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Orientation reversing finite abelian actions on $\mathbb{RP}^3$

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Abstract. *We classify, up to equivalence, the orientation-reversing finite abelian actions on* $\mathbb{RP}^3$ *and their quotient types. There are six different quotient types, and for each quotient type there is only one equivalence class. Descriptions of each action which represents an equivalence class are explicitly given.*

Keywords: Finite group action, lens space, orbifold, orbifold handlebody, Heegaard decomposition.
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1. Introduction

The symmetries of manifolds have been an increasingly ubiquitous topic of study in low-dimensional topology (See for example [8, 9, 11, 14, 15, 23]). In [8], a complete classification (up to conjugation) for symmetries of the orientable and nonorientable 3-dimensional handlebodies of genus one is obtained. A similar classification is obtained in [11] for I-bundles over the projective space. In [9], the finite group actions on the lens space $L(p, q)$ which preserve a Heegaard decomposition were classified up to equivalence for $p > 2$, by restricting these actions to an invariant Heegaard torus. However when $p = 1$ or 2, then an action on $L(p, q)$ may contain an element which when restricted to two different invariant Heegaard tori are not equivalent (See the examples in [9, p. 28]). To begin to address these questions for the case when $p = 2$, in [12] we initiated the study of orientation preserving primary cyclic group actions on $L(2, 1) = \mathbb{RP}^3$, and classified them up to equivalence. In [14], it was shown that the 3-sphere and $\mathbb{RP}^3$ are the only 3-dimensional lens spaces $L(p, q)$ which admit orientation-reversing PL maps of period $4k$ where $k \geq 1$, and in [15] no lens space other than the 3-sphere $\mathbb{S}^3$ and $\mathbb{RP}^3$ admits an orientation-reversing involution. In [10], a complete classification of orientation reversing geometric finite group actions on lens spaces $L(p, q)$ where $p > 2$ and $q^2 \equiv -1 \pmod{p}$ is obtained if the action leaves a Heegaard torus invariant whose sides are exchanged by an orientation-reversing element.

In this paper, continuing the study for $p = 2$, we consider the orientation-reversing abelian actions on the three-dimensional projective space $\mathbb{RP}^3 =$

$L(2,1)$, which is double covered by 3-sphere $\mathbb{S}^3$. Note that the special orthogonal group SO(3) is isomorphic to $\mathbb{RP}^3$ (See [7] for details). The finite orientation reversing abelian actions on $\mathbb{RP}^3$ leave a Heegaard torus invariant while preserving its sides. Using this, we are able to classify, up to equivalence, these actions and compute their quotient spaces. In addition, an explicit construction is given of a *standard action* representing each equivalence class. Note that $\mathbb{RP}^3$ is an elliptic 3-manifold with a geometric structure, and we may assume by [5, Theorem E], which follows from Perelman's results in [16, 17, 18], that a finite action on $\mathbb{RP}^3$ acts as a group of isometries. We work in the PL category.

A G-action on a manifold X is a homomorphism $\varphi\colon G \to \mathrm{Homeo}_{PL}(X)$ where $\mathrm{Homeo}_{PL}(X)$ is the group of PL-homeomorphisms of X and φ is an injection. Two G-actions φ and ψ are equivalent if their images are conjugate in $\mathrm{Homeo}_{PL}(X)$. When G is finite the quotient space is an orbifold which we denote by X/φ. We will assume G is always finite.

Let $\varphi\colon G \to \mathrm{Homeo}_{PL}(\mathbb{RP}^3)$ be an orientation-reversing abelian action. We show (See Corollary 4.2) that there is a Heegaard torus (a separating torus whose closure of the two complementary components are solid tori) which is left invariant by the action whose sides are also preserved. The restriction to each invariant solid torus determines an orbifold quotient whose Euler number is zero. In [8] there is a complete list of all the handlebody orbifolds whose Euler number is zero. For any positive integer n, the orientable orbifolds are denoted by $(A0,n)$ and $(B0,n)$, while the non-orientable ones are denoted by $(A1,n),...,(A3,n),(B1,n),...,(B8,n)$. The orbifolds in the main theorem are obtained by identifying the boundaries of the non-orientable orbifolds via explicitly defined homeomorphisms. If X and Y are orbifolds and $\xi\colon \partial X \to \partial Y$ is a homeomorphism, denote by $O_\xi(X,Y)$ the orbifold obtained by identifying ∂X to ∂Y via ξ. These orbifolds, together with the maps ξ and their fundamental groups are explicitly defined in the Appendix.

The main result in this paper, which appears as Theorem 6.1 in Section 6, is as follows:

THEOREM 1.1. *Let $\varphi\colon G \to Homeo_{PL}(\mathbb{RP}^3)$ be an orientation-reversing finite abelian action. Then one of the following cases is true*:

1) $G = \mathbb{Z}_{2^b m}$ where $b>1$, m is odd and $\mathbb{RP}^3/\varphi$ is $O_{h_1^{-1}}((B5, 2^{b-1}m), (A1, 2))$;

2) $G = \mathbb{Z}_{2m}$, m is odd and $\mathbb{RP}^3/\varphi$ is t $O_{h_2^{-1}}((B4, m), (A3, 1))$;

3) $G = \mathbb{Z}_m \times \mathbb{Z}_2$, m even and $\mathbb{RP}^3/\varphi$ is $O_{h_3}((A2, 2), (B3, , m))$;

4) $G = \mathbb{Z}_4 \times \mathbb{Z}_2$, and $\mathbb{RP}^3/\varphi$ is $O_{h_4}((B2, 2), (B2, 2))$;

5) $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{RP}^3/\varphi$ is $O_{h_5}((B6, 2), (B6, 2))$;

6) $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{RP}^3/\varphi$ is $O_{h_6}((B7, 1), (B7, 1))$.

Furthermore, in each individual case i), where $1 \le i \le 6$, φ is equivalent to the Standard Quotient Type i Action.

The paper is organized as follows. Section 2 is devoted to some preliminary remarks and definitions concerning orbifolds, the Euler number, and Heegaard decomposition. The orbifolds $A(0,n)$ and $B(0,n)$ which cover all the non-orientable orbifolds of Euler number zero are defined, and the non-orientable orbifolds which are the union of these orbifolds and have finite fundamental group are listed. In Section 3, we define the standard abelian actions on $\mathbb{RP}^3$, and identify their quotient types. We show in Section 4 that any orientation-reversing abelian action on $\mathbb{RP}^3$ preserves a Heegaard torus. In Section 5, we investigate which orbifolds defined in Section 2 and the Appendix have a $\mathbb{Z}_2$-normal subgroup of their fundamental groups with abelian quotient, and whether they are covered by $\mathbb{RP}^3$. Finally, we summarize the main results in Section 6. The Appendix contains the definition of each of the non-orientable orbifolds $(A1,n),...,(A3,n),(B1,n),...,(B8,n)$, the gluing maps identifying the boundaries of these orbifolds and their fundamental groups.

2. Orbifolds preliminaries, Heegaard decompositions with finite fundamental groups

Orbifolds were introduced and studied by Satake in [19, 20], and developed more fully by Thurston in [21]. Other good references include M.Yokoyama [22]; M. Boileau, S. Maillot and J. Porti [2]; S. Choi [3]; W. Dunbar [6]; D. Cooper, C.Hodgson and S. Kerchoff [4]. In this section we give brief preliminary notions about orbifolds, and refer the reader to the above references for more detail. We define the orientable orbifolds $(A0,n)$ and $(B0,n)$ which cover the non-orientable orbifolds of Euler number zero. In addition, we list which of the orbifolds having Euler number zero Heegaard decomposition have finite fundamental groups in Theorem 2.1.

An orbifold is a space which is the quotient space of $\mathbb{R}^n$ by a finite linear group. Consider $(\widetilde{U},G)$ where $\widetilde{U}$ is an open subset of $\mathbb{R}^n$ and G is a finite group of diffeomorphisms of $\widetilde{U}$. Let $U=\widetilde{U}/G$ be the quotient space and $\nu\colon \widetilde{U}\to U$ the quotient map. The quotient space U is called a *local model.* If $G_{\tilde{x}}$ is the stabilizer for any $\tilde{x}\in\widetilde{U}$ and $G_{\tilde{x}}\neq 1$, then $\nu(\tilde{x})$ is called an *exceptional point* in U; it may be labelled with the order of $G_{\tilde{x}}$. An *orbifold map* ψ between local models U and U' consists of a pair $(\widetilde{\psi},\gamma)$, where $\widetilde{\psi}\colon \widetilde{U}\to\widetilde{U}'$ is a smooth map and $\gamma\colon G\to G'$ is a group homomorphism such that $\widetilde{\psi}(g(\tilde{x}))=\gamma(g)\widetilde{\psi}(\tilde{x})$ for all $\tilde{x}\in\widetilde{U}$ and $g\in G$, and $\nu'\widetilde{\psi}=\psi\nu$. An *orbifold* is a space which consists of local models glued together by orbifold maps. The set of exceptional points is referred to as the *exceptional set* or the *singular locus.* An *orbifold O with boundary ∂O* is define similarly by replacing $\mathbb{R}^n$ with the closed half space $\mathbb{R}^n_+$ to obtain local models for $x\in\partial O$. If M is an n-manifold and G is a group of diffeomorphisms which acts properly discontinuously on M (for every compact subset $K\subset M$, the set $\{g\in G\mid g(K)\cap K\neq\emptyset\}$ is finite), then the quotient

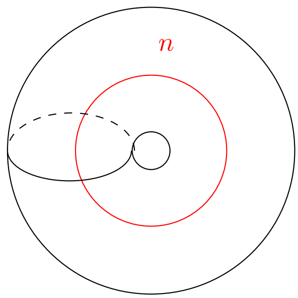

Figure 1: $(A0, n)$

space M/G is an orbifold. The orbifolds $(A0, n)$ and $(B0, n)$, defined below, are good examples of 3-dimensional orbifolds.

An orbifold handlebody O is formed by gluing together orbifold 0-handles (3-orbifolds covered by the 3-ball B^3) and orbifold 1-handles (products with 2-orbifolds covered by the disk D^2) so that the exceptional sets of the same type are identified. See [8] for more details. If the handlebody orbifold is orientable, then the underlying space is a handlebody. When there is a n-sheeted covering space $H \to O$ where H is a handlebody, then the Euler number $\chi(O) = \frac{1}{n}\chi(H)$. See [4] for a more detailed description of the Euler number. An *Euler number* $1 - g$ *Heegaard decomposition* of an orbifold O is an ordered triple (Σ, O_1, O_2) where $\Sigma \subset O$ is a closed 2-orbifold, O_i is an orbifold handlebody having Euler number $1 - g$, $\Sigma = \partial O_i = O_1 \cap O_2$ and $O = O_1 \cup O_2$.

In this paper we will be concerned with Euler number zero Heegaard decompositions where the orbifolds O_i, for $i = 1, 2$, will come from the list of the non-orientable orbifolds covered by $(A0, n)$ and $(B0, n)$. We now describe the orbifolds $(A0, n)$ and $(B0, n)$.

2.1. Orbifold $(A0, n)$

We begin with the unit disk D^2 parameterized by $\{\rho e^{i\theta} = v \mid 0 \leq \rho \leq 1\}$. Let V be the solid torus $S^1 \times D^2$ and define a $\mathbb{Z}_n$-action on V by $h(u, v) = (u, ve^{\frac{2\pi i}{n}})$. The orbifold quotient space $V/\langle h \rangle$ is denoted by $V(n)$ or $(A0, n)$. This quotient space is a torus with a core of exceptional points of order n (See Figure 1).

The orbifold fundamental group of $V(n)$ is

$$\pi_1(V(n)) = \langle l_1, m_1 \mid [l_1, m_1] = 1, m_1^n = 1 \rangle \simeq \mathbb{Z} \times \mathbb{Z}_n\,.$$

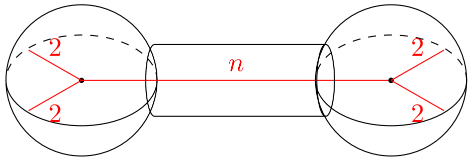

Figure 2: $(B0, n)$

2.2. Orbifold $(B0, n)$

Let $\tau\colon V(n) \to V(n)$ be the involution defined by $\tau(u, v) = (\overline{u}, \overline{v})$. The orbifold $V(n)/\langle\tau\rangle$ is denoted by $(B0, n)$. Its underlying space is a 3-ball which has an exceptional set consisting of an embedded tree with five edges, one edge labeled with n and the other four edges each labeled with 2. The boundary is a Conway sphere with four cone points of order 2 (See Figure 2).

We obtain a covering map $\nu\colon V(n) \to (B0, n) = V(n)/\langle\tau\rangle$ giving an exact sequence

$$1 \to \pi_1(V(n)) \to \pi_1((B0, n)) \to \mathbb{Z}_2 \to 1$$

which splits. Let $\nu_*(l_1) = l$ and $\nu_*(m_1) = m$. Since τ inverts both generators of $\pi_1(V(n))$, we obtain the following fundamental groups:

$$\begin{aligned}\pi_1((B0, n)) = \langle l, m, t \mid m^n = t^2 = 1, lm = ml, tlt^{-1} = l^{-1},\ tmt^{-1} = m^{-1}\rangle \\ = \mathrm{Dih}(\mathbb{Z} \times \mathbb{Z}_n)\end{aligned}$$

and

$$\begin{aligned}\pi_1(\partial(B0, n)) = \langle l, m, t \mid t^2 = 1,\ lm = ml, tlt^{-1} = l^{-1},\ tmt^{-1} = m^{-1}\rangle \\ = \mathrm{Dih}(\mathbb{Z} \times \mathbb{Z}) .\end{aligned}$$

In the Appendix we show that $(A0, n)$ will double cover the non-orientable orbifolds $(A1, n)$, $(A2, n)$, $(A3, n)$,$(B3, n)$, $(B4, n)$, and $(B5, n)$; and the orbifold $(B0, n)$ will double cover the non-orientable orbifolds $(B1, n)$, $(B2, n)$, $(B6, n)$, $(B7, n)$, $(B8, n)$. Furthermore these orbifolds are described there along with their fundamental groups. Recall that $O_\xi(X, Y)$ is the orbifold obtained by identifying ∂X to ∂Y via a homeomorphism $\xi\colon \partial X \to \partial Y$. The orbifolds X and Y will come from the list of non-orientable orbifolds whose boundaries are homeomorphic, and the gluing map $\xi = h_i$ for $1 \leq i \leq 7$ is defined in the Appendix. For groups A and B, we use the notation $A \circ B$ to denote the semidirect product $A \rtimes B$, and use $A \circ_{-1} B$ to represent the specific action

Orbifolds	Fundamental Group
$O_{h_1}((A1,n),(B5,m))$	$\langle a,b \mid a^2=b^2,\ a^{2m}=b^{2m}=(ba^{-1})^n=1\rangle \simeq \mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m}$
$O_{h_2}((A3,n),(B4,m))$	$\langle a,b,c \mid a^n=b^2=c^{2m}=1, bab^{-1}=a^{-1}, cac^{-1}=a^{-1}, cbc^{-1}=ba\rangle \simeq \mathrm{Dih}(\mathbb{Z}_n)\circ \mathbb{Z}_{2m}$
$O_{h_3}((A2,n),(B3,m))$	$\langle a,b,c \mid [a,b]=[a,c]=1, a^m=b^n=c^2=1, cbc^{-1}=b^{-1}\rangle \simeq \mathrm{Dih}(\mathbb{Z}_n)\times \mathbb{Z}_m$
$O_{h_4}((B2,n),(B2,m))$	$\langle a,b \mid a^{2n}=b^2=1, ba^2b^{-1}=a^{-2}, (ab)^{2m}=1\rangle \simeq (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m})\circ \mathbb{Z}_2$
$O_{h_5}((B6,n),(B6,m))$	$\langle a,b,c,d \mid a^2=b^2=c^2=(bc)^n=d^2=(ad)^m=1, a\leftrightarrow\{b,c\}, d\leftrightarrow\{b,c\}\rangle \simeq \mathrm{Dih}(\mathbb{Z}_n)\times \mathrm{Dih}(\mathbb{Z}_m)$
$O_{h_6}((B7,n),(B7,m))$	$\langle a,b,c \mid a^2=b^{2n}=(ab^{-1}ab)^m=c^2=1, a\leftrightarrow\{b^2,c\}, b^c=b^{-1}\rangle \simeq \mathrm{Dih}(\mathbb{Z}_m)\circ \mathrm{Dih}(\mathbb{Z}_{2n})$
$O_{h_7}((B1,n),(B8,m))$	$\langle a,b,c \mid a^n=b^2=c^2=1, bab^{-1}=a^{-1}, [a,c]=1, (cb)^{2m}=1\rangle \simeq \mathbb{Z}_n \circ \mathrm{Dih}(\mathbb{Z}_{2m})$

Table 1: Notation: $x^y = yxy^{-1}$, and if x and y commute we write $x \leftrightarrow y$.

$bab^{-1} = a^{-1}$ for every $a \in A$ and $b \in B$. Thus the dihedral group $\mathrm{Dih}(\mathbb{Z}_n) = \mathbb{Z}_n \circ_{-1} \mathbb{Z}_2$.

From [13], we have the following theorem:

Theorem 2.1. *Let X and Y be any of the orbifolds $(A1,n)$,...,$(A3,n)$,$(B1,n)$, ...,$(B8,n)$, and let $\xi\colon \partial X \to \partial Y$ be a homeomorphism. If $\pi_1(O_\xi(X,Y))$ is finite, then $O_\xi(X,Y)$ is homeomorphic to one of the orbifolds listed in Table 1 with the corresponding fundamental group.*

3. Standard orientation reversing abelian actions on $\mathbb{RP}^3$

In this section, we will define some standard orientation reversing abelian actions on $\mathbb{RP}^3$. In addition, we calculate the quotient spaces of these actions, and the quotient spaces for their orientation preserving subgroups. These actions will be sorted by their quotient types, Quotient Type i for $1 \le i \le 6$. A standard action with Quotient Type i will be called the *Standard Quotient Type i Action.* Since the later cases are similar to the previous cases, some of the details will be omitted.

We view $\mathbb{RP}^3 = V_1 \cup_\alpha V_2$ where the boundary ∂V_1 is identified with ∂V_2 by a homeomorphism $\alpha\colon \partial V_1 \to \partial V_2$ defined by $\alpha(u_1,v_1) = (u_2v_2^2, u_2v_2)$ for $(u_i,v_i) \in V_i$.

Consider two orbifold solid tori $V(a)$ and $V(b)$, let p and q be relatively prime positive integers and choose r, $s \in \mathbb{Z}$ such that $rq - ps = -1$. Let $h\colon \partial V(a) \to \partial V(b)$ be the homeomorphism defined by $h(u,v) = (u^r v^p, u^s v^q)$. The orbifold $W(p,q;a,b)$ is the orbifold obtained by identifying $\partial V(a)$ to $\partial V(b)$ via the homeomorphism h. The underlying space of $W(p,q,a,b)$, denoted by $|W(p,q,a,b)|$, is the lens space $L(p,q)$. As in the case of the lens space, the integers p, q, a and b determine the orbifold up to homeomorphism.

3.1. Quotient Type 1: $O_{h_1^{-1}}((B5, 2^{b-1}m), (A1, 2))$ with $b > 1$ and m odd.

Let $V_1 = S^1 \times D^2$, and define two homeomorphisms f and g on V_1 as follows: For m a positive odd integer

$$f(u_1, v_1) = \left(u_1, v_1 e^{\frac{2\pi i}{m}}\right), \quad \text{and} \quad g(u_1, v_1) = \left(\overline{u_1} e^{\frac{-2\pi i}{2^{b-2}}}, u_1 v_1 e^{\frac{3(2\pi i)}{2^b}}\right).$$

Note that

$$\begin{aligned} g^2(u_1, v_1) &= g\left(\overline{u_1} e^{\frac{-2\pi i}{2^{b-2}}}, u_1 v_1 e^{\frac{3(2\pi i)}{2^b}}\right) \\ &= \left(\overline{(\overline{u_1} e^{\frac{-2\pi i}{2^{b-2}}})} e^{\frac{-2\pi i}{2^{b-2}}}, (\overline{u_1} e^{\frac{-2\pi i}{2^{b-2}}}) u_1 v_1 e^{\frac{6(2\pi i)}{2^b}}\right) \\ &= \left(u_1, v_1 e^{\frac{-2\pi i}{2^{b-2}}} e^{\frac{3(2\pi i)}{2^{b-1}}}\right) \\ &= \left(u_1, v_1 e^{\frac{2\pi i}{2^{b-1}}}\right) \end{aligned}$$

It follows that g is an orientation reversing homeomorphism with finite order 2^b. Furthermore f and g commute, hence the two maps generate a $\mathbb{Z}_m \times \mathbb{Z}_{2^b} = \mathbb{Z}_{2^b m}$-action on V_1. We obtain an orbifold covering map $\eta_1 \colon V_1 \to V_1/\langle f \rangle = V_1(m)$ defined by $\eta_1(u_1, v_1) = (u_1, v_1^m)$. The homeomorphism g induces a homeomorphism g_1 on $V(m)$, and we may calculate g_1 as follows:

$$\begin{aligned} g_1(u_1, v_1) = \eta g\left(u_1, v_1^{\frac{1}{m}}\right) &= \eta\left(\overline{u_1} e^{\frac{-2\pi i}{2^{b-2}}}, u_1 v_1^{\frac{1}{m}} e^{\frac{3(2\pi i)}{2^b}}\right) \\ &= \left(\overline{u_1} e^{\frac{-2\pi i}{2^{b-2}}}, u_1^m v_1 e^{\frac{3m(2\pi i)}{2^b}}\right). \end{aligned}$$

We consider first the case where $b > 1$. Thus we have a $\mathbb{Z}_m \times \mathbb{Z}_{2^b} = \mathbb{Z}_{2^b m}$-action where $b > 1$ and m is odd. It also follows that $g_1^2(u_1, v_1) = (u_1, v_1 e^{\frac{2m\pi i}{2^{b-1}}})$ and $\langle g_1^2 \rangle = \mathbb{Z}_{2^{b-1}}$. We obtain an orbifold covering $\lambda_1 \colon V(m) \to V(2^{b-1}m) = V(m)/\langle g_1^2 \rangle$ defines by $\lambda_1(u_1, v_1) = (u_1, v_1^{2^{b-1}})$. Further, g_1 induces an orientation reversing involution g_2 on $V(2^{b-1}m)$ which may be computed as follows:

$$\begin{aligned} g_2(u_1, v_1) = \lambda_1 g_1\left(u_1, v_1^{\frac{1}{2^{b-1}}}\right) &= \lambda_1\left(\overline{u_1} e^{\frac{-2\pi i}{2^{b-2}}}, u_1^m v_1^{\frac{1}{2^{b-1}}} e^{\frac{3m(2\pi i)}{2^b}}\right) \\ &= \left(\overline{u_1} e^{\frac{-2\pi i}{2^{b-2}}}, -u_1^{2^{b-1}m} v_1\right). \end{aligned}$$

On the other hand, g_2 is an orientation reversing involution with two isolated fixed points, $(e^{\frac{-2\pi i}{2^{b-1}}}, 0)$ and $(-e^{\frac{-2\pi i}{2^{b-1}}}, 0)$. Thus by [13, Proposition 13] we see that $V(2^{b-1}m)/\langle g_2 \rangle$ is the orbifold $(B5, 2^{b-1}m)$, and we have the following lemma.

LEMMA 3.1. *For any orbifold of the form $(B5, 2^{b-1}m)$ where m is odd and $b > 1$, there exists a $\mathbb{Z}_m \times \mathbb{Z}_{2^b}$-action on the solid torus V_1 generated by $f(u_1, v_1) = (u_1, v_1 e^{\frac{2\pi i}{m}})$ and $g(u_1, v_1) = (\overline{u_1} e^{\frac{-2\pi i}{2^{b-2}}}, u_1 v_1 e^{\frac{3(2\pi i)}{2^b}})$. The quotient type $V_1/(\mathbb{Z}_m \times \mathbb{Z}_{2^b}) = (B5, 2^{b-1}m)$.*

At this point, we will extend f and g to $\mathbb{RP}^3$ and identify the quotient space. Let $V_2 = S^1 \times D^2$, and recall that $\mathbb{RP}^3 = V_1 \cup_\alpha V_2$ where $\alpha\colon \partial V_1 \to \partial V_2$ is a homeomorphism defined by $\alpha(u_1, v_1) = (u_2 v_2^2, u_2 v_2)$. Now $\alpha^{-1}(u_2, v_2) = (u_1^{-1} v_1^2, u_1 v_1^{-1})$. We have the following:

$$\begin{aligned}
\alpha f \alpha^{-1}(u_2, v_2) &= \alpha f(u_1^{-1} v_1^2, u_1 v_1^{-1}) \\
&= \alpha(u_1^{-1} v_1^2, u_1 v_1^{-1} e^{\frac{2\pi i}{m}}) \\
&= \big((u_2^{-1} v_2^2)(u_2 v_2^{-1} e^{\frac{2\pi i}{m}})^2, (u_1^{-1} v_2^2)(u_2 v_2^{-1} e^{\frac{2\pi i}{m}})\big) \\
&= (u_2 e^{\frac{4\pi i}{m}}, v_2 e^{\frac{2\pi i}{m}}).
\end{aligned}$$

Thus $f(u_2, v_2) = (u_2 e^{\frac{4\pi i}{m}}, v_2 e^{\frac{2\pi i}{m}})$ is a fixed-point free map on V_2. Similarly extend g to $\mathbb{RP}^3$ as follows:

$$\begin{aligned}
\alpha g \alpha^{-1}(u_2, v_2) &= \alpha g(u_1^{-1} v_1^2, u_1 v_1^{-1}) \\
&= \alpha\left(\overline{(u_1^{-1} v_1^2)}\, e^{\frac{-2\pi i}{2^{b-2}}}, (u_1^{-1} v_1^2)(u_1 v_1^{-1}) e^{\frac{3(2\pi i)}{2^b}}\right) \\
&= \alpha\left(\overline{(u_1^{-1} v_1^2)}\, e^{\frac{-2\pi i}{2^{b-2}}}, v_1 e^{\frac{3(2\pi i)}{2^b}}\right) \\
&= \left(\left(\overline{(u_2^{-1} v_2^2)} e^{\frac{-2\pi i}{2^{b-2}}}\right)\left(v_2 e^{\frac{3(2\pi i)}{2^b}}\right)^2, \left(\overline{(u_2^{-1} v_2^2)} e^{\frac{-2\pi i}{2^{b-2}}}\right)\left(v_1 e^{\frac{3(2\pi i)}{2^b}}\right)\right) \\
&= \left(u_2 e^{\frac{2\pi i}{2^{b-1}}}, u_2 \overline{v_2} e^{\frac{-2\pi i}{2^b}}\right).
\end{aligned}$$

In other words, $g(u_2, v_2) = \big(u_2 e^{\frac{2\pi i}{2^{b-1}}}, u_2 \overline{v_2} e^{\frac{-2\pi i}{2^b}}\big)$ where $b > 1$.

In the mean time, we extend η to V_2 to obtain a covering map $\eta_2\colon V_2 \to V_2/\langle f\rangle = V_2(1)$ defined by $\eta_2(u_2, v_2) = \big(u_2^m, u_2^{\frac{m-1}{2}} v_2\big)$. In addition, g induces g_1 on $V_2(1)$ which may be computed as follows:

$$\begin{aligned}
g_1(u_2, v_2) &= \eta_2 g\left(u_2^{\frac{1}{m}}, u_2^{\frac{1-m}{2m}} v_2\right) \\
&= \eta_2\left(u_2^{\frac{1}{m}} e^{\frac{2\pi i}{2^{b-1}}}, u_2^{\frac{1}{m}} (u_2^{\frac{1-m}{2m}} v_2)^{-1} e^{\frac{-2\pi i}{2^b}}\right) \\
&= \eta_2\left(u_2^{\frac{1}{m}} e^{\frac{2\pi i}{2^{b-1}}}, u_2^{\frac{m+1}{2m}} v_2^{-1} e^{\frac{-2\pi i}{2^b}}\right) \\
&= \left(u_2 e^{\frac{2\pi i m}{2^{b-1}}}, u_2^{\frac{m-1}{2m}} e^{\frac{2\pi i(m-1)}{2^b}} u_2^{\frac{m+1}{2m}} v_2^{-1} e^{\frac{-2\pi i}{2^b}}\right) \\
&= \left(u_2 e^{\frac{2\pi i m}{2^{b-1}}}, u_2 v_2^{-1} e^{\frac{2\pi i(m-2)}{2^b}}\right).
\end{aligned}$$

Hence

$$g_1(u_2,v_2)=\left(u_2e^{\frac{2\pi im}{2^{b-1}}},u_2v_2^{-1}e^{\frac{2\pi i(m-2)}{2^b}}\right)\text{ and }g_1^2(u_2,v_2)=\left(u_2e^{\frac{2\pi im}{2^{b-2}}},v_2e^{\frac{2\pi im}{2^{b-1}}}\right).$$

Note that $\langle g_1^2\rangle=\mathbb{Z}_{2^{b-1}}$ and $g_1^{2^{b-1}}$ has as its fixed-point set the core $S^1\times\{0\}$. We obtain an orbifold covering map $\lambda_2\colon V_2(1)\to V_1(1)/\langle g_1^2\rangle=V_2(2)$ defined by $\lambda_2(u_2,v_2)=(u_2^{2^{b-2}},u_2^{-1}v_2^2)$. Furthermore, g_1 induces an orientation reversing involution g_2 on $V_2(2)$ which we now compute below:

$$\begin{aligned}g_2(u_2,v_2)&=\lambda_2g_1\left(u_2^{\frac{1}{2^{b-2}}},u_2^{\frac{1}{2^{b-1}}}v_2^{\frac{1}{2}}\right)\\&=\lambda_2\left(u_2^{\frac{1}{2^{b-2}}}e^{\frac{2\pi im}{2^{b-1}}},u_2^{\frac{1}{2^{b-2}}}(u_2^{\frac{1}{2^{b-1}}}v_2^{\frac{1}{2}})^{-1}e^{\frac{2\pi i(m-2)}{2^b}}\right)\\&=\lambda_2\left(u_2^{\frac{1}{2^{b-2}}}e^{\frac{2\pi im}{2^{b-1}}},u_2^{\frac{1}{2^{b-1}}}v_2^{\frac{-1}{2}}e^{\frac{2\pi i(m-2)}{2^b}}\right)\\&=\left(-u_2,v_2^{-1}e^{\frac{-2\pi i}{2^{b-2}}}\right).\end{aligned}$$

Since g_2 is a fixed-point free orientation reversing involution on $V_2(2)$, it follows by [13, Proposition 13] that $V_2(2)/\langle g_2\rangle$ is the orbifold $(A1,2)$, and we have the following lemma.

Lemma 3.2. *For any orbifold of the form $(A1,2)$, there exists a $\mathbb{Z}_m\times\mathbb{Z}_{2^b}$-action on the solid torus V_2 where m is odd and $b>1$, generated by $f(u_2,v_2)=(u_2e^{\frac{4\pi i}{m}},v_2e^{\frac{2\pi i}{m}})$ and $g(u_2,v_2)=(u_2e^{\frac{2\pi i}{2^{b-1}}},u_2v_2^{-1}e^{\frac{-2\pi i}{2^b}})$, such that $V_2/(\mathbb{Z}_m\times\mathbb{Z}_{2^b})=(A1,2)$.*

The next step is to compute the quotient space for the covering $\eta_1\cup\eta_2\colon(V_1\cup_\alpha V_2)\to(V_1\cup_\alpha V_2)/\langle f\rangle=V_1(m)\cup_{\alpha_1}V_2(1)$. The matrix representations for η_1 and η_2 are $\begin{bmatrix}1&0\\0&m\end{bmatrix}$ and $\begin{bmatrix}m&0\\\frac{m-1}{2}&1\end{bmatrix}$ respectively. We compute the gluing map $\alpha_1\colon\partial V_1(m)\to\partial V_2(1)$ with matrix representation $\begin{bmatrix}\mathbf{x}&\mathbf{y}\\\mathbf{z}&\mathbf{w}\end{bmatrix}$, by solving the equation

$$\begin{bmatrix}\mathbf{x}&\mathbf{y}\\\mathbf{z}&\mathbf{w}\end{bmatrix}\begin{bmatrix}1&0\\0&m\end{bmatrix}=\begin{bmatrix}m&0\\\frac{m-1}{2}&1\end{bmatrix}\begin{bmatrix}1&2\\1&1\end{bmatrix}.$$

We see that $\mathbf{x}=m$, $\mathbf{y}=2$, $\mathbf{z}=\frac{m+1}{2}$ and $\mathbf{w}=1$. Thus $\alpha_1(u_1,v_1)=(u_2^mv_2^2,u_2^{\frac{m+1}{2}}v_2)$, the matrix representation for α_1 is $\begin{bmatrix}m&2\\\frac{m+1}{2}&1\end{bmatrix}$ and the quotient space $\mathbb{RP}^3/\langle f\rangle=W(2,1;m,1)$.

Finally, consider the orbifold covering $\lambda_1 \cup \lambda_2 \colon V_1(m) \cup_{\alpha_1} V_2(1) \to (V_1(m) \cup_{\alpha_1} V_2(1))/\langle g_1^2 \rangle = V_1(2^{b-1}m) \cup_{\alpha_2} V_2(2)$, and identify the quotient space by computing the gluing map α_2. The matrix representations for λ_1 and λ_2 are $\begin{bmatrix} 1 & 0 \\ 0 & 2^{b-1} \end{bmatrix}$ and $\begin{bmatrix} 2^{b-2} & 0 \\ -1 & 2 \end{bmatrix}$ respectively. Solving a matrix equation similar to that above, we obtain $\alpha_2(u_1, v_1) = (u_2^{2^{b-2}m} v_2, u_2)$ with its matrix representation $\begin{bmatrix} 2^{b-2}m & 1 \\ 1 & 0 \end{bmatrix}$. Thus $W(2,1;m,1)/\langle g_1^2 \rangle = W(1,0;2^{b-1}m,2)$. The underlying space of $W(1,0;2^{b-1}m,2)$ is the 3-sphere S^3 and the exceptional set is the Hopf link, with one exceptional set labeled with $2^{b-1}m$ and the other exceptional set labeled with 2.

Consequently, we can summarize the results above. Recall $\mathbb{RP}^3 = V_1 \cup_\alpha V_2$ where $\alpha \colon \partial V_1 \to \partial V_2$ is a homeomorphism defined by $\alpha(u_1, v_1) = (u_2 v_2^2, u_2 v_2)$ for $(u_i, v_i) \in V_i$. Define homeomorphisms f and g on $\mathbb{RP}^3$ as follows:

$$f(u_i, v_i) = \begin{cases} (u_1, v_1 e^{\frac{2\pi i}{m}}), & \text{if } i = 1 \\ (u_2 e^{\frac{4\pi i}{m}}, v_2 e^{\frac{2\pi i}{m}}), & \text{if } i = 2 \end{cases}$$

$$g(u_i, v_i) = \begin{cases} (\overline{u_1} e^{\frac{-2\pi i}{2^{b-2}}}, u_1 v_1 e^{\frac{3(2\pi i)}{2^b}}), & \text{if } i = 1 \\ (u_2 e^{\frac{2\pi i}{2^{b-1}}}, u_2 \overline{v_2} e^{\frac{-2\pi i}{2^b}}), & \text{if } i = 2 \end{cases}$$

THEOREM 3.3. *Let $\varphi \colon \mathbb{Z}_s \to Homeo_{PL}(\mathbb{RP}^3)$ be an action such that $s = 2^b m$ where $b > 1$ and m is odd. Then φ is equivalent to $\langle f \rangle \times \langle g \rangle = \mathbb{Z}_m \times \mathbb{Z}_{2^b} = \mathbb{Z}_{2^b m}$, and the quotient space $\mathbb{RP}^3/\varphi$ is homeomorphic to the orbifold $O_{h_1^{-1}}((B5, 2^{b-1}m), (A1, 2))$. Let $\varphi_0 \colon \mathbb{Z}_{s/2} \to Homeo_{PL}(\mathbb{RP}^3)$ represent the restriction of φ to the the orientation preserving subgroup. Then $\mathbb{RP}^3/\varphi_0$ is the orbifold $W(1,0;2^{b-1}m,2)$ whose underlying space is the 3-sphere S^3, and the exceptional set is the Hopf link with one exceptional set labeled with $2^{b-1}m$ and the other exceptional set labeled with 2.*

Proof. Let $\varphi \colon \mathbb{Z}_s \to \mathrm{Homeo}_{PL}(\mathbb{RP}^3)$ be an action such that $s = 2^b m$ where $b > 1$ and m is odd. By [14, Theorem A], there is only one such action up to equivalence. By construction, $\mathbb{RP}^3/\langle f, g \rangle = O_{\alpha_3}((B5, 2^{b-1}m), (A1, 2))$ for some gluing map α_3. Since $O_{\alpha_3}((B5, 2^{b-1}m), (A1, 2)) = O_{\alpha_3^{-1}}((A1, 2), (B5, 2^{b-1}m))$, which by [13, Lemma 21] is homeomorphic to $O_{h_1}((A1, 2), (B5, 2^{b-1}m))$, the result follows. ∎

Next, we will treat the case where $b = 1$, and so we will consider orientation reversing $\mathbb{Z}_m \times \mathbb{Z}_2 = \mathbb{Z}_{2m}$-actions on $\mathbb{RP}^3$ where m is odd.

3.2. Quotient Type 2: $O_{h_2^{-1}}((B4,m),(A3,1))$ and m odd

Substituting $b=1$ into the definition of g defined in Quotient Type 1, we obtain the involution $h\colon V_1 \to V_1$ defined by $h(u_1,v_1)=(\overline{u_1},-u_1v_1)$. It follows that h is an orientation reversing involution which commutes with f on V_1, and thus $\langle f\rangle\times\langle h\rangle=\mathbb{Z}_m\times\mathbb{Z}_2$. As above, if $\eta\colon V_1\to V_1/\langle f\rangle = V_1(m)$ is the covering, the induced map $\hat{h}$ on $V_1(m)$ is defined by $\hat{h}(u_1,v_1)=(\overline{u_1},-u_1^m v_1)$. The fixed-point set is $\{(1,0)\}\cup(\{-1\}\times D^2)\subseteq V_1(m)$. It follows by [13, Proposition 13] and the fixed-point set of $\hat{h}$, that $V_1(m)/\langle\hat{h}\rangle=(B4,m)$.

The involution h on V_2 is defined by $h(u_2,v_2)=(u_2,-u_2\overline{v_2})$, and the involution $\hat{h}$ on $V_2(1)$ is $\hat{h}(u_2,v_2)=(u_2,-u_2\overline{v_2})$. The fixed-point set consists of the set $\{(-e^{2i\theta},\rho e^{i\theta}) \mid 0\le\theta\le 2\pi,\ 0\le\rho\le 1\}\subseteq V_2(1)$, which is a Möbius band. Hence by [13, Proposition 13], $V_2(1)/\langle\hat{h}\rangle=(A3,1)$. We obtain the two lemmas below:

Lemma 3.4. *For any orbifold of the form $(B4,m)$ where m is odd, there exists a $\mathbb{Z}_m\times\mathbb{Z}_2$-action on the solid torus V_1, generated by $f(u_1,v_1)=(u_1,v_1e^{\frac{2\pi i}{m}})$ and $h(u_1,v_1)=(\overline{u_1},-u_1v_1)$, such that $V_1/(\mathbb{Z}_m\times\mathbb{Z}_2)=(B4,m)$.*

Lemma 3.5. *For any orbifold of the form $(A3,1)$, there exists a $\mathbb{Z}_m\times\mathbb{Z}_2$-action on the solid torus V_2 where m is odd, generated by $f(u_2,v_2)=(u_2e^{\frac{4\pi i}{m}},v_2e^{\frac{2\pi i}{m}})$ and $h(u_2,v_2)=(u_2,-u_2\overline{v_2})$, such that $V_2/(\mathbb{Z}_m\times\mathbb{Z}_2)=(A3,1)$.*

Let $f,h\colon\mathbb{RP}^3\to\mathbb{RP}^3$ be homeomorphisms defined as follows:

$$f(u_i,v_i)=\begin{cases}(u_1,v_1e^{\frac{2\pi i}{m}}), & \text{if } i=1\\ (u_2e^{\frac{4\pi i}{m}},v_2e^{\frac{2\pi i}{m}}), & \text{if } i=2\end{cases}$$

$$h(u_i,v_i)=\begin{cases}(\overline{u_1},-u_1v_1), & \text{if } i=1\\ (u_2,-u_2\overline{v_2}), & \text{if } i=2.\end{cases}$$

As a result of the above discussions, we obtain the following theorem.

Theorem 3.6. *Let $\varphi\colon\mathbb{Z}_s\to Homeo_{PL}(\mathbb{RP}^3)$ be an action such that $s=2m$ where m is odd. Then φ is equivalent to $\langle f\rangle\times\langle h\rangle=\mathbb{Z}_m\times\mathbb{Z}_2=\mathbb{Z}_{2m}$, and the quotient space $\mathbb{RP}^3/\varphi$ is homeomorphic to the orbifold $O_{h_2^{-1}}((B4,m),(A3,1))$. Let $\varphi_0\colon\mathbb{Z}_m\to Homeo_{PL}(\mathbb{RP}^3)$ represent the restriction of φ to the the orientation preserving subgroup. Then $\mathbb{RP}^3/\varphi_0$ is the orbifold $W(2,1;m,1)$, whose underlying space is $\mathbb{RP}^3$ with exceptional set a simple closed curve labeled with m.*

Proof. Let $\varphi\colon\mathbb{Z}_s\to \mathrm{Homeo}_{PL}(\mathbb{RP}^3)$ be an action such that $s=2m$ where m is odd. If $m>1$, then applying the Smith conjecture in [1] and [14, Theorem C], there is only one such action up to equivalence. If $m=1$, and hence the action is an involution, applying [6] there is only one such action up to equivalence. By the above construction $\mathbb{RP}^3/\langle f, h\rangle = O_\zeta((B4, m), (A3, 1)) = O_{\zeta^{-1}}((A3, 1), (B4, m))$ for some gluing map ζ. Since $O_{\zeta^{-1}}((A3, 1), (B4, m))$ is homeomorphic to $O_{h_2}((A3, 1), (B4, m))$ by [13, Lemma 23], the result follows. □

3.3. Quotient Type 3: $O_{h_3}((A2, 2), (B3, , m))$ where m is even

Define homeomorphisms f and g on $\mathbb{RP}^3$ as follows:

$$f(u_i, v_i) = \begin{cases} (u_1 e^{\frac{4\pi i}{m}}, v_1 e^{-\frac{2\pi i}{m}}), & \text{if } i = 1 \\ (u_2, v_2 e^{\frac{2\pi i}{m}}), & \text{if } i = 2 \end{cases}$$

$$g(u_i, v_i) = \begin{cases} (u_1, \overline{u_1 v_1}), & \text{if } i = 1 \\ (\overline{u_2}, \overline{u_2} v_2), & \text{if } i = 2 \end{cases}$$

A computation shows $fg = gf$, and so $\langle f, g\rangle$ defines a $\mathbb{Z}_m \times \mathbb{Z}_2$-action on $\mathbb{RP}^3$. Furthermore, it can be shown that $\mathbb{RP}^3/\varphi$ is the orbifold $O_{h'}((A2, 2), (B3, , m))$ for some homeomorphism h' between their boundaries.

THEOREM 3.7. *For m even, the maps f and g define an action $\varphi\colon \mathbb{Z}_m \times \mathbb{Z}_2 \to Homeo_{PL}(\mathbb{RP}^3)$ such that the quotient space $\mathbb{RP}^3/\varphi$ is $O_{h_3}((A2, 2), (B3, , m))$. Let $\varphi_0\colon \mathbb{Z}_m \to Homeo_{PL}(\mathbb{RP}^3)$ represent the restriction of φ to the the orientation preserving subgroup. Then $\mathbb{RP}^3/\varphi_0$ is the orbifold $W(1, \frac{m}{2}; 2, m)$ whose underlying space is the 3-sphere S^3, and the exceptional set is the Hopf link with one exceptional set labeled with 2 and the other exceptional set labeled with m.*

Proof. The proof is similar to Theorem 3.6, and uses the fact that by [13, Lemma 22], $O_{h'}((A2, 2), (B3, , m))$ is homeomorphic to $O_{h_3}((A2, 2), (B3, m))$. □

3.4. Quotient Type 4: $O_{h_4}((B2, 2), (B2, , 2))$

Define homeomorphisms θ and τ on $\mathbb{RP}^3$ as follows:

$$\theta(u_i, v_i) = \begin{cases} (-\overline{u_1}, u_1 v_1), & \text{if } i = 1 \\ (-u_2, -u_2 \overline{v_2}), & \text{if } i = 2 \end{cases}$$

$$\tau(u_i, v_i) = \begin{cases} (\overline{u_1}, \overline{v_1}), & \text{if } i = 1 \\ (\overline{u_2}, \overline{v_2}), & \text{if } i = 2 \end{cases}$$

A computation shows $\theta^4 = id = \tau^2$ and $\theta\tau = \tau\theta$, and so $\langle \theta, \tau\rangle$ defines a $\mathbb{Z}_4 \times \mathbb{Z}_2$-action on $\mathbb{RP}^3$. We remark that letting $b = 2$ in the definition of g in Quotient

Type 1 of this section, also gives a $\mathbb{Z}_4$-action which is conjugate to θ by the homeomorphism:

$$k(u_i, v_i) = \begin{cases} (iu_1, v_1), & \text{if } i = 1 \\ (iu_2, iv_2), & \text{if } i = 2 \end{cases}$$

Observe that $\theta^2(u_i, v_i) = (u_i, -v_i)$, and we have a covering map $\nu\colon \mathbb{RP}^3 \to \mathbb{RP}^3/\langle\theta^2\rangle = (A0,2) \cup_{\alpha_1} (A0,2)$ where $\nu(u_i, v_i) = (u_i, v_i^2)$; the matrix corresponding to α_1 is $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$. The induced maps $\overline{\theta}$ and $\overline{\tau}$ on $(A0,2) \cup_{\alpha_1} (A0,2)$ are defined by

$$\overline{\theta}(u_i, v_i) = \begin{cases} (-\overline{u_1}, u_1^2 v_1), & \text{if } i = 1 \\ (-u_2, u_2^2 \overline{v_2}), & \text{if } i = 2 \end{cases} \quad \text{and}$$
$$\overline{\tau}(u_i, v_i) = (\overline{u_i}, \overline{v_i})\,.$$

Moding out by the action of $\overline{\tau}$, we obtain a covering map $\nu_1\colon (A0,2) \cup_{\alpha_1} (A0,2) \to (B0,2) \cup_{\overline{\alpha_1}} (B0,2)$. Now $\overline{\theta}$ induces an involution on $(B0,2) \cup_{\overline{\alpha_1}} (B0,2)$, whose quotient is $O_{f'}((B2,2),(B2,2))$ for some homeomorphism $f'\colon \partial(B2,2) \to \partial(B2,2)$. We obtain the result below.

THEOREM 3.8. *Let $\varphi\colon \mathbb{Z}_4 \times \mathbb{Z}_2 \to Homeo_{PL}(\mathbb{RP}^3)$ be an action such that $\varphi(\mathbb{Z}_4 \times \mathbb{Z}_2) = \langle\theta, \tau\rangle$. Then the quotient space $\mathbb{RP}^3/\varphi$ is homeomorphic to the orbifold $O_{h_4}((B2,2),(B2,2))$. Let $\varphi_0\colon \mathbb{Z}_4 \times \mathbb{Z}_2 \to Homeo_{PL}(\mathbb{RP}^3)$ represent the restriction of φ to the the orientation preserving subgroup. Then $\mathbb{RP}^3/\varphi_0$ is the orbifold $O_{\overline{\alpha_1}}((B0,2),(B0,2))$, where $\overline{\alpha_1}$ is uniquely determined by the matrix*

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

Proof. The quotient space $\mathbb{RP}^3/\varphi$ has a finite fundamental group. The result now follows by the above construction, Theorem 11 and Lemma 25 in [13]. □

3.5. Quotient Type 5: $O_{h_5}((B6,2),(B6,2))$

Define homeomorphisms f, g and h on $\mathbb{RP}^3$ as follows:

$$f(u_i, v_i) = \begin{cases} (\overline{u_1}, \overline{v_1}), & \text{if } i = 1 \\ (\overline{u_2}, \overline{v_2}), & \text{if } i = 2 \end{cases}$$

$$g(u_i, v_i) = \begin{cases} (u_1, \overline{u_1 v_1}), & \text{if } i = 1 \\ (\overline{u_2}, \overline{u_2} v_2), & \text{if } i = 2 \end{cases}$$

$$h(u_i, v_i) = \begin{cases} (u_1, -v_1), & \text{if } i = 1 \\ (u_2, -v_2), & \text{if } i = 2 \end{cases}$$

It follows that $\langle f, g, h\rangle$ is a $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$-action on $\mathbb{RP}^3$. We may choose a covering map $\nu\colon \mathbb{RP}^3 \to \mathbb{RP}^3/\langle h\rangle$ defined by $\nu(u_1, v_1) = (u_1, -u_1v_1^2)$ and $\nu(u_2, v_2) = (u_2, u_2^{-1}v_2^2)$. Then $\mathbb{RP}^3/\langle h\rangle$ is the orbifold $V(2)\cup_{r_1} V(2) = W(1, 0; 2, 2)$ where $r_1\colon V(2) \to V(2)$ is defined by $r_1(u_1, v_1) = (-v_2, u_2)$. The induced maps f_1 and g_1 on $W(1, 0; 2, 2)$ are defined by

$$f_1(u_i, v_i) = \begin{cases} (\overline{u_1}, \overline{v_1}), & \text{if } i = 1\\ (\overline{u_2}, \overline{v_2}), & \text{if } i = 2\end{cases}$$

and

$$g_1(u_i, v_i) = \begin{cases} (u_1, \overline{v_1}), & \text{if } i = 1\\ (\overline{u_2}, v_2), & \text{if } i = 2\end{cases}$$

Now, $W(1, 0; 2, 2)/\langle f_1\rangle = O_{r_2}((B0, 2), (B0, 2))$ for some gluing map $r_2\colon \partial(B0, 2) \to \partial(B0, 2)$. The map r_2 is an order 4 rotation which permutes the cone points of order 2 on $\partial(B0, 2)$. It follows that for the induced map g_2 on the orbifold $O_{r_2}((B0, 2), (B0, 2))$ we obtain $O_{r_2}((B0, 2), (B0, 2))/\langle g_3\rangle = O_{r_3}((B6, 2), (B6, 2))$. Summarizing we have the following theorem:

Theorem 3.9. *The maps f, g and h define an action $\varphi\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \to Homeo_{PL}(\mathbb{RP}^3)$ such that the quotient space $\mathbb{RP}^3/\varphi$ is $O_{h_5}((B6, 2), (B6, 2))$. Let $\varphi_0\colon \mathbb{Z}_2\times\mathbb{Z}_2 \to Homeo_{PL}(\mathbb{RP}^3)$ represent the restriction of φ to the the orientation preserving subgroup. Then $\mathbb{RP}^3/\varphi_0$ is the orbifold $O_{r_2}((B0, 2), (B0, 2))$ where $\overline{r_2}$ is uniquely determined by the matrix $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.*

Proof. The quotient space $\mathbb{RP}^3/\varphi$ is the orbifold $O_{r_3}((B6, 2), (B6, 2))$ for some map r_3. By [13, Lemma 26], this orbifold is homeomorphic to $O_r((B6, 2), (B6, 2))$ which completes the proof. □

3.6. Quotient Type 6: $O_{h_6}((B7, 1), (B7, 1))$

Define homeomorphisms f and g on $\mathbb{RP}^3$ as follows:

$$f(u_i, v_i) = \begin{cases} (\overline{u_1}, -u_1v_1), & \text{if } i = 1\\ (u_2, -u_2\overline{v_2}), & \text{if } i = 2\end{cases}$$

$$g(u_i, v_i) = \begin{cases} (u_1, \overline{u_1v_1}), & \text{if } i = 1\\ (\overline{u_2}, \overline{u_2}v_2), & \text{if } i = 2\end{cases}$$

We see that $\langle f, g\rangle$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on $\mathbb{RP}^3$ and $fg(u_i, v_i) = (\overline{u_i}, -\overline{v_i})$. Let $\eta\colon \mathbb{RP}^3 \to \mathbb{RP}^3/\langle fg\rangle$ be an orbifold covering map and note that the quotient space is $O_{\widehat{\alpha}}((B0, 1), (B0, 1))$ for some homeomorphism $\widehat{\alpha}\colon \partial(B0, 1) \to \partial(B0, 1)$. Let $\widehat{g}$ be the induced involution on $O_{\widehat{\alpha}}((B0, 1), (B0, 1))$.

The fixed-point set of $fg|_{\partial V_i}$ is $\mathrm{Fix}(fg|_{\partial V_i})=\{(1,i),(1,-i),(-1,i),(-1,-i)\}$. It follows that $g|_{\partial V_i}$ fixes two elements of $\mathrm{Fix}(fg|_{\partial V_i})$, and exchanges the other two.

This implies that $O_{\widehat{\alpha}}((B0,1),(B0,1))/\langle\widehat{g}\rangle$ is the orbifold $O_{r'}((B7,1),(B7,1))$ for some homeomorphism $r'\colon \partial(B7,1)\to\partial(B7,1)$. As a result, we obtain the following theorem:

Theorem 3.10. *The maps f and g define an action* $\varphi\colon\mathbb{Z}_2\times\mathbb{Z}_2\to Homeo_{PL}(\mathbb{RP}^3)$ *such that the quotient space* $\mathbb{RP}^3/\varphi$ *is the orbifold* $O_{h_6}((B7,1),(B7,1))$. *Let* $\varphi_0\colon\mathbb{Z}_2\to Homeo_{PL}(\mathbb{RP}^3)$ *represent the restriction of* φ *to the the orientation preserving subgroup. Then* $\mathbb{RP}^3/\varphi_0$ *is the orbifold* $O_{\overline{\alpha}}((B0,1),(B0,1))$ *where* $\overline{\alpha}$ *is uniquely determined by the matrix* $\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}$.

Proof. The quotient space $\mathbb{RP}^3/\varphi$ is the orbifold $O_{r'}((B7,1),(B7,1))$ for some homeomorphism r'. Since the fundamental group of the quotient space is finite, it follows by [13, Lemma 27] that this orbifold is homeomorphic to $O_{h_6}((B7,1),(B7,1))$, completing the proof. □

4. Splitting orientation-reversing abelian actions on $\mathbb{RP}^3$

In this section, we will show that any abelian orientation reversing action on $\mathbb{RP}^3$ splits and preserves the sides of the splitting. An action $\varphi\colon G\to \mathrm{Homeo}_{PL}(\mathbb{RP}^3)$ is said to *split* if there is a Heegaard torus T such that $\varphi(g)(T)=T$ for all $g\in G$. If in addition, each complementary component of the Heegaard torus is invariant under the action, then we say φ *preserves the sides of the splitting.*

Theorem 4.1. *Let* $\varphi\colon G\to Homeo_{PL}(\mathbb{RP}^3)$ *be a finite action which contains an orientation reversing element* $j\in\varphi(G)$, *such that* $\langle j\rangle$ *is a normal subgroup of* $\varphi(G)$. *If j is an involution, assume* $\varphi(G)/\langle j\rangle$ *is not the symmetric group* S_4 *or the alternating groups* A_4 *and* A_5. *Then* φ *splits and preserves the sides of the splitting.*

Proof. The element j generates a cyclic group $\mathbb{Z}_{2^bm}$ where m is odd. Since the orientation preserving subgroup of $\mathbb{Z}_{2^bm}$ has index two and generated by j^2, it follows that $b\geq 1$.

Suppose first that $b>1$. By Theorem 3.3, $\langle j\rangle$ is conjugate to the group $\langle f\rangle\times\langle g\rangle=\mathbb{Z}_m\times\mathbb{Z}_{2^b}=\mathbb{Z}_{2^bm}$. Conjugating all of $\varphi(G)$ by this element, we may assume $\langle j\rangle=\langle f\rangle\times\langle g\rangle$. Furthermore, the quotient space $\mathbb{RP}^3/\langle j\rangle=(B5,2^{b-1}m)\cup_{h_1^{-1}}(A1,2)$. Let $\nu\colon\mathbb{RP}^3\to\mathbb{RP}^3/\langle j\rangle$ be the covering map. The core in V_2 is $S^1\times\{0\}$, and $\nu(S^1\times\{0\})$ is the exceptional set γ in $(A1,2)$, which is a simple closed curve labeled with the integer 2. The induced action $\varphi(G)/\langle j\rangle=H$ on $\mathbb{RP}^3/\langle j\rangle$ must leave γ invariant. Let U be an H-invariant regular neighborhood of γ. Now U is the orbifold $(A0, 2)$, which lifts to a $\varphi(G)$-invariant solid torus $\widetilde{U}$, containing the core. Its boundary $\partial\widetilde{U}$ is a $\varphi(G)$-invariant Heegaard torus whose sides are preserved by $\varphi(G)$.

Assume $b = 1$. By Theorem 3.6, $\langle j\rangle$ is conjugate to $\langle f\rangle \times \langle h\rangle = \mathbb{Z}_m \times \mathbb{Z}_2 = \mathbb{Z}_{2m}$, and we may assume as above that $\langle j\rangle = \langle f\rangle \times \langle h\rangle$. If $m \neq 1$, then $\mathbb{RP}^3/\langle f\rangle = (A0, m) \cup_{\alpha_1} (A0, 1)$, where the matrix for α_1 is $\begin{bmatrix} m & 2 \\ \frac{m+1}{2} & 1 \end{bmatrix}$. (See computation following Lemma 3.2.) The orbifold $(A0, m)$ contains an exceptional set consisting of a simple closed curve labeled with an m. Letting H be the quotient group $\varphi(G)/\langle f\rangle$, it follows that H must leave the exceptional set invariant. The exceptional set lifts to the core in V_1, and the proof follows as above.

Now suppose $m = 1$. In this case $\mathbb{RP}^3/\langle h\rangle = (B4, 1) \cup_{h_2^{-1}} (A3, 1)$, and we again let $\nu\colon \mathbb{RP}^3 \to (B4, 1) \cup_{h_2^{-1}} (A3, 1)$ be the covering map. The exceptional set consists of a point in $(B4, 1)$, a projective plane P with $P \cap (B4, 1)$ a mirrored disk and $P \cap (A3, 1)$ a mirrored Möbius band. For the core in $S^1 \times \{0\}$ in V_2, it follows that $\nu(S^1 \times \{0\})$ is an orientation reversing element in the mirrored Möbius band. Letting H be $\varphi(G)/\langle h\rangle$, the projective plane must be left invariant by H. Since $\varphi(G)/\langle h\rangle$ is neither S_4, A_4 nor A_5, it follows by [11, Theorem 7.2] that $H|_P$ leaves an orientation reversing loop invariant. Since the lift of this loop is isotopic to the core in V_2, the proof follows as above. □

We obtain the following corollary:

COROLLARY 4.2. *Let $\varphi\colon G \to Homeo_{PL}(\mathbb{RP}^3)$ be an orientation reversing abelian action. Then φ splits and preserves the sides of the splitting.*

5. Orbifolds covered by $\mathbb{RP}^3$

In this section, we will identify which of the non-orientable orbifolds listed in Theorem 2.1 as defined in the Appendix may be covered by $\mathbb{RP}^3$ and identify the subgroup corresponding to the covering. The orbifolds in Section 2 are: $O_{h_1}((A1, n), (B5, m))$, $O_{h_2}((A3, n), (B4, m))$, $O_{h_3}((A2, n), (B3, m))$, $O_{h_4}((B2, n), (B2, m))$, $O_{h_5}((B6, n), (B6, m))$, $O_{h_6}((B7, n), (B7, m))$, $O_{h_7}((B1, n), (B8, m))$.

It will be convenient to apply the following proposition and corollary, which essentially follow from orbifold covering space theory. The reader is referred to the paper of M. Yokoyama [22] for a good elucidation of orbifold theory.

PROPOSITION 5.1. *Let O be a 3-dimensional orbifold, W a 3-dimensional suborbifold and $i\colon W \hookrightarrow O$ the inclusion map. Suppose G is a subgroup of $\pi_1(O)$ and $H = i_*^{-1}(G) \leq \pi_1(W)$. Let $\eta\colon \widetilde{O} \to O$ and $\lambda\colon \widetilde{W} \to W$ be the cover-*

ings corresponding to G and H respectively. If $\widetilde{W}$ is an orbifold, then $\widetilde{O}$ is an orbifold.

Proof. Let L be a component of $\eta^{-1}(W)$ and note that $p = \eta|_L \colon L \to W$ is a covering map. From standard covering space theory $p_*(\pi_1(L)) = i_*^{-1}(\eta_*(\pi_1(\widetilde{O})))$. Since $i_*^{-1}(\eta_*(\pi_1(\widetilde{O}))) = H$, we have $p_*(\pi_1(L)) = H$, and this equals $\lambda_*(\pi_1(\widetilde{W}))$. Thus there is an orbifold homeomorphism $f \colon \widetilde{W} \to L \subset \widetilde{O}$, implying L and therefore $\widetilde{O}$ is an obifold. □

COROLLARY 5.2. *Let O be a 3-dimensional orbifold, W a 3-dimensional sub-orbifold and $i \colon W \hookrightarrow O$ the inclusion map. Let G be a subgroup of $\pi_1(O)$ containing an element $i_*(\alpha)$ where $\alpha \in \pi_1(W)$, and let $\widetilde{O}$ be the covering of O corresponding to G. Suppose the covering translation on the universal covering space of W associated with α has a fixed point. Then $\widetilde{O}$ is an orbifold.*

Proof. Let $H = i_*^{-1}(G) \leq \pi_1(W)$ and note that $\alpha \in H$. Let U be the universal covering space of W. Since the covering translation associated with α has a fixed point, it follows that $U/H = \widetilde{W}$ is an orbifold, which is the covering of W corresponding to H. The result now follows by Proposition 5.1. □

5.1. Quotient Type 1: $O_{h_1}((A1, n), (B5, m))$

From the Appendix the orbifold fundamental group of $O_{h_1}((A1, n), (B5, m))$ is

$$\begin{aligned}\pi_1(O_{h_1}((A1, n), (B5, m))) &= \langle a, b \mid a^2 = b^2,\ a^{2m} = b^{2m} = (ba^{-1})^n = 1\rangle \\ &= \langle ba^{-1}\rangle \circ_{-1} \langle a\rangle = \mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m}\,.\end{aligned}$$

Furthermore, the elements a and b in $\pi_1((A1, n))$ acting on the universal covering space $\mathbb{R} \times D^2$ of $(A1, n)$ are defined by $a(t, v) = (t - \frac{1}{2}, \overline{v})$ and $b(t, v) = (t - \frac{1}{2}, \overline{v}e^{\frac{2\pi i}{n}})$.

Note that as elements in either $\pi_1((A1, n))$ or $\pi_1(O_{h_1}((A1, n), (B5, m)))$ they are orientation reversing.

PROPOSITION 5.3. *Let H be a normal subgroup of $\pi_1(O_{h_1}((A1, n), (B5, m))) = \mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m}$ isomorphic to $\mathbb{Z}_2$, and let $Q = \pi_1(O_{h_1}((A1, n), (B5, m)))/H$ be the quotient group. Suppose $n \neq 1$ and Q is abelian. Then one of the following is true*:

1) $n = 2$, either $H = \langle ba^{m-1}\rangle$ or $\langle ba^{-1}\rangle$, and $Q = \mathbb{Z}_{2m}$;
2) $n = 2$, $H = \langle a^m\rangle$ and $Q = \mathbb{Z}_2 \times \mathbb{Z}_m$;
3) $n = 4$, $H = \langle (ba^{-1})^2\rangle$ and $Q = \mathbb{Z}_2 \times \mathbb{Z}_{2m}$.

Proof. Recall from Section 2 that $a(ba^{-1})a^{-1} = (ba^{-1})^{-1}$. Let $w = ba^{-1}$, and suppose $H = \langle w^s a^t\rangle$ where $0 \leq s < n$ and $0 \leq t < 2m$.

Assume first that s and t are both non-zero. Since H is normal, $w^s a^t = a(w^s a^t)a^{-1} = w^{-s}a^t$, which implies $w^{2s} = 1$ or $s = \frac{n}{2}$. Note that $1 = (w^{n/2}a^t)^2 = a^{2t}$, for either t even or odd. This implies $t = m$, and thus $H = \langle w^{n/2}a^m\rangle$.

Suppose m is odd. Then

$$w^{n/2}a^m = w(w^{n/2}a^m)w^{-1} = w^{(n/2+1)}a^m w^{-1}a^{-m}a^m = w^{(n/2+2)}a^m .$$

This implies $w^2 = 1$, and thus $n = 2$. We now suppose m is even. Consider the group $Q = \pi_1(O_{h_1}((A1,n),(B5,m)))/H$. Since Q is abelian, $wH = (aH)(wH)(aH)^{-1} = w^{-1}H$, which implies $w^2 \in \langle w^{n/2}a^m\rangle$. If $w^2 \neq 1$, then $w^2 = w^{n/2}a^m$, or $w^{(4-n)/2} = a^m$. But this contradicts the semi-direct product property that $\langle w\rangle \cap \langle a\rangle = \{1\}$, and so $w^2 = 1$ and $n = 2$. In either case $H = \langle (ba^{-1})^{n/2}a^m\rangle = \langle (ba^{-1})a^m\rangle = \langle ba^{m-1}\rangle$. Furthermore, $Q = \pi_1(O_{h_1}((A1,n),(B5,m)))/H = \langle a,b \mid a^2 = b^2,\ a^{2m} = b^{2m} = (ba^{-1})^2 = 1, ba^{m-1} = 1\rangle = \langle a \mid a^{2m} = 1\rangle \simeq \mathbb{Z}_{2m}$.

Suppose $s = 0$. Then $\mathbb{Z}_2 \simeq H = \langle a^t\rangle$, and $t = m$. A similar argument as above shows that if m is either even or odd, then $n = 2$ and $Q = \mathbb{Z}_2 \times \mathbb{Z}_m$. Now suppose $t = 0$ and $\mathbb{Z}_2 \simeq H = \langle w^s\rangle$. It follows that $s = n/2$, and $Q = \mathbb{Z}_{n/2} \circ_{-1} \mathbb{Z}_{2m}$. In order for Q to be abelian, either $n = 2$, $H = \langle w\rangle$ and $Q = \mathbb{Z}_{2m}$, or $n = 4$, $H = \langle w^2\rangle$ and $Q = \mathbb{Z}_2 \times \mathbb{Z}_{2m}$. □

PROPOSITION 5.4. *Let $\varphi\colon G \to Homeo_{PL}(\mathbb{RP}^3)$ be a finite action such that the quotient space $\mathbb{RP}^3/\varphi$ is the orbifold $O_{h_1}((A1,n),(B5,m))$. Then $n \neq 1$.*

Proof. Let $\nu\colon \mathbb{RP}^3 \to \mathbb{RP}^3/\varphi = O_{h_1}((A1,n),(B5,m))$ be the covering map, and note that $\nu_*(\pi_1(\mathbb{RP}^3))$ is a normal subgroup of $\pi_1(O_{h_1}((A1,n),(B5,m)))$ isomorphic to $\mathbb{Z}_2$ of finite index. Suppose $n = 1$, and therefore $\pi_1(O_{h_1}((A1,n),(B5,m))) \simeq \mathbb{Z}_{2m}$. This implies that $G \simeq \mathbb{Z}_m$, and therefore $m \neq 1$. Thus the maximum order of every exceptional point in $\mathbb{RP}^3/\varphi$ is m. However, $O_{h_1}((A1,n),(B5,m))$ has two cone points of order $2m$, giving a contradiction. Thus $n \neq 1$. □

COROLLARY 5.5. *Let $\varphi\colon G \to Homeo_{PL}(\mathbb{RP}^3)$ be a finite abelian action such that the quotient space $\mathbb{RP}^3/\varphi$ is the orbifold $O_{h_1}((A1,n),(B5,m))$. Then the following is true:*

1) *The action is conjugate to the Standard Quotient Type 1 Action;*
2) *$n = 2$, and $m = 2^{b-1}m_0$ where m_0 is odd and $b > 1$;*
3) *$G \simeq \mathbb{Z}_{m_0} \times \mathbb{Z}_{2^b} = \mathbb{Z}_{2m}$;*
4) *The covering corresponds to the subgroup $\langle ba^{m-1}\rangle$.*

Proof. By Proposition 5.4, $n \neq 1$.

Let $\nu\colon \mathbb{RP}^3 \to \mathbb{RP}^3/\varphi = O_{h_1}((A1,n),(B5,m))$ be the covering map, and note that $\nu_*(\pi_1(\mathbb{RP}^3)) = H$ is a $\mathbb{Z}_2$ normal subgroup. By assumption $Q =$

$\pi_1(O_{h_1}((A1,n),(B5,m)))/H = G$ is an abelian group. We now apply Proposition 5.3 and consider each case separately.

Suppose $n=2$, $H=\langle ba^{m-1}\rangle$ or $\langle ba^{-1}\rangle$, and $Q=\mathbb{Z}_{2m}$. Since both ba^{m-1} and a^m are orientation reversing elements when m is odd, and $\mathbb{RP}^3$ is orientable, it follows that $H=\langle ba^{m-1}\rangle$ or $\langle ba^{-1}\rangle$ and m is even. Viewing ba^{-1} as an element in $\pi_1((A1,n))$, we note that ba^{-1} has a fixed point as an action on the universal covering space. Therefore by Corollary 5.2, the covering of $O_{h_1}((A1,n),(B5,m))$ corresponding to $\langle ba^{-1}\rangle$ is not a manifold. The case where $n=2$, $H=\langle a^m\rangle$ and $Q=\mathbb{Z}_2\times\mathbb{Z}_m$ is eliminated in a similar way. This is done by recalling that a in $\pi_1((A1,n))$ is identified with x in $\pi_1((B5,m))$, where x acting on the universal covering space is defined by $x(t,v)=(-t,ve^{\frac{\pi i}{m}})$. Since this map has a fixed point, this case is also eliminated using Corollary 5.2. Now suppose $n=4$, $H=\langle (ba^{-1})^2\rangle$ and $Q=\mathbb{Z}_2\times\mathbb{Z}_{2m}$. Note that $(ba^{-1})^2(t,v)=(t,-v)$, and therefore has a fixed point eliminating this case also.

Thus, the only possible case is $n=2$, $H=\langle ba^{m-1}\rangle$ where m is even and $G=\mathbb{Z}_{2m}$. Write $m=2^{b-1}m_0$ where $b>1$ and m_0 is odd. By Theorem 3.3, φ is conjugate to the Standard Quotient Type 1 Action, which is a $\mathbb{Z}_{m_0}\times\mathbb{Z}_{2^b}=\mathbb{Z}_{2m}$-action on $\mathbb{RP}^3$ with quotient space $O_{h_1}((A1,n),(B5,m))$, completing the proof. □

5.2. Quotient Type 2 : $O_{h_2}((A3,n),(B4,m))$

The orbifold fundamental group of $\pi_1(O_{h_2}((A3,n),(B4,m)))$ is

$$\begin{aligned}\langle a,b,c \mid a^n=b^2=c^{2m}=1, bab^{-1}=a^{-1}, cac^{-1}=a^{-1}, cbc^{-1}=ba\rangle\\ =(\langle a\rangle\circ_{-1}\langle b\rangle)\circ\langle c\rangle\simeq Dih(\mathbb{Z}_n)\circ\mathbb{Z}_{2m}\end{aligned}$$

The maps a, b and c are defined on the universal covering space of $(A3,n)$ by $a(t,v)=(t,ve^{\frac{2\pi i}{n}})$, $b(t,v)=(t,\overline{v})$ and $c(t,v)=(t+\frac{1}{2},\overline{v}e^{\frac{-\pi i}{n}})$. Note that b and c are orientation reversing elements when viewed as elements of $\pi_1(O_{h_2}((A3,n),(B4,m)))$.

Proposition 5.6. *Let $H\simeq\mathbb{Z}_2$ be a normal subgroup of $\pi_1(O_{h_2}((A3,n),(B4,m)))$, and let $Q=\pi_1(O_{h_2}((A3,n),(B4,m)))/H$ be the quotient group. If Q is abelian, then one of the following is true:*

1) $n=1$, $H=\langle c^m\rangle$ and $Q=\mathbb{Z}_2\times\mathbb{Z}_m$;
2) $n=2$, $H=\langle a\rangle$, and $Q=\mathbb{Z}_2\times\mathbb{Z}_{2m}$;
3) $n=1$, $H=\langle b\rangle$ or $H=\langle bc^m\rangle$, and $Q=\mathbb{Z}_{2m}$.

Proof. Recall that $cac^{-1}=a^{-1}$, $cbc^{-1}=ba$ and c^2 commutes with both a and b. As the orbifold fundamental group is a semi-direct product, we may write $H=\langle a^s b^\epsilon c^t\rangle$ where $0\le s<n$, $\epsilon=0$ or 1, and $0\le t<2m$.

We first assume $\epsilon = 0$, and so $H = \langle a^s c^t \rangle$. Since H is normal, $a^s c^t = ca^s c^t c^{-1} = a^{-s} c^t$, which indicates $a^{2s} = 1$. Observe that this implies $1 = (a^s c^t)^2 = c^{2t}$ whether t is either even or odd. There are three cases to consider depending on the values of s and t.

Suppose $s = 0$, and therefore $t \neq 0$. Since $c^{2t} = 1$, it follows that $t = m$ and $H = \langle c^m \rangle$. If $n = 1$, then $Q = \mathbb{Z}_2 \times \mathbb{Z}_m$ proving 1) in the statement of the proposition. We now assume $n \neq 1$. If m is odd, then $bc^m b^{-1} = b(ba)^{-1} c^m = ac^m$, showing $H = \langle c^m \rangle$ is not a normal subgroup. If m is even, then $H = \langle c^m \rangle$ is a normal subgroup. However $Q = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \circ \mathbb{Z}_m$ is not abelian, removing this case from consideration.

Assume $t = 0$, and so $s \neq 0$ and $n \neq 1$. Furthermore since $a^{2s} = 1$, it follows that $s = n/2$ and $H = \langle a^{n/2} \rangle$. Obviously, $Q = (\mathbb{Z}_{n/2} \circ_{-1} \mathbb{Z}_2) \circ \mathbb{Z}_{2m}$ is abelian, if $n = 2$. Thus $H = \langle a \rangle$ and $Q = \mathbb{Z}_2 \times \mathbb{Z}_{2m}$ proving 2).

Next, we assume $s \neq 0$ and $t \neq 0$, and therefore $n \neq 1$ and $H = \langle a^{n/2} c^m \rangle$. We claim that the quotient Q is not abelian, and thus this case does not occur. If m is odd, then by normality of H, we have $a^{n/2} c^m = b(a^{n/2} c^m) b^{-1} = a^{-n/2} bc^m b^{-1} = a^{-n/2} b(ba)^{-1} c^m = a^{-n/2} ba^{-1} b^{-1} c^m = a^{-n/2} ac^m$. This implies $a = 1$ giving a contradiction. Thus m must be even which we now assume. Since c^m commutes with every element, it follows that the subgroup $L = \langle a^{n/2} c^m, c^m \rangle$ is also a normal subgroup of $\pi_1(O_{h_2}((A3, n), (B4, m)))$. We obtain an injection $\pi_1(O_{h_2}((A3, n), (B4, m)))/L \to \pi_1(O_{h_2}((A3, n), (B4, m)))/H = Q$. Now $\pi_1(O_{h_2}((A3, n), (B4, m)))/L$ is

$$\begin{aligned} \langle a, b, c \mid a^n = b^2 = 1, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = ba, c^m = 1, a^{n/2} = 1 \rangle \\ = (\mathbb{Z}_{n/2} \circ_{-1} \mathbb{Z}_2) \circ \mathbb{Z}_m \end{aligned}$$

If Q is abelian, then so is $\pi_1(O_{h_2}((A3, n), (B4, m)))/L$. This implies $n = 2$ and $H = \langle ac^m \rangle$. As a consequence, we must have

$$Q = \pi_1(O_{h_2}((A3, n), (B4, m)))/H = \langle b, c \mid b^2 = 1, c^{2m} = 1, cbc^{-1} = bc^m \rangle,$$

which is not abelian.

On the other hand, if $\epsilon = 1$, then our $\mathbb{Z}_2$ normal subgroup is written as $H = \langle a^s bc^t \rangle$. Assume first that $s = 0$, and thus $H = \langle bc^t \rangle$. By the normality condition, $bc^t = c(bc^t)c^{-1} = bac^t$, which implies $1 = a$ and hence $n = 1$. Furthermore, $1 = (bc^t)^2 = c^{2t}$. Hence $H = \langle bc^t \rangle$ where $t = 0$ or m. As a result, $\pi_1(O_{h_2}((A3, n), (B4, m))) = \mathbb{Z}_2 \times \mathbb{Z}_{2m}$ with $H = \langle b \rangle$ or $H = \langle bc^m \rangle$. In both cases, $Q = \mathbb{Z}_{2m}$ proving 3).

We now suppose $s \neq 0$, and so $n \neq 1$. Since $H \trianglelefteq \pi_1(O_{h_2}((A3, n), (B4, m)))$, we have $a^s bc^t = c(a^s bc^t)c^{-1} = a^{-s}(ba)c^t = a^{-s-1} bc^t$, which implies $a^{2s+1} = 1$. Thus n is odd and $s = (n-1)/2$. In addition, $1 = (a^s bc^t)^2 = c^{2t}$ whether t is even or odd. Hence $t = 0$ or m and $H = \langle a^{(n-1)/2} bc^t \rangle$. If t is even, then again by normality $a^{(n-1)/2} bc^t = b(a^{(n-1)/2} bc^t) b^{-1} = a^{(1-n)/2} bc^t$, showing $a^{n-1} = 1$.

However the order of a is n, giving a contradiction. Thus we may assume t is odd, therefore $t = m \geq 1$, and $H = \langle a^{(n-1)/2}bc^m \rangle$ with m odd. The group $Q = \pi_1(O_{h_2}((A3,n),(B4,m)))/H = \mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m}$, and since $n > 1$ and $m \geq 1$ are both odd, this group cannot be abelian. This completes the proof. ☐

COROLLARY 5.7. *Let $\varphi\colon G \to Homeo_{PL}(\mathbb{RP}^3)$ be a finite abelian action such that the quotient space $\mathbb{RP}^3/\varphi$ is the orbifold $O_{h_2}((A3,n),(B4,m))$. Then the following is true*:

1) The action is conjugate to the Standard Quotient Type 2 Action;
2) $n = 1$ and m is odd;
3) $G = \mathbb{Z}_{2m}$;
4) The covering corresponds to the subgroup $\langle bc^m \rangle$.

Proof. Let $\nu\colon \mathbb{RP}^3 \to \mathbb{RP}^3/\varphi = O_{h_2}((A3,n),(B4,m))$ be the covering map. Note that $\nu_*(\pi_1(\mathbb{RP}^3)) = H$ is a $\mathbb{Z}_2$ normal subgroup of $\pi_1(O_{h_2}((A3,n),(B4,m)))$. Furthermore, the quotient $Q = \pi_1(O_{h_2}((A3,n),(B4,m)))/H$ is isomorphic to the group G. Furthermore, 2) may also be excluded by Corollary 5.2 since the element $a \in \pi_1((A3,n))$ has a fixed point.

We now consider 1) of Proposition 5.6. Since c is orientation reversing, it follows that m must be even. Recall that c is identified with $zx \in \pi_1((B4,m))$, and $c^2 = (zx)^2 = y$. Thus $c^m = y^{\frac{m}{2}}$. The element $y \in \pi_1((B4,m))$ acts on the universal covering space as $y(t,v) = (t, ve^{\frac{2\pi i}{m}})$. Since this map has a fixed point, again by Corollary 5.2 we exclude this case. As for case 3), since b is an orientation reversing element, this leaves us with only $H = \langle bc^m \rangle$ and $G = \mathbb{Z}_{2m}$. Here m must be odd to guarantee an orientation preserving element. Applying Theorem 3.6, φ is conjugate to the Standard Quotient Type 2 Action, which is $\mathbb{Z}_{2m}$-action on $\mathbb{RP}^3$ with quotient type $O_{h_2}((A3,1),(B4,m))$. ☐

5.3. Quotient Type 3: $O_{h_3}((A2,n),(B3,m))$

From Section 2, the orbifold fundamental group of $O_{h_3}((A2,n),(B3,m))$ is

$$\begin{aligned}\pi_1(O_{h_3}((A2,n),(B3,m))) &= \langle a,b,c \mid [a,b] = [a,c] = 1, a^m = b^n = c^2 = 1, cbc^{-1} = b^{-1} \rangle \\ &= (\langle b \rangle \circ_{-1} \langle c \rangle) \times \langle a \rangle = \mathrm{Dih}(\mathbb{Z}_n) \times \mathbb{Z}_m\,.\end{aligned}$$

The elements a, b and c in $\pi_1((A2,n))$ acting on the universal covering space are defined by $a(t,v) = (t+1,v)$, $b(t,v) = (t, ve^{\frac{2\pi i}{n}})$ and $c(t,v) = (t,\overline{v})$.

PROPOSITION 5.8. *Let $H \simeq \mathbb{Z}_2$ be a normal subgroup of $\pi_1(O_{h_3}((A2,n),(B3,m)))$, and let $Q = \pi_1(O_{h_3}((A1,n),(B5,m)))/H$ be the quotient group. Then one of the following is true where $\epsilon = 0$ or 1:*

1) If m and n are both not equal to 1, then m, n are both even and $H = \langle b^{n/2}c^{\epsilon}a^{m/2} \rangle$. If either Q is abelian or $\epsilon = 1$, then $n = 2$ and $Q = \mathbb{Z}_2 \times \mathbb{Z}_m$;

2) If $n = 1$ and $m \neq 1$ is odd, then $H = \langle c \rangle$ and $Q = \mathbb{Z}_m$;

3) If $n = 1$ and $m \neq 1$ is even, then H is either $\langle c \rangle$, $\langle a^{m/2} \rangle$ or $\langle ca^{m/2} \rangle$, with quotient group Q isomorphic to $\mathbb{Z}_m$, $\mathbb{Z}_2 \times \mathbb{Z}_{m/2}$ or $\mathbb{Z}_m$;

4) If $n = 2$ and $m = 1$, then H is either $\langle b \rangle$, $\langle c \rangle$ or $\langle bc \rangle$ and $Q = \mathbb{Z}_2$;

5) If $n > 2$ and $m = 1$, then n is even, $H = \langle b^{n/2} \rangle$, and $Q = Dih(\mathbb{Z}_{n/2})$.

Proof. The subgroup $H = \langle b^s c^\epsilon a^t \rangle$ where $0 \leq s < n$, $\epsilon = 0$ or 1 and $0 \leq t < m$. We will assume first that m and n are both not equal to 1. Since H is normal, it follows that $b^s c^\epsilon a^t = c(b^s c^\epsilon a^t)c^{-1} = b^{-s} c^\epsilon a^t$. This implies $b^{2s} = 1$ or $s = n/2$. Observe that $1 = (b^s c^\epsilon a^t)^2$ is equal to a^{2t} if either $\epsilon = 0$ or 1. Hence $t = m/2$, and in either case $H = \langle b^{n/2} c^\epsilon a^{m/2} \rangle$. Suppose first that $\epsilon = 0$, and thus $H = \langle b^{n/2} a^{m/2} \rangle$. If follows that H is normal, and so no new information is obtained. If Q is abelian, then $bH = (cH)(bH)(cH)^{-1} = cbc^{-1}H = b^{-1}H$. Hence $b^2 \in \langle b^{n/2} c^\epsilon a^{m/2} \rangle$. If $b^2 \neq 1$, then $b^2 = b^{n/2} a^{m/2}$ or $b^{(4-n)/2} = a^{m/2}$, giving a contradiction. Thus $b^2 = 1$, $n = 2$ and $Q = \mathbb{Z}_2 \times \mathbb{Z}_m$. Suppose $\epsilon = 1$. Again by normality of H, it follows that $b^{n/2} c a^{m/2} = b(b^{n/2} c a^{m/2}) b^{-1}$ $= b^{n/2} b^2 c a^{m/2}$, implying again that $b^2 = 1$ and proving 1).

Since $\pi_1(O_{h_3}((A2, n), (B3, m)))$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_m$ in 2) and 3) and isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ in 4), the results follow easily. For 5), $\pi_1(O_{h_3}((A2, n), (B3, m))) = \mathrm{Dih}(\mathbb{Z}_n)$, $n > 2$ and $H = \langle b^s c^\epsilon \rangle$. If $\epsilon = 0$, it follows that $H = \langle b^{n/2} \rangle$. However if $\epsilon = 1$, it follows by normality that $b^s c = b(b^s c) b^{-1} = b b^s c$, which implies $b^2 = 1$ and $n = 2$. This contradicts $n > 2$. □

Corollary 5.9. *Let $\varphi\colon G \to Homeo_{PL}(\mathbb{RP}^3)$ be a finite abelian action such that the quotient space $\mathbb{RP}^3/\varphi$ is homeomorphic to $O_{h_3}((A2, n), (B3, m))$. Then the following is true:*

1) The action is conjugate to the Standard Quotient Type 3 Action;

2) $n = 2$ and m is even;

3) $G = \mathbb{Z}_2 \times \mathbb{Z}_m$;

4) The covering corresponds to the subgroup $\langle ba^{m/2} \rangle$.

Proof. Suppose $\varphi\colon G \to \mathrm{Homeo}_{PL}(\mathbb{RP}^3)$ is a finite abelian action such the quotient space $\mathbb{RP}^3/\varphi = O_{h_3}((A2, n), (B3, m))$, and let $\nu\colon \mathbb{RP}^3 \to O_{h_3}((A2, n), (B3, m))$ be the covering map with $\nu_*(\pi_1(\mathbb{RP}^3)) = H$. If $n = m = 1$, then $\pi_1(O_{h_3}((A2, 1), (B3, 1))) \simeq \mathbb{Z}_2$, giving a contradiction. Now H is a normal subgroup of $\pi_1(O_{h_3}((A2, n), (B3, m)))$ which is isomorphic to $\mathbb{Z}_2$ and corresponds to an orientation preserving element of order 2. The quotient group $Q = \pi_1(O_{h_3}((A2, n), (B3, m)))/H$ is abelian. The element $c \in \pi_1(O_{h_3}((A2, n), (B3, m)))$ is represented by an orientation reversing element, and therefore 2) in Proposition 5.8 is eliminated. Furthermore, since a and b are orientation preserving, H cannot be generated by $\langle b^{n/2} c a^{m/2} \rangle$, $\langle ca^{m/2} \rangle$ or $\langle b^{n/2} c \rangle$.

Suppose m and n are both not equal to 1. Then by 1) in Proposition 5.8, $n = 2$, $H = \langle ba^{m/2}\rangle$ and $Q = \mathbb{Z}_2 \times \mathbb{Z}_m$. We will show that the 3)-5) in Proposition 5.8 may also be eliminated.

Suppose $n = 1$, $m \neq 1$ is even, and since H must be an orientation preserving subgroup, $H = \langle a^{m/2}\rangle$ by 3) in Proposition 5.8. Recall that the element a is identified with the element y in $\pi_1((B3, m))$, and $y(t, v) = (t, e^{\frac{2\pi i}{m}})$ has a fixed point. This eliminates 3) by Corollary 5.2.

Suppose $n \neq 1$ is even, and $m = 1$. Thus by 4) and 5) in Proposition 5.8, $H = \langle b\rangle$ or $H = \langle b^{n/2}\rangle$, and $Q = \mathbb{Z}_2$ or $Q = \mathrm{Dih}(\mathbb{Z}_{n/2})$ respectively. In the latter case, in order for Q to be abelian, n must be 2 or 4. Since b has a fixed point, 4) and 5) are also eliminated by Corollary 5.2.

Since any regular covering of $O_{h_3}((A2, n), (B3, m))$ by $\mathbb{RP}^3$ corresponds to the subgroup $\langle ba^{m/2}\rangle$, any such action is conjugate to the Standard Quotient Type 3 Action on $\mathbb{RP}^3$, which is $\mathbb{Z}_2 \times \mathbb{Z}_m$. □

5.4. Quotient Type 4: The orbifold $O_{h_4}((B2, n), (B2, m))$

The orbifold fundamental group of $O_{h_4}((B2, n), (B2, m))$ is

$$\begin{aligned}\pi_1(O_{h_4}((B2, n), (B2, m))) &= \langle a, b \mid a^{2n} = b^2 = 1, ba^2b^{-1} = a^{-2}, (ab)^{2m} = 1\rangle \\ &= (\langle a^2\rangle \circ_{-1} \langle ab\rangle) \circ \langle b\rangle = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m}) \circ \mathbb{Z}_2 .\end{aligned}$$

The maps a and b are defined on the universal covering space of $(B2, n)$ by $a(t, v) = (-t + \frac{1}{2}, ve^{\frac{\pi i}{n}})$, $b(t, v) = (-t, \overline{v})$.

Proposition 5.10. *Let $H \simeq \mathbb{Z}_2$ be a normal subgroup of $\pi_1(O_{h_4}((B2, n), (B2, m)))$ generated by orientation preserving elements such that the quotient group $Q = \pi_1(O_{h_4}((B2, n), (B2, m)))/H$ is an abelian group. Then one of the following is true*:

1) $n = m = 2$, $H = \langle a^2(ab)^2\rangle$ and $Q = \mathbb{Z}_4 \times \mathbb{Z}_2$;
2) $n = 1$, $m = 2$, $H = \langle (ab)^2\rangle$ and $Q = \mathbb{Z}_2 \times \mathbb{Z}_2$;
3) $n = 2$, $m = 1$, $H = \langle a^2\rangle$ and $Q = \mathbb{Z}_2 \times \mathbb{Z}_2$;
4) $n = m = 1$, $H = \langle b\rangle$ and $Q = \mathbb{Z}_2$.

Proof. It is convenient to let $x = a^2$, $y = ab$ and $z = b$. Note that $yxy^{-1} = x^{-1}$, $zxz^{-1} = x^{-1}$ and $zyz^{-1} = x^{-1}y^{-1}$. Let $H = \langle x^s y^t z^\epsilon\rangle$ where $0 \le s < n$, $0 \le t < 2m$ and $\epsilon = 0$ or 1. Since y is orientation reversing, x and z are orientation preserving and H is generated by orientation preserving elements, it follows that t must be even. This implies $xy^tx^{-1} = y^t$ and $zy^tz^{-1} = y^{-t}$.

Case I: $H = \langle x^s y^t\rangle$.

Assume first that $s \neq 0$ and $t \neq 0$. Since $1 = (x^s y^t)^2 = x^{2s}y^{2t}$, we obtain $s = \frac{n}{2}$, $t = m$, and thus $H = \langle x^{\frac{n}{2}} y^m\rangle$. It follows that $\langle x^{\frac{n}{2}} y^m\rangle$ is a normal subgroup, and thus no new information is obtained. Being that

$\pi_1(O_{h_4}((B2,n),(B2,m)))/H$ is abelian, we have that $xH = yHxHy^{-1}H = x^{-1}H$. This implies $x^2 \in \langle x^{\frac{n}{2}}y^m\rangle$. The equation $x^2 = x^{\frac{n}{2}}y^m$ is impossible, and so $x^2 = 1$ showing $n = 2$. This shows that$\pi_1(O_{h_4}((B2,n),(B2,m))) = (\mathbb{Z}_2 \times \mathbb{Z}_{2m}) \circ \mathbb{Z}_2$, and so $Q = \pi_1(O_{h_4}((B2,n),(B2,m)))/H = \mathbb{Z}_{2m} \circ \mathbb{Z}_2$. The action given in the quotient group Q is $zyz^{-1} = y^{m-1}$. In order for Q to be abelian, $m = 2$, showing 1) in the statement of the proposition.

Suppose $s = 0$, $t \neq 0$, and so $H = \langle y^t\rangle$. We will show that this gives 2) in the proposition. It follows that $t = m$ which is even, and so H is always normal giving no new information. Now $Q = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_m) \circ \mathbb{Z}_2$ where $zyz^{-1} = x^{-1}y^{-1}$. In order for Q to be abelian, $x = 1$ and so $n = 1$, and $m = 2$.

Next assume that $s \neq 0$, $t = 0$ and so $H = \langle x^s\rangle$. We obtain $s = \frac{n}{2}$ and $H = \langle x^{\frac{n}{2}}\rangle$. Furthermore, $Q = (\mathbb{Z}_{\frac{n}{2}} \circ_{-1} \mathbb{Z}_{2m}) \circ \mathbb{Z}_2$. In order for Q to be abelian, $n = 2$ and $m = 1$, showing 3) in the statement of the proposition.

Case II: $H = \langle x^s y^t z\rangle$.

Suppose $t \neq 0$. It follows that $1 = (x^s y^t z)^2$, giving no new information. A computation shows $y(x^s y^t z)y^{-1} = x^{-s+1}y^{t+2}z$, which must equal $x^s y^t z$. Therefore $y^2 = 1$ and $m = 1$, contradicting t even.

Assume $t = 0$, $s \neq 0$ and $n \neq 1$, and thus $H = \langle x^s z\rangle$. A computation shows $y(x^s z)y^{-1} = x^{-s+1}y^2 z$, which must equal $x^s z$ implying $x^{2s-1} = 1$. In addition, we must also have $x^s z = z(x^s z)z^{-1} = x^{-s}z$, giving $x^{2s} = 1$. This implies $x = 1$, contradicting $n \neq 1$.

We now assume $t = 0$, $s = 0$, and thus $H = \langle z\rangle$. By normality, $z = yzy^{-1} = y^2xz$, which implies $x = 1$ and $y^2 = 1$. Thus $n = 1$ and $m = 1$, giving us 4) of the proposition and completing the proof. □

Corollary 5.11. *Let $\varphi\colon G \to Homeo_{PL}(\mathbb{RP}^3)$ be a finite abelian action such that the quotient space $\mathbb{RP}^3/\varphi$ is the orbifold $O_{h_4}((B2,n),(B2,m))$. Then the following is true:*

1) *The action is conjugate to the Standard Quotient Type 4 Action;*
2) *$n = m = 2$;*
3) *$G = \mathbb{Z}_4 \times \mathbb{Z}_2$;*
4) *The covering corresponds to the subgroup $\langle a^2(ab)^2\rangle$.*

Proof. Let $\nu\colon \mathbb{RP}^3 \to \mathbb{RP}^3/\varphi$ be an orbifold covering map and $\nu_*(\pi_1(\mathbb{RP}^3)) = H$, a $\mathbb{Z}_2$-normal subgroup of $\pi_1(O_{h_4}((B2,n),(B2,m)))$ with quotient $\pi_1(O_{h_4}((B2,n),(B2,m)))/H$ an abelian group. Applying Proposition 5.10, suppose 2) or 3) holds. Then $\mathbb{RP}^3/\varphi$ is either $O_{h_4}((B2,1),(B2,2))$ or $O_{h_4}((B2,2),(B2,1))$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. In either case, there is a cone point of order 4 in the quotient space $\mathbb{RP}^3/\varphi$. This would imply that there is an element in G of order 4, giving a contradiction. Since b defined on the universal covering space of $(B2,n)$ has a fixed point, 4) is also eliminated by Corollary 5.2. This leaves 1). Now any regular covering of $O_{h_4}((B2,n),(B2,m))$ by $\mathbb{RP}^3$ corresponds to the subgroup $\langle a^2(ab)^2\rangle$. Therefore, any such action is conjugate to the Standard Quotient Type 4 Action on $\mathbb{RP}^3$, which is $\mathbb{Z}_4\times\mathbb{Z}_2$. □

5.5. Quotient Type 5: The orbifold $O_{h_5}((B6,n),(B6,m))$

Recall the orbifold fundamental group of $\pi_1(O_{h_5}((B6,n),(B6,m)))$ is

$$\begin{aligned}\langle a,b,c,d \mid a^2=b^2=c^2=(bc)^n=d^2=1,\\ [a,b]=[a,c]=[b,d]=[c,d]=1,(ad)^m=1\rangle\\ =(\langle bc\rangle\circ_{-1}\langle c\rangle)\times(\langle ad\rangle\circ_{-1}\langle a\rangle)=(\mathbb{Z}_n\circ_{-1}\mathbb{Z}_2)\times(\mathbb{Z}_m\circ_{-1}\mathbb{Z}_2)\,.\end{aligned}$$

We note that the generators a, b, c and d are all orientation reversing elements. The maps on the universal covering space of $(B6,n)$ are defined as follows: $a(t,v)=(-t,v)$, $b(t,v)=(t,\overline{v})$, and $c(t,v)=(t,\overline{v}e^{\frac{-2\pi i}{n}})$ and $d(t,v)=(-t-1,v)$.

Proposition 5.12. *Let $H\simeq\mathbb{Z}_2$ be a normal subgroup of $\pi_1(O_{h_5}((B6,n),(B6,m)))$ generated by orientation preserving elements such that the quotient group $Q=\pi_1(O_{h_5}((B6,n),(B6,m)))/H$ is an abelian group. Then the following is true*:

1) $n=1$ or 2 and $m=2$ or 4, $H=\langle(ad)^{\frac{m}{2}}\rangle$ and Q is either $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ or $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$;
2) $n=2$ or 4 and $m=1$ or 2, $H=\langle(bc)^{\frac{n}{2}}\rangle$ and Q is either $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ or $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$;
3) $n=m=2$, $H=\langle(bc)(ad)\rangle$ and $Q=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$;
4) $n=m=2$, $H=\langle cd\rangle$ and $Q=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$;
5) $n=2$ and $m=1$ or 2, $H=\langle ba\rangle$ and Q is either $\mathbb{Z}_2\times\mathbb{Z}_2$ or $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$;
6) $n=1$ or 2, $m=1$ or 2, $H=\langle ca\rangle$ and Q is either $\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$, or $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$;
7) $n=m=2$, $H=\langle bd\rangle$ and $Q=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$.

Proof. The group $H=\langle(bc)^sc^{\epsilon_1}(ad)^ta^{\epsilon_2}\rangle$ where $0\le s<n$, $0\le t<m$ and $\epsilon_i=0$ or 1. Since both (bc) and (ad) are orientation preserving, a and c are both orientation reversing, the two cases that need to be considered are $\epsilon_1=\epsilon_2=0$ or $\epsilon_1=\epsilon_2=1$.

Case I: $H=\langle(bc)^s(ad)^t\rangle$.

Suppose $s=0$ and $t\ne0$. This implies that $H=\langle(ad)^{\frac{m}{2}}\rangle$ and the quotient group $Q\simeq(\mathbb{Z}_n\circ_{-1}\mathbb{Z}_2)\times(\mathbb{Z}_{\frac{m}{2}}\circ_{-1}\mathbb{Z}_2)$. If Q is abelian, we must have $n=1$ or 2 and $m=2$ or 4.

If $s\ne0$ and $t=0$, then $H=\langle(bc)^{\frac{n}{2}}\rangle$. The quotient group $Q\simeq(\mathbb{Z}_{\frac{n}{2}}\circ_{-1}\mathbb{Z}_2)\times(\mathbb{Z}_m\circ_{-1}\mathbb{Z}_2)$, and thus if Q is abelian $n=2$ or 4 and $m=1$ or 2.

We now assume $s\ne0$ and $t\ne0$. Since $H=\langle(bc)^s(ad)^t\rangle\simeq\mathbb{Z}_2$, and bc and ad commute, we have $1=((bc)^s(ad)^t)^2=(bc)^{2s}(ad)^{2t}$. This implies $s=\frac{n}{2}$,

$t = \frac{m}{2}$ and $H = \langle (bc)^{\frac{n}{2}} (ad)^{\frac{m}{2}} \rangle$. Clearly H is normal. In the abelian quotient Q, we have $bH = bcHcH = cHbcH = cbcH$, which implies $(bc)^2 \in H$. This could only happen if $(bc)^2 = 1$; hence $n = 2$. Similarly $dH = aHadH = adHaH = adaH$, which implies $(ad)^2 \in H$ and $m = 2$. Thus $H = \langle (bc)(ad) \rangle$ and $n = m = 2$.

Case II: $H = \langle (bc)^s c(ad)^t a \rangle$.

Suppose $s = 0$ and $t \neq 0$. In this case $H = \langle c(ad)^t a \rangle$. By normality, $c(ad)^t a = (ad)[c(ad)^t a](ad)^{-1} = c(ad)^t (ad)^2 a$, which implies $(ad)^2 = 1$, $m = 2$ and $t = 1$. Likewise, $c(ad)^t a = (bc)[c(ad)^t a](bc)^{-1} = (bc)^2 c(ad)^t a$ shows $(bc)^2 = 1$ and $n = 2$. Therefore in this case, $\pi_1(O_{h_5}((B6, 2), (B6, 2)))$ is abelian and $H = \langle cd \rangle$.

If $s \neq 0$ and $t = 0$, then $H = \langle (bc)^s ca \rangle$. Conjugating the generator by bc and using the argument from the previous case, shows that $(bc)^2 = 1$, and therefore $n = 2$, $s = 1$ and $H = \langle ba \rangle$. In order for Q to be abelian $m = 1$ or 2.

We consider the case where $s = t = 0$ and $H = \langle ca \rangle$. Suppose $n \neq 1$. By computing, we obtain $(bc)(ca)(bc)^{-1} = bcba$, and by normality this must equal ca. Hence we obtain $(bc)^2 = 1$ which implies $n = 2$. Similarly, if $m \neq 1$, then $(ad)(ca)(ad)^{-1} = ca(ad)^{-1}(da) = ca(ad)^{-2}$. By normality, this must equal ca, and thus $(ad)^{-2} = 1$ implying $m = 2$. We conclude that $n = 1$ or 2, and $m = 1$ or 2.

Finally consider the case $s \neq 0$ and $t \neq 0$ and $H = \langle (bc)^s c(ad)^t a \rangle$. By conjugating the generator by bc and ad, we conclude as above that $n = m = 2$. Thus the group is abelian and $H = \langle bd \rangle$. □

Corollary 5.13. *Let $\varphi \colon G \to Homeo_{PL}(\mathbb{RP}^3)$ be a finite abelian action such that the quotient space $\mathbb{RP}^3/\varphi$ is the orbifold $O_{h_5}((B6, n), (B6, m))$. Then the following is true:*

1) *The action is conjugate to the Standard Quotient Type 5 Action;*
2) *$n = m = 2$;*
3) *$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$;*
4) *The covering corresponds to the subgroup $\langle (bc)(ad) \rangle$.*

Proof. We obtain the following maps: $(bc)(t, v) = (t, ve^{\frac{2\pi i}{n}})$, $(cd)(t, v) = (-t - 1, \overline{v}e^{-\frac{2\pi i}{n}})$, $(ba)(t, v) = (-t, \overline{v})$, $(ca)(t, v) = (-t, \overline{v}e^{-\frac{2\pi i}{n}})$ and $(bd)(t, v) = (-t - 1, \overline{v})$. Note that all these maps have fixed points, and therefore 2) and 4) - 7) in Proposition 5.12 may be excluded by Corollary 5.2.

The element ad in $\pi_1((B6, n))$ is identified (See Appendix) with the element yz in $\pi_1((B6, m))$, and $(yz)(t, v) = (t, ve^{\frac{2\pi i}{n}})$ which has a fixed point. Thus 1) in Proposition 5.12 is excluded like the others above. Hence, the only remaining case in Proposition 5.12 is 3). Since any regular covering of $O_{h_5}((B6, n), (B6, m))$ by $\mathbb{RP}^3$ corresponds to the subgroup$\langle (bc)(ad) \rangle$, any such action is conjugate to the Standard Quotient Type 5 Action on $\mathbb{RP}^3$ which is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. □

5.6. Quotient Type 6: The orbifold $O_{h_6}((B7,n),(B7,m))$

The fundamental group of $O_{h_6}((B7,n),(B7,m))$ is

$$\begin{aligned}\langle a,b,c \mid a^2 = b^{2n} = (ab^{-1}ab)^m = c^2 = 1, [a,b^2] = [a,c] = 1, cbc^{-1} = b^{-1}\rangle \\ = (\langle ab^{-1}ab\rangle \circ_{-1} \langle a\rangle) \circ (\langle b\rangle \circ_{-1} \langle c\rangle) = (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2) \circ (\mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2).\end{aligned}$$

The maps a, b and c act on the universal covering space of $(B7,n)$ as follows: $a(t,v) = (-t-1,v)$, $b(t,v) = (-t, ve^{\frac{\pi i}{n}})$, $c(t,v) = (t,\overline{v})$. Note also that $bab^{-1} = b^{-1}ab$. For convenience if we let $d = ab^{-1}ab$, then $bdb^{-1} = d^{-1}$, $bab^{-1} = ad$ and $cdc^{-1} = d$.

Proposition 5.14. *Let $H \simeq \mathbb{Z}_2$ be a normal subgroup of $\pi_1(O_{h_6}((B7,n),(B7,m)))$ generated by orientation preserving elements such that the quotient group $Q = \pi_1(O_{h_6}((B7,n),(B7,m)))/H$ is an abelian group. Then the following is true*:

1) $n = 2$, $m = 1$, $H = \langle b^2\rangle$ and $Q = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$;
2) $n = 1$, $m = 2$, $H = \langle d\rangle$ and $Q = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$;
3) $n = m = 1$, H is one of the groups $\langle ac\rangle$, $\langle ab\rangle$, $\langle bc\rangle$ and $Q = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. The group $H = \langle d^s a^{\epsilon_1} b^t c^{\epsilon_2}\rangle \simeq \mathbb{Z}_2$ where $0 \le s < m$, $0 \le t < 2n$ and $\epsilon_i = 0$ or 1. Since a, b and c are orientation reversing elements, it follows that d is orientation preserving. Since H is an orientation preserving subgroup, we have the following cases to consider: I) t is even, and either $\epsilon_1 = \epsilon_2 = 0$ or $\epsilon_1 = \epsilon_2 = 1$, II) t is odd, and either $\epsilon_1 = 1$ and $\epsilon_2 = 0$ or $\epsilon_1 = 0$ and $\epsilon_2 = 1$.

Case I: t is even.

We consider first the situation when $\epsilon_1 = \epsilon_2 = 0$, and thus $H = \langle d^s b^t\rangle$. Assume $s \neq 0$ and $t \neq 0$. Since t is even, it follows that b^t commutes with d, and thus $1 = (d^s b^t)^2 = d^{2s} b^{2t}$. This implies $s = \frac{m}{2}$, $t = n$ and $H = \langle d^{\frac{m}{2}} b^n\rangle$. One can verify that H is indeed a normal subgroup. Since the quotient Q is abelian, we have $dH = (bH)(dH)(bH)^{-1} = d^{-1}H$, or $d^2 \in \langle d^{\frac{m}{2}} b^n\rangle$. This is impossible unless $d^2 = 1$. Thus $m = 2$, the fundamental group $\pi_1(O_{h_6}((B7,2),(B7,m))) = (\mathbb{Z}_2 \times \mathbb{Z}_2) \circ (\mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2)$ and $H = \langle db^n\rangle$. Now $Q = \mathbb{Z}_2 \times (\mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2)$, which is not abelian unless $n = 1$. However in this case $t = n$ is even giving a contradiction, and so this subcase cannot happen. Therefore, in this case either $s = 0$ or $t = 0$. Suppose $s = 0$, and thus $H = \langle b^t\rangle$. It follows that $t = n$ and H is always a normal subgroup. Furthermore, $Q = (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2) \circ (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2)$ being abelian implies m =1 and $n = 2$, and thus $\pi_1(O_{h_6}((B7,2),(B7,1))) = \mathbb{Z}_2 \times (\mathbb{Z}_4 \circ_{-1} \mathbb{Z}_2)$. A similar argument shows that if $t = 0$, then $n = 1$, $m = 2$, $H = \langle d\rangle$ and $\pi_1(O_{h_6}((B7,2),(B7,1))) = (\mathbb{Z}_2 \times \mathbb{Z}_2) \circ (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Assume $\epsilon_1 = \epsilon_2 = 1$ and hence $H = \langle d^s ab^t c\rangle$. If $s = 0$ and $H = \langle ab^t c\rangle$, then we always have $1 = (ab^t c)^2$ giving no new information. By normality, $ab^t c = b(ab^t c)b^{-1} = adb^{t+2}c$, which implies $m = 1$, $b^2 = 1$ and $n = 1$. Since

t is even, we must have $t = 0$. Thus $n = m = 1$, $\pi_1(O_{h_6}((B7,1),(B7,1))) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $H = \langle ac \rangle$. We now suppose $t = 0$ and $H = \langle d^s ac \rangle$. It always follows that $1 = (d^s ac)^2$. By normality, $d^s ac = b(d^s ac)b^{-1} = d^{-s-1}ab^2c = d^{-s-1}b^2ac$. This implies $b^2 = 1$, $n = 1$ and $s = \frac{m-1}{2}$. Conjugating by a yields $d^{\frac{m-1}{2}}ac = a(d^{\frac{m-1}{2}}ac)a^{-1} = d^{\frac{1-m}{2}}ac$, which implies d =1 and $m = 1$. The two outcomes give us 3) in the statement of the theorem. We now suppose $s \neq 0$ and $t \neq 0$. In this case it always follows that $1 = (d^s ab^t c)^2$, so we do not obtain any new information. By normality, we must have $d^s ab^t c = b(d^s ab^t c)b^{-1} = d^{-s-1}ab^{t+2}c$, which implies $s = \frac{m-1}{2}$, $b^2 = 1$ and $n = 1$. Since t is even, $t = 0$ giving a contradiction.

Case II: t is odd.

We suppose $\epsilon_1 = 1$ and $\epsilon_2 = 0$, and thus $H = \langle d^s ab^t \rangle$. If $s = 0$ and thus $H = \langle ab^t \rangle$, then $1 = (ab^t)^2 = db^{2t}$. This implies $d = 1$, $m = 1$ and $t = n$. Thus the orbifold fundamental group is $\mathbb{Z}_2 \times (\mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2)$ and $Q = \mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2$. Now Q is abelian only if $n = 1$. Thus $n = m = 1$, $\pi_1(O_{h_6}((B7,1),(B7,1))) = \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $H = \langle ab \rangle$. Suppose $s \neq 0$ and $H = \langle d^s ab^t \rangle$. Since $H \simeq \mathbb{Z}_2$, $1 = (d^s ab^t)^2 = d^{2s+1}b^{2t}$, so $s = \frac{m-1}{2}$ implying m is odd, and $t = n$ which is also odd. Thus $H = \langle d^{\frac{m-1}{2}}ab^n \rangle$, and one can check that this is always a normal subgroup. Suppose $m \neq 1$. Since Q is abelian, we have $dH = (bH)(dH)(bH)^{-1} = d^{-1}H$, implying $d^2 \in H$. It follows that $d^2 = 1$ and $m = 2$. However m is odd giving a contradiction. Thus $m = 1$, $\pi_1(O_{h_6}((B7,n),(B7,1))) = \mathbb{Z}_2 \times (\mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2)$ and $H = \langle ab^n \rangle$. Now $Q = \mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2$, which is abelian only if $n = 1$. Thus in this case $n = m = 1$ to obtain $\pi_1(O_{h_6}((B7,1),(B7,1))) = \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $H = \langle ab \rangle$.

Assume now that $\epsilon_1 = 0$ and $\epsilon_2 = 1$, and thus $H = \langle d^s b^t c \rangle$. If $s = 0$ and $H = \langle b^t c \rangle$, then it always follows that $1 = (b^t c)^2$. By normality, $b^t c = a(b^t c)a^{-1} = db^t c$. Thus $d = 1$, $m = 1$ and the orbifold fundamental group is $\mathbb{Z}_2 \times (\mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2)$. Again by normality, $b^t c = c(b^t c)c^{-1} = b^{-t}c$, implying $t = n$. Furthermore, $b^n c = b(b^n c)b^{-1} = b^{n+2}c$. This implies $b^2 = 1$ and $n = 1$. Thus $n = m = 1$ so that $\pi_1(O_{h_6}((B7,1),(B7,1))) = \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $H = \langle bc \rangle$. We now suppose $s \neq 0$. A computation shows that $(d^s b^t c)^2 = 1$ is always true. Suppose $m \neq 1$. By normality, $d^s b^t c = a(d^s b^t c)a^{-1} = d^{-s+1}b^t c$, and thus $s = \frac{m+1}{2} \neq 0$ and $H = \langle d^{\frac{m+1}{2}} b^t c \rangle$. Again by normality, we have $d^{\frac{m+1}{2}} b^t c = b(d^{\frac{m+1}{2}} b^t c)b^{-1} = d^{\frac{-m-1}{2}} b^{t+2}c$, which implies $d^{m+1} = 1$. Hence $d = 1$ and $m = 1$, contradicting the fact that $m \neq 1$. Hence $m = 1$ and $H = \langle b^t c \rangle$. Using normality, we have $b^t c = b(b^t c)b^{-1} = b^{t+2}c$, or $b^2 = 1$. Thus $n = 1$, $\pi_1(O_{h_6}((B7,1),(B7,1))) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $H = \langle bc \rangle$. □

Corollary 5.15. *Let $\varphi\colon G \to Homeo_{PL}(\mathbb{RP}^3)$ be a finite abelian action such that the quotient space $\mathbb{RP}^3/\varphi$ is the orbifold $O_{h_6}((B7,n),(B7,m))$. Then the following is true*:

1) The action is conjugate to the Standard Quotient Type 6 Action;

2) $n = m = 1$;
3) $G = \mathbb{Z}_2 \times \mathbb{Z}_2$;
4) The covering corresponds to the subgroup $\langle ab\rangle$.

Proof. Note that in 1) of Proposition 5.14, $b^2(t,v) = (t,-v)$, and in 3) $ac(t,v) = (-t-1,\overline{v})$ and $bc(t,v) = (-t,-\overline{v})$. Since they have fixed points, these cases are excluded by Corollary 5.2. In 2) of Proposition 5.14, $d = ab^{-1}ab \in \pi_1((B7,1))$ is identified with $y^2 \in \pi_1((B7,2))$ (See Appendix). Since $y(t,v) = (-t, ve^{\frac{\pi i}{m}})$, we see that y^m has a fixed point, and we may exclude this case. This leaves only the 3) where $n = m = 1$ and the subgroup $\langle ab\rangle$. Since any regular covering of $O_{h_6}((B7,n),(B7,m))$ by $\mathbb{RP}^3$ corresponds to the subgroup$\langle ab\rangle$, any such action is conjugate to the Standard Quotient Type 6 Action on $\mathbb{RP}^3$ which is $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

5.7. Quotient Type 7: The orbifold $O_{h_7}((B1,n),(B8,m))$

Recall that the orbifold fundamental group is

$$\begin{aligned}&\pi_1(O_{h_7}((B1,n),(B8,m)))\\ &\quad = \langle a,b,c \mid a^n = b^2 = c^2 = 1, bab^{-1} = a^{-1}, [a,c] = 1, (cb)^{2m} = 1\rangle\\ &\qquad = \langle a\rangle \circ (\langle cb\rangle \circ_{-1} \langle c\rangle) = \mathbb{Z}_n \circ \mathrm{Dih}(\mathbb{Z}_{2m}).\end{aligned}$$

It follows that $(cb)a(cb)^{-1} = a^{-1}$. From the Appendix, that maps a, b, c on the universal covering space of $(B1,n)$ are defined as follows: $a(t,v) = (t, ve^{2\pi i/n})$, $b(t,v) = (-t,\overline{v})$, and $c(t,v) = (\frac{1}{2}-t, v)$.

Proposition 5.16. *Let $H \simeq \mathbb{Z}_2$ be a normal subgroup of $\pi_1(O_{h_7}((B1,n),(B8,m)))$ generated by orientation preserving elements such that the quotient group $Q = \pi_1(O_{h_7}((B1,n),(B8,m)))/H$ is an abelian group. Then the following is true:*
1) $n = 2$, $m = 1$, H is $\langle a\rangle$, $\langle b\rangle$ or $\langle ab\rangle$ and Q is $\mathbb{Z}_2 \times \mathbb{Z}_2$;
2) $n = 4$, $m = 1$, $H = \langle a^2\rangle$ and Q is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$;
3) $n = 1$ or 2, $m = 2$, $H = \langle (cb)^2\rangle$ and Q is either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. The subgroup $H = \langle a^s(cb)^t c^\epsilon\rangle$ where $0 \le s < n$, $0 \le t < 2m$ and $\epsilon = 0$ or 1. Since only c is orientation reversing, the elements cb and c are orientation reversing. Thus there are two cases to consider, t even and $\epsilon = 0$, or t odd and $\epsilon = 1$.

<u>Case I</u>: t is even, $\epsilon = 0$, and thus $H = \langle a^s(cb)^t\rangle$.

Since t is even, we have $1 = (a^s(cb)^t)^2 = a^{2s}(cb)^{2t}$. Suppose first that $t = 0$ and $s \neq 0$, and thus $H = \langle a^s\rangle$. It follows that $s = \frac{n}{2}$, H is normal and $Q = \mathbb{Z}_{\frac{n}{2}} \circ (\mathbb{Z}_{2m} \circ_{-1} \mathbb{Z}_2)$. In order for Q to be abelian, we must have $\frac{n}{2} = 1$ or 2 and $m = 1$. This gives us 1) and 2) in the statement of the proposition.

Suppose $t \neq 0$ and $s = 0$, and so $H = \langle (cb)^t \rangle$. We see that $t = m$ which is even, and thus a and $(cb)^m$ commute. This implies H is always normal. Now $Q = \mathbb{Z}_n \circ (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2)$, which is only abelian if both n and m equal 1 or 2. Since m is even, $m = 2$. and this gives 3). Assume next that $t \neq 0$ and $s \neq 0$. It follows that $s = \frac{n}{2}$ and $t = m$ where m is even. A check shows that $H = \langle a^{\frac{n}{2}}(cb)^m \rangle$ is always normal. Since Q is abelian, we must have $a^{-1}H = (cbH)(aH)(cbH)^{-1} = aH$, which implies $a^2 \in \langle a^{\frac{n}{2}}(cb)^m \rangle$. This is impossible unless $a^2 = 1$, and thus $n = 2$. This shows $H = \langle a(cb)^m \rangle$, and $Q = \mathbb{Z}_{2m} \circ_{-1} \mathbb{Z}_2$. This can only be abelian if $m = 1$, which contradicts m being even. So this sub-case cannot happen.

Case II: t is odd, $\epsilon = 1$, and thus $H = \langle a^s(cb)^t c \rangle$.

It is always the case that $(a^s(cb)^t c)^2 = 1$, since t is odd. Suppose $s = 0$, and so $H = \langle (cb)^t c \rangle$. Now $a(cb)^t c a^{-1} = a^2(cb)^t c$, which must equal $(cb)^t c$ by normality. This implies $n = 2$. Furthermore, $(cb)((cb)^t c)(cb)^{-1} = (cb)^{t+2}c$, which by normality must equal $(cb)^t c$. This implies $(cb)^2 = 1$ and $m = 1$. Thus $\pi_1(O_{h_7}((B1,2),(B8,1))) = \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $H = \langle (cb)c \rangle = \langle b \rangle$, giving 1). We now assume $s \neq 0$, and thus $H = \langle a^s(cb)^t c \rangle$. A computation shows $a(a^s(cb)^t c)a^{-1} = a^{s+2}(cb)^t c$ and $(cb)((a^s(cb)^t c)(cb)^{-1} = a^{-s}(cb)^{t+2}c$. By normality, it must be the case that $a^2 = 1$ and $(cb)^2 = 1$. Thus $n = 2$ and $m = 1$, which implies $H = \langle a(cb)c \rangle = \langle ab \rangle$, giving us 1). □

COROLLARY 5.17. *There is no abelian action on $\mathbb{RP}^3$, whose quotient space is the orbifold $O_{h_7}((B1,n),(B8,m))$.*

Proof. By Proposition 5.16, we need only consider the subgroups listed there. Observe that the maps a, a^2, b and ab have fixed points in the universal cover of $(B1,n)$. So these cases may be excluded by Corollary 5.2. The remaining case to consider is the subgroup generated by $(cb)^2$ where $n = 1$ or 2 and $m = 2$. From the Appendix, we see that the elements c and b in $\pi_1((B1,n))$ are identified with the elements y and yz in $\pi_1((B8,2))$ respectively. Thus $(cb)^2$ is identified with $(y^2z)^2$. The maps y and z acting on the universal cover $\mathbb{R} \times D^2$ of $(B8,2)$ are defined by $y(t,v) = (t,\overline{v})$ and $z(t,v) = (1-t, ve^{\frac{\pi i}{2}})$. Since $y^2 = 1$, we have $(cb)^2$ identified with z^2. Note that z^2 has a fixed point. Again applying Corollary 5.2 proves the result. □

6. Main results

In the last section of this paper, we summarize the main results.

THEOREM 6.1. *Let $\varphi\colon G \to Homeo_{PL}(\mathbb{RP}^3)$ be an orientation reversing finite abelian action. Then one of the following cases is true:*

1) $G = \mathbb{Z}_{2^b m}$ where $b>1$, m is odd and $\mathbb{RP}^3/\varphi$ is $O_{h_1^{-1}}((B5, 2^{b-1}m),(A1,2))$;

2) $G = \mathbb{Z}_{2m}$, m is odd and $\mathbb{RP}^3/\varphi$ is t $O_{h_2^{-1}}((B4,m),(A3,1))$;

3) $G = \mathbb{Z}_m \times \mathbb{Z}_2$, m even and $\mathbb{RP}^3/\varphi$ is $O_{h_3}((A2,2),(B3,,m))$;
4) $G = \mathbb{Z}_4 \times \mathbb{Z}_2$, and $\mathbb{RP}^3/\varphi$ is $O_{h_4}((B2,2),(B2,2))$;
5) $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{RP}^3/\varphi$ is t $O_{h_5}((B6,2),(B6,2))$;
6) $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{RP}^3/\varphi$ is t $O_{h_6}((B7,1),(B7,1))$.
Furthermore, in each individual case i), where $1 \leq i \leq 6$, φ is equivalent to the Standard Quotient Type i Action.

Proof. Let $\varphi\colon G \to Homeo_{PL}(\mathbb{RP}^3)$ be an orientation reversing finite abelian action. By Corollary 4.2, φ splits and preserves the sides of the splitting. Write $\mathbb{RP}^3 = V_1 \cup V_2$, where each V_i for $i = 1,2$ is a $\varphi(G)$-invariant solid torus. The non-orientable 3-orbifold $V_i/\varphi(G)$ is one of the orbifolds $(A1,n)$,...,$(A3,n)$, $(B1,n)$,...,$(B8,n)$. This implies that if $\nu\colon \mathbb{RP}^3 \to \mathbb{RP}^3/\varphi(G)$ is the orbifold covering, then $\mathbb{RP}^3/\varphi(G)$ is $O_\xi(X,Y)$ where X and Y are any of the orbifolds $(A1,n)$,...,$(A3,n)$,$(B1,n)$,...,$(B8,n)$ and $\xi\colon \partial X \to \partial Y$ is some homeomorphism. Since $\nu_*(\pi_1(\mathbb{RP}^3))$ has finite index in $\pi_1(O_\xi(X,Y))$, it follows that $\pi_1(O_\xi(X,Y))$ is finite. By Theorem 2.1, $O_\xi(X,Y)$ is one of the seven orbifolds listed in the chart. Corollary 5.17 states there is no orientation reversing finite abelian action on $\mathbb{RP}^3$ whose quotient space is the orbifold $O_{h_7}((B1,n),(B8,m))$, thus excluding the seventh orbifold in the chart. Applying Corollaries 5.5, 5.7, 5.9, 5.11, 5.13 and 5.15 to the first six orbifolds proves the result. □

Appendix

In this Appendix, we will define the orbifolds $(A1,n),...,(A3,n),(B1,n),...,$ $(B8,n)$ along with their fundamental groups. Since the fundamental groups of each boundary surjects onto the fundamental group of their orbifolds, we use the same letters for both presentations of the fundamental groups. In addition, if X and Y are orbifolds from this list having homeomorphic boundaries, we will identify the orbifolds $O_\xi(X,Y)$ obtained by identifying ∂X to ∂Y via ξ which have finite fundamental groups.

In describing the orbifolds $O_\xi(X,Y)$, we will give the details for $X = (A1,n)$ and $Y = (B5,m)$ by providing the definition of the lift of the gluing map $\xi\colon \partial X \to \partial Y$ to the universal cover of each boundary component. In addition, we obtain a description of the lift of the gluing map on the orientable covers $\partial V(n)$ and $\partial V(m)$ of $\partial(A1,n)$ and $\partial(B5,m)$ respectively. For subsequent orbifolds, we just describe the lift of the gluing map to the covers $\partial V(n)$ and $\partial V(m)$ and refer the reader to [13] for the details.

We start by considering the orbifolds which are double covered by $(A0,n)$. It will be convenient to define the 2-dimensional orbifolds $D^2(n)$ and $\Delta(n)$. Let r_o be a rotation and r_e be a reflection on D^2, defined by $r_o(\rho e^{i\theta}) = \rho e^{i(\theta+2\pi/n)}$ and $r_e(\rho e^{i\theta}) = \rho e^{-i\theta}$. Now $D^2/\langle r_o\rangle$ is the orbifold $D^2(n)$ whose underlying space is a disk, and has a cone point of order n in its center. The map r_e

induces a reflection $\overline{r_e}$ on $D^2(n)$, and $D^2(n)/\langle\overline{r_e}\rangle$ is the orbifold denoted by $\Delta(n)$. We may be parameterize it as $\{\rho e^{i\theta} \mid 0 \le \rho \le 1,\ 0 \le \theta \le \pi\}$ where the point $(0,0)$ is a coner-reflector point of order n, and $\{\rho e^{i\theta} \mid \theta = 0 \text{ or } \pi\}$ is the set of mirror points.

In addition, we need to define the 3-dimensional orbifolds $B^3(n)$, $C(\mathbb{P}^2, 2n)$ and Z_n^h. Let $\mathbb{B}^3 = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$ and for any point $(x,y,z) \in \mathbb{B}^3$ using spherical coordinates, we have $x = \rho\ \sin\phi\cdot\cos\theta$, $y = \rho\ \sin\phi\cdot\sin\theta$ and $z = \rho\ \cos\phi$ where $0 \le \rho \le 1$. We begin by defining a rotation of order n on $\mathbb{B}^3$ as follows:

$$r(x,y,z) = (\rho\ \sin\phi\cdot\cos(\theta+\tfrac{2\pi}{n}),\ \rho\ \sin\phi\cdot\sin(\theta+\tfrac{2\pi}{n}),\ \rho\ \cos\phi)\,.$$

Note that r fixes the line segment $\{(x,y,z) \in \mathbb{B}^3 \mid x = 0, y = 0, -1 \le z \le 1\}$.

We define the antipodal map i on $\mathbb{B}^2$ by $i(x,y,z) = (-x,-y,-z)$. In terms of the spherical coordinate system, $i(x,y,z) = (\rho\ \sin(\phi+\pi)\cdot\cos\theta,\ \rho\ \sin(\phi+\pi)\cdot\sin\theta,\ \rho\ \cos(\phi+\pi))$. Observe that $i\circ r\circ i^{-1} = r$.

Let $\mathbb{B}^3(n)$ be the orbifold $\mathbb{B}^3/\langle r\rangle$, which is a 3-ball with an arc of exceptional points of order n. The induced involution on $\mathbb{B}^3(n)$ is designated by $\bar{i}$, and denote $C(\mathbb{P}^2, 2n)$ to be the 3-orbifold $\mathbb{B}^3(n)/\langle\bar{i}\rangle$. The underlying space of $C(\mathbb{P}^2, 2n)$ is the cone over the projective plane $\mathbb{P}^2$, which is $\mathbb{P}^2\times[0,1]/(w,0) \simeq *$, where $*$ indicates a point. The exceptional set consists of an arc where all points except one endpoint have order n, and the other endpoint has order $2n$. The boundary of this orbifold, $\partial(C(\mathbb{P}^2, 2n))$, consists of a projective plane with one cone point of order n.

Let Z_n^h be the orbifold $\mathbb{B}^3(n)/\langle r_e\rangle$ where the reflection $r_e\colon \mathbb{B}^3(n) \to \mathbb{B}^3(n)$ is defined by $r_e(x,y,z) = (x,y,-z)$. The underlying space of Z_n^h is a 3-ball, with a half of its boundary is a mirrored disk, together with an arc of exceptional points each of order n except for one endpoint meeting this mirrored disk at a point of order $2n$. Let $s\colon B^3 \to B^3$ be the spin involution about the y-axis which we defined by $s(x,y,z) = (\rho\sin(\phi+\pi)\cdot\cos(-\theta),\ \rho\sin(\phi+\pi)\cdot\sin(-\theta),\ \rho\cos(\phi+\pi))$. Notice that $srs^{-1} = r^{-1}$, and thus s induces an involution $\overline{s}$ on $B^3(n)$. Let $B^3(n,2,2) = B^3(n)/\langle\overline{s}\rangle$. The underlying space of $B^3(n,2,2)$ is a ball with a properly embedded tree having three edges meeting at one point of order $2n$, with two of the edges labeled with a 2 and the remaining edge labeled with an n.

Orbifold $(A1,n)$**:** It will be convenient to view the elements of the fundamental groups as acting on the universal covering spaces of $\partial(A1,n)$ and $(A1,n)$. Let $\mathbb{R}^2$ be the plane and define $\tilde{a},\tilde{b}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\tilde{a}(t,s) = (t-\frac{1}{2}, -s)$ and $\tilde{b}(t,s) = (t-\frac{1}{2}, -s+\frac{1}{n})$. Note that $\tilde{a}^2 = \tilde{b}^2$, and $(\tilde{a}\tilde{b}^{-1})^n(t,s) = (t, s-1)$. If $\eta\colon \mathbb{R}^2 \to \mathbb{R}^2/\langle(\tilde{a}\tilde{b}^{-1})^n\rangle = \mathbb{R}\times S^1$, then $\eta(t,s) = (t, e^{2\pi i s})$ and the induced maps $a,b\colon \mathbb{R}\times S^1 \to \mathbb{R}\times S^1$ are defined by $a(t,v) = (t-\frac{1}{2}, \overline{v})$ and $b(t,v) = (t-\frac{1}{2}, \overline{v}e^{\frac{2\pi i}{n}})$. These maps extend to $\mathbb{R}\times D^2$, which is the universal covering of $(A1,n)$, and we will use the same labels for the extensions. We obtain a covering $p_1 \colon \mathbb{R} \times D^2 \to \mathbb{R} \times D^2/\langle a^2, ab^{-1}\rangle = V(n)$ defined by $p_1(t, \rho e^{i\theta}) = (e^{2\pi i t}, \rho e^{in\theta})$. The induced map a_1 on $V(n)$ is defined by $a_1(u,v) = (-u, \overline{v})$ and $V(n)/\langle a_1\rangle = (A1, n)$. The orbifold $(A1,n)$ is a solid Klein bottle with a simple closed curve core of exceptional points of type n. The boundary $\partial(A1,n)$ is a Klein bottle with fundamental group $\pi_1(\partial(A1,n)) = \langle a, b \mid a^2 = b^2\rangle$. Since $a(ab^{-1})a^{-1} = a^2b^{-1}a^{-1} = b^2b^{-1}a^{-1} = ba^{-1} = (ab^{-1})^{-1}$, it follows that ab^{-1} is a meridian curve. Thus the orbifold fundamental group of $(A1,n)$ is

$$\pi_1((A1,n)) = \langle a, b \mid a^2 = b^2, (ab^{-1})^n = 1\rangle \quad \text{and}$$
$$\pi_1(\partial(A1,n)) = \langle a, b \mid a^2 = b^2\rangle\,.$$

Orbifold $(B5,m)$: Let D_1 and D_2 be two disjoint disks in $\partial\mathbb{B}^3(m)$ containing the exceptional points. We consider the orbifold $C(\mathbb{P}^2, 2m) \cup \mathbb{B}^3(m) \cup C(\mathbb{P}^2, 2m) = (B5,m)$ where we glue D_1 to the boundary of one copy of $C(\mathbb{P}^2, 2m)$ and D_2 to the boundary of the other copy of $C(\mathbb{P}^2, 2m)$ so that the exceptional sets match up. Furthermore, $\partial(B5,m)$ is a Klein bottle whose fundamental group surjects to the orbifold fundamental group of $(B5,m)$. It can be seen that if $f \colon V(m) \to V(m)$ is the map defined by $f(u,v) = (\overline{u}, -v)$, then $V(m)/\langle f\rangle = (B5,m)$.

We view the generators of the fundamental groups acting on the universal covering space. Let $\tilde{x}, \tilde{y} \colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\tilde{x}(t,s) = (-t, s - \frac{1}{2m})$ and $\tilde{y}(t,s) = (-t+1, s - \frac{1}{2m})$. Observe that $\tilde{x}^2 = \tilde{y}^2$ and $\tilde{x}^{-2m}(t,s) = (t, s+1)$. We obtain a covering $\eta \colon \mathbb{R}^2 \to \mathbb{R}^2/\langle \tilde{x}^{-2m}\rangle = \mathbb{R} \times S^1$ defined by $\eta(t,s) = (t, e^{2\pi i s})$. The induced maps x and y on $\mathbb{R} \times S^1$ are defined by $x(t,v) = (-t, ve^{\frac{-\pi i}{m}})$ and $y(t,v) = (-t+1, ve^{\frac{-\pi i}{m}})$. These maps extend to $\mathbb{R} \times D^2$, which is the universal covering of $(B5,m)$. Let $p_1 \colon \mathbb{R} \times D^2 \to \mathbb{R} \times D^2/\langle yx^{-1}, x^{-2}\rangle = V(m)$ be the covering map defined by $p_1(t,v) = (e^{2\pi i t}, v^m)$. The induced map $x_1 \colon V(m) \to V(m)$ is defined by $x_1(u,v) = (\overline{u}, -v)$, and $V(m)/\langle x_1\rangle = (B5,m)$. The orbifold fundamental group of $(B5,m)$ is

$$\pi_1((B5,m)) = \langle x, y \mid x^2 = y^2,\ x^{2m} = y^{2m} = 1\rangle \quad \text{and}$$
$$\pi_1(\partial(B5,m)) = \langle x, y \mid x^2 = y^2\rangle\,.$$

Orbifold $O_{h_1}((A1,n),(B5,m))$: Recall that the maps defining the fundamental groups $\tilde{a}, \tilde{b}, \tilde{x}, \tilde{y} \colon \mathbb{R}^2 \to \mathbb{R}^2$ are defined as follows: $\tilde{a}(t,s) = (t - \frac{1}{2}, -s)$, $\tilde{b}(t,s) = (t - \frac{1}{2}, -s + \frac{1}{n})$, $\tilde{x}(t,s) = (-t, s - \frac{1}{2m})$ and $\tilde{y}(t,s) = (-t+1, s - \frac{1}{2m})$. We obtain covering maps $\lambda_1 \colon \mathbb{R}^2 \to \mathbb{R}^2/\langle \tilde{a}^2, \tilde{a}\tilde{b}^{-1}\rangle = T_1 = \partial(A0,n)$ and $\lambda_2 \colon \mathbb{R}^2 \to \mathbb{R}^2/\langle \tilde{x}^{-2}, \tilde{y}\tilde{x}^{-1}\rangle = T_2 = \partial(B5,m)$ defined by $\lambda_1(t,s) = (e^{2\pi i t}, e^{2\pi i n s})$ and $\lambda_2(t,s) = (e^{2\pi i t}, e^{2\pi i m s})$ respectively. The induced maps a_1 on T_1 and x_1 on T_2 are defined by $a_1(u,v) = (-u, \overline{v})$ and $x_1(u,v) = (\overline{u}, -v)$ respectively. We obtain covering maps $\mu_1 \colon T_1 \to T_1/\langle a_1\rangle = \partial(A1,n)$ and $\mu_2 \colon T_2 \to T_2/\langle x_1\rangle = \partial(B5,m)$.

Define a map $\tilde{h}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\tilde{h}(t,s) = (ns, \frac{t}{m})$, and note that $\tilde{h}^{-1}(t,s) = (ms, \frac{t}{n})$. We compute $\tilde{h}\tilde{a}\tilde{h}^{-1}(t,s) = \tilde{h}\tilde{a}(ms, \frac{t}{n}) = \tilde{h}(ms - \frac{1}{2}, -\frac{t}{n}) = (-t, s - \frac{1}{2m}) = \tilde{x}(t,s)$. A similar computation shows that $\tilde{h}\tilde{b}\tilde{h}^{-1} = \tilde{y}$. Thus $\tilde{h}$ projects to maps h_1 and h making the following diagram commute:

$$
\begin{array}{ccc}
\mathbb{R}^2 & \xrightarrow{\tilde{h}} & \mathbb{R}^2 \\
\Big\downarrow{\scriptstyle \lambda_1} & & \Big\downarrow{\scriptstyle \lambda_2} \\
T_1 & \xrightarrow{\tilde{h}_1} & T_2 \\
\Big\downarrow{\scriptstyle \mu_1} & & \Big\downarrow{\scriptstyle \mu_2} \\
\partial(A1,n) & \xrightarrow{h_1} & \partial(B5,m)
\end{array}
$$

A computation shows that $\tilde{h}_1(u,v) = (v,u)$ for any $(u,v) \in T_1$.

When we identify $\partial(A1,n)$ to $\partial(B5,m)$ via h_1, the generators are identified by $a = x$ and $b = y$. It follows that the orbifold fundamental group of $O_{h_1}((A1,n),(B5,m))$ is

$$
\begin{aligned}
\pi_1(O_{h_1}((A1,n),(B5,m))) &= \langle a, b \mid a^2 = b^2,\ a^{2m} = b^{2m} = (ba^{-1})^n = 1 \rangle \\
&= \langle ba^{-1} \rangle \circ_{-1} \langle a \rangle = \mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m} \,.
\end{aligned}
$$

We note that both a and b are orientation reversing elements.

Orbifold $(A3,n)$**:** The orbifold $(A3,n)$ is

$$
(\Delta(n) \times [0,1])/(\rho e^{i\theta}, 0) \simeq (\rho e^{i(-\theta-\pi)}, 1)\,.
$$

and the underlying space of $(A3,n)$ is a solid Klein bottle. The boundary of the underlying space consists of two Mobius strips, one of which is mirrored containing an orientation reversing circle of cone points of orders n.

The universal covering space of $(A3,n)$ is $\mathbb{R} \times D^2$, and the covering transformation maps a, b, c on $\mathbb{R} \times D^2$ are defined as follows: $a(t, \rho e^{i\theta}) = (t, \rho e^{i(\theta + \frac{2\pi}{n})})$, $b(t, \rho e^{i\theta}) = (t, \rho e^{-i\theta})$ and $c(t, \rho e^{i\theta}) = (t + \frac{1}{2}, \rho e^{i(-\theta - \frac{\pi}{n})})$. A computation shows the following: $cac^{-1} = a^{-1}$, $cbc^{-1} = ba$, $bab^{-1} = a^{-1}$, $a^n = b^2 = 1$. Hence the group generated by these elements is $\mathrm{Dih}(\mathbb{Z}_n) \circ \mathbb{Z} = (\langle a \rangle \circ_{-1} \langle b \rangle) \circ \langle c \rangle$.

Define an orbifold covering map $p_1\colon \mathbb{R} \times D^2 \to \mathbb{R} \times D^2/\langle a, c^2 \rangle = \widetilde{V}(n)$ by $p_1(t, \rho e^{i\theta}) = (e^{2\pi i t}, \rho e^{in\theta})$. The maps b and c induce maps b_1 and c_1 respectively on $\widetilde{V}(n)$, and it can be shown using the covering map p_1 that $b_1(u,v) = (u, \overline{v})$ and $c_1(u,v) = (-u, -\overline{v})$. Observe that $b_1 c_1(u,v) = (-u,-v)$. Let $p_2\colon \widetilde{V}(n) \to \widetilde{V}(n)/\langle b_1 c_1 \rangle = V(n)$ be the orbifold covering map, and note that $p_2(u,v) =$

(u^2, uv). We see that b_1 induces a map b_2 on $V(n)$ defined by $b_2(u,v) = p_2 b_1(u^{1/2}, u^{-1/2}v) = (u, u\overline{v})$. It follows that $V(n)/\langle b_2\rangle = (A3,n)$ and the fundamental group $\pi_1((A3,n)) = (\langle a\rangle \circ_{-1} \langle b\rangle) \circ \langle c\rangle = \mathrm{Dih}(\mathbb{Z}_n) \circ \mathbb{Z}$.

The boundary of $(A3,n)$ is a mirrored Mobius band $m\ddot{M}$, and its fundamental group $\pi_1(m\ddot{M}) = (\langle a\rangle \circ_{-1} \langle b\rangle) \circ \langle c\rangle = (\mathbb{Z} \circ_{-1} \mathbb{Z}_2) \circ \mathbb{Z}$. It may be convenient to write $\pi_1(m\ddot{M}) = (\langle b\rangle * \langle ba\rangle) \circ \langle c\rangle = (\mathbb{Z}_2 * \mathbb{Z}_2) \circ \mathbb{Z}$ where $cbc^{-1} = ba$ and $c(ba)c^{-1} = b$. Thus, the orbifold fundamental group of $(A3,n)$ is

$$\begin{aligned}
\pi_1((A3,n)) &= \langle a,b,c \mid a^n = b^2 = 1, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = ba\rangle \\
&= (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \circ \mathbb{Z} \quad \text{and} \\
\pi_1(\partial(A3,n)) &= \langle b, ba, c \mid cbc^{-1} = ba, c(ba)c^{-1} = b\rangle = (\mathbb{Z}_2 * \mathbb{Z}_2) \circ \mathbb{Z}\,.
\end{aligned}$$

Orbifold $(B4,m)$: The orbifold $(B4,m)$ is $C(\mathbb{P}^2, 2m) \cup \mathbb{B}^3(m) \cup Z^h_m$ where the exceptional sets of order m match up. The boundary $\partial(B4,m)$ is a mirrored Mobius band. The covering translations on the universal covering space $\mathbb{R}\times\mathbb{D}^2$ of $(B4,m)$ are defined as follows: $x(t,v) = (t+1, v)$, $y(t,v) = (t, ve^{\frac{2\pi i}{m}})$, $z(t,v) = (-t, ve^{\frac{2\pi it}{m}})$. The element z is an orientation reversing element. Define an orbifold covering map $p_1\colon \mathbb{R}\times D^2 \to \mathbb{R}\times D^2/\langle x,y\rangle = V(m)$ by $p_1(t, \rho e^{i\theta}) = (e^{2\pi it}, \rho e^{im\theta})$. Then z induces a map $z_1\colon V(m) \to V(m)$ defined by $z_1(u,v) = (\overline{u}, uv)$. The quotient space $V(m)/\langle z_1\rangle$ is the orbifold $(B4,m)$ and its fundamental group is

$$\begin{aligned}
\pi_1((B4,m)) &= \langle x,y,z \mid [x,y] = 1, y^m = z^2 = 1, zxz^{-1} = x^{-1}y, zyz^{-1} = y\rangle \\
&= (\mathbb{Z}\times\mathbb{Z}_m) \circ \mathbb{Z}_2 \quad \text{and} \\
\pi_1(\partial(B4,m)) &= \langle x^{-2}yz, z, zx \mid (zx)(x^{-2}yz)(zx)^{-1} = z, (zx)z(zx)^{-1} = x^{-2}yz\rangle \\
&= (\langle x^{-2}yz\rangle * \langle z\rangle) \circ \langle zx\rangle = (\mathbb{Z}_2 * \mathbb{Z}_2) \circ \mathbb{Z}\,.
\end{aligned}$$

Orbifold $O_{h_2}((A3,n),(B4,m))$: Recall that $\partial V(n)/\langle b_2\rangle = \partial(A3,n)$ and $V(m)/\langle z_1\rangle = \partial(B4,m)$. Define $\tilde{h}_2\colon \partial V(n) \to \partial V(m)$ by $\tilde{h}_2(u,v) = (\overline{u}v^2, u\overline{v})$ and observe that $\tilde{h}_2^{-1}(u,v) = (uv^2, uv)$. Since $\tilde{h}_2 b_2 \tilde{h}_2^{-1} = z_1$, we obtain the following commutative diagram:

$$\begin{array}{ccc}
\partial V(n) & \xrightarrow{\ \tilde{h}_2\ } & \partial V(m) \\
\big\downarrow{\scriptstyle \mu_1} & & \big\downarrow{\scriptstyle \mu_2} \\
\partial(A3,n) & \xrightarrow{\ h_2\ } & \partial(B4,m)
\end{array}$$

where μ_i are the quotient maps.

We use the map $h_2\colon \partial(A3,n) \to \partial(B4,m)$ to define $O_{h_2}((A3,n),(B4,m))$. It follows by [13], that the generators are identified by $b = z$, $ba = x^{-2}yz$, $c = zx$

and $c^2 = y$, and so the orbifold fundamental group $\pi_1(O_{h_2}((A3,n),(B4,m)))$ is

$$\begin{aligned}\langle a,b,c|a^n = b^2 = c^{2m} = 1, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = ba\rangle \\ = (\langle a\rangle \circ_{-1} \langle b\rangle) \circ \langle c\rangle = \mathrm{Dih}(\mathbb{Z}_n) \circ \mathbb{Z}_{2m}\,.\end{aligned}$$

The elements b and c are orientation reversing elements in the fundamental group.

Orbifold $(A2,n)$**:** Define the maps $a,b,c\colon \mathbb{R}\times D^2 \to \mathbb{R}\times D^2$ as follows: $a(t,v) = (t+1,v)$, $b(t,v) = (t, ve^{\frac{2\pi i}{n}})$, $c(t,v) = (t,\overline{v})$. We obtain an orbifold covering map $p_1\colon \mathbb{R}\times D^2 \to \mathbb{R}\times D^2/\langle a,b\rangle = V(n)$ defined by $p_1(t,\rho e^{i\theta}) = (e^{2\pi i t}, \rho e^{in\theta})$. Then c induces an involution $c_1\colon V(n) \to V(n)$ defined by $c_1(u,v) = (u,\overline{v})$. There is an orbifold covering map $\mu_1\colon V(n) \to V(n)/\langle c_1\rangle = (A2,n)$. The orbifold $(A2,n) = S^1 \times \Delta(n)$, has underlying space a solid torus with boundary $\partial(A2,n)$ a mirrored annulus. The orbifold fundamental group of $(A2,n)$ is

$$\begin{aligned}\pi_1((A2,n)) &= \langle a,b,c \mid [a,b] = [a,c] = 1, b^n = 1, cbc^{-1} = b^{-1}, c^2 = 1\rangle \\ &= (\langle b\rangle \circ_{-1} \langle c\rangle) \times \langle a\rangle = \mathrm{Dih}(\mathbb{Z}_n) \times \mathbb{Z} \quad \text{and} \\ \pi_1(\partial(A2,n)) &= \langle a,b,c \mid [a,b] = [a,c] = 1, cbc^{-1} = b^{-1}, c^2 = 1\rangle \\ &= \mathrm{Dih}(\mathbb{Z}) \times \mathbb{Z}\,.\end{aligned}$$

Orbifold $(B3,m)$**:** On $\mathbb{R}\times D^2$ define maps x, y and z by $x(t,v) = (t+1,v)$, $y(t,v) = (t, ve^{\frac{2\pi i}{m}})$ and $z(t,v) = (-t,v)$. As above, we obtain an orbifold covering map $p_2\colon \mathbb{R}\times D^2 \to \mathbb{R}\times D^2/\langle x,y\rangle = V(m)$ defined by $p_1(t,\rho e^{i\theta}) = (e^{2\pi i t}, \rho e^{im\theta})$.

The induced involution $z_1\colon V(m) \to V(m)$ is defined by $z_1(u,v) = (\bar{u},v)$. There is an orbifold covering map $\mu_2\colon V(m) \to V(m)/\langle z_1\rangle = (B3,m)$. The orbifold quotient $(B3,m)$, has underlying space $D^2\times I$ with both $D^2\times\{0\}$ and $D^2\times\{1\}$ being mirrored, and an exceptional set $\{0\}\times I$ of order m. The boundary $\partial(B3,m)$ is a mirrored annulus. The orbifold fundamental group of $(B3,m)$ is

$$\begin{aligned}\pi_1((B3,m)) &= \langle x,y,z \mid [x,y] = 1, y^m = 1, [y,z] = 1, zxz^{-1} = x^{-1}, z^2 = 1\rangle \\ &= (\langle x\rangle \circ_{-1} \langle z\rangle) \times \langle y\rangle = \mathrm{Dih}(\mathbb{Z}) \times \mathbb{Z}_m \quad \text{and} \\ \pi_1(\partial(B3,m)) &= \langle x,y,z \mid [x,y] = [y,z] = 1, zxz^{-1} = x^{-1}, z^2 = 1\rangle \\ &= \mathrm{Dih}(\mathbb{Z}) \times \mathbb{Z}\,.\end{aligned}$$

Orbifold $O_{h_3}((A2,n),(B3,m))$**:** Recall that $\partial V(n)/\langle c_1\rangle = \partial(A2,n)$ and $\partial V(m)/\langle z_1\rangle = \partial(B3,m)$. Define $\tilde{h}_3\colon \partial V(n) \to \partial V(m)$ by $\tilde{h}_3(u,v) = (v,u)$ and observe that $\tilde{h}_3^{-1} = \tilde{h}_3$. Since $\tilde{h}_3 c_1 \tilde{h}_3^{-1} = z_1$, we obtain the following commutative diagram:

$$\begin{array}{ccc} \partial V(n) & \xrightarrow{\tilde{h}_3} & \partial V(m) \\ \downarrow \mu_1 & & \downarrow \mu_2 \\ \partial(A2,n) & \xrightarrow{h_3} & \partial(B3,m) \end{array}$$

where μ_i are the quotient maps. Identify $\partial(A2,n)$ to $\partial(B3,m)$ via h_3 to obtain the orbifold $O_{h_3}((A2,n),(B3,m))$. It follows by [13] that that the generators are identified by $a=y$, $b=x$ and $c=z$. Hence the orbifold fundamental group is

$$\begin{aligned} &\pi_1(O_{h_3}((A2,n),(B3,m))) \\ &\qquad = \langle a,b,c | [a,b]=[a,c]=1, a^m=b^n=c^2=1, cbc^{-1}=b^{-1}\rangle \\ &\qquad\qquad = (\langle b\rangle \circ_{-1} \langle c\rangle)\times\langle a\rangle = \mathrm{Dih}(\mathbb{Z}_n)\times\mathbb{Z}_m\,. \end{aligned}$$

The element c is an orientation reversing element.

Orbifold $(B2,n)$**:** The orbifold $(B2,n)=B^3(n,2,2)\cup C(\mathbb{P}^2,2n)$ where a disk in $\partial(B^3(n,2,2))$ containing the exceptional point of order n is identified to a disk in $\partial(C(\mathbb{P}^2,2n))$ containing the exceptional point of order n.

Define maps a and b on $\mathbb{R}\times D^2$ by $a(t,\rho e^{i\theta})=(-t+\frac{1}{2},\rho e^{i(\theta+\frac{\pi}{n})})$ and $b(t,\rho e^{i\theta})=(-t,\rho e^{-i\theta})$. The map a is orientation reversing. It is easy to check that $a^2(t,\rho e^{i\theta})=(t,\rho e^{i(\theta+\frac{2\pi}{n})})$ and $(ab)^2(t,\rho e^{i\theta})=(t+1,\rho e^{i\theta})$, hence we have relations $a^{2n}=b^2=1$ and $ba^2b^{-1}=a^{-2}$. The manifold $\mathbb{R}\times D^2$ is the universal covering of $(B2,n)$, which can be seen by means of the following sequence of coverings. First, let $p\colon \mathbb{R}\times D^2\to\mathbb{R}\times D^2/\langle a^2,(ab)^2\rangle=V(n)$ be defined by $p(t,\rho e^{i\theta})=(e^{2\pi i t},\rho e^{in\theta})$. The induced maps a_1 and b_1 on $V(n)$ are defined by $a_1(u,v)=(-\overline{u},-v)$ and $b_1(u,v)=(\overline{u},\overline{v})$. Secondly, we have a covering map $p_1\colon V(n)\to V(n)/\langle b_1\rangle=(B0,n)$, and a_1 induces the anti-podal map a_2 on $(B0,n)$. Finally we obtain a covering map $\mu\colon (B0,n)\to(B0,n)/\langle a_2\rangle=(B2,n)$. The orbifold fundamental group of $(B2,n)$ is

$$\begin{aligned} \pi_1((B2,n)) &= \langle a,b \mid a^{2n}=b^2=1, ba^2b^{-1}=a^{-2}\rangle \\ &= (\langle a^2\rangle\circ\langle ab\rangle)\circ\langle b\rangle = (\mathbb{Z}_n\circ\mathbb{Z})\circ\mathbb{Z}_2 \quad\text{and} \\ \pi_1(\partial(B2,n)) &= \langle a,b \mid b^2=1, ba^2b^{-1}=a^{-2}\rangle \\ &= (\langle a^2\rangle\circ\langle ab\rangle)\circ\langle b\rangle = (\mathbb{Z}\circ\mathbb{Z})\circ\mathbb{Z}_2\,. \end{aligned}$$

The boundary of $(B2,n)$ is a projective plane with two cone points each of order 2 (See Figure 3).

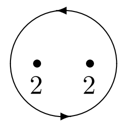

Figure 3: $\partial(B2,n)$

Orbifold $O_{h_4}((B2,n),(B2,m))$**:** We use the letters x and y to denote the generators of $\pi_1((B2,m))$, and note that the definitions are identical with n replaced by m. Thus $\mathbb{R}\times D^2/\langle a^2,(ab)^2\rangle = V(n)$ and $\mathbb{R}\times D^2/\langle x^2,(xy)^2\rangle = V(m)$; and $V(n)/\langle b_1\rangle = (B0,n)$, and $V(m)/\langle y_1\rangle = (B0,m)$. Define a map $\tilde{h}_4\colon \partial V(n)\to\partial V(m)$ by $\tilde{h}_4(u,v) = (-\overline{v},u)$ and observe that $\tilde{h}_4^{-1}(u,v) = (v,-\overline{u})$. A computation shows $\tilde{h}_4 a_1\tilde{h}_4^{-1} = y_1x_1^{-1}$ and $\tilde{h}_4 b_1\tilde{h}_4^{-1} = y_1$. Thus $\tilde{h}_4$ induces maps $\widehat{h}_4$ and h_4 making the following diagram commute:

$$\begin{array}{ccc}
\partial V(n) & \xrightarrow{\tilde{h}_4} & \partial V(m)\\
\big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle p_2}\\
\partial(B0,n) & \xrightarrow{\widehat{h_4}} & \partial(B0,m)\\
\big\downarrow{\scriptstyle \mu_1} & & \big\downarrow{\scriptstyle \mu_2}\\
\partial(B2,n) & \xrightarrow{h_4} & \partial(B2,m)
\end{array}$$

By identifying $\partial(B2,n)$ to $\partial(B2,m)$ via h_4, it follows by [13] that the generators are related by $a = yx^{-1}$ and $b = xyx$. It follows that $ab = x$ and $a^2b = y$. Thus the orbifold fundamental group is

$$\begin{aligned}\pi_1(O_{h_4}((B2,n),(B2,m))) &= \langle a,b \mid a^{2n} = b^2 = 1, ba^2b^{-1} = a^{-2}, (ab)^{2m} = 1\rangle\\ &= (\langle a^2\rangle \circ_{-1} \langle ab\rangle)\circ\langle b\rangle = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m})\circ\mathbb{Z}_2\,.\end{aligned}$$

The element a is an orientation reversing element.

Orbifold $(B6,n)$**:** Define maps on $\mathbb{R}\times D^2$ by $a(t,\rho e^{i\theta}) = (-t,\rho e^{i\theta})$, $b(t,\rho e^{i\theta}) = (t,\rho e^{-i\theta})$, $c(t,\rho e^{i\theta}) = (t,\rho e^{i(-\theta-\frac{2\pi}{n})})$ and $d(t,\rho e^{i\theta}) = (-t-1,\rho e^{i\theta})$. Let $p\colon \mathbb{R}\times D^2\to\mathbb{R}\times D^2/\langle ad,bc\rangle = V(n)$ be defined by $p(t,\rho e^{i\theta}) = (e^{2\pi it},\rho e^{in\theta})$. Then a and b induce maps a_1 and b_1 on $V(n)$ defined by $a_1(u,v) = (\overline{u},v)$ and $b_1(u,v) = (u,\overline{v})$. Furthermore, there is a covering map $p_1\colon V(n)\to V(n)/\langle a_1b_1\rangle = (B0,n)$ and b_1 induces a reflection b_2 on $(B0,n)$ through a disk containing the exceptional set. Modding out by b_2 we obtain the final covering map $\mu\colon (B0,n)\to(B0,n)/\langle b_2\rangle = (B6,n)$. The orbifold fundamental

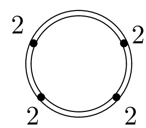

Figure 4: $\partial(B6, n)$

group of $(B6, n)$ is

$$\begin{aligned}
\pi_1((B6,n)) &= \langle a,b,c,d \mid a^2 = b^2 = c^2 = (bc)^n = d^2 = 1, \\
&\qquad\qquad [a,b] = [a,c] = [b,d] = [c,d] = 1\rangle \\
&= (\langle bc\rangle \circ_{-1} \langle c\rangle) \times (\langle a\rangle * \langle d\rangle) = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \times (\mathbb{Z}_2 * \mathbb{Z}_2) \quad \text{and} \\
\pi_1(\partial(B6,n)) &= \langle a,b,c,d \mid a^2 = b^2 = c^2 = d^2 = 1, \\
&\qquad\qquad [a,b] = [a,c] = [b,d] = [c,d] = 1\rangle \\
&= (\langle bc\rangle \circ_{-1} \langle c\rangle) \times (\langle a\rangle * \langle d\rangle) = (\mathbb{Z} \circ_{-1} \mathbb{Z}_2) \times (\mathbb{Z}_2 * \mathbb{Z}_2) .
\end{aligned}$$

The boundary of $(B6, n)$ is a mirrored disk with four cone points of order two on the mirror (See Figure 4).

Orbifold $O_{h_5}((B6,n),(B6,m))$**:** As above, we use the letters x, y, z and w to denote the generators of $\pi_1((B6,m))$ where the definitions are identical with m replacing n. Thus $\mathbb{R}\times D^2/\langle ad, bc\rangle = V(n)$ and $\mathbb{R}\times D^2/\langle xw, yz\rangle = V(m)$. Define a homeomorphism $\tilde{h}_5 \colon \partial V(n) \to \partial V(m)$ by $\tilde{h}_5(u,v) = (-v,u)$. A computation shows $\tilde{h}_5 a_1 \tilde{h}_5^{-1} = y_1$ and $\tilde{h}_5 b_1 \tilde{h}_5^{-1} = x_1$. The map $\tilde{h}_5$ induces maps $\widehat{h}_5$ and h_5 making the following diagram commute:

$$\begin{array}{ccc}
\partial V(n) & \xrightarrow{\tilde{h}_5} & \partial V(m) \\
\big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle p_2} \\
\partial(B0,n) & \xrightarrow{\widehat{h}_5} & \partial(B0,m) \\
\big\downarrow{\scriptstyle \mu_1} & & \big\downarrow{\scriptstyle \mu_2} \\
\partial(B6,n) & \xrightarrow{h_5} & \partial(B6,m)
\end{array}$$

Thus when identifying $\partial(B6,n)$ to $\partial(B6,m)$ via h_5, it follows by [13] that the generators are identified by $a = y$, $b = xwx$, $c = x$ and $d = z$, and the orbifold

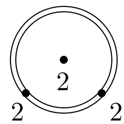

Figure 5: $\partial(B7,n)$

fundamental of group is

$$\begin{aligned}
&\pi_1(O_{h_5}((B6,n),(B6,m)))\\
&\qquad = \langle a,b,c,d \mid a^2=b^2=c^2=(bc)^n=d^2=1,\\
&\qquad\qquad [a,b]=[a,c]=[b,d]=[c,d]=1,(ad)^m=1\rangle\\
&\qquad = (\langle bc\rangle \circ_{-1} \langle c\rangle)\times(\langle ad\rangle \circ_{-1}\langle a\rangle)=(\mathbb{Z}_n\circ_{-1}\mathbb{Z}_2)\times(\mathbb{Z}_m\circ_{-1}\mathbb{Z}_2)\,.
\end{aligned}$$

Note that the elements a, b, c and d are orientation reversing.

Orbifold $(B7,n)$**:** Define maps a,b,c on $\mathbb{R}\times D^2$ as follows: $a(t,\rho e^{i\theta})=(-t-1,\rho e^{i\theta})$, $b(t,\rho e^{i\theta})=(-t,\rho e^{i(\theta+\frac{\pi}{n})})$ and $c(t,\rho e^{i\theta})=(t,\rho e^{-i\theta})$. Let $p\colon \mathbb{R}\times D^2\to\mathbb{R}\times D^2/\langle b^2,(ba)\rangle=V(n)$ be the covering map defined by $p(t,\rho e^{i\theta})=(e^{2\pi it},\rho e^{i(n\theta+\pi t)})$. The induced maps a_1 and c_1 on $V(n)$ are $a_1(u,v)=(\overline{u},-\overline{u}v)$ and $c_1(u,v)=(u,u\overline{v})$. Observe that $a_1c_1(u,v)=(\overline{u},-\overline{v})$, and thus there is a covering map $p_1\colon V(n)\to V(n)/\langle a_1c_1\rangle=(B0,n)$. If a_2 be the induced map on $(B0,n)$, then we have another covering $\mu\colon(B0,n)\to(B0,n)/\langle a_2\rangle=(B7,n)$. Since a and b^2 commute, we have $b^{-1}ab=bab^{-1}$, hence the orbifold fundamental group of $(B7,n)$ is

$$\begin{aligned}
\pi_1((B7,n)) &= \langle a,b,c\mid a^2=b^{2n}=c^2=1,[a,b^2]=[a,c]=1,cbc^{-1}=b^{-1}\rangle\\
&= \langle a,bab^{-1}\rangle\circ(\langle b\rangle\circ_{-1}\langle c\rangle)=(\mathbb{Z}_2*\mathbb{Z}_2)\circ(\mathbb{Z}_{2n}\circ_{-1}\mathbb{Z}_2)\quad\text{and}\\
\pi_1(\partial(B7,n)) &= \langle a,b,c\mid a^2=c^2=1,[a,b^2]=[a,c]=1,cbc^{-1}=b^{-1}\rangle\\
&= \langle a,bab^{-1}\rangle\circ(\langle b\rangle\circ_{-1}\langle c\rangle)=(\mathbb{Z}_2*\mathbb{Z}_2)\circ(\mathbb{Z}\circ_{-1}\mathbb{Z}_2)\,.
\end{aligned}$$

The boundary of $(B7,n)$ is a mirrored disk with two cone points on the mirror and one cone point in the interior (See Figure 5).

Orbifold $O_{h_6}((B7,n),(B7,m))$**:** We use the letters x, y and z to denote the generators of $\pi_1((B7,m))$ where the definitions are identical with n replaced with m. As above we obtain covering maps $\mathbb{R}\times D^2\to\mathbb{R}\times D^2/\langle b^2,(ba)\rangle=V(n)$ and $\mathbb{R}\times D^2\to\mathbb{R}\times D^2/\langle y^2,(yx)\rangle=V(m)$. The induced maps on $V(n)$ and $V(m)$ are denoted by a_1, c_1 and x_1, z_1 respectively.

Define a homeomorphism $\tilde{h}_6\colon\partial V(n)\to\partial V(m)$ by $\tilde{h}_6(u,v)=(-\overline{u}v^2,\overline{u}v)$.

The map $\tilde{h}_6$ induces maps $\widehat{h}_6$ and h_6 making the following diagram commutes.

$$\begin{array}{ccc}
\partial V(n) & \xrightarrow{\tilde{h}_6} & \partial V(m) \\
\Big\downarrow {\scriptstyle p_1} & & \Big\downarrow {\scriptstyle p_2} \\
\partial(B0,n) & \xrightarrow{\widehat{h}_6} & \partial(B0,m) \\
\Big\downarrow {\scriptstyle \mu_1} & & \Big\downarrow {\scriptstyle \mu_2} \\
\partial(B7,n) & \xrightarrow{h_6} & \partial(B7,m)
\end{array}$$

Identifying $\partial(B7,n)$ to $\partial(B7,m)$ via h_6 to obtain the orbifold $O_{h_6}((B7,n),(B7,m))$, it follows by [13] that the generators are identified by $a = z$, $b = zyx$, and $c = yxy^{-1}$. Furthermore, the fundamental group $\pi_1(O_{h_6}((B7,n),(B7,m)))$ is

$$\begin{aligned}
&\langle a,b,c \mid a^2 = b^{2n} = (ab^{-1}ab)^m = c^2 = 1, [a,b^2] = [a,c] = 1, cbc^{-1} = b^{-1}\rangle \\
&\qquad = (\langle ab^{-1}ab\rangle \circ_{-1} \langle a\rangle) \circ (\langle b\rangle \circ_{-1} \langle c\rangle) = (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2) \circ (\mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2)\,.
\end{aligned}$$

Note that the elements a, b and c are orientation reversing.

Orbifold $(B1,n)$**:** The orbifold $B^3(n,2,2) \cup Z_n^h$ where a disk in $\partial(B^3(n,2,2))$ containing the exceptional point of order n is identified with a disk in $\partial(Z_n^h)$ containing the exceptional point of order n.

Define maps a, b, c on $\mathbb{R} \times D^2$ as follows: $a(t,v) = (t, ve^{2\pi i/n})$, $b(t,v) = (-t, \overline{v})$, and $c(t,v) = (\frac{1}{2} - t, v)$. The manifold $\mathbb{R} \times D^2$ is the universal cover of $(B1,n)$. Let $p_1\colon \mathbb{R} \times D^2 \to \mathbb{R} \times D^2/\langle (cb)^2, a\rangle = V(n)$ be defined by $p_1(t,\rho e^{i\theta}) = (e^{2\pi i t}, \rho e^{in\theta})$. Then b and c induce involutions b_1 and c_1 respectively on $V(n)$, where $b_1(u,v) = (\overline{u}, \overline{v})$ and $c_1(u,v) = (-\overline{u}, v)$. Now $V(n)/\langle b_1\rangle = (B0,n)$; and c_1 induces an orientation reversing involution c_2 on $(B0,n)$. The quotient space $(B0,n)/\langle c_2\rangle$ is the orbifold $(B1,n)$. The orbifold fundamental group of $(B1,n)$ is

$$\begin{aligned}
\pi_1((B1,n)) &= \langle a,b,c \mid a^n = b^2 = c^2 = 1, bab^{-1} = a^{-1}, cac^{-1} = a\rangle \\
&= \langle a\rangle \circ (\langle b\rangle * \langle c\rangle) = \mathbb{Z}_n \circ (\mathbb{Z}_2 * \mathbb{Z}_2) \quad \text{and} \\
\pi_1(\partial(B1,n)) &= \langle a,b,c \mid b^2 = c^2 = 1, bab^{-1} = a^{-1}, cac^{-1} = a\rangle \\
&= \langle a\rangle \circ (\langle b\rangle * \langle c\rangle) = \mathbb{Z} \circ (\mathbb{Z}_2 * \mathbb{Z}_2)\,.
\end{aligned}$$

Orbifold $(B8,m)$**:** Define orientation reversing maps x, y and z on $\mathbb{R} \times D^2$ by $x(t,v) = (-t,,ve^{\frac{\pi i}{m}})$, $y(t,v) = (t,\overline{v})$ and $z(t,v) = (1-t, ve^{\frac{\pi i}{m}})$. Now $\mathbb{R} \times D^2$ is the universal covering of $(B8,m)$. Note that $xz^{-1}(t,v) = (t-1,v)$. Let $p_1\colon \mathbb{R} \times D^2 \to \mathbb{R} \times D^2/\langle xz^{-1}, z^2\rangle = V(m)$ be defined by $p_1(t,\rho e^{i\theta}) =$

$(e^{2\pi it}, \rho e^{im\theta})$. We obtain induced maps x_1 and y_1 on $V(m)$ defined as follows: $x_1(u,v) = (\overline{u}, -v)$ and $y_1(u,v) = (u, \overline{v})$. Since $x_1y_1(u,v) = (\overline{u}, -\overline{v})$, this implies $V(m)/\langle x_1y_1\rangle = (B0, m)$. Furthermore, if y_2 is the induced map on $(B0,m)$, then y_2 is a reflection through a disk that does not contain the cone points of order 2 in the boundary. The orbifold $(B8,m) = (B0,m)/\langle y_2\rangle$. The orbifold fundamental group of $(B8,m)$ is

$$\begin{aligned}\pi_1((B8,m)) &= \langle x,y,z \mid x^{2m} = y^2 = z^{2m} = 1, yxy^{-1} = x^{-1}, \\ &\qquad\qquad yzy^{-1} = z^{-1}, x^2 = z^2\rangle \\ &= \langle xz^{-1}\rangle \circ (\langle x\rangle \circ_{-1} \langle y\rangle) = \mathbb{Z} \circ (\mathbb{Z}_{2m} \circ_{-1} \mathbb{Z}_2) \quad \text{and} \\ \pi_1(\partial(B8,m)) &= \langle x,y,z \mid y^2 = 1, yxy^{-1} = x^{-1}, yzy^{-1} = z^{-1}, x^2 = z^2\rangle \\ &= \langle xz^{-1}\rangle \circ (\langle x\rangle \circ_{-1} \langle y\rangle) = \mathbb{Z} \circ (\mathbb{Z} \circ_{-1} \mathbb{Z}_2) \\ &= \langle xz^{-1}\rangle \circ (\langle xy\rangle * \langle y\rangle) = \mathbb{Z} \circ (\mathbb{Z}_2 * \mathbb{Z}_2)\,.\end{aligned}$$

Orbifold $O_{h_7}((B1,n),(B8,m))$: As above we obtain covering maps $\mathbb{R}\times D^2 \to \mathbb{R}\times D^2/\langle (cb)^2, a\rangle = V(n)$ and $\mathbb{R}\times D^2 \to \mathbb{R}\times D^2/\langle xz^{-1}, z^2\rangle = V(m)$. The induced maps on $V(n)$ are a_1, b_1 and c_1, and the induced maps on $V(m)$ are x_1 and y_1.

Define a homeomorphism $\tilde{h}_7\colon V(n) \to V(m)$ by $\tilde{h}_7(u,v) = (v, -iu)$. The map $\tilde{h}_7$ induced maps $\widehat{h}_7$ and h_7 making the following diagram commute:

$$\begin{array}{ccc} \partial V(n) & \xrightarrow{\tilde{h}_7} & \partial V(m) \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle p_2} \\ \partial(B0,n) & \xrightarrow{\widehat{h}_7} & \partial(B0,m) \\ \big\downarrow{\scriptstyle \mu_1} & & \big\downarrow{\scriptstyle \mu_2} \\ \partial(B1,n) & \xrightarrow{h_7} & \partial(B8,m) \end{array}$$

When we identify $\partial(B1,n)$ to $\partial(B8,m)$ via h_7, we obtain the orbifold $O_{h_7}((B1,n),(B8,m))$. By [13] the generators are identified by $a = (xz^{-1})^{-1} = zx^{-1}$, $b = yx$ and $c = y$ and the fundamental group is

$$\begin{aligned}\pi_1(O_h((B1,n),(B8,m))) &= \langle a,b,c \mid a^n = b^2 = c^2 = 1, bab^{-1} = a^{-1}, \\ &\qquad\qquad [a,c] = 1, (cb)^{2m} = 1\rangle \\ &= \langle a\rangle \circ (\langle b\rangle * \langle c\rangle/\langle (cb)^{2m}\rangle) \\ &= \langle a\rangle \circ (\langle cb\rangle \circ_{-1} \langle c\rangle/\langle (cb)^{2m}\rangle) \\ &= \mathbb{Z}_n \circ \mathrm{Dih}(\mathbb{Z}_{2m})\,.\end{aligned}$$

The elements c is orientation reversing.

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