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Analytic Semigroups Generated by Square Hörmander Operators

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SUMMARY. - *We show that degenerate elliptic operators of Hörmander type realized in* $\mathbf{R}^n$ *generate analytic semigroups in proper Sobolev spaces and we characterize some real interpolation spaces related to the original problem.*

1. Introduction

The semigroup approach in the study of parabolic evolution equations is widely known. The two basical steps in this method consist usually first in showing that the realization of the operator in suitable Banach spaces generates an analytic semigroup and then in characterizing the interpolation spaces that are necessary to obtain optimal regularity results for the problem considered

In the following we deal with a degenerate elliptic operator with continuous coefficients of Hörmander type and extend to this new setting generation results originally proved in [14].

Some differences with respect to [14] are to be underlined. First of all we consider proper Sobolev spaces (related to Hörmander vector fields) consisting of functions defined in the whole $\mathbf{R}^n$ and not just in a bounded domain Ω: this is due to the fact that boundary estimates for operators defined by vector fields are not yet available,

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except to our knowledge for some partial results by Franchi and others about trace theorems ([9])(see Section 2 for the general definitions and Section 3 for a theorem we need in the successive proofs).

Moreover Poincaré and Sobolev inequality hold only on (intrinsic) balls and not in general domains, even highly regular: therefore special care has to be paid in the localization procedure when proving generation results (see Section 4).

Finally a fundamental feature of Hörmander vector fields is that they do not commute: therefore a characterization of Hölder continuous functions spaces in terms of interpolation theory, as considered, for example, in [12] or in [13] seems to be not so straightforward here. As a consequence we were not able to complete characterize interpolation spaces and had to restrict ourselves to a more specific situation (see Section 5).

Our hope is to fill this last gap and then apply our results, as well as the others considered in [10], to evolution problems, even nonlinear ones.

2. Functional Spaces

2.1 Let us consider $C^\infty(\mathbf{R}^n)$ bounded vector fields $X_i,\ i=1,\dots,m$ that satisfy a uniform Hörmander condition of order k: at any point the vectors and their commutators up to order k span $\mathbf{R}^n$. In this case an intrinsic distance d_X associated to the X_i can be defined. Namely

$$d_X(x,y)=\inf\Big\{b;\ \gamma:[0,b]\to\mathbf{R}^n \text{ admissible path with } \gamma(0)=x,\ \gamma(b)=y\Big\}$$

where an admissible path is a Lipschitz curve such that

$$\gamma'(t)=\sum_{i=1}^m d_i(t)X_i(\gamma(t)) \quad \text{with} \quad \sum_{i=1}^m |d_i(t)|^2\le 1$$

(see also [15]). It is well-known that $\forall x,\ y\in\mathbf{R}^n$ $d_X(x,y)$ satisfies the condition

$$\frac{1}{c}|x-y|\le d_X(x,y)\le c|x-y|^\epsilon$$

with $\epsilon = \frac{1}{k+1}$ and $c > 1$ suitable constant. We can then define balls in the usual way relying on the distance d_X. In the following we will always deal with so-called *intrinsic* balls.

In [11] it is proved that the following duplication property holds:

$$0 < m(B(2r, y)) \le c_0\, m(B(r, y)) \tag{2.1}$$

for every ball with center at $y \in \mathbf{R}^n$ and radius $r < R_0$, with the constant c_0 possibly depending only on R_0. We can then say that the space $\mathbf{R}^n$ with distance d_X and Lebesgue measure dx gets the structure of homogeneous space (see [7]).

As a consequence of (2.1) there exists a constant $\nu = \log_2 c_0$ such that

$$m(B(r, y)) \le 2\, m(B(s, y)) \left(\frac{r}{s}\right)^{\nu}$$

for every $0 < s < r \le \frac{R_0}{2}$. On the other hand it is easy to see that

$$m(B(2r, y)) \ge c^*\, m(B(r, y)) \tag{2.2}$$

(where $c^* > 1$ is a constant that depends only on c_0) and

$$m(B(s, y)) \le m(B(r, y))(\frac{s}{r})^{\sigma} \tag{2.3}$$

where $0 < s < \frac{r}{2} < \frac{R_0}{2}$ and $\sigma = 1 - \frac{1}{c^*}$.

The numbers ν and σ give an upper and lower bound on the intrinsic dimension of $\mathbf{R}^n$ and are in general different. However there are special cases in which they can coincide. Moreover a further dimension can be defined, the so-called homogeneous dimension. We refer to [10] for a general discussion of this situation and a comparison between the different quantities.

In any case the intrinsic dimension ν is usually different from n. Just to make an example, let us see the subelliptic Laplacian on the Heisenberg group. Consider the space $\mathbf{R}^{2n+1}$, whose coordinates we denote by (x, y, z), $x, y \in \mathbf{R}^n$, $z \in \mathbf{R}$; then the Heisenberg group of degree n is the Lie group whose underlying manifold is $\mathbf{R}^{2n+1}$ endowed with the group law

$$(x, y, z) \circ (x', y', z') = (x + x', y + y', z + z' + 2(x'y - xy')).$$

If we define on the group the vector fields (which actually are not bounded)

$$X_j = D_{x_j} + 2y_j D_z, \qquad Y_j = D_{y_j} - 2x_j D_z,$$

the subelliptic laplacian is given by $\triangle_H \equiv \sum_{i=1}^n X_j^2 + Y_j^2$; it is easy to see that X_j, Y_j satisfy a Hörmander condition of order 1. In this case the dimension of the manifold is (obviously) $2n+1$, while the intrinsic dimension (which incidentally is the same as the homogeneous one we referred to above) is $\nu = 2n+2$.

2.2 Let us consider suitable spaces of continuous functions. Namely, if $B(\mathbf{R}^n)$ denotes the space of bounded functions defined in $\mathbf{R}^n$, we put

$$C_b^0(\mathbf{R}^n) = C^0(\mathbf{R}^n) \cap B(\mathbf{R}^n)$$

and

$$C_b^m(\mathbf{R}^n) = \{f \in C^m(\mathbf{R}^n): \ X^J f \in C_b^0(\mathbf{R}^n) \ \forall \ |J| \le m\}$$

where J is a multi index and X^J is an ordered monomial of order $|J|$ built with the vector fields X_i. If we set

$$[u]_{l,0}^X = \sup_{|J|=l} \sup_{x \in \mathbf{R}^n} |X^J u(x)|,$$

then $C_b^m(\mathbf{R}^n)$ is a Banach space with the norm

$$\|u\|_{C_b^m} = \sum_{j=0}^{l} [u]_{j,0}^X.$$

Moreover we can introduce a new class of Hölder continuous functions with respect to the intrinsic distance d_X. For $0 < \alpha < 1$ we define

$$C^{0,\alpha}(\mathbf{R}^n) = \Big\{f \in C_b^0(\mathbf{R}^n): \ [f]_\alpha^X = \sup \frac{|f(x) - f(y)|}{d_X(x,y)^\alpha} < +\infty\Big\}$$

and for $l \in \mathbf{N}, \ 0 \le \alpha < 1$ we define

$$C^{l,\alpha}(\mathbf{R}^n) = \{u \in C^{0,\alpha}(\mathbf{R}^n): \ X^J u \in C^{0,\alpha}(\mathbf{R}^n) \ \ \forall |J| \le l\}.$$

If we set

$$[u]^X_{l,\alpha} = \sup_{|J|=l} [X^J u]^X_\alpha\,,$$

then, as before, $C^{l,\alpha}(\mathbf{R}^n)$ is a Banach space with the norm given by

$$\|u\|_{C^{l,\alpha}} = \sum_{j=0}^{l} [u]^X_{j,0} + [u]^X_{l,\alpha}.$$

The proof of the previous results, as well as of other similar ones which usually hold for classical $C^{l,\alpha}(\mathbf{R}^n)$ Hölder spaces, can be given as in [19], where the case of functions defined in a bounded domain Ω is treated in details. Actually a second condition is assumed, besides the uniform Hörmander's one discussed at the beginning of this Section. Namely it is also supposed that

> for each $j \leq k$ the dimension of the space spanned by the commutators of length $\leq j$ at each point is constant in a neighbourhood (by convention, we define X_i to be of length one),

but, as it is also stated in Section 2 of [19], this is not relevant in the construction of Hölder spaces and in the proof of the properties we need (see also [16] for other properties of these spaces).

Moreover, as in the case of Sobolev spaces considered below, we can define $C^{l,\alpha}_{loc}(\mathbf{R}^n)$ as the space of functions $u \in C^{l,\alpha}(\bar{\Omega}_i)$ for every $\Omega_i \subset\subset \mathbf{R}^n$. Let us remark that thanks to the $C^\infty(\mathbf{R}^n)$ coefficients in the definition of the vector fields, $C^{l,\alpha}$ regular functions can be far less regular than "classical" $C^{l,\alpha}$ functions. Finally we can further define $C^{l,\alpha}_j(\mathbf{R}^n)$ (and its obvious local version) as the space of functions whose Hölder continuity is considered only in the j "direction" with j fixed.

In the following, dealing with (Hölder) continuous functions defined in the whole $\mathbf{R}^n$, we will always consider them bounded and therefore we will suppress the suffix b. Moreover Hölder continuity will always be considered in the degenerate sense if not otherwise stated.

2.3 We define $W^{1,p}(\mathbf{R}^n, X)$ as the space of all $u \in L^p(\mathbf{R}^n)$ such that we have $X_i(u) \in L^p(\mathbf{R}^n) \quad \forall\, i = 1, \ldots, m$. It is easy to see that

$W^{1,p}(\mathbf{R}^n, X)$ is a Banach space if it is endowed with the norm

$$\|u\|_{1,p} = \Big(\|u\|_p^p + \sum_{i=1}^m \|X_i(u)\|_p^p\Big)^{\frac{1}{p}}.$$

If $p \in]1, \infty[$, then $W^{1,p}(\mathbf{R}^n, X)$ is reflexive. The proof is absolutely analogous to the classical one (see also [4]). As in the usual situation, we can also deal with the *local* version of the previous spaces or with spaces of functions defined not in the whole $\mathbf{R}^n$ but just in a bounded domain Ω.

Let us now consider the dual spaces. Namely we define

$$W^{-1,p'}(\mathbf{R}^n, X) \equiv (W^{1,p}(\mathbf{R}^n, X))', \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

We can then prove

PROPOSITION 2.1. *Let $p \in]1, \infty[$. Then $f \in W^{-1,p'}(\mathbf{R}^n, X)$ if and only if there exists $(f_0, f_1, \dots, f_m) \in (L^{p'}(\mathbf{R}^n))^{m+1}$ s. t.*

$$< f, \varphi >= \int_{\mathbf{R}^n} f_0 \varphi \, dx + \sum_{i=1}^m \int_{\mathbf{R}^n} f_i X_i(\varphi) \, dx \quad \forall \, \varphi \in W^{1,p}(\mathbf{R}^n, X).$$

Proof. As in [4], Lemma 2.4.

We define

$$\|f\|_{W^{-1,p'}} = \inf\{(\|f_0\|_{p'}^{p'} + \sum_{i=1}^m \|f\|_{p'}^{p'})^{\frac{1}{p'}}\}$$

where $(f_0, f_1, \dots, f_m)$ realizes the representation of f.
Furthermore we define

$$E_0 = \Big\{ f \in \bigcap_{p>1} (W^{1,p}(\mathbf{R}^n, X))' : \\ \exists f_0, \dots, f_m \in C^0(\mathbf{R}^n) \text{ s.t. } f = f_0 + \sum_{i=1}^m X_i(f_i) \Big\}$$

with $\|f\|_{E_0} = \inf\{\|f_0\|_\infty + \sum_{i=1}^m \|f_i\|_\infty\}$ where, as usual, $\{f_0, \dots, f_m\}$ represents f.

REMARK 2.2. In the following we will sometimes deal with Sobolev spaces $W^{2,p}(\mathbf{R}^n, X)$. After what we said above, the definition should be obvious.

3. Analytic Semigroup Generation in Spaces of Continuous Functions

Let us consider the differential operator

$$\Delta_X u = \sum_{i=1}^{m} X_i^2; \tag{3.1}$$

it will act in the special sets of functions

$$C_0^0(\mathbf{R}^n) = \{u \in C^0(\mathbf{R}^n) : \ u \to 0 \text{ as } |x| \to \infty\}$$

$$D_0^q(\Delta_X) = \{u \in C_0^0(\mathbf{R}^n) \text{ with } \Delta_X u \in C_0^0(\mathbf{R}^n), \ u \in W_{loc}^{2,q}(\mathbf{R}^n, X)\}.$$

We get the following estimate

THEOREM 3.1. *Let $q > \nu$. There exist positive constants ϵ, M_0, λ_0, r_0 depending on q such that for $u \in D_0^q(\Delta_X)$ we have*

$$\sum_{0 \le |J| < 2} |\lambda|^{-|J|/2} \|X^J u\|_{C^0} + \sum_{I=2} |\lambda|^{-1+\theta} \sup_{\mathbf{R}^n} \|X^I u\|_{L^q(B(r_\lambda, x_0))} \le$$

$$\le M_0 |\lambda|^{-1+\theta} \sup_{\mathbf{R}^n} \|(\lambda - \Delta_X) u\|_{L^q(B(r_\lambda, x_0))}$$

for $|\lambda| \ge \lambda_0$, $|\arg \lambda| \le \frac{\pi}{2} + \epsilon$, $\theta = \frac{\nu}{2q}$, $r_\lambda = r_0 |\lambda|^{-1/2}$ with J, I multi-indices and $X^J u$, $X^I u$ as above.

Sketch of the proof. The proof is essentially the same as given in [17] once the proper quantities are considered and we won't repeat it here. Let's just remark that Agmon - Douglis - Nirenberg estimates are here replaced by the analogous ones proved in [8]. Many of the difficulties in Stewart's proof, linked with the study of the behaviour at the boundary, are obviously missing here, since we limited ourselves at the whole $\mathbf{R}^n$ (see also what we said in the introduction).

As a direct consequence of the previous theorem we obtain

COROLLARY 3.2. *For every $p > \nu$ there are $R' > 0$, $M' > 0$, $\theta' > \frac{\pi}{2}$ s.t. if $|\lambda| > R'$, $|\arg \lambda| < \theta'$ and $h \in L^p(\mathbf{R}^n)$, then the problem*

$$\lambda u - \Delta_X u = h \qquad \text{in } \mathbf{R}^n$$

has a unique solution $u \in W^{2,p}(\mathbf{R}^n, X)$ *and*

$$\|u\|_{L^\infty(\mathbf{R}^n)} \le M'|\lambda|^{-1} \sup\{|\lambda|^{\frac{\nu}{2p}} \|h\|_{L^p(B(|\lambda|^{-1/2},x))} : \quad x \in \mathbf{R}^n\}.$$

REMARK 3.3. Here we have considered the same formulation as given in [14]; see also [1] for a somewhat different statement of the same result. Corollary 3.2 will be crucial in the characterization of $D_A(\beta, \infty)$ (see Section 5 for the precise definition).

4. Analytic Semigroup Generation in $W^{-1,p}(\mathbf{R}^n, X)$ and in E_0

Let us consider a $C^0(\mathbf{R}^n)$ matrix $[a_{ij}]$ with $i, j = 1, \ldots, m$ which satisfies a uniform ellipticity condition, that is

$$\sum_{i,j=1}^{m} a_{ij}\xi_i\xi_j \ge \delta|\xi|^2$$

with δ proper positive constant, and the $C^0(\mathbf{R}^n)$ functions a_i, b_i, c, with $i = 1, \ldots, m$ as above. We can then define the sesquilinear form

$$a(u, v) = \int_{\mathbf{R}^n} [a_{ij}X_i(u)X_j(v) \; + \; a_iX_i(u)v \; + \; b_iuX_i(v) \; + \; c\,uv] \, dx$$

(from now on summation over repeated indeces is implied if not otherwise stated) in $W^{1,p}(\mathbf{R}^n, X) \times W^{1,p'}(\mathbf{R}^n, X)$ with $1 < p < \infty$. We now introduce the operator $A_p : D(A_p) = W^{1,p}(\mathbf{R}^n, X) \to W^{-1,p}(\mathbf{R}^n, X)$ by setting

$$< A_pu, v >= a(u, v) \qquad \forall\, u \in W^{1,p}(\mathbf{R}^n, X), \; \forall\, v \in W^{1,p'}(\mathbf{R}^n, X)$$

and relying on the previous definition of E_0 we consider a further operator A by

$$D(A) = \{u \in \bigcap_{p>1} W^{1,p}(\mathbf{R}^n, X) : \; A_pu \in E_0\} \quad Au = A_pu \;\; \forall\, p > 1.$$

We state now the main results of this Section with the relative proofs.

THEOREM 4.1. *Let $p \in [2,\infty[$. Then the operator $A_p : D(A_p) \to W^{-1,p}(\mathbf{R}^n, X)$ generates an analytic semigroup in $W^{-1,p}(\mathbf{R}^n, X)$. There are C_p, $R_p > 0$, $\theta_p \in]\frac{\pi}{2}, \pi[$ depending on the ellipticity constant δ, such that the resolvent set of A_p contains the sector $S_p = \{\lambda \in \mathbf{C} : \ |\lambda| \geq R_p, \ |\arg \lambda| < \theta_p\}$ and*

$$|\lambda| \|u\|_{W^{-1,p}(\mathbf{R}^n,X)} + |\lambda|^{1/2} \|u\|_{L^p(\mathbf{R}^n)} + \|u\|_{W^{1,p}(\mathbf{R}^n,X)} \leq$$

$$\leq C_p \|\lambda u - A_p u\|_{W^{-1,p}(\mathbf{R}^n,X)} \qquad \forall\, u \in D(A_p). \tag{4.1}$$

Proof. For $f_i \in L^p(\mathbf{R}^n)$ existence and uniqueness of the solution of

$$\lambda\, u - (X_i(a_{ij} X_j(u)) + a_i X_i(u) + X_i(b_i u) + cu) = f_0 + X_i(f_i) \tag{4.2}$$

for $\mathrm{Re}\,\lambda$ large can be proved using monotone operators technique (see, for example, [18], chapter 4.2). To prove estimate (4.1) it is enough to choose $|u|^{p-2}u$ as test function in both members of (4.2). Existence, uniqueness and the estimate for λ in a sector follow in a standard way.

THEOREM 4.2. *The operator $A : D(A) \to E_0$ generates an analytic semigroup in E_0. For every $p > \nu$ there are C, $R > 0$, $\theta \in]\frac{\pi}{2}, \pi[$ depending on the ellipticity constant δ, such that the resolvent set of A contains the sector $S = \{\lambda \in \mathbf{C} : \ |\lambda| \geq R, \ |\arg \lambda| < \theta\}$ and*

$$|\lambda| \|u\|_{E_0} + |\lambda|^{1/2} \|u\|_{L^\infty(\mathbf{R}^n)} + \sup_{\mathbf{R}^n} |\lambda|^{\nu/2p} \|u\|_{W^{1,p}(B(|\lambda|^{-1/2},x),X)} \leq$$

$$\leq C \|\lambda u - Au\|_{E_0} \qquad \forall\, u \in D(A). \tag{4.3}$$

Before proving this last result, we first need the following introductory Lemma:

LEMMA 4.3. *Let $p \geq 2$, $\lambda \in S_p$, $f \in W^{-1,p}(\mathbf{R}^n, X)$ and let $u \in W^{1,p}(\mathbf{R}^n, X)$ be the solution of $\lambda u - A_p u = f$. Then there is a constant $K_p > 0$ (independent of λ and f) such that*

$$\sum_{i=1}^m \|X_i(u)\|_{L^p(B(r,x_0))} \leq$$

$$\leq K_p \Big[\sum_{i=0}^m \|f_i\|_{L^p(B(2r,x_0))} + \frac{1}{r} \|u\|_{L^p(B(2r,x_0))} \Big] \tag{4.4}$$

for every $x_0 \in \mathbf{R}^n$ *and* $r > 0$.

Proof. It is exactly as in [14], once we substitute classical derivatives with vector fields and the dimension n with the intrinsic dimension ν. The fundamental tool in the proof is the Sobolev inequality for $v \in W^{1,p}(B(r,x_0),X)$, whose proof can be found, for example, in [3] (see also the following Remark).

REMARK 4.4. In the general framework of Dirichlet forms (of which square Hörmander operators can be seen as a particular case) Biroli and Mosco have recently proved Sobolev imbeddings which extend usual classical ones (see [2] and [3]). In particular Theorem 3 of [2] implies the imbedding

$$W^{1,p}(B(r,x_0),X) \hookrightarrow L^{\infty}(B(r,x_0))$$

when $p > \nu$. Even if their estimate is essentially local because everything is given in terms of intrinsic balls, a global result can easily be obtained, due to the fact that the constants do not depend on the point. Therefore we can conclude that

$$W^{1,p}(\mathbf{R}^n,X) \hookrightarrow L^{\infty}(\mathbf{R}^n)$$

if $p > \nu$, a fact we will need in the forthcoming proof.

We can now finally conclude.

Proof of Theorem 4.2. The aim is the proof of (4.3) in a proper sector. As in [14], we will divide it in three parts.

I) *Estimate of* $\|u\|_{L^{\infty}(\mathbf{R}^n)}$. - Let $p > \nu$, $\lambda \in S_p$, $f \in E_0$ and $u \in W^{1,p}(\mathbf{R}^n,X)$ be the solution of

$$\lambda u - A_p u = f_0 + \sum_{i=1}^{m} X_i(f_i) \qquad \text{in } \mathbf{R}^n.$$

Consider $x_0 \in \mathbf{R}^n$ and $r > 0$ and let θ be a $C_0^{\infty}(\mathbf{R}^n)$ function satisfying

$$\theta = 1 \qquad \text{in } B(r,x_0);$$

$$\theta = 0 \qquad \text{in } \mathbf{R}^n \backslash B(\sqrt{2}r,x_0);$$

$$|X_i(\theta)| \leq \frac{3}{r} \qquad i = 1, \dots, m \text{ in } \mathbf{R}^n$$

(for the existence of such a function, see [2] and [3]). Then $v = \theta u$ satisfies

$$\lambda v - A_p v = X_i(f_i\theta + a_{ij}uX_j(\theta)) + f_0\theta - f_iX_i(\theta) + \\ + a_{ij}X_j(u)X_i(\theta) + (a_i + b_i)uX_i(\theta).$$

Thanks to Theorem 4.1, we have

$$\begin{aligned} &|\lambda|\|\theta u\|_{W^{-1,p}(\mathbf{R}^n,X)} + |\lambda|^{1/2}\|\theta u\|_{L^p(\mathbf{R}^n)} + \|\theta u\|_{W^{1,p}(\mathbf{R}^n,X)} \\ &\leq C_p\|X_i(f_i\theta + a_{ij}uX_j(\theta)) + f_0\theta - f_iX_i(\theta) + a_{ij}X_j(u)X_i(\theta) + \\ &\qquad + (a_i + b_i)uX_i(\theta)\|_{W^{-1,p}(\mathbf{R}^n,X)} \\ &\leq C_p\Big[\sum_{i=0}^m \|f_i\|_{L^p(B(2r,x_0))} + \\ &\quad + \frac{2}{r}\Big(\sum_{i,j=1}^m \|a_{ij}\|_\infty + \sum_{i=1}^m \|a_i + b_i\|_\infty\Big)\|u\|_{L^p(B(2r,x_0))} + \\ &\quad + \|f_iX_i(\theta)\|_{W^{-1,p}(\mathbf{R}^n,X)} + \|a_{ij}X_j(u)X_i(\theta)\|_{W^{-1,p}(\mathbf{R}^n,X)}\Big]. \end{aligned}$$

If we now take into account Sobolev imbedding theorem and Lemma 4.3, (where we replace p by $\frac{\nu p}{\nu+p}$), we obtain

$$\begin{aligned} \|f_iX_i(\theta)\|_{W^{-1,p}(\mathbf{R}^n,X)} &\leq \frac{2}{r}k_{p'}\sum_{i=1}^m \|f_i\|_{L^{\frac{p\nu}{p+\nu}}(B(2r,x_0))} \\ &\leq \frac{2}{r}k^*k_{p'}m(B(2r,x_0))^{1/\nu+1/p}\sum_{i=1}^m \|f_i\|_{L^\infty(B(2r,x_0))} \\ &\leq \frac{2}{r}k^*k_{p'}m(B(2r,x_0))^{1/\nu+1/p}\sum_{i=1}^m \|f_i\|_{L^\infty(\mathbf{R}^n)}. \end{aligned}$$

$$\begin{aligned} &\|a_{ij}X_j(u)X_i(\theta)\|_{W^{-1,p}(\mathbf{R}^n,X)} \leq \\ &\qquad \leq \frac{2}{r}k_{p'}\sum_{i,j=1}^m \|a_{ij}\|_\infty \sum_{i=1}^m \|X_i(u)\|_{L^{\frac{p\nu}{p+\nu}}(B(2r,x_0))} \\ &\leq \frac{2}{r}k^{**}k_{p'}\sum_{i,j=1}^m \|a_{ij}\|_\infty \left(\sum_{i=0}^m \|f_i\|_{L^{\frac{p\nu}{p+\nu}}(B(4r,x_0))} + \frac{1}{r}\|u\|_{L^{\frac{p\nu}{p+\nu}}(B(4r,x_0))}\right) \end{aligned}$$

$$\leq \frac{2}{r} k^{**} k_{p'} m(B(4r,x_0))^{1/\nu+1/p} \Big[\sum_{i=0}^{m} \|f_i\|_{L^\infty(B(4r,x_0))} + \\ + \frac{1}{r}\|u\|_{L^\infty(B(4r,x_0))} \Big]$$

$$\leq \frac{2}{r} k^{**} k_{p'} m(B(4r,x_0))^{1/\nu+1/p} \left[\sum_{i=0}^{m} \|f_i\|_{L^\infty(\mathbf{R}^n)} + \frac{1}{r}\|u\|_{L^\infty(\mathbf{R}^n)} \right].$$

We can then conclude

$$|\lambda| \|\theta u\|_{W^{-1,p}(\mathbf{R}^n,X)} + |\lambda|^{1/2} \|\theta u\|_{L^p(\mathbf{R}^n)} + \|\theta u\|_{W^{1,p}(\mathbf{R}^n,X)}$$

$$\leq \frac{M}{r} m(B(4r,x_0))^{1/\nu+1/p} \left[\sum_{i=0}^{m} \|f_i\|_{L^\infty(\mathbf{R}^n)} + \frac{1}{r}\|u\|_{L^\infty(\mathbf{R}^n)} \right].$$

Let us now recall that, since $p > \nu$, for each $\epsilon > 0$ there exists $C(\epsilon) > 0$ (independent of x_0 and r) such that

$$\|\varphi\|_{L^\infty(B(r,x_0))} \leq C(\epsilon) \frac{1}{m(B(r,x_0))^{1/p}} \|\varphi\|_{L^p(B(r,x_0))} + \\ + \epsilon \frac{r}{m(B(r,x_0))} \|\varphi\|_{W^{1,p}(B(r,x_0),X)}$$

for any $\varphi \in W_0^{1,p}(B(r,x_0),X)$. Therefore, if $x_0 \in \mathbf{R}^n$ is a maximum point for $|u|$, we have

$$\begin{aligned}
\|u\|_{L^\infty(\mathbf{R}^n)} &= \|\theta u\|_{L^\infty(B(2r,x_0))} \\
&\leq C(\epsilon) \frac{1}{m(B(2r,x_0))^{1/p}} \|\theta u\|_{L^p(B(2r,x_0))} + \\
&\quad + \epsilon \frac{2r}{m(B(2r,x_0))^{1/p}} \|\theta u\|_{W^{1,p}(B(2r,x_0),X)} \\
&\leq C(\epsilon) |\lambda|^{-1/2} \frac{M}{r} \frac{m(B(4r,x_0))^{1/\nu+1/p}}{m(B(2r,x_0))^{1/p}} \cdot \\
&\quad \cdot \left(\sum_{i=0}^{m} \|f_i\|_{L^\infty(\mathbf{R}^n)} + \frac{1}{r}\|u\|_{L^\infty(\mathbf{R}^n)} \right) + \\
&\quad + \epsilon \frac{M}{r} \frac{2r}{m(B(2r,x_0))^{1/p}} m(B(4r,x_0))^{1/\nu+1/p} \cdot \\
&\quad \cdot \left(\sum_{i=0}^{m} \|f_i\|_{L^\infty(\mathbf{R}^n)} + \frac{1}{r}\|u\|_{L^\infty(\mathbf{R}^n)} \right)
\end{aligned}$$

$$\leq \quad M^* \left(C(\epsilon)|\lambda|^{-1/2} + \epsilon r\right) \sum_{i=0}^{m} \|f_i\|_{L^\infty(\mathbf{R}^n)} + \\ + M^* \left(C(\epsilon)|\lambda|^{-1/2} r^{-1} + \epsilon\right) \|u\|_{L^\infty(\mathbf{R}^n)}.$$

If we choose $r = 2M^*C(\epsilon)|\lambda|^{-1/2}$ and $\epsilon = \frac{1}{4M}$, we obtain

$$\|u\|_{L^\infty(\mathbf{R}^n)} \leq 6MC(\epsilon)|\lambda|^{-1/2} \sum_{i=0}^{m} \|f_i\|_{L^\infty(\mathbf{R}^n)}$$

and we are finished.

II) *Estimate of* $\|u\|_{W^{1,p}(B(|\lambda|^{-1/2},x),X)}$. - If we take $x_0 \in \mathbf{R}^n$ and consider a test function θ as before, repeating the same procedure as before, we have

$$\|u\|_{W^{1,p}(B(r,x_0),X)} \leq \|\theta u\|_{W^{1,p}(\mathbf{R}^n,X)}$$

$$\leq \frac{M}{r} m(B(4r,x_0))^{1/\nu+1/p} \left(\sum_{i=0}^{m} \|f_i\|_{L^\infty(\mathbf{R}^n)} + \frac{1}{r}\|u\|_{L^\infty(\mathbf{R}^n)}\right)$$

$$\leq \frac{M}{r} m(B(4r,x_0))^{1/\nu+1/p} \Bigg(\sum_{i=0}^{m} \|f_i\|_{L^\infty(\mathbf{R}^n)} + \\ + \frac{6}{r} MC(\epsilon)|\lambda|^{-1/2} \sum_{i=0}^{m} \|f_i\|_{L^\infty(\mathbf{R}^n)}\Bigg)$$

$$\leq \frac{M}{r} m(B(4r,x_0))^{1/\nu+1/p} \left(1 + r^{-1}C_1|\lambda|^{-1/2}\right) \sum_{i=0}^{m} \|f_i\|_{L^\infty(\mathbf{R}^n)}$$

$$\leq M^{**} r^{\nu/p} \left(1 + r^{-1}C_1|\lambda|^{-1/2}\right) \sum_{i=0}^{m} \|f_i\|_{L^\infty(\mathbf{R}^n)}.$$

If we choose $r = |\lambda|^{-1/2}$ we can further estimate by

$$M^{**} r^{\nu/p}(1 + C_1) \sum_{i=0}^{m} \|f_i\|_{L^\infty(\mathbf{R}^n)}$$

and conclude

$$\|u\|_{W^{1,p}(B(|\lambda|^{-1/2},x_0),X)} \leq M^{**} \sum_{i=0}^{m} \|f_i\|_{L^\infty(\mathbf{R}^n)}$$

since r is small and $\nu < p$.

III) *Estimate of* $\|u\|_{E_0}$. - Let us choose $\lambda \in S_p$ such that $|\lambda| > R'$, $|\arg\lambda| < \theta'$ with R' and θ' as in Corollary 3.2. Now let θ be defined as in Step I and consider $v = \theta u$. Then v satisfies

$$\begin{aligned}\lambda v - \sum_{i=1}^{m} X_i^2(v) &= X_i(f_i\theta + a_{ij}uX_j(\theta)) + f_0\theta - f_iX_i(\theta) + \\ &+ a_{ij}X_j(u)X_i(\theta) + (a_i + b_i)uX_i(\theta) + A_p v - \sum_{i=1}^{m} X_i^2(v) \\ &= h_0 + \textstyle\sum_{i=1}^{m} X_i(h_i) \qquad \text{in } \mathbf{R}^n.\end{aligned} \tag{4.5}$$

The right-hand side of (4.5) belongs to $W^{-1,p}(\mathbf{R}^n, X)$. Thanks to the previous step, we have

$$\|h_i\|_{L^p(B(|\lambda|^{-1/2},x))} \leq C|\lambda|^{\frac{-\nu}{2p}} \left[\|f_0\|_\infty + \sum_{i=1}^{m} \|f_i\|_\infty\right]$$

for any $j = 0, 1, \dots, m$ and $x \in \mathbf{R}^n$. If we define $v_0, v_1, \dots, v_m \in W^{1,p}(\mathbf{R}^n, X)$ as the solutions of

$$\lambda v_j - \Delta_X v_j = h_j \qquad j = 1, \dots, m; \tag{4.6}$$

(4.7)
$$\begin{aligned} v_0 - \Delta_X v_0 &= h_0 + \sum_{i,j=1}^{m} [X_i, X_j]X_i(v_j) + \sum_{i,j=1}^{m} X_i([X_i, X_j]v_j) \\ &= h_0^* \end{aligned}$$

we easily obtain

$$v = v_0 + \sum_{i=1}^{m} X_i(v_i). \tag{4.8}$$

With respect to the original case of usual derivatives dealt with in [14], in (4.7) we have to take into account the non commutativity of our vector fields. Now, thanks to Corollary 3.2 applied to (4.6), $v_j \in W^{2,p}(\mathbf{R}^n, X)$, $\ j = 1, \dots, m$ and satisfy

$$\begin{aligned}\|v_j\|_\infty &\leq \frac{M_0}{|\lambda|} \sup\{|\lambda|^{\frac{\nu}{2p}} \|h_j\|_{L^p(B(|\lambda|^{-1/2},x))} : \ x \in \mathbf{R}^n\} \\ &\leq \frac{C}{|\lambda|} \left[\|f_0\|_\infty + \sum_{i=1}^{m} \|f_i\|_\infty\right].\end{aligned} \tag{4.9}$$

As a consequence the right - hand side of (4.7) belongs to $W^{-1,p}(\mathbf{R}^n, X)$, which in turn assures that $v_0 \in W^{1,p}(\mathbf{R}^n, X)$ (as already stated above), but nothing more under the point of view of Sobolev spaces regularity. However we can now work as in Theorem 2.12 of [4] and obtain that

$$\|v_0\|_\infty \leq \frac{M_0}{|\lambda|} \sup\{|\lambda|^{\frac{\nu}{2p}} \|h_0^*\|_{L^p(B(|\lambda|^{-1/2},x))} : \quad x \in \mathbf{R}^n\}. \tag{4.10}$$

Since h_0^* depends on h_0 and v_j, $j = 1, \dots, m$, we can further estimate and conclude as in (4.9) that

$$\|v_0\|_\infty \leq \frac{C}{|\lambda|} \left[\|f_0\|_\infty + \sum_{i=1}^{m} \|f_i\|_\infty \right]. \tag{4.11}$$

Thanks to (4.8) we have $\|v\|_{E_0} \leq \|v_0\|_\infty + \sum_{i=1}^m \|v_i\|_\infty$ and taking into account (4.9) and (4.11) we are done.

5. Characterization of Some Interpolation Spaces

Let us consider the operator $A : D(A) \to E_0$ and let e^{tA} denote the analytic semigroup that A generates.

We can indifferently define the interpolation space $D_A(\beta, \infty)$ between $D(A)$ and E_0 as

$$D_A(\beta, \infty) = \{x \in E_0 : \sup_{t \geq T} \|t^\beta \, A \, R(t, A)x\|_{E_0} < \infty\}$$

for any $t \geq T$ such that $R(t, A) \equiv (t - A)^{-1}$ exists, or as

$$D_A(\beta, \infty) = \{x \in E_0 : \sup_{t \in]0,1[} \|t^{-\beta}(e^{tA} - I)x\|_{E_0} < \infty\}.$$

We put

$$\|x\|_{D_A(\beta,\infty)} = \|x\|_{E_0} + \sup_{t \geq T} \|t^\beta \, A \, R(t, A)x\|_{E_0} < \infty$$

or

$$\|x\|_{D_A(\beta,\infty)} = \|x\|_{E_0} + \sup_{t \in]0,1[} \|t^{-\beta}(e^{tA} - I)x\|_{E_0} < \infty.$$

Moreover

$$D_A(\beta, \infty) = (D(A), E_0)_{1-\beta,\infty}$$

according to the usual definition of real interpolation spaces (see [5] or [6]).

REMARK 5.1. The possibility to characterize $D_A(\beta,\infty)$ in two different ways is a direct consequence of the general theory of strongly contractive semigroups (see also [12]).

We have the following

THEOREM 5.2. *If* $\frac{1}{2}<\beta<1$ *we have*

$$D_A(\beta,\infty)=C^{2\beta-1}(\mathbf{R}^n).$$

Proof. It is similar to the analogous proof of [14]; we give it here for the sake of completeness, trying to highlighten the main differences. Let us show that:

a) $C^{2\beta-1}(\mathbf{R}^n)\supset D_A(\beta,\infty)$

b) $C^{2\beta-1}(\mathbf{R}^n)\subset D_A(\beta,\infty)$.

The proof relies on Theorems 4.1 and 4.2 and the relative generation estimates.

a) If we take R as given in Theorem 4.2, $\forall\ \gamma\in D_A(\beta,\infty)$ and $\forall\ t>s\geq R$ we have

$$tR(t,A)\gamma-sR(s,A)\gamma=\int_s^t\frac{\partial}{\partial\tau}(\tau R(\tau,A)\gamma)\,d\tau=$$
$$=-\int_s^t R(\tau,A)AR(\tau,A)\gamma\,d\tau$$

where we have explicitely taken the derivative with respect to τ. Relying on the generation estimate of Theorem 4.2, we know that

$$|\lambda|^{1/2}\|(\lambda-A)^{-1}f\|_\infty\leq C\|f\|_{E_0}$$

which implies that

$$\|(\lambda-A)^{-1}\|_{\mathcal{L}(E_0,L^\infty)}\leq\frac{C}{|\lambda|^{1/2}}.$$

Furthermore, practically by definition, $\|AR\|_{\mathcal{L}(D_A(\beta,\infty),E_0)}\leq K\,\tau^{-\beta}$; therefore, if we take the L^∞-norm both on the left hand side and on the right hand one, we have

$$\|tR(t,A)\gamma-sR(s,A)\gamma\|_\infty\leq\frac{KC}{\beta-1/2}(s^{1/2-\beta}-t^{1/2-\beta})\|\gamma\|_{D_A(\beta,\infty)}.$$

Since $\gamma = \lim_{t\to\infty} tR(t,A)\gamma$, the previous inequality tells us that $\{tR(t,A)\gamma\}_t$ is a Cauchy sequence with the sup-norm; therefore, due to the completeness, γ is a continuous function. Moreover

$$\|tR(t,A)\gamma - \gamma\|_\infty \le C(\beta) t^{1/2-\beta} \|\gamma\|_{D_A(\beta,\infty)}$$

We finish showing that γ is not only continuous but also Hölder continuous (of course in the vector fields sense). For any $x,\ y \in \mathbf{R}^n$ with $d_X(x,y) < R^{-1/2}$ (where again R is given by Theorem 4.2) and taken $t = d_X(x,y)^{-2}$ we have

$$\begin{aligned}
|\gamma(x) - \gamma(y)| &\le |\gamma(x) - (tR(t,A)\gamma)(x)| + \\
&\quad + |(tR(t,A)\gamma)(x) - (tR(t,A)\gamma)(y)| + |(tR(t,A)\gamma)(y) - \gamma(y)| \\
&\le 2\|tR(t,A)\gamma - \gamma\|_\infty + [tR(t,A)\gamma]_{C^{2\beta-1}(B(t^{-1/2},x)} d_X(x,y)^{2\beta-1} \\
&\le 2C(\beta) d_X(x,y)^{2\beta-1} \|\gamma\|_{D_A(\beta,\infty)} + \\
&\quad + [tR(t,A)\gamma]_{C^{2\beta-1}(B(t^{-1/2},x))} d_X(x,y)^{2\beta-1}.
\end{aligned}$$

We can now apply the Sobolev imbedding for $q > p = \frac{\nu}{2-2\beta}$ proved in [2] and conclude

$$\begin{aligned}
&[tR(t,A)\gamma]_{C^{2\beta-1}(\bar{B}(t^{-1/2},x))} \le \\
&\quad \le K_q t^{-\nu/2(1-p/q)} \|tR(t,A)\gamma\|_{W^{1,q}(B(t^{-1/2},x))}.
\end{aligned}$$

For any fixed s we have

$$tR(t,A)\gamma = sR(s,A)\gamma - \int_s^t R(\tau,A)AR(\tau,A)\gamma\, d\tau.$$

We can then take $s = R$ (as usual with R as in Theorem 4.2) and obtain

$$\|tR(t,A)\gamma\|_{W^{1,q}} \le \|sR(s,A)\gamma\|_{W^{1,q}} + \|\int_s^t R(\tau,A)AR(\tau,A)\gamma\, d\tau\|_{W^{1,q}}.$$

Now we rely on the fact that

$$\begin{aligned}
&\|R(\tau,A)AR(\tau,A)\gamma\|_{W^{1,q}} \le \\
&\quad \le \|R(t,A)\|_{\mathcal{L}(E_0,W^{1,q})} \|AR(t,A)\|_{\mathcal{L}(D_A(\beta,\infty),E_0)} \|\gamma\|_{D_A(\beta,\infty)}
\end{aligned}$$

We use Theorem 4.2 once again to estimate the first term on the right-hand side. We have

$$\|tR(t,A)\gamma\|_{W^{1,q}} \leq \int_s^t \|R(\tau,A)\|_{\mathcal{L}(E_0,W^{1,q})}\tau^{-\beta}\,d\tau\|\gamma\|_{D_A(\beta,\infty)} + \\ + s\|R(s,A)\gamma\|_{W^{1,q}}$$

$$\leq \int_s^t \|R(\tau,A)\|_{\mathcal{L}(E_0,W^{1,q})}\tau^{-\beta}\|\gamma\|_{D_A(\beta,\infty)}\,d\tau + s\|R(s,A)\gamma\|_{W^{1,q}}$$

$$\leq C\int_s^t \tau^{-\beta-\nu/2q}\,d\tau\|\gamma\|_{D_A(\beta,\infty)} + RC_q\|\gamma\|_{W^{-1,q}}$$

$$\leq C(1-\frac{\nu}{2q}-\beta)^{-1}t^{1-\frac{\nu}{2q}-\beta}\|\gamma\|_{D_A(\beta,\infty)} + RC_q\|\gamma\|_{E_0}$$

where we used $s = R$. Recovering the previous estimate we obtain

$$[tR(t,A)\gamma]_{C^{2\beta-1}(\bar{B}(t^{-1/2},x))} \leq C\|\gamma\|_{D_A(\beta,\infty)}$$

and we conclude

$$|\gamma(x)-\gamma(y)| \leq C\, d_X^{2\beta-1}(x,y)\|\gamma\|_{D_A(\beta,\infty)}.$$

b) Let us now suppose that $u \in C^{2\beta-1}(\mathbf{R}^n)$. Then the solution of the following problem

$$\begin{cases} v \in W^{1,2}(\mathbf{R}^n, X) \\ tv - \sum_{i=1}^m X_i^2(v) = u \qquad \text{in } \mathbf{R}^n \end{cases} \tag{5.1}$$

is highly regular: namely $v \in C^{2\beta+1}(\mathbf{R}^n)$ and solves the equation in the classical sense (again we can adapt to our situation the results given in [19]). Therefore $v \equiv R(t,\Delta_X)u \in D(A)$ and

$$\|v\|_{C^{2\beta+1}(\mathbf{R}^n)} \leq C\|u\|_{C^{2\beta-1}(\mathbf{R}^n)}$$

which implies

$$\|\Delta_X R(t,\Delta_X)u\|_{C^{2\beta-1}(\mathbf{R}^n)} \leq C\,\|u\|_{C^{2\beta-1}(\mathbf{R}^n)} \tag{5.2}$$

for some proper constant C. Moreover, relying directly on (5.1), we have

$$tR(t,\Delta_X)u(x) - \Delta_X R(t,\Delta_X)u(x) = u(x)$$

$$tR(t, \Delta_X)u(y) - \Delta_X R(t, \Delta_X)u(y) = u(y).$$

Subtracting and using (5.2) we easily obtain

$$\|R(t, \Delta_X)u\|_{C^{2\beta-1}(\mathbf{R}^n)} \leq C\, t^{-1} \|u\|_{C^{2\beta-1}(\mathbf{R}^n)}$$

for t large, say $t \geq T$. If we now take $\varphi \in C^{2\beta+1}(\mathbf{R}^n)$ we have

$$\|\varphi\|_{C^1} \leq C(\|\varphi\|_{C^{2\beta+1}})^{1-\beta}(\|\varphi\|_{C^{2\beta-1}})^{\beta}.$$

Therefore

$$\|R(t, \Delta_X)u\|_{C^1} \leq C(\|R(t, \Delta_X)u\|_{C^{2\beta+1}})^{1-\beta}(\|R(t, \Delta_X)u\|_{C^{2\beta-1}})^{\beta}$$
$$\leq C(t^{-1}\|u\|_{C^{2\beta-1}})^{\beta}(\|u\|_{C^{2\beta-1}})^{1-\beta}$$

where we have used the analogue of Xu's estimate which allows us in this degenerate context to conclude about $C^{2\beta+1}$ continuity. Finally

$$\|R(t, \Delta_X)u\|_{C^1} \leq c\; t^{-\beta}\|u\|_{C^{2\beta-1}}$$

with $t \geq T$. Starting from this inequality and from the generation estimate given in Theorem 4.2, we can argue as in [14] and conclude that $u \in D_A(\beta, \infty)$.

REMARK 5.3. As in the classical case, we cannot expect $D_A(1/2, \infty)$ to be made up just of continuous functions. As for $0 < \beta < 1/2$, this is still an open problem, mainly due to two difficulties: first of all the definition of Hölder spaces with negative exponent has to be done with some care in the case of Hörmander vector fields; on the other hand, there are a certain number of difficulties in characterizing Hölder spaces by means of interpolation, due to the lack of commutativity. Therefore the original method used in [14] does not seem to apply here.

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