragraph we consider the correlations between the characterization of optimal solutions for not necessarily total preorders obtained in the previous chapter and the sup-convolution problem introduced in the thesis-introduction.

It is clear that a solution to the sup-convolution problem 5.1.1 is also a solution to the multi-objective optimization problem 4.8. From Proposition 4.3.18, statement 1, and this latter consideration, we get the following proposition.

Proposition 5.2.1. *If U_i is an order-preserving function for $\precsim_i$ for every $i \in \{1, ..., m\}$, then a solution $(Y_1^*, ..., Y_m^*)$ to the sup-convolution problem (5.1) is optimal.*

A simple characterization of an order preserving function based on the Choquet integral is provided in the following example (see Bosi and Zuanon).[15])

Example 5.2.2. *Consider the following example concerning decision theory under uncertainty. Let $\mathbf{M} = \{\mu_n : n \in \{1, \ldots, n^*\}\}$ be a finite family of* concave capacities *on a measurable space $(\Omega, \mathcal{A})$, with Ω the* state space*, and $\mathcal{A}$ a* σ-algebra *of subsets of Ω. We recall that a* capacity *μ on $\mathcal{A}$ (i.e., a function from $\mathcal{A}$ into $[0, 1]$ such that $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$, and $\mu(A) \leq \mu(B)$ for all $A \subseteq B$, $A, B \in \mathcal{A}$) is said to be concave if for all sets $A, B \in \mathcal{A}$,*

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$$

(see e.g. Chateauneuf[19]). Consider the normed space $L^1(\Omega, \mathcal{A}, \mu_n)$ of all the real random variables x such that the Choquet integral

$$\int_\Omega x d\mu = \int_0^\infty \mu(\{x \geq t\})dt + \int_{-\infty}^0 (\mu(\{x \geq t\}) - 1)dt$$

is finite (see e.g. Denneberg[27]). Define a binary relation on L^1 as follows:

$$Y_i \precsim_i Y_i' \text{ if and only if } \int_\Omega Y_i d\mu_n \leq \int_\Omega Y_i' d\mu_n \text{ for all } n \in \{1, \ldots, n^*\}.$$

It is clear that $\precsim_i$ is a preorder and that $\precsim_i$ is not complete in general. Then the real-valued function u defined by

$$u(x) = \sum_{n=1}^{n^*} \int_\Omega Y_i d\mu_n$$

is an order-preserving function *for $\precsim_i$.*

A simple adaptation of the arguments above leads to the following proposition concerning the case when every individual preorder has a finite multi-utility representation.

Proposition 5.2.3. *Assume that for every $i \in \{1, ..., m\}$ there exists a finite multi-utility representation $\mathcal{U}_i = \{U_{i,j} : j = 1, ..., k_i\}$ for $\precsim_i$. Then a solution to the sup-convolution problem*

$$U_{1,1}\square...\square U_{1,k_1}\square U_{2,1}...\square U_{m,1}\square...\square U_{m,k_m}(Y_1, ..., Y_m) = \\ = \sup \sum_{i=1}^{m}\sum_{j=1}^{k_i} U_{i,j}(Y_i); \tag{5.3}$$

is a Pareto optimal allocation.

Proof. By contraposition, let the solution $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ of the sup-convolution problem be not Pareto Optimal. Then there exists an allocation $(Z_1, ..., Z_m) \in \mathcal{A}(X)$ such that $Y_i^* \precsim_i Z_i$ for $i \in \{1, .., m\}$ and $Y_i^* \prec_i Z_i$ for some i, with:

$$Y_i^* \precsim_i Z_i \Leftrightarrow U_{i,j}(Y_i^*) \leq U_{i,j}(Z_i) \ j \in \{1, ..., k_i\}$$

and

$$Y_i^* \prec_i Z_i \Leftrightarrow U_{i,j}(Y_i^*) \leq U_{i,j}(Z_i) \quad for\ j \in \{1, ..., k_i\} \quad and \quad U_{i,\bar{j}}(Y_i^*) < U_{i,\bar{j}}(Z_i).$$

Hence,

$$\sum_{i=1}^{m}\sum_{j=1}^{k_i} U_{i,j}(Z_i) > \sum_{i=1}^{m}\sum_{j=1}^{k_i} U_{i,j}(Y_i^*)$$

that is a contraddiction because $(Y_1^*, ..., Y_m^*)$ is the solution of the sup-convolution problem. □

Remark 5.2.4. A finite multi-utility (or equivalently multi-risk) setting is useful to fully describe agents behaviour over multiple regulatory requirements. In such a context, in fact, each agent is equipped with multiple individual functions where some of them may reflect her own preferences and other are regulatory requirements.

Example 5.2.5. *Note that Example 5.2.2 outlines a finite multi-utility representation for the preorder $\precsim_i$ based on a finite family of concave capacities. In* *fact if we call*

$$U_{i,j} = \int_\Omega Y_i d\mu_j$$

Then we have:

$$Y_i \precsim_i Y_i' \Leftrightarrow U_{i,j}(Y_i) \leq U_{i,j}(Y_i') \quad \forall\, U_{i,j} \in \mathcal{U}_i$$

In the particular case when the distortion function is the identity function, the previous finite multi-utility representation is in particular a finite expected multi-utility representation. For a complete characterization of the existence of an expected multi-utility representation see Dubra et al.[28] *and Evren.*[30]

We want now to use the previous considerations in parallel with the assumption of compactness of the set $\mathcal{S}$ of all the feasible allocations for which each agent is at least as well as under the initial exposure in order to characterize the existence of optimal solutions by using the sup-convolution.

As an easy consequence of Proposition 5.2.1 and Theorem 4.3.11, since the sum of upper semicontinuous functions is itself upper semicontinuous, we have that the following proposition holds.

Proposition 5.2.6. *If for every $i \in \{1, ..., m\}$ there exists an upper semicontinuous order-preserving function U_i for $\precsim_i$, then there exists an optimal solution that is obtained as a solution $(Y_1^*, ..., Y_m^*)$ to the sup-convolution problem (5.1), provided that the induced topology $\tau_{\mathcal{S}}^m$ on $\mathcal{S}$ is compact.*

Further, the following proposition holds, which concerns the case when every individual prorder has a finite upper semicontinuous multi-utility representation.

Proposition 5.2.7. *If for every $i \in \{1, ..., m\}$ there exists a finite upper semicontinuous multi-utility representation $\mathcal{U}_i = \{U_{i,j} : j = 1, ..., k_i\}$ for $\precsim_i$, then there exists an optimal solution that is obtained as a solution $(Y_1^*, ..., Y_m^*)$ to the sup-convolution problem*

$$\begin{aligned} &U_{1,1}\square ... \square U_{1,k_1}\square U_{2,1}\square ... \square U_{2,k_2}\square .. \square U_{m,1}\square ... \square U_{m,k_m}(Y_1, ..., Y_m) = \\ &= \sup \sum_{i=1}^{m} \sum_{j=1}^{k_i} U_{i,j}(Y_i), \end{aligned} \tag{5.4}$$

provided that the induced topology $\tau_{\mathcal{S}}^m$ *on* $\mathcal{S}$ *is compact.*

5.2.2 Optimal solutions for total preorders

As we have already justified in the previous chapter, the case of individual total preorders $\precsim_i$ with translation invariant utility functions U_i is particularly favorable since it characterize the equivalence between pareto optimal allocations and optimal solutions.

If in addition we consider the case that the utility function U_i is also comonotone superadditive, then Pareto optima and the solutions to the sup-convolution problem coincide, in this way completing a well known result from the literature according to which the two problems conicide in case that all the functions are translation invariant (see for instance Acciaio[2]). Indeed, the following theorem holds true.

Proposition 5.2.8. *Assume that* $\precsim_i$ *is a total preorder for every* $i \in \{1, ..., m\}$ *Then the following conditions are equivalent:*

1. $(Y_1^*, ..., Y_m^*)$ *is a solution to the sup-convolution problem (5.1);*
2. $(Y_1^*, ..., Y_m^*)$ *is a Pareto optimal allocation.*

provided for every $i \in \{1, ..., m\}$ *there is a utility function* U_i *for* $\precsim_i$ *satisfying one of the following two conditions:*

1. U_i *is translation invariant;*
2. U_i *is comonotone superadditive, positively homogeneous and normalized.*

Proof. 1. $\Rightarrow$ 2.. Obvious.

2. $\Rightarrow$ 1.. By contraposition, assume that $(Y_1^*, ..., Y_m^*)$ is not a solution to the sup-convolution problem (5.4). Therefore, there exists a feasible allocation $(Y_1', ..., Y_m')$ such that

$$\sum_{i=1}^{m} U_i(Y_i^*) < \sum_{i=1}^{m} U_i(Y_i').$$

Define $\alpha = \sum_{i=1}^{m} U_i(Y_i') - \sum_{i=1}^{m} U_i(Y_i^*)$. Further, define, for all indexes i, $\alpha_i = U_i(Y_i') - U_i(Y_i^*)$. Let κ be the cardinality of $\{i : U_i(Y_i') - U_i(Y_i^*) \neq 0\}$.

Consider now a new feasible allocation $(Y_1'', ..., Y_m'')$ defined as follows:

$$Y_i'' = \begin{cases} Y_i' & \text{if } \alpha_i = 0 \\ Y_i' - \alpha_i + \frac{\alpha}{\kappa} & \text{if } \alpha_i \neq 0 \end{cases}.$$

Then we have that

$$U_i(Y_i'') = \begin{cases} U_i(Y_i^*) = U_i(Y_i') & \text{if } \alpha_i = 0 \\ U_i(Y_i' - \alpha_i + \frac{\alpha}{\kappa}) \geq U_i(Y_i^*) + \frac{\alpha}{\kappa} > U_i(Y_i^*) & \text{if } \alpha_i \neq 0 \end{cases}.$$

Therefore the allocation $(Y_1^*, ..., Y_m^*)$ is not Pareto optimal. □

Example 5.2.9. *As we have already justified in chapter 3, a classical example of translation invariant, comonotone superadditive, positively homogeneous and normalized functional U on $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ is provided by the Choquet integral with respect to a convex probability distortion (i.e., with respect to $\mu = g \circ \mathcal{P}$ with g convex).*

The following theorem also concerns the case of translation invariant total preorders with translation invariant utilities. We show that in this case the upper semicontinuity of the individual preorders implies the upper semicontinuity of the utilities.

Theorem 5.2.10. *Let $\precsim$ be an upper semicontinuous total preorder on $\mathcal{L}$. If U is any translation invariant utility function for $\precsim$, then U is upper semicontinuous.*

Proof. Let U be any translation invariant utility function for $\precsim$, and consider any $X \in \mathcal{L}$ and $\alpha > 0$ such that $U(X) < \alpha$. The proposition is proved as soon as we are able to find an open subset O of X such that $U(Z) < \alpha$ for all $Z \in O$. Then there exists $c \in \mathbb{R}$ such that

$$U(X) < U(X) + c < \alpha$$

, which, from translation invariance of U, is equivalent to

$$U(X) < U(X+c) < \alpha$$

. Hence,

$$l_{\precsim}(X+c) = \{Z \in \mathcal{L} \mid Z \prec X+c\}$$

is an open subset of $\mathcal{L}$ such that $U(Z) < U(X+c) < \alpha$ for every $Z \in l_{\precsim}(X+c)$. This consideration finishes the proof. □

As a consequence of Proposition 5.2.8 and Theorem 5.2.10, we get the following nice result.

Theorem 5.2.11. *Assume that, for every $i \in \{1, ..., m\}$, the preorder $\precsim_i$ is total, translation invariant and upper semicontinuous. If U_i is any translation invariant utility function for $\precsim_i$ ($i \in \{1, ..., m\}$, and the feasible set $\mathcal{A}(X)$ is compact, then for any initial allocation $(X_1, ..., X_m) \in \mathcal{A}(X)$ the set of all the optimal solutions is nonempty and it coincides with the solution of the associated sup-convolution problem*

$$U_1 \square U_2 \square ... \square U_m(Y_1, ..., Y_m) = \sup \sum_{i=1}^{m} U_i(Y_i). \tag{5.5}$$

.

We end this paragraph considering the particularly favorable case of assuming the feasible set $\mathcal{A}(X)$ to be a metric space. This allows us to guarantee that upper semicontinuous total preorder on a subset of $A(X)$ admits an upper semicontinuous utility representation.

Theorem 5.2.12. *Assume that, for every $i \in \{1, ..., m\}$, the preorder $\precsim_i$ is total, upper semicontinuous on S and $\mathcal{A}(X)$ is a compact metric space. Then every preorder $\precsim_i$ admits an upper semicontinuous utility representation U_i and every solution of the problem*

$$U_1 \square U_2 \square ... \square U_m(Y_1, ..., Y_m) = \sup \sum_{i=1}^{m} U_i(Y_i). \tag{5.6}$$

is optimal.

Proof. $\mathcal{A}(X)$ is a compact metric space, and therefore it is in particular a separable metric space (see e.g. Engelking [29 Theorem 4.1.18]). Then the subset $\mathcal{S}$ of $\mathcal{A}(X)$ can be metrized as a separable metric space, and therefore as a second countable metric space (see e.g. Engelking [29 Corollary 4.1.16]). Since each preorder $\precsim_i$ is an upper semicontinuous total preorder on $\mathcal{S}$, then $\precsim_i$ admits an upper semicontinuous utility function U_i by Rader's theorem (see Rader [46 Theorem 1]). Therefore, every solution to the sup-convolution problem is optimal. □

Remark 5.2.13. The ball $B(0, \|X\|)$ is metrizable in the weak* topology. (see e.g. J.B. Conway[20] Exercise 4, p.136 on else the proof of 6.34 Theorem, p.254, in Aliprantis and Border[5])

5.2.3 The inf-convolution problem

We finish this section by observing that Pareto optimal risk sharing is consider from the point of view of *risk minimization* better that *utility maximization*. This means that individual risk measures ρ_i are considered instead of individual utilities U_i. In order to use the previous arguments and results, we only have to define $U_i = -\rho_i$ $(i = 1, ..., m)$.

In this framework, the definition of a *Pareto optimal allocation under risk* is perfectly symmetrical with respect to the definition of a Pareto optimal allocation (see Definition 4.2.2).

Definition 5.2.14 (Pareto optimal allocation under risk)**.** An allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ is said to be *Pareto optimal under risk* if for no other allocation $(Y_1, ..., Y_m) \in \mathcal{A}(X)$ it occurs that $Y_1 \precsim_i Y_1^*, ..., Y_m \precsim_m Y_m^*$ with at least one index i such that $Y_1 \prec_i Y_i^*$.

Finally, the sup-convolution problem 5.1.1 is therefore replaced by the following *inf-convolution problem.*

Definition 5.2.15 (inf-convolution problem)**.** *The inf-convolution problem* relative to the functions $\rho_1, ..., \rho_m$ on $\mathcal{S}$ is defined as follows

$$\rho_1 \Box \rho_2 \Box ... \Box \rho_m(Y_1, ..., Y_m) = \inf \sum_{i=1}^{m} \rho_i(Y_i). \tag{5.7}$$

As we have anticipated in the previous introduction, there is an extensive literature related to risk-redistribution problems obtained by inf-convolution of the associated risk measures of the m agents. The introduction of the Basel II regulation and the Swiss Solvency Test (SST), in fact, has increased the use of risk measures to evaluate financial or insurance risk.

In particular, the inf-convolution problem can traduce the maximization of the overall utility of the agents. For example in a risk-redistribution context where agents preferences are represented by distortion risk measures, each distortion risk measure can represented as a vN-M expected utility function if and only if the distortion function is given by the identity function, i.e., if the risk measure is risk-neutral.

In such a context (see for instance Boonen[13]), given a finite number m of agents (agencies), each of them is endowed with a risk X_i and a distortion risk measure ρ_i that depends on a distortion function g_i. The set of risk redistribution is as usual the set of all the feasible shares $(Y_1, ..., Y_m)$ such that $\sum X_i = \sum Y_i$. The set of risk redistribution allows for example proportional or stop loss contract on the aggregate risk.

Following definition 5.2.15, a risk redistribution is called Pareto optimal if there does not exist another feasible redistribution that is weakly better for all firms, and strictly better for at least one firm. Then, as usual in the inf-convolution problems, the set of Pareto optimal risk redistributions is given by the set of all feasible risk redistributions such that the aggregate risk value is minimal.

It is clear that in the case of Pareto optimality under risk we state the equivalence between optimal allocations under risk and minimal elements with respect to the coalition preorder. Therefore, in this case we need to provide the lower semicontinuity of the individual preorders.

5.3 Preferences over different risky outcomes

5.3.1 Introduction

Until now we considered the case of m agents (agencies) partecipating in sharing a risk X and we reduced the problem of characterizing optimal solutions to that of maximizing a single function (the sup convolution problem).

We are now interested on considering the case in which agents have to make a choice between different risky outcomes, and then share the risks of the selected outcome. For example, consider the case of cooperative investments projects in construction or agriculture under the risks of natural hazards and disasters. In this case, agents with different risky attitudes have to make a choice over different risky projects and then share the potential losses (earthquakes, hurricanes,..) of the selected project.

For every risk X^h on $\mathcal{L}_+$, we define as usual the set of all the possible feasible allocations of X^h shared by the m agents.

Definition 5.3.1 (feasible allocations)**.** For every risk X^h, denote by $\mathcal{A}(X^h)$ the set of all the X^h-*feasible allocations* of X^h, i.e. the set

$$\mathcal{A}(X^h) = \{(Y_1, ..., Y_m) \mid X^h = \sum_{j=1}^{m} Y_j\}. \tag{5.8}$$

It is necessary now to define a way the coalition of m agents make a decision over different risks comparing all the possible feasible allocations of the risks.

5.3.2 Coalition preorder

We shall refer to the work of Grechuk et al.[36] modifying the definition of coalition preorder to incorporate the social preorder (4.3). Given an arbitrary set of random variables(risky outcomes) from $\mathcal{L}_+$ we can now introduce a preference relation $\precsim_C$ for the coalition of m-agencies in this way:

Definition 5.3.2 (coalition preorder over risky outcomes)**.** Given two risky-outcomes X^1 and X^2, we say that $X^1 \precsim_C X^2$ if for every X^1-feasible allocation $(Y_1^1, ..., Y_m^1) \in \mathcal{A}(X^1)$ there exists a X^2-feasible allocation $(Y_1^2, ..., Y_m^2) \in \mathcal{A}(X^2)$ such that $Y_i^1 \precsim_i Y_i^2$ for all $i \in \{1, ..m\}$.

It is clear that

$$(X^1 \prec_C X^2) \Leftrightarrow (X^1 \precsim_C X^2) \wedge \neg(X^2 \precsim_C X^1)$$

where $\neg(X^2 \precsim_C X^1)$ implies that there exists a X^2 feasible allocation $(Y_1^2, ..., Y_m^2) \in \mathcal{A}(X^2)$ that is not dominated by any X^1-feasible allocation $(Y_1^1, ..., Y_m^1) \in \mathcal{A}(X^1)$. So, we say $(X^1 \prec_C X^2)$ if for every X^1-feasible allocation $(Y_1^1, ..., Y_m^1) \in \mathcal{A}(X^1)$ there exists a X^2-feasible allocation $(Y_1^2, ..., Y_m^2) \in \mathcal{A}(X^2)$ such that $(Y_1^1, ..., Y_m^1) \precsim (Y_1^2, ..., Y_m^2)$ and there exists a X^1-feasible allocation $(Y_1^1, ..., Y_m^1) \in \mathcal{A}(X^1)$ and a X^2-feasible allocation $(Y_1^2, ..., Y_m^2) \in \mathcal{A}(X^2)$ such that $(Y_1^1, ..., Y_m^1) \prec (Y_1^2, ..., Y_m^2)$.

By definition, $\precsim_C$ is clearly not total in general, so we can traduce the problem of making a choice between different risky outcomes by studying the conditions that guarantee the existence of maximal elements for a not necessary total preorder, i.e. the coalition preorder $\precsim_C$.

It is possible now to define a maximal element with respect to the coalition preorder $\precsim_C$.

Definition 5.3.3 (maximal elements)**.** We say that X^h is maximal with respect to the coalition preorder $\precsim_C$, if for no other X^k it occurs that:

$$X^h \prec_C X^k$$

It is clear that a maximal element X^h is the best preferred risky outcome by the coalition of m agents.

It is intuitive to define a comparison between different risky outcomes considering the associated optimal solution provided optimal solution exists. In fact we know that, for every X^h, a solution of the sup-convolution problem

$$U_C(X^h) = \sup \sum_{i=1}^{m} U_i(Y_i^i)$$
$$\text{sub}$$
$$(Y_1^h, .., Y_m^h) \in \mathcal{A}(X^h) \tag{5.9}$$

is an optimal allocation for X^h.

So, we can now introduce the following optimization problem:

$$sup\ U_C(X) \tag{5.10}$$

A solution of (5.10) is the outcome $X^h \in \mathcal{L}_+$ which has the greatest sup-convolution.

In the following proposition we are going to use the previous concept in order to find a maximal element for the coalition preorder $\precsim_C$.

Proposition 5.3.4. *Let U_i be a translation invariant utility function for the individual total preorder $\precsim_i$ ($i \in \{1, ..., m\}$). Then the following statements are equivalent:*

1. *X^h is a solution to the problem (5.10);*
2. *X^h is maximal w.r.t. the coalition preorder $\precsim_C$.*

Proof. $2 \Rightarrow 1$. Assume that X^h is not a solution to the problem (5.10). Then there exists a risky outcome X^k such that:

$$U_C(X^h) < U_C(X^k)$$

This implies that there exists a X^h-feasible allocation $(Y_1^h, ..., Y_m^h) \in \mathcal{A}(X^h)$ and a X^k-feasible allocation $(Y_1^k, ..., Y_m^k) \in \mathcal{A}(X^k)$ such that:

$$\sum_{i=1}^{m} U_i(Y_i^h) < \sum_{i=1}^{m} U_i(Y_i^k)$$

It is possible now to define a X^k-feasible allocation $(Z_1, ...Z_m)$ that dominates $(Y_1^h, ..., Y_m^h)$, clearly implying X^h is not maximal w.r.t the coalition preorder $\precsim_C$:

$$Z_i = Y_i^k - U_i(Y_i^k) + U_i(Y_i^h) + \frac{\sum_{i=1}^m U_i(Y_i^k) - \sum_{i=1}^m U_i(Y_i^h)}{m} \quad i \in \{1, ..., m\}$$

Note that $\sum_{i=1}^m Z_i = \sum_{i=1}^m Y_i^k = X^k$. Then,

$$U_i(Z_i) = U_i(Y_i^h) + \frac{\sum_{i=1}^m U_i(Y_i^k) - \sum_{i=1}^m U_i(Y_i^h)}{m} > U_i(Y_i^h) \quad i \in \{1, ..., m\}$$

Since U_i is a utility function for the total preorder $\precsim_i$, we have that:

$$U_i(Z_i) > U_i(Y_i^h) \Leftrightarrow Y_i^h \prec_i Z_i, \; i \in \{1, ..., m\}$$

Then X^h is not maximal w.r.t the coalition preorder $\precsim_C$:

$1 \Rightarrow 2$. Consider now the case X^h is a solution to the problem (5.10). Then, for every risk X^k there exists a X^h-feasible allocation $(Y_1^h, ..., Y_m^h)$ such that:

$$\sum_{i=1}^m U_i(Y_i^h) \geq U_C(X^k)$$

Consider now the allocation $(Y_1^{*k}, ..., Y_m^{*k})$ that is solution to $U_C(X^k)$. Then we can define a X^h-feasible allocation $(Z_1, ..., Z_m)$ in this way:

$$Z_i = Y_i^h - U_i(Y_i^h) + U_i(Y_i^{*k}) + \frac{\sum_{i=1}^m U_i(Y_i^h) - U_C(X^k)}{m} \quad i \in \{1, ..., m\}$$

Note that $\sum_{i=1}^m Z_i = \sum_{i=1}^m Y_i^k = X^k$. Then,

$$U_i(Z_i) = U_i(Y_i^{*k}) + \frac{\sum_{i=1}^m U_i(Y_i^h) - U_C(X^k)}{m} \geq U_i(Y_i^{*k}) \quad i \in \{1, ..., m\}$$

Then, for every X^k-feasible allocation $(Y_1^k, ..., Y_m^k)$ the relation:

$$U_i(Y_i^k) > U_i(Y_i^{*k}) \; i \in \{1, ..., m\}$$

is an absurd and

$$U_i(Y_i^k) \geq U_i(Y_i^{*k}) \; i \in \{1, ..., m\}$$

clearly implies

$$U_i(Y_i^k) = U_i(Y_i^{*k})$$

Then, no X^k feasible allocation can dominate $(Z_1, ..., Z_m)$. □

From these considerations, we can introduce the conditions that allow the coalition preorder $\precsim_C$ to be total. Note that $\precsim_C$ in not necessary total in general as we described in the previous paragraph.

Proposition 5.3.5. *Let U_i be a translation invariant utility function for the individual total preorder $\precsim_i$ ($i \in \{1, ..., m\}$). Then, given*

$$\begin{aligned} & U_C(X^h) = \sup \sum_{i=1}^{m} U_i(Y_i^h) \\ & sub \\ & (Y_1^h, .., Y_m^h) \in \mathcal{A}(X^h) \end{aligned} \tag{5.11}$$

for every risky outcomes X^h, we have that

$$X^h \precsim_C X^k \Leftrightarrow U_C(X^h) \leq U_C(X^k) \tag{5.12}$$

for every risky outcomes X^h and X^k.

Proof. Let $U_C(X^h) \leq U_C(X^k)$. Then, for every X^h-feasible allocation $(Y_1^h, ..., Y_m^h) \in \mathcal{A}(X^h)$ there exists a X^k-feasible allocation $(Y_1^k, ..., Y_m^k) \in \mathcal{A}(X^k)$ such that

$$\sum_{i=1}^{m} U_i(Y_i^h) \leq \sum_{i=1}^{m} U_i(Y_i^k)$$

Consider now the X^k-feasible allocation $(Z_1, ..., Z_m)$ such that $\sum_{i=1}^m Z_i = X^k$, defined in this way:

$$Z_i = Y_i^k - U_i(Y_i^k) + U_i(Y_i^h) + \frac{\sum_{i=1}^m U_i(Y_i^k) - \sum_{i=1}^m U_i(Y_i^h)}{m} \quad i \in \{1, ..., m\}$$

Note that $\sum_{i=1}^{m} Z_i = \sum_{i=1}^{m} Y_i^k = X^k$. Then,

$$U_i(Z_i) = U_i(Y_i^h) + \frac{\sum_{i=1}^{m} U_i(Y_i^k) - \sum_{i=1}^{m} U_i(Y_i^h)}{m} \geq U_i(Y_i^h) \quad i \in \{1, ..., m\}$$

Since U_i is a utility function for the total preorder $\precsim_i$, we have that:

$$U_i(Z_i) \geq U_i(Y_i^h) \Leftrightarrow Y_i^h \precsim_i Z_i,\ i \in \{1, ..., m\}$$

With

$$Y_i^h \precsim_i Z_i,\ i \in \{1, ..., m\} \Leftrightarrow (Y_1^h, .., Y_m^h) \precsim (Z_1, ..., Z_m)$$

So, we prooved that for every X^h-feasible allocation $(Y_1^h, ..., Y_m^h) \in \mathcal{A}(X^h)$ there exists a X^k-feasible allocation $(Z_1, ..., Z_m)$ such that $(Y_1^h, .., Y_m^h) \precsim (Z_1, ..., Z_m)$. Then $X^h \precsim_C X^k$.

We want to prove now that $X^h \precsim_C X^k \Rightarrow U_C(X^h) \leq U_C(X^k)$. By contraposition, suppose $U_C(X^h) > U_C(X^k)$. Then, there exists a X^k-feasible allocation $(Y_1^k, ..., Y_m^k) \in \mathcal{A}(X^k)$ and a X^h-feasible allocation $(Y_1^h, ..., Y_m^h) \in \mathcal{A}(X^h)$ such that

$$\sum_{i=1}^{m} U_i(Y_i^h) > \sum_{i=1}^{m} U_i(Y_i^k)$$

Let consider the latter 2 feasible allocations for which the inequility of the associated sup convolution is strict. Consider now the X^h-feasible allocation $(Z_1, ..., Z_m)$ such that $\sum_{i=1}^{m} Z_i = X^h$, defined in this way:

$$Z_i = Y_i^h - U_i(Y_i^h) + U_i(Y_i^k) + \frac{\sum_{i=1}^{m} U_i(Y_i^h) - \sum_{i=1}^{m} U_i(Y_i^k)}{m} \quad i \in \{1, ..., m\}$$

Note that $\sum_{i=1}^{m} Z_i = \sum_{i=1}^{m} Y_i^h = X^h$. Then,

$$U_i(Z_i) = U_i(Y_i^k) + \frac{\sum_{i=1}^{m} U_i(Y_i^h) - \sum_{i=1}^{m} U_i(Y_i^k)}{m} > U_i(Y_i^k) \quad i \in \{1, ..., m\}$$

Since U_i is a utility function for the total preorder $\precsim_i$, we have that:

$$U_i(Z_i) > U_i(Y_i^k) \Leftrightarrow Y_i^k \prec_i Z_i,\ i \in \{1, ..., m\}$$

Then:

$$\neg(X^h \precsim_C X^k)$$

□

6

Comonotonicity and efficient risk sharing

In this chapter we are going to re-adapt the main propositions and theorems of the previous chapters restricting our attention to the set of comonotone allocations in order to study the existence of individually rational pareto optimal comonotone allocations.

6.1 Introduction

The concept of comonotonicity is actually a robust tool for solving several research and practical problems in capital allocation and risk sharing.

The main result related to application of comonotonicity in risk sharing is originally due to Landsberger and Meilijson[42] who states that any allocation is dominated by a comonotone one if agents' preferences agree with second stochastic dominance. This result was originally obtained for the discrete case of two agents and then extended to more general cases. This domination result could be expressed by the following Proposition (see Dana[22]):

Proposition 6.1.1. *Any allocation in $\mathcal{A}(X)$ is $\precsim_{SSD}$ dominated by a comonotone allocation in $\mathcal{A}(X)$. If the allocation is not comonotone, then there exists a comonotone allocation that strictly dominates it.*

Recall the following definition for the case of consistency with respect to second stochastic dominance.

Definition 6.1.2. $\precsim_i$ is (strictly) consistent with respect to second stochastic dominance (namely risk averter) if

$$(Y_i \precsim_{SSD} Y_i' \Rightarrow Y_i \precsim_i Y_i') \wedge (Y_i \prec_{SSD} Y_i' \Rightarrow Y_i \prec_i Y_i')$$

As we will see in this chapter, if we introduce the set of comonotone and feasible allocations:

$$\mathcal{C} = \{(Y_1, ..., Y_m) \in \mathcal{A}(X) : (Y_1, ..., Y_m) \ \ \textit{comonotone}\}$$

the domination result (Proposition 6.1.1) allows us to reformulate the *multi-objective maximization problem* (4.8) in the following form:

$$\begin{aligned} &\sup \ (U_1(Y_1), U_2(Y_2), ..., U_m(Y_m)) \\ &\text{sub} \\ &(Y_1, .., Y_m) \in \mathcal{C}' \end{aligned} \tag{6.1}$$

where

- $\mathcal{C}' = \{(Y_1, ..., Y_m) \in \mathcal{C} : (X_1, ..., X_m) \precsim (Y_1, ..., Y_m)\}$ is the set of individually rational comonotone and feasible allocations,
- $\precsim = \bigcap_{i=1}^m \precsim_i$ is the coalition preorder and $\precsim_i$ preserves second stochastic dominance for all $i \in \{1, .., m\}$,
- $U_i(Y_i)$ is an order preserving function for the individual preorder $\precsim_i$ for all $i \in \{1, .., m\}$.

6.2 Existence of optimal solutions

In this paragraph we study the existence of individually rational Pareto optimal allocations (optimal solutions) restricting our attention on comonotone allocations.

From Proposition 4.3.8 we know that the problem concerning the existence of Pareto optimal allocations can be related to the problem concerning the existence of maximal elements for the coalition preorder $\precsim = \bigcap_{i=1}^m \precsim_i$. This result was obtained for every risk X and for every feasible allocation $(Y_1^*, .., Y_m^*) \in \mathcal{A}(X)$. We just want now to readapt this proposition to the case of finding maximal elements for the coalition preorder $\precsim = \bigcap_{i=1}^m \precsim_i$ defined on the set $\mathcal{C}' = \{(Y_1, ..., Y_m) \in \mathcal{A}(\mathcal{X}) : (X_1, ..., X_m) \precsim (Y_1, ..., Y_m), (Y_1, ..., Y_m) \quad \textit{comonotone}\}$ of individually rational comonotone and feasible allocations, with the further assumption that $\precsim_i$ preserves second stochastic dominance for all $i \in \{1, .., m\}$. We start from the following proposition:

Proposition 6.2.1. *For every risk X and for every feasible allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ the following condition holds:*

(i) *if $(Y_1^*, ..., Y_m^*)$ is maximal with respect to the coalition preorder $\precsim = \bigcap_{i=1}^m \precsim_i$ defined on the set $\mathcal{C} = \{(Y_1, ..., Y_m) \in \mathcal{A}(\mathcal{X}) : (Y_1, ..., Y_m) \quad \textit{comonotone}\}$*

of comonotone and feasible allocations, and $\precsim_i$ *preserves second stochastic dominance for all* $i \in \{1,..,m\}$*, then* $(Y_1^*, ..., Y_m^*)$ *is Pareto optimal.*

Proof. By contraposition, consider a feasible allocation $(Y_1^*, ..., Y_m^*) \in \mathcal{A}(X)$ which is not Pareto optimal. Then, by the domination result, (Proposition 6.1.1), there exists $(Y_1', ..., Y_m') \in \mathcal{C}(X)$ such that:

$$[Y_i^* \precsim_{SSD} Y_i' \; i \in \{1, ..., m\}] \wedge [\; \exists \; \bar{i} \in \{1, ..., m\} \; s.t. \; Y_{\bar{i}}^* \prec_{SSD} Y_{\bar{i}}']$$

Since $\precsim_i$ preserves second stochastic dominance for all $i \in \{1,..,m\}$, then:

$$[(Y_i^* \precsim_{SSD} Y_i') \Rightarrow (Y_i^* \precsim_i Y_i') \;] \wedge [(Y_{\bar{i}}^* \prec_{SSD} Y_{\bar{i}}') \Rightarrow \; (Y_i^* \prec_i Y_i')]$$

Therefore, there exists $(Y_1', ..., Y_m') \in \mathcal{C}(X)$ such that

$$[(Y_1^*, ..., Y_m^*) \precsim (Y_1', ..., Y_m')] \wedge [\neg((Y_1', ..., Y_m') \precsim (Y_1^*, ..., Y_m^*)]$$

clearly implies that $(Y_1^*, ..., Y_m^*) \prec (Y_1', ..., Y_m')$. Hence, $(Y_1^*, ..., Y_m^*)$ is not maximal in C for $\precsim$. □

This proposition allows us to study the existence of Pareto Optimal allocations by finding maximal elements for the coalition preorder $\precsim = \bigcap_{i=1}^m \precsim_i$ with $\precsim_i$ preserving second stochastic dominance for all $i \in \{1,..,m\}$.

In particular, we are interested on finding an individually rational Pareto optimal comonotone allocation. Therefore, we can adapt the previous proposition to the case of studiyng the existence of maximal elements for the set:

$$\mathcal{C}' = \{(Y_1, ..., Y_m) \in \mathcal{C} : (X_1, ..., X_m) \precsim (Y_1, ..., Y_m)\}$$

with $\precsim = \bigcap_{i=1}^m \precsim_i$ and $\precsim_i$ preserves second stochastic dominance for all $i \in \{1,..,m\}$.

From Proposition 6.2.1, finding maximal elements for $\mathcal{C}'$ is equivalent to finding an individually rational Pareto optimal comonotone allocation.

The following theorem provides sufficient topological conditios for the existence of individually rational pareto optimal comonotone allocations. The following conditions will be assumed:

A1: for every i and every $Z \in \mathcal{A}(X)$, $i_{\precsim_i}(Z) = \{Y \in \mathcal{A}(X) \mid Z \precsim_i Y\}$ is τ-closed (i.e., $\precsim_i$ is *upper semicontinuous* for every i),

A2: $\precsim_i$ preserves second stochastic dominance for all $i \in \{1, .., m\}$.

Theorem 6.2.2. *There exists a Pareto optimal and individually rational element* $(Y_1^*, ..., Y_m^*)$ *of* $\mathcal{C}' = \{(Y_1, ..., Y_m) \in \mathcal{C} : (X_1, ..., X_m) \precsim (Y_1, ..., Y_m)\}$, *where* $\precsim = \bigcap_{i=1}^m \precsim_i$ *is the coalition preorder on* $\mathcal{C}'$, *provided that* $\precsim_i$ *is an upper semicontinuous preorder and preserves second stochastic dominance for every* i, *and the induced topology* $\tau_{\mathcal{C}'}^m$ *on* $\mathcal{C}'$ *is compact.*

As we already anticipated in the introduction of this chapter, we can reformulate the so called *multi-objective maximization problem* (4.8) and the sup-convolution problems (5.1) in the following forms:

$$
\begin{aligned}
&\sup \ (U_1(Y_1), U_2(Y_2), ..., U_m(Y_m)) \\
&\text{sub} \\
&(Y_1, .., Y_m) \in \mathcal{C}'
\end{aligned}
\tag{6.2}
$$

where

- $\mathcal{C}' = \{(Y_1, ..., Y_m) \in \mathcal{C} : (X_1, ..., X_m) \precsim (Y_1, ..., Y_m)\}$ is the set of individually rational comonotone and feasible allocations,
- $\precsim = \bigcap_{i=1}^m \precsim_i$ is the coalition preorder and $\precsim_i$ preserves second stochastic dominance for all $i \in \{1, .., m\}$,
- $U_i(Y_i)$ is an order preserving function for the individual preorder $\precsim_i$ for all $i \in \{1, .., m\}$.

Definition 6.2.3. The sup-convolution problem relative to the functions $U_1, ..., U_m$ on $\mathcal{C}'$ is defined as follows

$$
U_1 \Box U_2 \Box ... \Box U_m(Y_1, ..., Y_m) = \sup \sum_{i=1}^m U_i(Y_i). \tag{6.3}
$$

Then, we can adapt the main propositions of the thesis with the previous concepts in order to produce sufficient conditions for the existence of an optimal solution in the risk-sharing setting under comonotone allocations.

Proposition 6.2.4. *Let U_i be an order-preserving function for the individual preorder $\precsim_i$ ($i \in \{1, ..., m\}$) on $\mathcal{C}'$, and let $\precsim_i$ preserves second stochastic dominance for every i . Then the following statement is valid:*

1. *If $(Y_1^*, ..., Y_m^*)$ is a solution to the problem (6.2), then it is an optimal solution.*

Proof. By contraposition, assume that $(Y_1^*, ..., Y_m^*)$ is not an optimal solution. Then there exists $(Y_1', ..., Y_m') \in \mathcal{C}'$ such that $Y_i^* \precsim_{SSD} Y_i'$ for all $i \in \{1, ..., m\}$ with one strict inequality. Since $\precsim_i$ preserves second stochastic dominance for every i we have that $Y_i^* \precsim_i Y_i'$ for all $i \in \{1, ..., m\}$ with one strict inequality. Since U_i is an order-preserving function for $\precsim_i$ for all $i \in \{1, ..., m\}$, it is clear that $U_i(Y_i^*) \leq U_i(Y_i')$ for all $i \in \{1, ..., m\}$ with one strict inequality, contradicting the fact that $(Y_1^*, ..., Y_m^*)$ is a solution to the problem (6.2). □

Proposition 6.2.5. *If U_i is an order-preserving function for $\precsim_i$ for every $i \in \{1, ..., m\}$ and $\precsim_i$ preserves second stochastic dominance for every i , then a solution $(Y_1^*, ..., Y_m^*)$ to the sup-convolution problem (6.3) is optimal.*

Proposition 6.2.6. *If for every $i \in \{1, ..., m\}$ there exists an upper semicontinuous order-preserving function U_i for $\precsim_i$, and $\precsim_i$ preserves second stochastic dominance for every i, then there exists an optimal solution that is obtained as a solution $(Y_1^*, ..., Y_m^*)$ to the sup-convolution problem (6.3), provided that the induced topology $\tau_{\mathcal{C}'}$ on $\mathcal{C}'$ is compact.*

We can recall some results from the literature used to solve the previous problems in particular topological context. In this sense, the case of non-atomic space is particularly favorable. Recall the definition of non-atomic probability space as follows:

Definition 6.2.7. We say that a probability space $(\Omega; \mathcal{F}; \mathcal{P})$ is non-atomic, or alternatively call P non-atomic if $P(A) > 0$ implies the existence of $B \in \mathcal{F}$, $B \subset A$ with $0 < P(B) < P(A)$.

These are the results obtained in Carlier et al.[18] and Dana[22] for the case of non atomic space:

Lemma 6.2.8. *(Carlier et al.[18]) If the state space is non-atomic, then the set of comonotone allocations of X is convex and compact in L^∞ up to zero-sum translations (which means that it can be written as):*

$$C = \{(\lambda_1, ..., \lambda_m) \ s.t. \sum_{i=1}^{m} \lambda_i = 0\} + A_0$$

with A_0 compact in L^∞. In particular, the set of comonotone allocations of X is closed in L^∞.

Proposition 6.2.9. *(Dana[22]) Let the state space be non-atomic, and $u : L^\infty \to R$ be concave and $\|\|_\infty$ upper semicontinuous. Then:*

- *u is $\sigma(L^\infty, L^1)$ upper semicontinuous*
- *u is SSD preserving if and only if u is law invariant and monotone.*

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Conclusions

The existence of optimal solutions to the problem of optimal risk sharing is generally treated in the literature by considering the usual requirement of completeness over decision makers' preferences. Optimality in our context stands for Pareto optimality and individual rationality. This means that there is no other allocation such that all agents are better off with respect to their initial exposures and at least one agent is strictly better off.

In this work we present several conditions for the existence of optimal solutions starting from the assessment of the individual preferences expressed by not necessarily total preorders $\precsim_i$. In particular we define a coalition preorder (4.3) representing the attitude of all the agents to prefer an allocation to another one, and we prove (Proposition 4.3.10) the equivalence between optimality and maximality with respect to the coalition preorder.

Proposition 4.3.10 does not require any restrictive assumption on the preorders $\precsim_i$, and this consideration validates our assessment of the individual preferences expressed by not necessarily total preorders. In particular, Proposition 4.3.10 allows us to traduce the problem of finding optimal solutions to that of studying the existence of maximal elements for a not necessarily total coalition preorder.

The concepts of upper semicontinuity of a preorder (Definition 2.3.2) on a topological space is in this sense fundamental in order to prove the existence of maximal elements for the coalition preorder. We proved the so called "Folk theorem" (Theorem 2.3.30) based on the Zorn's Lemma (Lemma 2.3.29), which guarantees the existence of a maximal element for every (not necessarily total) preorder on a compact set provided that the preorder is upper semicontinuous. These considerations are traduced in Theorem 4.3.11, that guarantees the existence of an optimal solution provided that $\precsim_i$ is an upper semicontinuous preorder for every i and the induced topology τ_S on the set $\mathcal{S}$ (of all the feasible allocations for which each agent is at least as well as under the initial exposure) is compact.

Then we refer to the optimal risk sharing functional approaches identified with the multi-objective maximization problem associated to m assigned real-valued functions $U_1, ..., U_m$ (Definition 4.3.16):

$$(1) \qquad \sup \ (U_1(Y_1), U_2(Y_2), ..., U_m(Y_m))$$
$$\text{sub}$$
$$(Y_1, .., Y_m) \in \mathcal{S}.$$

and the sup-convolution problem (Definition 5.1.1)

$$(2) \quad U_1 \square U_2 \square ... \square U_m(Y_1, ..., Y_m) = \sup \sum_{i=1}^{m} U_i(Y_i).$$

with the aim of incorporating the representation of not-necessarily total preorders $\precsim_i$, essentially defined by order-preserving functions (Definition 2.2.6) and multi-utility representations (Definition 2.2.8).

In Proposition 4.3.18 and Proposition 5.2.1 we prove that if $\precsim_i$ is represented by an order preserving function U_i for every i, then a solution to the multi-objective maximization problem (or equivalently to the sup-convolution problem in 5.2.1) is optimal. Proposition 5.2.3 extends the previous considerations for the case of preorders represented by a finite multi-utility representation. In this context, the concepts of upper semicontinuous real-valued functions (Definition 2.3.5) and upper semicontinuous multi-utility representations (Definition 2.3.31) are introduced in order to determine optimal solutions. In particular, Theorem 2.3.32, proved by considering lexicographic arguments, guarantees the existence of a maximal element relative to a preorder $\precsim$ which admits a finite upper semicontinuous multi-utility representation.

A relevant example of a (upper-semi)continuous functional is provided by the *Choquet integral*, when we consider the topology L^∞ of (essentially) bounded functions on a common probability space. The case of a finite multi-utility representation based on the Choquet integrals is of interest since each agent may be equipped with multiple individual reward (risk) functionals where some of them may reflect her own preferences and other are regulatory requirements.

In the case of individual translation invariant total preorders, Proposition 4.3.4 and Proposition 4.3.6 guarantee that determining Pareto optimal allocations is in fact equivalent to determining optimal solutions for every choice of the initial exposure. If in addition we consider the case of individual translation invariant preorders with comonotone super-additive utility functions, then Pareto optima and solutions to the sup-convolution problem coincide (Proposition 5.2.8). Since we often deal with metric spaces, the case of a compact metric feasible set of allocations allows us to apply Rader's theorem (Theorem 2.3.18) in order to guarantee the existence of an upper semicontinuous utility representation for every upper semicontinuous total preorder.

In the case of individual preorders $\precsim_i$ (strictly) monotone with respect to second order stochastic dominance (Definition 6.1.2), it is of help a well known improvement theorem (Proposition 6.1.1) that is at the base of applications of comonotonicity in risk sharing. We apply the aforementioned theorem in order to prove Proposition 6.2.1 that traduces the problem of finding optimal solutions to that of studying the existence of maximal elements for a not necessarily total coalition preorder with the individual preorders $\precsim_i$ monotone with respect to second order stochastic dominance. Then, we incorporate functional representations of not necessarily total preorders to the functional approaches (1) and (2) restricted to the set of comonotone allocations.

In addition to our framework that is essentially related to problems concerning risk sharing in the presence of a single risk X, we consider also the case of risk sharing in the presence of different risky outcomes. In particular we define a coalition preorder over different risky outcomes (Definition 5.3.2) that incorporates also the social preorder (4.3), in this way traducing the problem of making a choice among different initial risks to that of comparing all the possible feasible allocations of the (different) initial risks. Under particular assumptions provided in Proposition 5.3.5, the coalition preorder is total and the related utility function is the associated sup-convolution.

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