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Six Lectures on Translation-Invariant Operators and Subspaces

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Summary. - *This paper represents the text of a minicourse delivered by the author for participants of the "Workshop on Measure Theory and Real Analysis", Grado, September-97. We consider topics on multiplier theory and translation-invariant subspaces in function spaces on groups $\mathbb{T}$ and $\mathbb{Z}$. Our goal is to give an introduction to the subject from the very beginning up to some recent results. The presentation is aimed at graduate students; no preliminaries in Fourier Analysis are supposed.*

Lecture 1. The Space $M(\mathbb{T})$

Elements of the space are finite (complex) Borel measures on the circle group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The set of all such measures μ constitutes a Banach space with respect to natural linear operations and the norm

$$\|\mu\| = \sup_{\{E_j\}} \Sigma|\mu(E_j)|$$

where sup is taken over all finite Borel partitions of $\mathbb{T}$.

1.1. Riesz representation

According to the Riesz theorem one can identify the space $M(\mathbb{T})$ with the dual space C^* to the space $C(\mathbb{T})$ of continuous functions on

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the circle. Namely, *every measure* μ *produces a linear functional*

$$\Phi_\mu : f(t) \mapsto \int_{\mathbb{T}} f d\mu \quad , \quad \|\Phi_\mu\|_{C^*} = \|\mu\|_M \ ,$$

and every functional Φ *has such a form* .

1.2. Lebesgue decomposition

By m we denote the normalized Lebesgue measure on $\mathbb{T}$. With every function $f \in L(\mathbb{T})$ one can associate a measure μ_f with density f:

$$\mu_f(E) = \int_E f dm \ .$$

Such measures are called *absolutely continuous* (notation: $\mu \in M_{a.c.}$). This correspondence stands the canonical *isometrical* embedding of the space $L(\mathbb{T})$ into the space $M(\mathbb{T})$. Another important class of measures is the family M_d of *discrete* measures. The simplest example is so called ≪delta-functions≫ δ_a supported at the point $a \in \mathbb{T}$: $\delta_a(E) = \begin{cases} 1, & \text{if } a \in E \\ 0 & \text{otherwise} \ . \end{cases}$. The general discrete measure is defined as

$$\mu = \sum_{k \in \mathbb{N}} \lambda_k \delta_{a_k} \ , \quad a_k \in \mathbb{T} \ , \quad \lambda_k \in \mathbb{C} \ , \quad \Sigma |\lambda_k| < \infty \ .$$

The last important class M_s of singular measures μ is in a way intermediate between two previous. It is defined by the following conditions:

$$(i) \qquad \mu(a) = 0 \qquad \forall a \in \mathbb{T}$$

(which means that μ is not as concentrated as the discrete one), but it still has ≪a small support≫:

$$(ii) \quad \exists E_0 \ , \quad mE_0 = 0 : \quad \mu(E') = 0 \quad \forall E' \quad \text{disjoint with} \quad E_0 \ .$$

The fundamental Lebesgue Theorem says that *every* $\mu \in M(\mathbb{T})$ *can be decomposed as*

$$\mu = \mu_1 + \mu_2 + \mu_3 \ , \quad \mu_1 \in M_{a.c.}, \mu_2 \in M_d, \mu_3 \in M_s$$

and this decomposition is unique.

1.3. Weak convergence

Convergence of a given sequence $\{\mu_n\}$ of measures in sense of Banach norm is too strong a condition. More relaxed and more important is the notion of weak convergence which appears from the duality. One says that μ_n *weakly* converges to μ if

$$\lim_{n\to\infty} \int_{\mathbb{T}} f d\mu_n = \int_{\mathbb{T}} f d\mu \qquad \forall f \in C(\mathbb{T}) \ .$$

The following simple but important result gives sufficient conditions for such a convergence:

Let $\{\mu_u\}$ be a sequence of measures s.t.

(i) $\|\mu_n\| < \text{const}$

(ii) $\exists$ a dense set $F \subset C(\mathbb{T})$ *such that* $\forall f \in F$ *there exists a limit* $\lim_{n\to\infty} \int_{\mathbb{T}} f d\mu_n$.

Then μ_n converges weakly to some measure μ.

REMARK 1.1. *Both conditions are also necessary ones. This is obvious for (ii); for (i) it comes from the Banach-Steinhaus Theorem.*

1.4. Weak compactness

The following compactness property comes directly from the previous result:

If $\|\mu_n\| < \text{const}(n = 1, 2, \dots)$ *then there exists a subsequence $\{\mu_{n_j}\}$ which converges weakly to some measure μ.*

1.5. Fourier transform of measures

With every $\mu \in M(\mathbb{T})$ one can associate a sequence of numbers:

$$\widehat{\mu}(n) = \int_{\mathbb{T}} e^{-int} d\mu(t) \qquad (n \in \mathbb{Z}) \ ,$$

so-called *Fourier coefficients*, (or Fourier transform) of the measure μ. Obviously:

$$\|\widehat{\mu}\|_\infty \equiv \sup_{n\in\mathbb{Z}} |\widehat{\mu}(n) \leq \|\mu\|_{M(\mathbb{T})} \ .$$

The series

$$\mu \sim \sum_{n\in\mathbb{Z}} \widehat{\mu(n)} e^{int}$$

is called the *Fourier series* of the measure μ.

EXAMPLE 1.2. *The series $\sum_{n\in\mathbb{Z}} e^{int}$ is the Fourier series of delta-function supported at zero.*

REMARK 1.3. *If $\mu \in M_{a.c.}$, i.e. $d\mu = f dm$ then $\{\widehat{\mu}\}$ is the same as Fourier transform of the function f:*

$$\widehat{f}(n) = \int_{\mathbb{T}} f(t) e^{-int} \frac{dt}{2\pi} \qquad (n \in \mathbb{Z}) \; .$$

In this case $\widehat{\mu}(n) \to 0(|n| \to \infty)$.

1.6. Exercises

EXERCISE 1.4: Prove that:

(i) M_d is a linear *closed* subspace in $M(\mathbb{T})$;

(ii) on the other hand it is dense with respect to weak convergence: for every $\mu \in M(\mathbb{T})$ one can construct a sequence

$$\mu_n = \sum_{k=1}^{n} \lambda_k^{(n)} \delta_{2\pi \frac{k}{n}} \; ,$$

weakly convergent to μ.

EXERCISE 1.5: Prove the theorem of section 1.4

EXERCISE 1.6: Prove the uniqueness theorem: $\widehat{\mu} \equiv 0 \Rightarrow \mu = 0$.

EXERCISE 1.7: Is it true that $\widehat{\mu}(n) \to 0$ $(|n| \to \infty) \Rightarrow \mu \in M_{a.c.}$?

Lecture 2. Convolution

2.1. Algebra of measures

There exists an important operation – a sort of multiplication in the space $M(\mathbb{T})$, which makes it a Banach algebra. It is called – *convolution*.

First we define it for absolutely continuous measures (or equivalently – for L^1-functions). For $f, g \in L^1(\mathbb{T})$ we set:

$$h(t) = \int_{\mathbb{T}} f(t-x)g(x)dm(x) \ .$$

Using the Foubini theorem one can check that h is defined almost everywhere and belongs to L^1. We say that h is the convolution of f and g, and write: $h = f * g$. It follows (for any $\varphi \in C(\mathbb{T})$):

$$\int_{\mathbb{T}\times\mathbb{T}} \varphi(x+y)f(x)g(y)dm(x)dm(y) = \int_{\mathbb{T}} \varphi(t)\cdot(f*g)(t)dm(t) \ .$$

This suggests the following

Definition 2.1. *Let $\mu, \nu \in M(\mathbb{T})$. Consider the linear functional*

$$\Phi : \varphi \mapsto \int_{\mathbb{T}\times\mathbb{T}} \varphi(x+y)d\mu(x)d\nu(y) \qquad \varphi \in C(\mathbb{T})$$

According to 1.2 it can be written as:

$$\Phi(\varphi) = \int_{\mathbb{T}} \varphi(t)d\theta(t) \ ,$$

where θ is a uniquely defined measure on $\mathbb{T}$. This measure is called the convolution of μ and $\nu : \theta = \mu * \nu$.

properties 2.2. *(directly from the definition):*

(i) $\mu * \nu = \nu * \mu$

(ii) $(\mu_1 + \mu_2) * \nu = \mu_1 * \nu + \mu_2 * \nu; (\lambda\mu) * \nu = \lambda(\mu * \nu) \quad (\lambda \in \mathbb{C}).$

(iii) $\|\mu * \nu\| \leq \|\mu\| \cdot \|\nu\|$

Example 2.3. *(convolution of discrete measures).*

If $\mu = \Sigma\alpha_j\delta_{a_j}$, $\nu = \Sigma\beta_k\delta_{b_k}$ then $\mu * \nu = \sum_{j,k} \alpha_j\beta_k\delta_{a_j+b_k}$.

Exercise 2.4: If μ is a *continuous* measure (i.e. no atoms) then the convolution with any ν is also continuous.

Exercise 2.5: If μ is absolutely continuous: $d\mu = fdm, f \in L^1$, then for any $\nu \in M$ the convolution $\theta = \mu * \nu$ is also absolutely continuous: $d\theta = hdm$, and the function $h \in L^1$ is given by the formula

$$h(t) = \int_{\mathbb{T}} f(t-y)d\nu(y)$$

(it is called the convolution of f and ν, $h = f * \nu$).

2.2. Convolution and Fourier transform

We already mentioned that the convolution is ≪like multiplication≫. It appears exactly as multiplication after Fourier transform.

THEOREM 2.6. *If* $\mu, \nu \in M(\mathbb{T})$ *then*

$$\widehat{\mu * \nu}(n) = \widehat{\mu}(n) \cdot \widehat{\nu}(n) \qquad (n \in \mathbb{Z})$$

Proof.

$$\widehat{\mu * \nu}(n) = \int_{\mathbb{T}} e^{-int} d(\mu \times \nu) = \int_{\mathbb{T}\times\mathbb{T}} e^{-in(x+y)} d\mu(x) d\mu(y) = \widehat{\mu}(n)\widehat{\nu}(n)$$

□

EXAMPLE 2.7. *The partial sum* $S_N(\mu)$ *of the Fourier series of a measure* μ*:*

$$S_N = \sum_{|n|\leq N} \widehat{\mu}(n) e^{int}$$

can be written as the convolution of μ *with the function*

$$D_N(t) = \sum_{|n|\leq N} e^{int} \equiv \frac{\mathrm{Sin}(N + 1/2)t}{\mathrm{Sin} t/2}$$

(so-called Dirichlet kernel).

EXERCISE 2.8: (i) For $\nu \in M(\mathbb{T})$ prove: $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\leq N} \widehat{\nu}(n) = \nu\{0\}$;

(ii) For $\mu \in M(\mathbb{T})$ difine $\mu^{\#} \in M(\mathbb{T})$ by the equality $\mu^{\#}(E) = \overline{\mu(-E)}$.
Prove: $|\widehat{\mu}(n)|^2 = (\mu * \mu^{\#})(n) \qquad (n \in \mathbb{Z})$.

(iii) For $\mu \in M(\mathbb{T})$ prove: $(\mu * \mu^{\#})\{0\} = \sum_a |\mu\{a\}|^2$, where summation is taken over all atoms.

(iv) Deduce a Wiener theorem:

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\le N} |\widehat{\mu}(n)|^2 = \sum_a |\mu\{a\}|^2$$

(it gives a simple characterization of continuous measures in terms of its Fourier transform: $\sum_{|n|\le N} |\widehat{\mu}(n)|^2 = o(N)$. Unfortunately there is no characterization of this type for absolutely continuous measures).

EXERCISE 2.9: Construct a continuous measure with $\widehat{\mu}(n) \not\to 0$.

EXERCISE 2.10: (i) Find a sequence $c(n) = 0, 1$ $(n \in \mathbb{Z})$ which is not a Fourier transform of measure.

(ii)* Show that $c(n) = \begin{cases} 1, & n \ge 0 \\ 0 & n < 0 \end{cases}$ gives such an example .

The natural question appears: how to characterize sequences $\{c(n)\}$ which are Fourier transforms of measures. Unfortunately no simple condition (like in L^2-theory) does exist. Still a very useful answer can be given in terms of *Cesaro means*.

2.3. Cesaro means

For every trigonometric series

$$\sum_{n\in\mathbb{Z}} c(n)e^{int} \tag{1}$$

(with arbitrary, not necessary Fourier, coefficients) one can consider the sequence of arithmetical (or Cezaro) averages of its partial sums S_n:

$$\sigma_N = \frac{1}{N+1} \sum_{0\le n\le N} S_n \equiv \sum_{|k|\le N} c(n)\Big(1 - \frac{|k|}{N+1}\Big)e^{ikt}$$

Clearly if (1) is a Fourier series of some measure μ then σ_N are convolutions of this measure with polynomials

$$K_N(t) = \sum_{|k|\le N} \Big(1 - \frac{|k|}{N+1}\Big)e^{ikt} \qquad \Big(= \frac{1}{N+1} \sum_{0\le n\le N} D_n\Big) ,$$

so called *Fejer kernels*. A remarkable property of those kernels (contrary to the Dirichlet ones) is that *they are positive*. To see this rewrite K_N as follows:

$$K_N(t) = \frac{1}{N+1}|\sum_{0\le k\le N} e^{ikt}|^2$$

(the last equality comes from the calculation:

$$|\sum e^{ikt}|^2 = \sum e^{ikt} \cdot \sum e^{i\ell t} = \sum_{|s|\le N} \alpha(s)e^{ist} ,$$

where $\alpha(s)$ is a number of representations $s = j - \ell$, $0 \le j \le N$, $0 \le \ell \le N$, which is equal to $N + 1 - |s|$).

THEOREM 2.11. *The series (1) is a Fourier series of some measure μ if and only if the corresponding Cesaro means σ_N are bounded in L^1-norm.*

Proof. If $c(n) = \widehat{\mu}(n)$ then $\sigma_N = \mu * K_N$ and

$$\|\sigma_N\|_1 \le \|\mu\|_M \cdot \|K_N\|_1 = \|\mu\|_M$$

Let (1) is given and $\|\sigma_N\|_1 < C$. Then according to 1.5 there exists a sequence $\{N_j\}$ such that σ_{N_j} weakly converge to some measure μ (more precisely corresponding measures μ_{N_j}, $d\mu_N = \sigma_N dm$, converge to μ). So $\forall n \in \mathbb{Z}$

$$\int_{\mathbb{T}} e^{-int}\sigma_{N_j} dm \to \int_{\mathbb{T}} e^{-int} d\mu \equiv \widehat{\mu}(n) .$$

But

$$\int_{\mathbb{T}} e^{-int}\sigma_{N_j} dm = \Big(1 - \frac{|n|}{N_j}\Big)c(n) \to c(n) \quad (j \to \infty) .$$

□

EXERCISE 2.12: (Herglotz Theorem). Prove that the following conditions are equivalent:

(i) (1) is a Fourier series of some *positive* measure;

(ii) $\sigma_N(t) \geq 0 \ \forall N, t$;

(iii) the sequence $\{c(n)\}$ is *positively-definite*, i.e.

$$\sum_{-N \leq n,m \leq N} c(n-m) z_n \overline{z}_m \geq 0 \qquad \forall N, \{z_n\} \ .$$

EXERCISE 2.13: A sequence of vectors $\{\xi_k\}$ $(k \in \mathbb{Z})$ in a Hilbert space H is called *stationary* if the scalar product depends only on the difference:

$$\langle \xi_k, \xi_m \rangle = c(k-m) \ .$$

Example: $\{e^{ikt}\}$ in $L^2_\mu(\mathbb{T})$ (μ is a given positive measure). Prove that this example is *universal* in the following sense: *every* stationary sequence one can get from the example (with appropriate μ) by some isometry $U : L^2_\mu \to H$. (This result has an important probabilistic interpretation).

2.4. Convolution operators

With every $\mu \in M(\mathbb{T})$ one can associate the *convolution operator* in the space $L^1(\mathbb{T})$:

$$Q_\mu : f \mapsto (f * \mu)(t) = \int_{\mathbb{T}} f(t-x) d\mu(x)$$

with obvious estimate of the norm: $\|Q_\mu\| \leq \|\mu\|$.

EXERCISE 2.14: Prove that in fact the equality is true.

The same estimate (not the equality) holds in L^p-spaces:

$$\|f * \mu\|_p \leq \|\mu\|_{M(\mathbb{T})} \|f\|_p \qquad (1 \leq p \leq \infty) \ .$$

This is clear for $p = \infty$ and follows for other p by general interpolation argument, but also by the elementary proof, using Hölder inequality (below $1/p + 1/q = 1$):

$$\begin{aligned} \|f * \mu\|_p &= \sup_{g:\|g\|_q=1} \left|\int_{\mathbb{T}} (f * \mu) \cdot g dm\right| = \\ &= \sup_g \left|\int_{\mathbb{T}\times\mathbb{T}} f(t-x)g(t)d\mu(x)dm(t)\right| \\ &= \sup_g \left|\int_{\mathbb{T}} \Big[\int_{\mathbb{T}} f(t-x)g(t)dm(t)\Big] d\mu(x)\right| \leq \\ &\leq \|\mu\| \cdot \|f\|_p \ . \end{aligned}$$

An important particular case of convolution operators is *translation* (or *shift*):

$$S_\tau : f(t) \mapsto f(t-\tau)$$

which corresponds to $\mu = \delta_\tau$ $(\tau \in \mathbb{T})$.

On the other hand a convolution operator can be ≪composed≫ using those ≪elementary convolutions≫ – translations.

To state this in a precise form we need the following

Definition 2.15. *A sequence $\{A_n\}$ of operators in a Banach space X is called convergent pointwisely to an operator A if $\forall x \in X$ $\|A_n x - Ax\|_X \to 0$ $(n \to \infty)$.*

We use

Theorem 2.16. *A sequence $\{A_n\}$ converges pointwisely iff*

(i) $\|A_n\| < C$ $(n \in \mathbb{Z})$

(ii) $\exists$ a dense set $X_0 \subset X$ s.t. $\forall x \in X_0$ the sequence $\{A_n x\}$ converges in X.

(Clearly the result of 1.4 is a particular case of the theorem). In what follows we mean $X = L^p(\mathbb{T})$, $1 \leq p < \infty$, or $C(\mathbb{T})$.

Theorem 2.17. *Every convolution operator Q_μ in X can be obtained as the pointwise limit of a sequence $\{A_n\}$ of linear combinations of translations.*

LEMMA 2.18. *If $\mu_n \to \mu$ weakly then $Q_{\mu_n} \to Q_\mu$ pointwisely in X.*

Indeed, weak convergence implies: $\|\mu_n\| < C$, so $\|Q_{\mu_n}\|_{X\to X}$ are bounded. Now, for a fixed $k \in \mathbb{Z}$. $Q_{\mu_n}(e^{ikt}) \equiv \mu_n * e^{ikt} = \widehat{\mu}_n(k)e^{ikt}$ in X, so the condition (ii) of Theorem 2.16 is fulfilled for $X_0 = \{$the set of trigonometric polynomials$\}$.

For a given $\mu \in M(\mathbb{T})$ we use the exercise 1.4 (ii) and the lemma gives the Theorem 2.17

EXERCISE 2.19: Prove that a convolution operator commutes with translations. A stronger form: any two convolution operators commute.

Lecture 3. Invariant Subspaces & Operators on $\mathbb{T}$

3.1. Translation-invariant subspaces

DEFINITION 3.1. *A closed linear subspace E in the space $X(= L^p(\mathbb{T})$, $1 \le p < \infty$, or $C(\mathbb{T}))$ is called (translation)-invariant if*

$$f \in E \Rightarrow S_\tau f \in E \qquad \forall \tau \in \mathbb{T}\ .$$

EXERCISE 3.2: Prove that every such subspace is automatically invariant with respect to a convolution operator, i.e.

$$\forall \mu \in M(\mathbb{T}), \quad f \in E \Rightarrow \mu * f \in E\ .$$

EXAMPLES 3.3.

(i) $\forall k \in \mathbb{Z}$ *one-dimensional subspace* $E_k = \{ce^{ikt}, c \in \mathbb{C}\}$ *is invariant.*

(ii) Fix a set $\Lambda \subset \mathbb{Z}$. *The subspace*

$$E_\Lambda = \{f \in X : \widehat{f}(n) = 0 \quad \forall n \notin \Lambda\} \quad \textit{is invariant}\ .$$

EXERCISE 3.4: Prove that E_Λ is the closure in X of the set of trigonometric polynomials of the form $\sum_{k\in\Lambda} c_k e^{ikt}$.

Hint. Show that $f * K_n \to f$. (Fejer theorem). The set Λ is called *the spectrum* of E_Λ.

It turns out that E_Λ are the only translation-invariant subspaces.

THEOREM 3.5. *Every (closed) invariant subspace E in X has the form: $E = E_\Lambda$.*

Proof. For a given E we set:

$$\Lambda = \{n \in \mathbb{Z} : \exists f \in E, \quad \widehat{f}(n) \neq 0\} \ .$$

Obviously $E \subset E_\Lambda$. To see the converse inclusion fix $n \in \Lambda$ and take $f \in E$, $\widehat{f}(n) \neq 0$. According to Exercise 3.2 $f * e^{int} = \widehat{f}(n)e^{int} \in E$, so $e^{int} \in E \quad \forall n \in \Lambda$. However, $\{e^{int}\}_{n\in\Lambda}$ generates all the space E_Λ (Exercise 3.4). □

3.2. Translation-invariant operators

DEFINITION 3.6. *A bounded linear operator $T : X \mapsto X$ is called (translation) invariant if it commutes with translations:*

$$TS_\tau = S_\tau T \qquad (\forall \tau \in \mathbb{T}) \ . \tag{1}$$

Example: The convolution operator (see Exercise 2.19).

PROPOSITION 3.7. *It follows from (1) that T commutes with the convolution:*

$$(Tf) * \mu = T(f * \mu) \qquad \forall \mu \in M(\mathbb{T}) \ . \tag{2}$$

Proof. Take a sequence μ_n – linear combinations of δ-functions, such that $\mu_n \to \mu$ weakly, (Exercise 1.4). The convolution with μ_n is a linear combination of translates, so

$$(TF) * \mu_n = T(f * \mu_n) \ .$$

Passing to the limit ($n \to \infty$) and using the Lemma 2.18 we get the result. □

PROPOSITION 3.8. *Let $T : X \to X$ be a bounded linear operator. Then the following two properties are equivalent:*

(i) T is invariant

(ii) $T(e^{int}) = m(n)e^{int}, \quad \forall n \in \mathbb{Z}$.

Proof. (i)→(ii).
Denote Q_n – the convolution operator $f \mapsto f * e^{int}$. We have:

$$T(e^{int}) = T(Q_n e^{int}) = Q_n T(e^{int}) = m(n)e^{int} \ .$$

(ii)→(i).
It is enough to check (1) on a dense set $\subset X$.

$$(TS_\tau)e^{int} = T(e^{-in\tau}e^{int}) = e^{-in\tau}m(n)e^{int} = S_\tau T(e^{int}) \ ,$$

so (1) holds on a set of trigonometric polynomials. □

3.3. Multipliers

Let $\{m(n)\}$ be a function: $\mathbb{Z} \to \mathbb{C}$. It yields an operator

$$T_m : f \mapsto [\widehat{f}(n) \cdot m(n)]^\vee \ , \tag{3}$$

$\vee$ – be a symbol of inverse Fourier transform, so (3) means that

$$(\widehat{T_m f})(n) = m(n)\widehat{f}(n) \qquad (n \in \mathbb{Z}) \ .$$

By this way, T_m is defined at least on a set $\mathcal{P}$ of trigonometric polynomials, which is dense in X. If it is bounded:

$$\|T_m f\| \leq \mathcal{K}\|f\| \qquad f \in \mathcal{P}$$

then T_m has a *unique extension* as a bounded linear operator in X. According to Proposition 3.7 this operator is translation-invariant, and moreover: *every invariant bounded operator can be obtained by this way.*

DEFINITION 3.9. *A function $m : \mathbb{Z} \to \mathbb{C}$ is called an X-multiplier if it yields a bounded operator T_m in X.*

We denote the set of all X-multipliers by $\mathfrak{m}_X$, or $\mathfrak{m}_p(\mathbb{Z})$ (for $p=\infty$ this notation corresponds to $X=C(\mathbb{T})$).

Clearly $\mathfrak{m}_p$ is a Banach algebra with respect to standard linear operations, operator norm:

$$\|m\|_{\mathfrak{m}_p} = \|T_m\|_{X\to X}$$

and pointwise multiplication.

It follows from above that to study the algebra of multipliers $\mathfrak{m}_p(\mathbb{Z})$ $(1 \le p \le \infty)$ is the same that to study the algebra of translation-invariant operators in $L^p(\mathbb{T})$ $(C(\mathbb{T}))$.

EXERCISE 3.10: Prove:

$$\mathcal{F}(M(\mathbb{T})) \subset \mathfrak{m}_p \subset \ell^\infty(\mathbb{Z}) \qquad (\forall_p)$$

(by $\mathcal{F}(M)$ the class of Fourier transform of measures is denoted).

3.4. Special cases: $p=2,1,\infty$

In general this is an extremely difficult problem for a given sequence m to recognize whether it belongs to $\mathfrak{m}_p$ and to estimate its multiplier norm. But there are some special cases when the solution is available.

The simplest case is $p=2$.

In this case the multiplier algebra is the maximal possible:

EXERCISE 3.11: Prove that $\mathfrak{m}_2 = \ell^\infty$.

The opposite extreme case is realized for $p=1,\infty$.

THEOREM 3.12. *Every translation invariant operator in $L^1(\mathbb{T})$ has the form: $f \mapsto f * \mu$, $\mu \in M(\mathbb{T})$.*

Proof. Let $m \in \mathfrak{m}_1$. Taking $f = K_N$ (Fejer kernel) we have

$$\|T_m f\|_1 = \Big\| \sum_{|n|\le N} m(n)\Big(1-\frac{|n|}{N+1}\Big)e^{int}\Big\|_1 \le \|m\|_{\mathfrak{m}_1} \qquad (\forall N)$$

and Theorem 2.11 gives that $m=\widehat{\mu}$ for some $\mu \in M(\mathbb{T})$. □

EXERCISE 3.13: Prove the same result for $C(\mathbb{T})$.

3.5. Duality result

The last effect is of a more general nature.

THEOREM 3.14. *For every* $1 < p < \infty$ $\mathfrak{m}_p = \mathfrak{m}_q$ $(1/p + 1/q = 1)$.

Proof. Fix $f \in \mathcal{P}$. Then:

$$\|Tf\|_p = \sup_{g \in \mathcal{P},\ \|g\|_q = 1} |\int_{\mathbb{T}} (Tf \cdot g) dm| = \sup_{g \in \mathcal{P},\ \|g\|_q = 1} |\int_{\mathbb{T}} (f \cdot Tg)_{dm}| \leq$$
$$\leq \|f\|_p \cdot \|T\|_{L^q \to L^q} \ .$$

It follows: $\|\mathfrak{m}\|_{\mathfrak{m}_p} \leq \|\mathfrak{m}\|_{\mathfrak{m}_q}$. Changing p and q we get the result. □

EXERCISE 3.15: Show that if $2 \leq p < p' < \infty$ then $\mathfrak{m}_p \supset \mathfrak{m}_{p'}$.

Hint. Use the Riesz-Torin interpolation theorem.

Lecture 4. Complementable Subspaces

4.1. Decompositions and projectors

Let E, E' be a pair of (closed linear) subspaces in a Banach space X. One says that X is a direct sum of E and E',

$$X = E \oplus E' \ , \tag{1}$$

if every $x \in X$ can be decomposed as

$$x = g + g' \ , \quad g \in E,\ g' \in E' \tag{2}$$

and such a decomposition is unique.

A space E is called *complementable* if there exists E' such that (1) holds.

This notion can be defined in an equivalent way in terms of *projectors*. One says that a (bounded linear) operator

$$P : X \mapsto E \subset X$$

is a projector of X on a subspace E if $Pg = g\ \forall g \in E$. The class of projectors has an algebraic characterization:

$$P^2 = P \tag{3}$$

Every projector satisfies the condition and vice versa if (3) is fulfilled then P is a projector on the subspace $E = \{g : Pg = g\}$.

Proposition 4.1. *A subspace E is complementable iff $\exists$ a projection $P : X \mapsto E$.*

Indeed, if P is a projector then $P' = I - P$ is a projector too, on some subspace E'. Taking $x \in X$ we have (2) with $g = Px, g' = P'x$, and clearly the decomposition is unique. On the other hand having (1) we define a linear operator $P : X \mapsto E$, $Px = g$. According to the closed graph theorem P is bounded.

Exercise 4.2: (i) Show that every finite dimensional subspace E in X is complementable;

(ii) describe all projectors of the space $C(\mathbb{T})$ on the subspace E of constants.

In a Hilbert space every subspace is complementable. For example one can consider orthogonal projector. But in general this is not true. An instructive example is considered below.

4.2. Averaging over translations

Consider in the space $C(\mathbb{T})$ subspace E of functions f having only positive harmonics in the Fourier expansion: $f \sim \sum_{n\geq 0} \widehat{f}(n)e^{int}$.

Is this subspace complementable? At first glance the answer seems to be positive: the natural candidate for the role of the projector is:

$$R : f \to \sum_{n\geq 0} \widehat{f}(n)e^{int} ,$$

the so-called Riesz projector, which is an orthogonal projector of $L^2(\mathbb{T})$ on the subspace generated by $\{e^{int}\}_{n\geq 0}$. But this operator is not well-defined in $C(\mathbb{T})$.

EXERCISE 4.3: (i) Show that $P_n(t) = \sum_{k=1}^{n} \frac{\sin kt}{k}$ satisfies the inequality:

$$\|P_n\|_{C(\mathbb{T})} < K \qquad (n = 1, 2, \dots) ;$$

(ii) Show that the operator R, defined on the set $\mathcal{P}$ (of trigonometric polynomials), is unbounded as an operator in $C(\mathbb{T})$;

(iii) Show that for some $f \in C(\mathbb{T})$ the function $Rf \notin C(\mathbb{T})$.

So the ≪natural candidate≫ failed. But this does not exclude the possibility that some other projector may exists. We will see that this is not the case. Suppose P is a projector: $C(\mathbb{T}) \to E$.

Now consider a new operator $\widetilde{P}$ obtained from P by the averaging with respect to translations of the circle:

$$\widetilde{P} : f \mapsto \int_{\mathbb{T}} (S_{-\tau} P S_{\tau}) f \, dm(\tau) \tag{4}$$

The meaning of the integral follows from

EXERCISE 4.4: Let $\tau \to f_\tau$ be a continuous mapping of $\mathbb{T}$ into a Banach space X. Show that the integral $\int_{\mathbb{T}} f_\tau dm(\tau)$ exists as a limit in X of Riemannian sums.

The definition (4) implies that $\widetilde{P}$ is a projector on the space E. Since the space is translation-invariant $\widetilde{P}$ commutes with translations. The Proposition 3.2 implies:

$$\widetilde{P}(e^{int}) = \begin{cases} e^{int}, & n \geq 0 \\ 0 & n < 0 \, . \end{cases}$$

So, $\widetilde{P} = R$. But it is unbounded! Contradiction. Conclusion: *E is not a complementable subspace in $C(\mathbb{T})$.*

Actually the argument above works in a more general setting and gives:

THEOREM 4.5. *Let E be a complementable translation invariant subspace in $L^p(\mathbb{T})$ $1 \leq p < \infty$, or $C(\mathbb{T})$. Then there exists a translation-invariant projector on E.*

In other words (remembering Theorem 3.5):

$$E_\Lambda \quad \text{is complemented} \quad \iff \mathbb{1}_\Lambda \in \mathfrak{m}_p(\mathbb{Z}) \qquad (1 \le p \le \infty)$$

(by $\mathbb{1}_\Lambda$ we denote the indicator-function of a set $\Lambda \subset \mathbb{Z}$). Therefore the problem of characterizing all complementable invariant subspaces in $L^p(\mathbb{T})$ is the same that to describing all *idempotent multipliers* (i.e. taking values 0 and 1 only). For $p = 2$ the problem is trivial. For $p = 1, \infty$ it is solved by the remarkable

THEOREM 4.6. *(Helson)* $\mathbb{1}_\Lambda \in \mathfrak{m}_1$ *if and only if* $\Lambda = \Lambda_0 \cup \Lambda_1$, *where* Λ_0 *is finite and* Λ_1 *is periodic set.*

EXERCISE 4.7: Prove the part ≪if≫ in the theorem.

4.3. Classical examples

For other values of p the problem seems to be extremely difficult. Some classical examples follow.

EXAMPLE 4.8. *(M. Riesz)* $\mathbb{1}_{\mathbb{Z}+} \in \mathfrak{m}_p(\mathbb{Z})$, $1 < p < \infty$.
In other words the Riesz projector is bounded for all p strictly between 1 *and* ∞.

EXERCISE 4.9: (i) Deduce that $\|\mathbb{1}_{[-N,N]}\|_{\mathfrak{m}_p} < C(p)$ $(N = 1, 2, \dots), 1 < p < \infty$;

(ii) prove the M.Riesz theorem: for any $f \in L^p(\mathbb{T})$, $1 < p < \infty$ the Fourier series $\sum \widehat{f}(n)e^{int}$ converges to f in the norm;

(iii) show that the result fails for $C(\mathbb{T})$ (Dubois Reimond) and for $L^1(\mathbb{T})$ (Banach, Steinhaus).

A much more general and delicate example was given by Littlewood and Paley:

EXAMPLE 4.10. *Let* Λ *be a union of any subfamily of diadical intervals* $[2^k, 2^{k+1}[(k \in \mathbb{Z})$. *Then*

$$\mathbb{1}_\Lambda \in \mathfrak{m}_p \qquad \forall p \in]1, \infty[\ .$$

On the other hand there are idempotent sequences $\{m(n)\}$ which do not belong to the multipliers algebras $\mathfrak{m}_p$ for any $p \neq 2$. Moreover, this situation is *typical*; it is realized for a *randomly chosen* sequence $\{m(n)\}$ with probability 1.

4.4. Random multipliers

THEOREM 4.11. *Let* $\{\varepsilon(n)\}$ *be a sequence of random independent variables taking the values* $0, 1$ *with equal probabilities. Then* $\forall p \neq 2$ $\{\varepsilon(n)\} \notin \mathfrak{m}_p$ *almost surely.*

Consider the so-called *Rademacher system* of functions $\{r_n(x)\}$: $r_n(x) = r(2^n x)$, $n = 1, 2, \dots$, where $r(x) = sign x$, $|x| < 1/2$, and periodical with period 1. An important property of this system is: *on the linear span of* $\{r_n\}_1^\infty$ *all* $L^p[0,1]$ *norms* $(1 \leq p < \infty)$ *are equivalent*. This is the *Khintchine inequality*. We prove the following particular case, which is technically the simplest:

$$\|\sum_{n=1}^{N} c_n r_n\|_{L^4[0,1]} \leq C\sqrt{\sum_{n=1}^{N} |c_n|^2} \quad (c_n \in \mathbb{C}\,,\ 1 \leq n \leq N; N = 1, 2, \dots)\,.$$

Indeed:

$$\int_{[0,1]} |\sum c_n r_n(x)|^4 dx = \sum c_n c_m \bar{c}_\ell \bar{c}_k \int_0^1 r_n r_m r_\ell r_k dr =$$
$$= \sum |c_n|^4 + \sum_{n \neq \ell} c_n^2 \bar{c}_\ell^2 + 2 \sum_{n \neq m} |c_n|^2 |c_m|^2 \leq 3(\Sigma |c_n^2|)^2\ .$$

The inequality (5) implies:

$$(6) \qquad \int_{[0,1]} \int_{\mathbb{T}} |\sum_{n=1}^{N} r_n(x) e^{int}|^4 dm(t) dx \leq C^4 N^2\ .$$

By choosing x we get the following

PROPOSITION 4.12. *For every* N $\exists$ *a polynomial*

$$(7) \qquad P_N(t) = \sum_{n=1}^{N} \pm e^{int}\ , \quad \|P_N\|_4 \leq C\sqrt{N}\ .$$

EXERCISE 4.13: Prove that

$$(8) \qquad \|\sum_{n=1}^{N} e^{int}\|_p \geq C_1 N^{1-1/p} \qquad \forall p > 2$$

Exercise 4.14: Deduce from (7) and (8) the Theorem 4.11.

In conclusion we mention again that to recognize *for an individual* $\Lambda \subset \mathbb{Z}$ whether $\mathbb{1}_\Lambda \in \mathfrak{m}_p,\ 1 < p < \infty, \neq 2$, is usually a very difficult problem. For example the complete answer is unknown for $\Lambda = \{k^2, \quad k = 1, 2, \dots\}$.

Lecture 5. Noncompact Case

We wish to investigate analogous problems in function spaces on the group $\mathbb{Z}$. At first glance, this case seems to be simpler (for example, measure theory on $\mathbb{Z}$ is really much easier than on $\mathbb{T}$). However, the fundamental obstacle is *the noncompactness* of $\mathbb{Z}$. So the invariant measure ($E \subset \mathbb{Z} \to |E|$) is infinite and ≪characters≫ $\psi_t : n \to e^{int}$ do not belong to L^p spaces (which we denote as usually $\ell_p(\mathbb{Z})$) for any $p < \infty$. The Fourier transform:

$$\{c(n)\} \in \ell_p \longrightarrow \check{c}(t) = \sum_{n \in \mathbb{Z}} c(n) e^{int} \qquad (t \in \mathbb{T})$$

(as a function on the ≪character group≫) is defined *everywhere* only for $p = 1$, and *almost everywhere* for $1 < p \leq 2$. However for $p > 2$ there is no reasonable way to define $\check{c}$ even a.e. on $\mathbb{T}$ (it can be treated only in framework of Schwartz distributions).

Our main problems are, as before:

1. To describe (translation) invariant subspaces in $\ell_p(\mathbb{Z})$; we are especially interested in *complementable invariant* subspaces.

2. To describe invariant operators T in these spaces i.e. operators, commuting with translations

$$S_k : c(n) \mapsto c(n-k) \ .$$

In spite of the fact that there is no possibility of repeating the arguments above, to a great extent the results are similar; on the other hand, some unexpected phenomena appear.

5.1. Invariant subspaces in $\ell_2(\mathbb{Z})$

For a given Borel set $\Lambda \subset \mathbb{T}$ we set:

$$E_\Lambda = \{c \in \ell_2(\mathbb{Z}) : \overset{\vee}{c}(t) = 0 \quad \text{a.e. outside of } \Lambda\} \ .$$

EXERCISE 5.1: Prove:

(i) E_Λ is a closed invariant subspace in $\ell_2(\mathbb{Z})$;

(ii) The correspondence $\Lambda \to E_\Lambda$ is one-to-one (here Λ and Λ' are identified if they differ on a set of measure zero)

THEOREM 5.2. *(Ditkin, Wiener). None other than E_Λ invariant spaces exist in $\ell_2(\mathbb{Z})$.*

Proof. For a given (closed) invariant E define Λ as a *minimal support* of the family $\{\overset{\vee}{c}; c \in E\}$, i.e. a set of minimal measure such that $\forall c \in E\ \overset{\vee}{c}(t) = 0$ a.e. outside Λ. □

EXERCISE 5.3: Prove that such a Λ exists and is uniquely defined (up to a set of measure zero).

Obviously $E \subset E_\Lambda$. To show the converse inclusion consider $h \in E^\perp$. Because of invariance:

$$\langle h, S_k c\rangle = 0 \qquad c \in E, k \in \mathbb{Z} \ ,$$

so

$$\int_{\mathbb{T}} \overset{\vee}{c}(t) e^{int} \overline{\overset{\vee}{h}(t)} dm = 0 \ .$$

It follows: $\overset{\vee}{c}\overline{\overset{\vee}{h}} = 0$ a.e. on $\mathbb{T}$ $\forall c \in E$, and finally:

$$\overset{\vee}{h}(t) = 0 \quad \text{a.e. on } \Lambda \ , \quad \text{so} \quad E^\perp \subset (E_\Lambda)^\perp$$

□

It is reasonable to call Λ the spectrum of the corresponding invariant subspace.

5.2. Schwartz-Malliavin phenomenon

The situation for $p \neq 2$ is much harder. Consider $p = 1$. In this case $\overset{\vee}{c} \in C(\mathbb{T})$, so the set of zeros is closed. The role of E_Λ is now played by the following

EXAMPLE 5.4. *For a compact $K \subset \mathbb{T}$ denote:*

$$E(K) = \{c \in \ell_1 : \overset{\vee}{c}(t) = 0 \quad t \in K\} \ .$$

Obviously this is a closed invariant subspace. We expect that no others exist, but surprisingly this is false. Denote by

$$\mathcal{E}(K) = \mathrm{Clos}\{c \in \ell_1 : \overset{\vee}{c}(t) = 0 \quad \textit{in a neighbourhood of } K\} \ .$$

It appears that $E(K) = \mathcal{E}(K)$ (≪every function vanishing on K can be approximated by a function vanishing in a neighbourhood≫). But even in the simplest cases this is nontrivial to prove.

EXERCISE 5.5: Let $K = \{0\}$. Prove: $E(K) = \mathcal{E}(K)$.

Hint. (i) for φ_ε ,which is piecelinear ,
$= \ 1$ on $[-\varepsilon, \varepsilon]$,
$= \ 0$ outside of $[-2\varepsilon, 2\varepsilon]$

prove: $\|\widehat{\varphi_\varepsilon}\|_{\ell_1(\mathbb{Z})} \leq const.$ (not depending on ε.)

(ii) Prove that for piecewise-smooth $f \in C(\mathbb{T})$

$$\|\widehat{f}\|_{\ell_1(\mathbb{Z})} \leq |f(0)| + \|f'\|_{L^2(\mathbb{T})} \ ;$$

(iii) approximate $f \in \overset{\vee}{\ell_1}$ by $f(1 - \varphi_\varepsilon)$.

□

It turns out that in general (for quite complicated compact K's) the subspace $\mathcal{E}(K)$ is a *proper part* of $E(K)$ and between them there are uncountably many different invariant subspaces of ℓ_1. All of them have *the same* set of common zeroes of Fourier transforms – namely, K. This shows the difficulty of characterizing all invariant subspaces. At present, this problem is unsolved (also for other $p \neq 2$).

5.3. Invariant operators

The main idea coming from Section 3.3 is: translation-invariant operators appear in Fourier images as *operators of multiplication*.

EXERCISE 5.6: For a given $m \in L^2(\mathbb{T})$ consider the operator $f \mapsto mf$. Prove:

(i) if the operator is bounded $L^2 \to L^2$ on the set of trigonometric polynomials then $m \in L^\infty(\mathbb{T})$;

(ii) if $m \in L^\infty(\mathbb{T})$ then the operator is bounded in $L^2(\mathbb{T})$, and its norm equals to $\|m\|_\infty$.

It follows:

EXAMPLE 5.7. *Every $m \in L^\infty(\mathbb{T})$ yields a bounded translation invariant operator in $\ell_2(\mathbb{Z})$:*

$$T_m : c \mapsto (\check{c} \cdot m)^\wedge \ . \tag{1}$$

THEOREM 5.8. *Every bounded invariant operator T in $\ell_2(\mathbb{Z})$ has the form (1).*

Proof. Denote: $\{e_k\}$ – the canonical basis in $\ell_2(\mathbb{Z})$;

$$Te_0 := g \ . \quad \text{Then :}$$
$$Te_k = TS_k e_0 = S_k g \ ,$$
$$(T\check{e}_k)(t) = e^{ikt}\check{g}(t) \ .$$

So for any $c = \sum_{|k|\leq N} c(k)e_k$ we have:

$$(\check{Tc})(t) = \check{g}(t)\check{c}(t) \ .$$

This means that on a set of trigonometric polynomials the operator T has the form (1) with $m = \check{g}(t)$.

Using Exercise 5.6(i) we get the result. □

From Section 5.3 one may expect problems for $p \neq 2$. Fortunately, this is not the case.

THEOREM 5.9. *Let T be a bounded invariant operator in $\ell_p(\mathbb{Z})$, $1 \le p \le \infty$. Then there exists a function $m \in L^\infty(\mathbb{T})$ such that*

$$Tc = (\overset{\vee}{c} \cdot m)^\wedge \quad c \in \Phi \equiv \{ \sum_{|k| \le N} c(k) e_k \ , \quad N = 1, 2, \dots \} \ . \tag{2}$$

Denote as before $Te_0 = g$, we have:

$$(Tc)(n) = \sum_k g(n-k)c(k) \equiv (g * c)(n) \quad , \quad c \in \Phi \ .$$

The crucial point: the convolution operator $c \to g * c$ has equal norms in ℓ_p and ℓ_q $(1/p + 1/q = 1)$. Indeed,

$$\|T\|_{p\to p} = \sup_{\substack{c \in \Phi \\ \|c\|_p = 1}} \|Tc\|_p = \sup_{\substack{c, d \in \Phi \\ \|c\|_p = \|d\|_q = 1}} |\sum_n \sum_k g(n-k)c(k)d(n)| =$$

$$= \sup_d \|\{\sum_n g(n-k)d(n)\}\|_q = \sup_d \|g * d\|_q = \|T\|_{q \to q} \ .$$

Now the result follows from Theorem 5.8 and Riesz-Torin interpolation theorem.

EXERCISE 5.10: Formulate and prove an analog of Theorem 3.12.

5.4. Multipliers algebra

It is not true that every $m \in L^\infty$ yields a *bounded* operator (2). If such an m does it is called an p-multiplier. The set of all such multipliers forms a Banach algebra $\mathfrak{m}_p(\mathbb{T})$ with respect to pointwise multiplication and operator norm. For $p = 2$ this coincides with $L^\infty(\mathbb{T})$. The minimal algebra correspond to $p = 1$. This is the famous *Wiener algebra* of absolutely convergent Fourier series:

$$W(\mathbb{T}) = \{m : \widehat{m} \in \ell_1(\mathbb{Z})\} \ .$$

If p goes from 1 up to 2 the corresponding algebra $\mathfrak{m}_p$ strictly increases. For $2 < p < \infty$ the duality result holds: $\mathfrak{m}_p = \mathfrak{m}_q$.

The idempotent elements $m = \mathbb{1}_\Lambda$ of the algebra $\mathfrak{m}_p$ are of special interest. Every such an m yields an *invariant projector* $T = P_\Lambda$ on a complemented subspace

$$E_\Lambda = P_\Lambda(\ell_p(\mathbb{Z})) \ .$$

THEOREM 5.11. *(H. Rosenthal [9]) For $1 < p < \infty$ every complemented translation invariant subspace $E \subset \ell_p(\mathbb{Z})$ has the above form.*

This is an analogue of Theorem 4.5, which requires another proof in the noncompact case. (for $p = 1$ the result fails; one can easily see that $\mathfrak{m}_1(\mathbb{T})$ contains only trivial idempotents $m = 0$ and $m = 1$). Thus in spite of the phenomena discussed in 5.2, *complemented* invariant subspaces again can be characterized in terms of spectra Λ. The study of such subspaces in ℓ_p, $1 < p < \infty$ is the same as the study of Borel subsets $\Lambda \subset \mathbb{T}$ yielding idempotent multipliers $\mathbb{1}_\Lambda$. Again (for $p \neq 2$) this is an extremely difficult problem. Classical examples (as in Section 4.4) are: an interval on $\mathbb{T}$; a union of diadical intervals. It was unknown for a long time whether a Cantor set Λ (of positive measure) can yield a multiplier. The negative answer was obtained in [7] We discuss the main points of this result in the last lecture.

Lecture 6. Idempotent multipliers on the circle

The subject of this lecture is a recent joint result of Lebedev and Olevskiĭ which shows that the *spectrum of a complemented invariant subspace* in $\ell_p(\mathbb{Z})$ $(p \neq 2)$ *has quite a simple structure*.

THEOREM 6.1. *[7] If $\mathbb{1}_\Lambda \in \mathfrak{m}_p(\mathbb{T})$ for some $p \neq 2$ then Λ is an open set on $\mathbb{T}$ (up to a set of measure zero), as well as $^c\Lambda$.*

6.1. De Leeuw theorem

We start with a theorem making a connection between multipliers on $\mathbb{T}$ and on $\mathbb{Z}$.

THEOREM 6.2. *[1] Let $t_k = a + kh$, $1 \leq k \leq k_0$, be Lebesgue points of a given function $m \in L^\infty(\mathbb{T})$. Then the following inequality holds:*

$$\|\Sigma m(t_n) c(k) e^{ikt}\|_{L^p(\mathbb{T})} \leq C\|m\|_{\mathfrak{m}_p} \|\sum c(k) e^{ikt}\|_{L^p(\mathbb{T})} \quad \forall \{c(k)\} \ .$$

Here C is an absolute constant (which can be taken $= 1$). Remember that t is a *Lebesgue point of* m if

$$\int_{-\varepsilon}^{+\varepsilon} |m(t+h) - m(t)| dh = o(\varepsilon) \ .$$

We sketch a proof.

Proof. 1°. By convolution with the Fejer kernel K_N we can reduce this to a specific case: $m \in C^1(\mathbb{T}) \subset W(\mathbb{T})$.

(Note that the convolution $(f * K_N)(t) \to f(t)$ at every Lebesgue point; this is a classical result which is especially easy to prove for $f \in L^\infty$).

2°. Any $m \in W(\mathbb{T})$ can be approximated in the norm W, and so – in $\mathfrak{m}_p$, by a function constant in a neighbourhood of any t_k (see ex. 5.5). Thus we may assume this property. Suppose also that h/π – is irrational.

3°. Consider ≪the triangle function≫

$$\Delta_\varepsilon(t) = \max\{0, 1 - |t|/\varepsilon\} \ , \quad -\pi \le t \le \pi \ .$$

A direct calculation shows: $\widehat{\Delta}_\varepsilon$ is positive, decreases in a regular way and $\sum_n \widehat{\Delta}_\varepsilon^p(n) \approx \varepsilon^{p-1}$

4°. Fix $c(k)$, $1 \le k \le k_0$. We denote:

$$\mu = \Sigma c(k)\delta_{t_k}, \ m\mu = \Sigma m(t_k)c(k)\delta_{t_k},$$
$$\varphi(t) = \Sigma c_k e^{ikt}, \ f_\varepsilon = \Delta_\varepsilon * \mu$$

2° implies: $mf_\varepsilon = \Delta_\varepsilon * (m\mu)$.

5°. Now we claim: if $\varepsilon > 0$ is small enough then

$$\|\widehat{f_\varepsilon}\|_{\ell_p(\mathbb{Z})} \simeq \varepsilon^{\frac{p-1}{p}} \|\Sigma c_k e^{ikt}\|_{L^P(\mathbb{T})}$$

($\alpha \simeq \beta$ means that the ratio α/β is contained between two absolute positive constants).

Indeed,

$$\|\widehat{f}_\varepsilon\|^p_{\ell_p} = \|\widehat{\Delta}_\varepsilon \widehat{\mu}\|^p_{\ell_p} \equiv \sum_n |\widehat{\Delta}_\varepsilon(n)|^p \ |\widehat{\mu}(n)|^p =$$

$$= \sum |\widehat{\Delta}_\varepsilon(n)|^p | \sum_k c(k) e^{-iknh}|^p = \sum_n |\widehat{\Delta}_\varepsilon(n)|^p |\varphi(-nk)|^p$$

and the estimate follows from 3° and uniform distribution of the sequence $\{nh\}$ on $\mathbb{T}$.

6°. The same argument shows that

$$\|\widehat{mf_\varepsilon}\|_{\ell_p(\mathbb{Z})} \simeq \varepsilon^{\frac{p-1}{p}} \| \sum m(t_k) c(k) e^{ikt}\|_{L^p(\mathbb{T})} \ .$$

Now (1) comes immediately from the inequality:

$$\|\widehat{mf_\varepsilon}\|_{\ell_p} \leq \|m\|_{\mathfrak{m}_p} \cdot \|\widehat{f}_\varepsilon\|_{\ell_p} \ .$$

□

Exercise 6.3: Recover lacunas in the proof above.

6.2. A packing lemma

The result of this section (cp. [7, 8]) is crucial for the proof of Theorem 6.1

Below we denote the Lebesgue measure of a Borel set $A \subset \mathbb{R}$ by $|A|$.

Definition 6.4. *We say that x belongs to the essential boundary $\partial(A, B)$ of given sets A, B if any neighbourhood of x contains portions of positive measure from both sets.*

Lemma 6.5. *For any pair A, B with $|\partial(A, B)| > 0$ and for any disjoint finite sets $\mathfrak{a}, \mathcal{B} \subset \mathbb{R}$ there exists an affine mapping*

$$\varphi : x \mapsto \ell x + c$$

such that:

$$\varphi(\mathfrak{a}) \subset A \qquad \varphi(\mathcal{B}) \subset B \ .$$

Proof. 1°. We leave a well-known particular case for the readers:

> EXERCISE 6.6: Prove that every set of positive measure contains an affine copy of any finite set.

2°. For a given Borel $\mathcal{D} \subset \mathbb{R}$ we use the following notation:
$\mathcal{D}^\circ$ is the set of all *density points* of $\mathcal{D}$, belonging to $\mathcal{D}$. (It is well known that $|\mathcal{D}^\circ| = |\mathcal{D}|$); for $x \neq y$ we denote:

$$\psi_{x,y} : \ell \mapsto \ell(x-y) + y \qquad \ell \in \mathbb{R} ;$$
$$L_{x,y} = \psi_{x,y}^{-1}(\mathcal{D}^\circ).$$

3°. One can easily see that for every interval $I \subset \mathbb{R}$

$$|L_{x+h,y+h} \cap I| \to |L_{x,y} \cap I| \qquad (h \to 0) .$$

4°. If $x \in \mathcal{D}^\circ$ then for $I_\delta = (1-\delta, 1+\delta)$

$$\frac{1}{2\delta}|L_{x,y} \cap I_\delta| \to 1 \qquad (\delta \to 0) .$$

(Since an affine mapping $\psi_{x,y}$ transforms I_δ onto a neighbourhood of x and $L_{x,y}$ onto $\mathcal{D}^\circ$).

5°. If $x \in \mathcal{D}^\circ$ then:

$$\forall \varepsilon > 0 \qquad \exists \delta_0 > 0 \qquad \forall \delta, \qquad 0 < \delta < \delta_0 \qquad \exists \gamma > 0$$
$$|h| < \gamma \Longrightarrow \frac{1}{2\delta}|L_{x+h,y+h} \cap I_\delta| > 1 - \varepsilon .$$

It comes from 3° and 4°.

6°.

LEMMA 6.7. *If X is a finite subset of $\mathcal{D}^\circ$, $y \notin X$, then for any sufficiently small h there exists an affine mapping $\psi_h : x \mapsto \ell x + c$ such that:*

(i) $y + h$ is a fixed point of ψ_h;

(ii) $\psi_h(x+h) \in \mathcal{D}^\circ \qquad \forall x \in X.$

Indeed for each $x = x_j \in X$, $1 \le j \le N = cardX$ we use 5° with $\varepsilon < 1/N$. For a small $\delta > 0$ it gives a choice of γ such that for $h, |h| < \gamma$, we have:

$$\frac{1}{2\delta}|L_{x+h,y+h} \cap I_\delta| > 1 - 1/N \qquad \forall x \in X \ ,$$

so the intersection

$$L = \bigcap_{x \in X} L_{x+h,y+h}$$

is nonempty. Taking any $\ell \in L$ and the corresponding $c = (1-\ell)(y+h)$ one gets the result.

7° Now to finish the proof of Lemma 6.5 we use induction based on each step of Lemma 6.7. We start with an affine copy X_0 of $\mathfrak{a} \cup \mathcal{B}$ contained in $\mathcal{D}_0 = [\partial(A,B)]^\circ$ (see Exercise 6.6). Then by a small translation h of X_0 we move some $x \in X_0$ into an appropriate set A^0, or B^0 – as required and by dilation and by keeping it fixed we return all other points into $\mathcal{D}_0$. Continuing this way we get, after $N = cardX_0$ steps, an affine copy of X_0 packed inside A and B as required.

□

6.3. Proof of Theorem 6.1

Let $m = 1\!\!1_\Lambda \in \mathfrak{m}_p(\mathbb{T})$ for some $p \neq 2$. Denote

$$\mathcal{K} = \partial(\Lambda, {}^c\Lambda) \ , \qquad {}^c\Lambda = \mathbb{T} \backslash \Lambda \ .$$

We claim: $|\mathcal{K}| = 0$. If not, using Lemma 6.5, we can arrange an arbitrary long arithmetical progression $g_k = a + kh$ $(1 \le k \le k_0)$ inside $]-\pi, \pi[$ such that for any pregiven family $\{\varepsilon(k) = 0, 1\}$ the following inclusions hold:

$$t_k \in \begin{cases} \Lambda^\circ, & \varepsilon(k) = 1 \\ ({}^c\Lambda)^\circ \ , & \varepsilon(k) = 0 \ . \end{cases}$$

All those points are Lebesgue points of m, so by de Leeuw's Theorem

$$\|\sum_{k=1}^{k_0} \varepsilon(k)c(k)e^{ikt}\|_p \leq C\|\mathfrak{m}\|_{\mathfrak{m}_p}\|\sum_{k=1}^{k_0} c(k)e^{ikt}\|_p \ .$$

The last inequality, being true with an absolute constant C *for any* k_0, $\{\varepsilon(k)\}, \{c(k) \in \mathbb{C}\}$ contradicts the result of Section 4.4. Thus $|\mathcal{K}| = 0$. On the other hand every interval Δ contiguous to $\mathcal{K}$ either belongs entirely to Λ or to ${}^{c}\Lambda$ (up to a set of measure zero). Thus, Λ (and ${}^{c}\Lambda$ as well) is the union of a subfamily of Δ.

An equivalent form of the result is: *if* $1\!\!1_\Lambda \in \mathfrak{m}_p$ *then the function is continuous a.e.* (more precisely, it is equivalent to such a function). Actually this result is true in the general situation – not only for idempotents.

THEOREM 6.8. *If* $m \in \mathfrak{m}_p(\mathbb{T})$ *for some* $p \neq 2$ *then it is continuous almost everywhere.*

The proof is based on a similar approach ,see [8].

In conclusion, here are a few comments.

1. Let E be an invariant complemented subspace in $\ell_p(\mathbb{Z})$, $p \in]1,\infty[, \neq 2$. *Then* E *contains (nontrivial) vectors* $c = \{c(n)\}$ *with fast decreasing components*: $c(n) = o(1/|n|^\ell)$, $\ell = 1, 2, \ldots$.

   Indeed: according to Theorem 6.1 the spectrum Λ of E *contains an interval* Δ; consider a function $\varphi \in C^\infty(\mathbb{T})$ supported on Δ. Clearly $\widehat{\varphi} \in E$.

   EXERCISE 6.9: Show that the result for $p = 2$ is false.

   For $p > 2$ one can say more: Theorem 6.1 implies that *the set of vectors with fast decreasing components is dense in* E.

   It is unknown whether the result holds for $p < 2$.

2. Another interesting open problem is: *to investigate more deeply a structure of* $\mathcal{K}$ (the essential boundary of spectrum Λ of complemented invariant subspaces). Which compacts of measure zeros can appear as $\mathcal{K}$? How does this ability depend on p?

3. Some other applications of Theorem 6.1 one can find in [8].

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