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Local Overdetermined Linear Elliptic Problems in Lipschitz Domains with Solutions Changing Sign

Bruno Canuto and Diego Rial

Abstract. *We prove that the only domain* Ω *such that there exists a solution to the following overdetermined problem* $\Delta u + \omega^2 u = -1$ *in* Ω, $u = 0$ *on* $\partial\Omega$, *and* $\partial_{\mathbf{n}} u = c$ *on* $\partial\Omega$, *is the ball* B_1, *independently on the sign of* u, *if we assume that the boundary* $\partial\Omega$ *is a perturbation (no necessarily regular) of the unit sphere* ∂B_1 *of* $\mathbb{R}^n$. *Here* $\omega^2 \neq (\lambda_n)_{n\geq 1}$ *(the eigenvalues of* $-\Delta$ *in* B_1 *with Dirichlet boundary conditions), and* $\omega \notin \Lambda$, *where* Λ *is a enumerable set of* $\mathbb{R}^+$, *whose limit points are the values* λ_{1m}, *for some integer* $m \geq 1$, λ_{1m} *being the* m^{th}*-zero of the first-order Bessel function* I_1.

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1. Introduction

The objective of the present paper is to give an answer to the following problem: for $\omega \in \mathbb{R}$, is it true that the only domain Ω such that there exists a solution to the overdetermined problem

$$\begin{cases} \Delta u + \omega^2 u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

$$\partial_{\mathbf{n}} u = c \text{ on } \partial\Omega, \tag{2}$$

is a ball? Here Ω is a sufficiently smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$, $\partial_{\mathbf{n}} u$ is the external normal derivative to the boundary $\partial\Omega$, and c is a given constant. As application of the problem, we consider a uniform membrane, plane at rest, covering a region Ω. Let the deformation normal to the equilibrium be denoted by $\psi(x,t)$. Neglecting higher powers of ψ and its derivatives, the forced motion of membrane is described by the wave equation

$$-\mu\Delta\psi + \rho\partial_t^2\psi = p,$$

where μ is the elastic modulus, ρ mass density and p is the pressure over the membrane. For the case of a uniform periodic pressure of the form $p = p_0 e^{i\alpha t}$, we obtain a solution $\psi(x,t) = (p_0/\mu)\, u(x) e^{i\alpha t}$ where u solves (1), with $\omega = \alpha\sqrt{\rho/\mu}$. The normal derivative represents the line density force on the boundary. The question we ask is the following: if the line density force on the membrane boundary is the same at all points, is then the shape circular?

By using the method of moving planes J. Serrin [6] has given a positive answer, in the case where the solution u has a sign in Ω (for example for $\omega = 0$, by the maximum principle it follows that u is positive in Ω). For the particular case $\omega = 0$ see also the results of M. Choulli, A. Henrot [1], which use the technique of the domain derivative. We point out that Serrin in [6] has studied the same problem for more general nonlinear elliptic equations. All these proofs need hypothesis on the sign of u.

Let $(\lambda_n)_{n\geq 1}$ be the sequence, in increasing order, of eigenvalues of $-\Delta$ in B_1 with Dirichlet boundary conditions, where B_1 is the ball of radius 1 in $\mathbb{R}^n$ centered at zero. We observe that if $\omega^2 \neq (\lambda_n)_{n\geq 1}$, and $\Omega = B_1$, the solution to (1) is unique and radial, and then it satisfies (2). More precisely, by a simple calculation, one can verify that it is given by

$$u_0(x) = \frac{1}{\omega^2}\left(\frac{I_0(\omega r)}{I_0(\omega)} - 1\right), \tag{3}$$

for $\omega \neq 0$, and

$$u_0(x) = \frac{1}{2n}\left(1 - r^2\right), \tag{4}$$

for $\omega = 0$, where $r = |x|$, and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. Here and in what follows I_ℓ, $\ell \geq 0$ integer, is the so-called n-dimensional

ℓ-order Bessel function of first kind, and is given by

$$I_\ell(s) = s^{-\nu} J_{\nu+\ell}(s),$$

where $\nu = \frac{n}{2} - 1$, and $J_{\nu+\ell}$ is the well-known $\nu + \ell$-order Bessel function of the first kind (see Section 2 for more details). We have that the constant c in (2) is equal to $\frac{I_0'(\omega)}{\omega I_0(\omega)}$ for $\omega \neq 0$, and to $-1/n$ for $\omega = 0$, since $\partial_{\mathbf{n}} u_{0|\partial B_1} = \frac{I_0'(\omega)}{\omega I_0(\omega)}$ for $\omega \neq 0$, and $\partial_{\mathbf{n}} u_{0|\partial B_1} = -\frac{1}{n}$ for $\omega = 0$ (the symbol $'$ denoting the ordinary derivative). In the rest of the paper we will assume $\omega \geq 0$, and $\omega^2 \neq (\lambda_n)_{n\geq 1}$. The same conclusions hold true for $\omega < 0$, since the coefficient ω^2 is even in (1). One can verify easily (by using that $I_0' = -I_1$) that if the constant ω is smaller or equal than λ_{11}, λ_{11} is the first zero of I_1, the solution u_0 is positive in B_1, while if ω is bigger than λ_{11}, then u_0 changes sign. So for this values of ω we cannot expect to study the above problem by Alexandrov-Serrin method of moving planes, and nothing can be said about this question.

We can formulate the problem in the following manner. Let us define by E the vector space of sufficiently regular functions defined on ∂B_1 (for example $E = C^{2,\alpha}(\partial B_1)$, see Section 3 and 4 for more details). For $k \in E$, let us denote by Ω_k the domain whose boundary $\partial\Omega_k$ can be written as perturbation of the sphere ∂B_1, i.e.

$$\partial\Omega_k = \{x = (1 + k(y))y, y \in \partial B_1\} \tag{5}$$

(in particular for $k = 0$, $\partial\Omega_0 = \partial B_1$). For a fixed $\omega \geq 0$, $\omega^2 \neq (\lambda_n)_{n\geq 1}$, we can find a neighborhood $\mathcal{U}$ of 0 in E such that for every $k \in \mathcal{U}$ the kernel of the operator $\Delta + \omega^2$ is equal to zero in Ω_k. For such values of k there exists a unique solution u to (1), when $\Omega = \Omega_k$. Now let Φ_ω be the following (nonlinear) Neumann-type operator

$$\Phi_\omega : E \mapsto F$$

defined by

$$\Phi_\omega(k) = \partial_n u \circ \varphi, \tag{6}$$

where φ is the parametrization of $\partial\Omega_k$ defined in (5) (F will be a space of functions defined on ∂B_1, whose regularity will depend on the regularity class of E). We have that Φ_ω is well-defined in

$\mathcal{U}$. We point out that Φ_ω is not injective. In fact, by observing that the sphere of radius one, centered at the point $x_0 \in \mathbb{R}^n$, is parametrized by

$$\partial B_1(x_0) = \{x = (1 + k_0(y))y, y \in \partial B_1\},$$

where k_0 is given by

$$k_0(y) = x_0 \cdot y - 1 + \sqrt{1 + |x_0 \cdot y|^2 - |x_0|^2},$$

we have that

$$\Phi_\omega(k_0) = c,$$

for every k_0 such that the center x_0 verifies $1 + |x_0 \cdot y|^2 - |x_0|^2 \geq 0$ on ∂B_1.

Now here and in what follows we will denote by $Y_{\ell m}$ the spherical harmonics of degree ℓ (where $m = 1, \cdots, d_\ell$, and d_ℓ is the dimension of the space of spherical harmonics $Y_{\ell m}$ of degree ℓ, see (3)), and we will use the following convention: we say that a function f has the frequency ℓ, if the Fourier-coefficients of order ℓ of f, i.e. $f_{\ell m} = \int_{\partial B_1} f Y_{\ell m}$, are different to zero, for some $m \in \{1, \cdots, d_\ell\}$. And similarly we say that a function f doesn't have the frequency ℓ, if the Fourier-coefficients of order ℓ of f vanish for all $m = 1, \cdots, d_\ell$.

By going back to the parametrization of the sphere $\partial B_1(x_0)$, we point out that the Fourier's series expansion of the function k_0 has the frequency 1, which is equal to x_0. In fact we have that the function

$$h(y) = \sqrt{1 + |x_0 \cdot y|^2 - |x_0|^2}$$

is even in the variable y, and then the function hY_{1m} is odd, which implies that $\int_{\partial B_1} hY_{1m} = 0$, for all $m = 1, \cdots, n$. Then we have that for every $x_0 \neq 0$, the function k_0 has the frequency 1, which is equal to x_0. So the best one can aspect is that the operator Φ_ω is injective in a neighborhood of 0 in E_0, where E_0 denotes the space of functions $k \in E$ which don't have the frequency 1, i.e.

$$E_0 = \left\{ k \in E; k = \sum_{\ell \neq 1} \sum_{m=1}^{d_\ell} k_{\ell m} Y_{\ell m} \right\},$$

or equivalently

$$E_0 = \left\{ k \in E; \int_{\partial B_1} kY_{1m} = 0,\, m = 1, \cdots, n \right\}.$$

By studying the behavior of the differential of Φ_ω at zero, $\mathrm{d}\Phi_\omega(0)$, we prove that, for $\omega \notin \Lambda$, this operator is bijective from E_0 into F_0 (see Theorem 3.1; F_0 is a subspace of F whose functions don't have the frequency 1). Here the set Λ is defined at Definition 3.9 (see pp. 20). More precisely, see Lemma 3.10, we prove that Λ is a enumerable set of $\mathbb{R}^+$, whose limit points are the values λ_{1m}, for some integer $m \geq 1$ (λ_{1m} is the m^{th}-zero of the Bessel function I_1). Then, by defining a new operator Ψ_ω, which coincides with Φ_ω in E_0, we prove that it is bijective from a neighborhood of 0 in E into a neighborhood of c in F. This yields in particular that Φ_ω is injective in a neighborhood of 0 in E_0 (see Theorem 5.1). What happens for $\omega_* \in \Lambda$? In this case we have that there exists at least a integer $\ell_0 \geq 2$, such that Φ_{ω_*} is injective in a neighborhood of 0 in E_{0*}, where E_{0*} is the space of functions k which don't have both the frequency 1 and ℓ_0.
By going back to the overdetermined problem (1), (2), we observe that, in order to give a positive answer to the question, it is sufficient to prove that, if there exists a function k such that $\Phi_\omega(k) = c$, then k is equal to k_0. This is the content of the following

Theorem 1.1. *For $\omega^2 \neq (\lambda_n)_{n\geq 1}$, and $\omega \notin \Lambda$, there exists a neighborhood $\mathcal{U}$ of 0 in E such that if, for $k \in \mathcal{U}$, $\Phi_\omega(k) = c$, then $k = k_0$.*

Let us make some remarks about the theorem. Our proof doesn't require hypothesis on the sign of the solution u. In fact, as we have observed previously, by choosing $\omega > \lambda_{11}$, u_0 changes sign in B_1. On the other hand the result is local, i.e. holds true only for domains which are "small" perturbations of the unit sphere. In particular, since the $\inf \Lambda = \lambda_{11}$, and u_0 is positive for $\omega \leq \lambda_{11}$, Theorem 1.1 gives a new "local" proof of the Serrin's result. If $\omega_* \in \Lambda$ it remains an open question to know if there exists a domain Ω_k different from B_1 such that $\Phi_{\omega_*}(k) = c$.
The paper is organized as follows: in Section 3 we consider regular perturbations of the unit sphere, i.e. we assume that E is the space of functions of class $C^{2,\alpha}$, while in Section 4 we study perturbations

of Lipschitz class $C^{0,1}$. This implies that the domain Ω in (1), (2) can be Lipschitz (we recall that in [1] and [6] the domain Ω is of class $C^{2,\alpha}$, $\alpha \in (0,1]$). In Section 5 we prove Theorem 1.1. Since the more interesting case is which where $\omega \neq 0$, we omit the case $\omega = 0$. Obviously the same conclusions hold true for $\omega = 0$, mutatis mutandis.

2. Preliminaries and Notations

Let us denote by B_1 and $B_1(x_0)$ the ball of radius 1 in $\mathbb{R}^n$ centered at zero, and at the point x_0 respectively. By $\overline{B_1}$ we define the Euclidean closure of B_1. Let us denote by I_ℓ the so-called n-dimensional ℓ-order Bessel function of first kind, i.e.

$$I_\ell(s) = s^{-\nu} J_{\nu+\ell}(s), \tag{1}$$

where $\nu = \frac{n}{2} - 1$, and $J_{\nu+\ell}$ is the well-known $\nu + \ell$-order Bessel function of the first kind (we observe that for $n = 2$, I_ℓ coincides with the ℓ-order Bessel function J_ℓ). We have that I_ℓ verifies the following Bessel-type equation

$$I_\ell'' + \frac{n-1}{s} I_\ell' + \left(1 - \frac{\ell^2}{s^2}\right) I_\ell = 0 \quad \text{in } \mathbb{R}. \tag{2}$$

Let $(\lambda_n)_{n\geq 1}$ be the sequence of eigenvalues of $-\Delta$ in B_1 with Dirichlet boundary conditions. We have that the eigenvalue λ_n, for some $n \in \mathbb{N}$, coincides, for some integers $\ell \geq 0$ and $m \geq 1$, with $\lambda_{\ell m}^2$, where $\lambda_{\ell m}$ is the m-zero of the ℓ-order Bessel function I_ℓ. The eigenfunctions of $-\Delta$ in B_1 can be written as (in polar coordinates)

$$\varphi_{\ell m}(r,\theta) = I_\ell(\lambda_{\ell m} r) \sum_{k=1}^{d_\ell} a_k Y_{\ell k}(\theta)$$

where $Y_{\ell k}$ are the spherical harmonics of degree ℓ. The dimension of the space of spherical harmonics with degree ℓ is given by

$$d_\ell = \frac{(2\ell + n - 2)\,(\ell + n - 3)!}{\ell!\,(n-2)!}. \tag{3}$$

We observe that $d_1 = n$. Let us write the boundary $\partial\Omega_k$ in local coordinates $u = (u_1, \cdots, u_{n-1})$, i.e.

$$\partial\Omega_k = \{x = (1 + k(y(u)))y(u)\}.$$

Let assume that the boundary is sufficiently regular at the point $x \in \partial\Omega_k$ such that we can define the external normal vector at this point. Then we have that, for $i = 1, \cdots, n-1$, the tangent vector $\boldsymbol{\tau}_i$ at the point x is given by

$$\boldsymbol{\tau}_i = \sum_{j=1}^{n} \partial_{y_j} k(\partial_i y_j) y + (1 + k)\mathbf{t}_i,$$

where $\partial_i\cdot$ denotes the partial derivative with respect to the variable u_i, and $\mathbf{t}_i = \partial_i y$ is the tangent vector to the sphere ∂B_1 (we assume that $\boldsymbol{\tau}_i \neq 0$, and the function k is, for example, at least Lipschitz on ∂B_1). Let us call A the Jacobian matrix of change of variables

$$x = (1 + k(y))y, \quad y \in \overline{B_1}. \tag{4}$$

The matrix A is given by

$$A_{ij} = \begin{bmatrix} 1 + k + y_1\partial_1 k & y_1\partial_2 k & \cdots & y_1\partial_n k \\ y_2\partial_1 k & 1 + k + y_2\partial_2 k & \cdots & y_2\partial_n k \\ \vdots & \vdots & \vdots & \vdots \\ y_n\partial_1 k & \cdots & \cdots & 1 + k + y_n\partial_n k \end{bmatrix}. \tag{5}$$

We have that $\boldsymbol{\tau}_i = A\mathbf{t}_i$. Let $\boldsymbol{\nu}$ be the external normal vector at the point x. Then we have that

$$0 = \boldsymbol{\nu} \cdot \boldsymbol{\tau}_i = \boldsymbol{\nu} \cdot A\mathbf{t}_i = A^T\boldsymbol{\nu} \cdot \mathbf{t}_i,$$

where A^T is the transpose matrix of A. This yields that the vector $A^T\boldsymbol{\nu}$ is normal to the sphere, i.e.

$$A^T\boldsymbol{\nu} = \alpha y,$$

for some $\alpha \neq 0$. Then we obtain that the external unit normal vector at the point $x \in \partial\Omega_k$ is given by

$$\mathbf{n}(1 + k(y)) = \frac{(A^T)^{-1}y}{\sqrt{G^{-1}y \cdot y}}, \tag{6}$$

where G^{-1} is the inverse of the matrix G, and $G = A^T A$ (we observe that $(A^T)^{-1}y \cdot (A^T)^{-1}y = A^{-1}(A^T)^{-1}y \cdot y = G^{-1}y \cdot y$). We point out that since $G^{-1}(0)y \cdot y = 1$, we can suppose that for $k \in \mathcal{U}$, reducing $\mathcal{U}$ if it is necessary, $G^{-1}(k)y \cdot y \geq \alpha$, α a positive constant. In such way we have that the boundary $\partial\Omega_k$ doesn't have turning points.

3. The Regular Case

In this section we study the case where the domain Ω in (1) is of class $C^{2,\alpha}$, $\alpha \in (0, 1]$. More precisely let us define by

$$E = \left\{k \in C^{2,\alpha}(\partial B_1)\right\},$$

where by $C^{2,\alpha}(\partial B_1)$ we denote the restriction on ∂B_1 of functions of class $C^{2,\alpha}$ in $\overline{B_1}$. Let ω be fixed, and $\omega^2 \neq (\lambda_n)_{n\geq 1}$. For $k \in \mathcal{U}$, by well-known results of elliptic boundary value problems (see for example Gilbarg, Trudinger [4], Theorem 6.14, pp. 107), we have that there exists a unique solution $u \in C^{2,\alpha}(\overline{\Omega_k})$ to (1), when $\Omega = \Omega_k$. The operator Φ_ω (defined in (6)) is well-defined in $\mathcal{U}$, and

$$\Phi_\omega : \mathcal{U} \mapsto F,$$

where F is the space

$$F = \left\{f \in C^{1,\alpha}(\partial B_1)\right\}.$$

Let E_0 and F_0 be the following vector subspaces of E and F respectively, defined by

$$E_0 = \left\{ k \in E; k = \sum_{\ell \neq 1} \sum_{m=1}^{d_\ell} k_{\ell m} Y_{\ell m} \right\}, \tag{7}$$

$$F_0 = \left\{ f \in F; f = \sum_{\ell \neq 1} \sum_{m=1}^{d_\ell} f_{\ell m} Y_{\ell m} \right\}, \tag{8}$$

where $k_{\ell m} = \int\limits_{\partial B_1} k Y_{\ell m}$ and $f_{\ell m} = \int\limits_{\partial B_1} f Y_{\ell m}$ are the Fourier-coefficients of the function k and f respectively. E_0 and F_0 are spaces of functions whose Fourier series expansions don't have the frequency 1. The main result of the present section is the following

THEOREM 3.1. *Under the hypothesis of Theorem 1.1, the operator* $d\Phi_\omega(0)$ *is an isomorphism from* E_0 *into* F_0.

Here $d\Phi_\omega(0)$ denotes the differential of the operator Φ_ω at zero. Before proving Theorem 3.1 we need some preliminary lemmas. First of all we observe that, by changing the coordinates x into the new y, the change of coordinates is given by (4), and denoting by $\widetilde{u}(k)$ the function defined by

$$\widetilde{u}(k)(y) = u((1+k)y),$$

we have that $u(k) \in C^{2,\alpha}(\overline{B_1})$ (we have denoted $\widetilde{u}(k)$ by $u(k)$) solves

$$\begin{cases} \frac{1}{\sqrt{g(k)}}\operatorname{div}(\sqrt{g(k)}G^{-1}(k)\nabla u(k)) + \omega^2 u(k) & = & -1 & \text{in } B_1, \\ u(k) & = & 0 & \text{on } \partial B_1, \end{cases} \tag{9}$$

where $g = |\det G|$. By observing that $g(0) = 1$, reducing $\mathcal{U}$ if it is necessary, we can suppose that, for $k \in \mathcal{U}$, $g(k) \geq \alpha$, α a positive constant. We have that in the new coordinates the operator Φ_ω becomes

$$\Phi_\omega(k) = (A^T)^{-1}\nabla u \cdot \mathbf{n} \quad \text{on } \partial B_1, \tag{10}$$

where the external unit normal vector $\mathbf{n}$ to $\partial\Omega_k$ is given by (6).
Then $\Phi_\omega(k)$ can be written as

$$\Phi_\omega(k) = \frac{G^{-1}\nabla u \cdot x}{\sqrt{G^{-1}x \cdot x}} \quad \text{for } x \in \partial B_1, \tag{11}$$

where we have denoted the new variables y by x. We can write the matrix $G = A^T A$, as

$$G = I_n + G_1 + G_2,$$

where I_n is the n-order identity matrix, G_1 depends linearly on k and ∇k, and G_2 depends quadratically on k and ∇k. By (5), one can verify that the entries G_{1ij} of the matrix G_1 are given by

$$G_{1ij} = 2kI_n + \tag{12}$$
$$+ \begin{bmatrix} 2x_1\partial_1 k & x_1\partial_2 k + x_2\partial_1 k & \cdots & x_1\partial_n k + x_n\partial_1 k \\ x_1\partial_2 k + x_2\partial_1 k & 2x_2\partial_2 k & \cdots & x_2\partial_n k + x_n\partial_2 k \\ \vdots & \vdots & \vdots & \vdots \\ x_1\partial_n k + x_n\partial_1 k & \cdots & \cdots & 2x_n\partial_n k \end{bmatrix}.$$

Lemma 3.2. *For $\omega^2 \neq (\lambda_n)_{n\geq 1}$, there exists a neighborhood $\mathcal{U}$ of zero in E such that the operator $\Phi_\omega \in C^1(\mathcal{U}, F)$.*

Proof. Let us denote by

$$L(k) : C^{2,\alpha}(\overline{B_1}) \cap C_0(\overline{B_1}) \mapsto C^{0,\alpha}(\overline{B_1})$$

the operator defined by

$$L(k)\cdot = \frac{1}{\sqrt{g(k)}}\operatorname{div}(\sqrt{g(k)}G^{-1}(k)\nabla\cdot), \tag{13}$$

where $C_0(\overline{B_1})$ denotes the space of continuous functions in $\overline{B_1}$ which are zero on ∂B_1. Since G_1 and G_2 are linear and quadratic, in the variables k and ∇k, respectively, it is easy to verify that $G \in C^1(E, C^{1,\alpha}(\overline{B_1}, \mathbb{R}^{n\times n}))$. Using that $G(0) = I_n$, we can see that there exists a neighborhood of the origin $\mathcal{U}$ in E such that G^{-1} is a continuously differentiable map in $\mathcal{U}$. It follows immediately that the operator L is a continuously differentiable map from $\mathcal{U}$ to $\mathcal{L}(C^{2,\alpha}(\overline{B_1}) \cap C_0(\overline{B_1}), C^{0,\alpha}(\overline{B_1}))$. Assuming that ω^2 is not a eigenvalue, $\Delta + \omega^2$ is a isomorphism, and then, reducing $\mathcal{U}$ if it is necessary, $(L(\cdot) + \omega^2)^{-1}$ is a continuously differentiable map from $\mathcal{U}$ to $\mathcal{L}(C^{0,\alpha}(\overline{B_1}), C^{2,\alpha}(\overline{B_1}) \cap C_0(\overline{B_1}))$. We note that

$$u(k) = -(L(k) + \omega^2)^{-1}1.$$

We consider the map T of class C^1 from $\mathcal{U}$ to $\mathcal{L}(C^{2,\alpha}(\overline{B_1}) \cap C_0(\overline{B_1}), F)$, defined by

$$T(k)\cdot = \frac{G^{-1}(k)\nabla\cdot\cdot x}{\sqrt{G^{-1}(k)x\cdot x}}.$$

Writing $\Phi_\omega(k) = -T(k)\left(L(k) + \omega^2\right)^{-1}1$, we obtain the result. □

Let us denote by

$$\delta\Phi_\omega(0) = < \mathrm{d}\Phi_\omega(0) \mid k >$$

the first variation of the operator Φ_ω at zero. In the next lemma we give a explicit expression of $\delta\Phi_\omega(0)$.

LEMMA 3.3. *We have that*

$$\delta\Phi_\omega(0) = \partial_{\mathbf{n}} u_1 - \partial_{\mathbf{n}} u_0 (k + \partial_{\mathbf{n}} k) \quad \text{in } C^{1,\alpha}(\partial B_1), \tag{14}$$

where $u_1 \in C^{2,\alpha}(\overline{B_1})$ *solves*

$$\begin{cases} \Delta u_1 + \omega^2 u_1 &= f \quad \text{in } B_1, \\ u_1 &= 0 \quad \text{on } \partial B_1, \end{cases} \tag{15}$$

and

$$\begin{aligned} f \;=\; & -\frac{2I_0(\omega r)}{I_0(\omega)}(k + x\cdot\nabla k) + r\frac{I_0'(\omega r)}{\omega I_0(\omega)}\Delta k \\ & -2(n-2)\frac{I_0'(\omega r)}{\omega r I_0(\omega)} x\cdot\nabla k. \end{aligned} \tag{16}$$

Proof. We divide the proof into three steps.

Step 1 Let us consider the matrix G as function of independent variables k and ∇k. In this step we give the first-order Taylor's expansion, as function of k and ∇k, in a neighborhood of 0, of the matrix G^{-1}. We have that $g = |\det G|$ (as function of k and ∇k) can be written as

$$g = 1 + 2nk + 2x\cdot\nabla k + g_2,$$

where the function g_2 depends quadratically on k and ∇k. We have that the first-order Taylor's expansion of $\sqrt{g}$ is given by

$$\sqrt{g} = 1 + nk + x\cdot\nabla k + o\big(\|k\|_{C^1(\overline{B_1})}\big). \tag{17}$$

Let us write the matrix $\sqrt{g}G^{-1}$ in (9) as

$$\sqrt{g}G^{-1} = I_n + K. \tag{18}$$

We have that

$$\sqrt{g}I_n - G = KG.$$

By taking the linear part of K, noted by K_1, we have that

$$K_1 = (nk + x\cdot\nabla k)I_n - G_1, \tag{19}$$

where G_1, the linear part of G, is given by (13). From (18), (19) we obtain that the matrix G^{-1} can be written as

$$\begin{aligned} G^{-1} &= \frac{I_n}{\sqrt{g}} + \frac{1}{\sqrt{g}} K_1 + K_2 \\ &= I_n - G_1 + K_2, \end{aligned} \tag{20}$$

where the matrix K_2 depends at least quadratically on k and ∇k, and in the last step we use that $\frac{1}{\sqrt{g}} = 1 - nk - x \cdot \nabla k + o(\|k\|_{C^1(\overline{B_1})})$.

Step 2 By writing the function u as

$$u = u_0 + u_1 + o\big(\|k\|_{C^1(\overline{B_1})} \big),$$

where u_0 is given by (4) (i.e. solves (9) for $k = 0$), and u_1 depends linearly on k and ∇k, in this step we prove that the operator $\Phi_\omega(k)$ can be written as

$$\Phi_\omega(k) = \partial_{\mathbf{n}} u_0 + \partial_{\mathbf{n}} u_1 - \partial_{\mathbf{n}} u_0 (k + \partial_{\mathbf{n}} k) + o\big(\|k\|_{C^1(\overline{B_1})} \big) \quad \text{on } \partial B_1. \tag{21}$$

By using (20), we have

$$\begin{aligned} \Phi_\omega(k) &= (G^{-1} x \cdot x)^{-1/2} G^{-1} \nabla u \cdot x \\ &= (1 - G_1 x \cdot x + K_2 x \cdot x)^{-1/2} (\partial_{\mathbf{n}} u - G_1 \nabla u \cdot x) \\ &\qquad + o\big(\|k\|_{C^1(\overline{B_1})} \big). \end{aligned}$$

Using (13), by a direct calculation we have

$$\begin{aligned} G_1 x \cdot x &= 2k + 2 \sum_{i \geq 1} x_i^3 \partial_i k + 2 \sum_{i \geq 1} \sum_{j > i} x_i x_j^2 \partial_i k \\ &= 2k + 2 \sum_{i \geq 1} x_i \partial_i k \sum_{j \geq 1} x_j^2 \\ &= 2(k + \partial_{\mathbf{n}} k). \end{aligned}$$

By writing the function u as

$$u = u_0 + u_1 + o\Big(\|k\|_{C^1(\overline{B_1})} \Big),$$

we obtain

$$\begin{aligned} \Phi_\omega(k) &= (1 - 2k - 2\partial_{\mathbf{n}} k + K_2 x \cdot x)^{-1/2} (\partial_{\mathbf{n}} u_0 + \partial_{\mathbf{n}} u_1 - G_1 \nabla u_0 \cdot x) \\ &\qquad + o\big(\|k\|_{C^1(\overline{B_1})} \big). \end{aligned} \tag{22}$$

Now we have that the first factor in the previous product can be written as

$$(1-2k-2\partial_{\mathbf{n}}k+K_2x\cdot x)^{-1/2}=1+k+\partial_{\mathbf{n}}k+o(\|k\|_{C^1(\overline{B_1})}),$$

and the second factor as

$$\begin{aligned} G_1\nabla u_0\cdot x &= \partial_{\mathbf{n}}u_0G_1x\cdot x \\ &= 2\partial_{\mathbf{n}}u_0(k+\partial_{\mathbf{n}}k). \end{aligned}$$

So (22) becomes

$$\begin{aligned} \Phi_\omega(k) &= (1+k+\partial_{\mathbf{n}}k)(\partial_{\mathbf{n}}u_0+\partial_{\mathbf{n}}u_1-2\partial_{\mathbf{n}}u_0(k+\partial_{\mathbf{n}}k)) \\ &\qquad +o\big(\,\|k\|_{C^1(\overline{B_1})}\big) \\ &= \partial_{\mathbf{n}}u_0+\partial_{\mathbf{n}}u_1-\partial_{\mathbf{n}}u_0(k+\partial_{\mathbf{n}}k)+o\big(\,\|k\|_{C^1(\overline{B_1})}\big). \end{aligned}$$

Step 3 In this step we prove the assertion of the lemma. By step 2 we have that

$$\Phi_\omega(k)-\Phi_\omega(0)=\partial_{\mathbf{n}}u_1-\partial_{\mathbf{n}}u_0(k+\partial_{\mathbf{n}}k)+o\big(\,\|k\|_{C^1(\overline{B_1})}\big) \tag{23}$$

Let $w=u-u_0$. We have that w solves

$$\begin{cases} \Delta w+\omega^2 w = -\sqrt{g}(\omega^2u+1)-\operatorname{div}(K\nabla u)+\omega^2u+1 & \text{in } B_1, \\ w = 0 & \text{on } \partial B_1, \end{cases} \tag{24}$$

where in (9) we have written the matrix $\sqrt{g}G^{-1}=I_n+K$. By using (17), we can write the right hand side of (24) as follows

$$\begin{aligned} &-\sqrt{g}(\omega^2u+1)-\operatorname{div}(K\nabla u)+\omega^2u+1 \\ &\quad =-(nk+x\cdot\nabla k)(1+\omega^2u)-\operatorname{div}(K\nabla u)+o\big(\,\|k\|_{C^1(\overline{B_1})}\big) \\ &\quad =-(nk+x\cdot\nabla k)(1+\omega^2u_0)-\operatorname{div}(K\nabla u_0)+o\big(\,\|k\|_{C^1(\overline{B_1})}\big). \end{aligned}$$

So we have that u_1 solves

$$\begin{cases} \Delta u_1+\omega^2 u_1 = -(nk+x\cdot\nabla k)(1+\omega^2u_0)-\operatorname{div}(K_1\nabla u_0) & \text{in } B_1, \\ u_1 = 0 & \text{on } \partial B_1, \end{cases} \tag{25}$$

where we recall that K_1 is the linear part of the matrix K. In order to compute the term $\operatorname{div}(K_1\nabla u_0)$, by (19) we have that

$$K_1 = (nk + x\cdot\nabla k)I_n - G_1, \tag{26}$$

or equivalently

$$K_1 = (n-2)I_n k + (2k + x\cdot\nabla k)I_n - G_1.$$

Let us denote by M the matrix $(2k + x\cdot\nabla k)I_n - G_1$. We have that the entries M_{ij} of the matrix M are given by

$$M_{ij} = \begin{bmatrix} -x_1\partial_1 k + \sum\limits_{i\neq 1} x_i\partial_i k & -x_1\partial_2 k - x_2\partial_1 k & \cdots & -x_1\partial_n k - x_n\partial_1 k \\ -x_1\partial_2 k - x_2\partial_1 k & -x_2\partial_2 k + \sum\limits_{i\neq 2} x_i\partial_i k & \cdots & -x_2\partial_n k - x_n\partial_2 k \\ \vdots & \vdots & \vdots & \vdots \\ -x_1\partial_n k - x_n\partial_1 k & \cdots & \cdots & -x_n\partial_n k + \sum\limits_{i\neq n} x_i\partial_i k \end{bmatrix}.$$

By a direct calculation we have that

$$K_1\nabla u_0 = u_0' K_1 \frac{x}{r} = (n-2)\frac{u_0'}{r}kx - u_0' r\nabla k.$$

Finally we obtain that

$$\operatorname{div}(K_1\nabla u_0) = (n-2)\operatorname{div}\left(\frac{u_0'}{r}kx\right) - \operatorname{div}(u_0' r\nabla k).$$

We have that

$$\begin{aligned} \operatorname{div}\left(\frac{u_0'}{r}kx\right) &= u_0''k + \frac{n-1}{r}u_0'k + \frac{u_0'}{r}x\cdot\nabla k \\ &= -(\omega^2 u_0 + 1)k + \frac{u_0'}{r}x\cdot\nabla k, \end{aligned}$$

where in the second step we use that $u_0'' = -\frac{n-1}{r}u_0' - \omega^2 u_0 - 1$. Similarly we have

$$\begin{aligned} \operatorname{div}(u_0' r\nabla k) &= -r\Delta k u_0' - \frac{u_0'}{r}x\cdot\nabla k - u_0'' x\cdot\nabla k \\ &= -r\Delta k u_0' + \frac{n-2}{r}u_0' x\cdot\nabla k + (\omega^2 u_0 + 1)x\cdot\nabla k. \end{aligned}$$

Then the right hand side in (25) becomes

$$\begin{aligned} &-(nk + x\cdot\nabla k)(1+\omega^2 u_0) - \operatorname{div}(K_1\nabla u_0) \\ &\quad = r\Delta k u_0' - 2\frac{n-2}{r}u_0' x\cdot\nabla k - 2(k + x\cdot\nabla k)\omega^2 u_0 - 2x\cdot\nabla k - 2k. \end{aligned} \tag{27}$$

By recalling that $u_0 = \frac{1}{\omega^2}(\frac{I_0(\omega r)}{I_0(\omega)} - 1)$, and substituting into (27), we obtain (16). The proof of Lemma 3.3 is complete □

In the next lemma we give an explicit expression of the solution u_1 to (15).

LEMMA 3.4. *The solution u_1 to (15) can be written as*

$$u_1 = \frac{I_0'(\omega r)}{\omega I_0(\omega)} rk + \widetilde{k} \quad \text{in } B_1,$$

where $\widetilde{k} \in C^{2,\alpha}(\overline{B_1})$ solves

$$\begin{cases} \Delta\widetilde{k} + \omega^2\widetilde{k} = 0 & \text{in } B_1, \\ \widetilde{k} = -\frac{I_0'(\omega)}{\omega I_0(\omega)}k & \text{on } \partial B_1. \end{cases} \tag{28}$$

Proof. It is clear that the boundary condition $u_1 = 0$ on ∂B_1 is satisfied. Let us call

$$\overline{u}_1 = \frac{I_0'(\omega r) r}{\omega I_0(\omega)} k,$$

and

$$F(r) = I_0'(\omega r) r.$$

Then we have

$$\begin{aligned} F' &= \omega r I_0''(\omega r) + I_0'(\omega r) \\ &= \omega r I_0''(\omega r) + (n-1)I_0'(\omega r) - (n-2)I_0'(\omega r) \\ &= -\omega r I_0(\omega r) - (n-2)I_0'(\omega r), \end{aligned}$$

where in the third step we use that I_0 solves (2) for $\ell = 0$, and

$$F'' = -\omega I_0(\omega r) - \omega^2 r I_0'(\omega r) - \omega(n-2)I_0''(\omega r).$$

We obtain that

$$\begin{aligned}\Delta\overline{u}_1+\omega^2\overline{u}_1 &= \frac{1}{\omega I_0(\omega)}\times \\ &\times\left(\left(F''+\frac{n-1}{r}F'+\omega^2F\right)k+F\Delta k+2F'x\cdot\nabla k/r\right).\end{aligned}\tag{29}$$

We have

$$2F'x\cdot\nabla k/r=-2\left(\omega rI_0(\omega r)+(n-2)I_0'(\omega r)\right)x\cdot\nabla k/r.$$

A straightforward calculation shows that

$$\begin{aligned}F''+\frac{n-1}{r}F' &= -\omega I_0(\omega r)-\omega^2rI_0'(\omega r)-\omega(n-2)I_0''(\omega r)\\ &\quad+\frac{n-1}{r}\left(-\omega rI_0(\omega r)-(n-2)I_0'(\omega r)\right)\\ &= -\omega(n-2)I_0''(\omega r)-\omega^2rI_0'(\omega r)\\ &\quad-n\omega I_0(\omega r)-(n-2)(n-1)I_0'(\omega r)/r.\end{aligned}$$

Then we obtain

$$\begin{aligned}&\frac{1}{\omega I_0(\omega)}\left(\left(F''+\frac{n-1}{r}F'+\omega^2F\right)k+2F'x\cdot\nabla k/r\right)\\ &= -\frac{n-2}{I_0(\omega)}\left(I_0''(\omega r)+\frac{n-1}{\omega r}I_0'(\omega r)\right)k-n\frac{I_0(\omega r)}{I_0(\omega)}k\\ &\qquad-2\left(\frac{I_0(\omega r)}{I_0(\omega)}+(n-2)\frac{I_0'(\omega r)}{\omega rI_0(\omega)}\right)x\cdot\nabla k\\ &= \frac{n-2}{I_0(\omega)}I_0(\omega r)k-n\frac{I_0(\omega r)}{I_0(\omega)}k\\ &\qquad-2\left(\frac{I_0(\omega r)}{I_0(\omega)}+(n-2)\frac{I_0'(\omega r)}{\omega rI_0(\omega)}\right)x\cdot\nabla k\\ &= -2\frac{I_0(\omega r)}{I_0(\omega)}k-2\left(\frac{I_0(\omega r)}{I_0(\omega)}+(n-2)\frac{I_0'(\omega r)}{\omega rI_0(\omega)}\right)x\cdot\nabla k.\end{aligned}$$

Then we have that $\overline{u}_1$ solves the equation in (15), and $\overline{u}_1=\frac{I_0'(\omega)}{\omega I_0(\omega)}k$ on ∂B_1. Since $\widetilde{k}$ solves (28), we have that u_1 verifies (15). □

We observe that the first variation $\delta\Phi_\omega(0)$ of the functional Φ_ω at zero doesn't depend on the extension of k in B_1. More precisely we have the following

LEMMA 3.5. *We have that*

$$\delta\Phi_\omega\,(0) = -\frac{I_1'(\omega)}{I_0(\omega)}k + \partial_{\mathbf{n}}\widetilde{k} \quad \textit{on } \partial B_1, \tag{30}$$

where $\widetilde{k}$ *solves (28).*

Proof. The following equality holds true

$$I_0' = -I_1 \quad \text{in } \mathbb{R}, \tag{31}$$

where I_0, I_1 are defined in (2), when $\ell = 0$, 1 respectively. In fact, by (1) we have

$$I_0' = -\nu s^{-\nu-1}J_\nu + s^{-\nu}J_\nu'. \tag{32}$$

Since J_ν' can be written as (see Courant, Hilbert [2], pp. 486)

$$J_\nu' = \frac{\nu}{s}J_\nu - J_{\nu+1}, \tag{33}$$

inserting in (32), we obtain (31). Then the solution u_1 to (15) can be written as

$$u_1 = -\frac{I_1(\omega r)}{\omega I_0(\omega)}rk + \widetilde{k} \quad \text{in } B_1.$$

By a simple calculation we have that

$$\partial_{\mathbf{n}}u_{1|\partial B_1} = -\frac{I_1'(\omega)}{I_0(\omega)}k - \frac{I_1(\omega)}{\omega I_0(\omega)}k - \frac{I_1(\omega)}{\omega I_0(\omega)}\partial_{\mathbf{n}}k + \partial_{\mathbf{n}}\widetilde{k}.$$

Recalling that $\partial_{\mathbf{n}}u_{0|\partial B_1} = -\frac{I_1(\omega)}{\omega I_0(\omega)}$, substituting into (14), we obtain (30). □

In the next lemma we give the Fourier's series expansion of $\delta\Phi_\omega\,(0)$.

LEMMA 3.6. *We have that*

$$\delta\Phi_\omega\,(0) = \sum_{\ell\geq 0}\sum_{m=1}^{d_\ell} k_{\ell m}\frac{I_1(\omega)I_\ell'(\omega) - I_1'(\omega)I_\ell(\omega)}{I_0(\omega)I_\ell(\omega)}Y_{\ell m}. \tag{34}$$

Proof. By writing $\widetilde{k}$ in polar coordinates, we have

$$\widetilde{k}(r,\theta) = \frac{I_1(\omega)}{\omega I_0(\omega)}\sum_{\ell\geq 0}\sum_{m=1}^{d_\ell} k_{\ell m}I_\ell(\omega r)/I_\ell(\omega)Y_{\ell m}(\theta),$$

By inserting in (30), we obtain (34). □

Instead of the operator Φ_ω, let us define the new operator

$$\widetilde{\Phi}_\omega(k) := \Phi_\omega(k) - \frac{1}{|\partial B_1|}\int_{\partial B_1} \Phi_\omega(k). \tag{35}$$

Obviously we have that $\delta\widetilde{\Phi}_\omega(0) = \delta\Phi_\omega(0) - \frac{1}{|\partial B_1|}\int_{\partial B_1} \delta\Phi_\omega(0)$, and then the constant term in the Fourier expansion of $\delta\widetilde{\Phi}_\omega(0)$ disappears. We observe that the first variation $\delta\widetilde{\Phi}_\omega(0)$ can be written in the following form.

LEMMA 3.7. *We have that*

$$\delta\widetilde{\Phi}_\omega(0) = \frac{J_{\nu+1}(\omega)}{J_\nu(\omega)} \sum_{\ell\geq 2}\sum_{m=1}^{d_\ell} a_\nu(\ell,\omega) k_{\ell m} Y_{\ell m}, \tag{36}$$

where

$$a_\nu(\ell,\omega) = \frac{\ell-1}{\omega} + \frac{J_{\nu+2}(\omega)}{J_{\nu+1}(\omega)} - \frac{J_{\nu+\ell+1}(\omega)}{J_{\nu+\ell}(\omega)}. \tag{37}$$

Proof. By using (33), we have that

$$I'_\ell = \frac{\ell}{s} I_\ell - I_{\ell+1}.$$

By inserting in (34), we have

$$\begin{aligned}
\delta\widetilde{\Phi}_\omega(0) &= \sum_{\ell\geq 1}\sum_{m=1}^{d_\ell} k_{\ell m}\left(\frac{I_1(\omega)I'_\ell(\omega)}{I_0(\omega)I_\ell(\omega)} - \frac{I'_1(\omega)}{I_0(\omega)}\right) Y_{\ell m} \\
&= \sum_{\ell\geq 1}\sum_{m=1}^{d_\ell} k_{\ell m}\left(\frac{\ell}{\omega} - \frac{I_{\ell+1}(\omega)}{I_\ell(\omega)} - \frac{I'_1(\omega)}{I_0(\omega)}\frac{I_0(\omega)}{I_1(\omega)}\right)\frac{I_1(\omega)}{I_0(\omega)} Y_{\ell m} \\
&= \sum_{\ell\geq 1}\sum_{m=1}^{d_\ell} k_{\ell m}\left(\frac{\ell-1}{\omega} - \frac{I_{\ell+1}(\omega)}{I_\ell(\omega)} + \frac{I_2(\omega)}{I_1(\omega)}\right)\frac{I_1(\omega)}{I_0(\omega)} Y_{\ell m}.
\end{aligned}$$

By observing that $a_\nu(1,\omega) = 0$, and recalling that $I_\ell = s^{-\nu} J_{\nu+\ell}$, we obtain (36). ☐

LEMMA 3.8. *For any $\omega > 0$, there exists a asymptotic series expansion*

$$\frac{J_{\nu+\ell+1}(\omega)}{J_{\nu+\ell}(\omega)} \sim \sum_{j\geq 1} b_j(\nu,\omega)\,\ell^{-j}. \tag{38}$$

Proof. We only give the main ideas of the proof (see [3] and [5] for details). Let F be the function defined by

$$F(z) = J_{\nu+z+1}(\omega)\,/J_{\nu+z}(\omega)\,.$$

From Corollary 5.6 in [5], we obtain a Mittag-Leffler expansion

$$F(z) = \sum_{k=1}^{\infty} \frac{A_k}{z-\zeta_k},$$

where $\zeta_1 > \zeta_2 > \cdots > -\infty$ are the zeros of $J_{\nu+z}(\omega)$ as function of the variable z, and A_k is the residue of F in ζ_k. By Lemmas 1, 2 in [3], we obtain

$$\begin{aligned} \zeta_k &= -\nu - k + O(1/k!)\,, \\ A_k &= \frac{(\omega/2)^{2k-1}}{(k-1)!^2}\,(1+O(1/k))\,. \end{aligned}$$

Let us define $b_j(\nu,\omega) = \sum_{k\geq 1} A_k \zeta_k^{j-1}$. From

$$\frac{A_k}{z-\zeta_k} = \sum_{j=1}^{N} A_k\zeta_k^{j-1} z^{-j} + \frac{A_k\,\zeta_k^N}{z-\zeta_k} z^{-N},$$

we have $F(\ell) = \sum_{j=1}^{N} b_j(\nu,\omega)\,\ell^{-j} + O\left(\ell^{-N-1}\right)$ for any $N \geq 1$, which completes the proof. □

We remark that we can compute the first terms of the asymptotic expansion (38). In fact, by recalling the following relation (see Courant, Hilbert [2], pp. 488)

$$\frac{J_{\nu+\ell+1}(\omega)}{J_{\nu+\ell}(\omega)} = \frac{1}{\frac{2(\nu+\ell+1)}{\omega} - \frac{J_{\nu+\ell+2}(\omega)}{J_{\nu+\ell+1}(\omega)}}, \tag{39}$$

multiplying both sides of the equality by ℓ, taking the limit as $\ell \to +\infty$, and using (38), we obtain

$$b_1 = \frac{\omega}{2}.$$

Similarly, multiplying by ℓ^2, and taking the limit as $\ell \to +\infty$, we obtain $b_2 = -\frac{(\nu+1)\omega}{2}$. Then we can write (39) like

$$\begin{aligned} \frac{J_{\nu+\ell+1}(\omega)}{J_{\nu+\ell}(\omega)} \sim \; & \frac{\omega}{2}\ell^{-1} - \frac{(\nu+1)\omega}{2}\ell^{-2} + \frac{4(\nu+1)^2\omega + \omega^3}{8}\ell^{-3} \\ & - \frac{4(\nu+1)^3\omega + (3\nu+4)\omega^3}{8}\ell^{-4} + \cdots . \end{aligned} \tag{40}$$

Now we give the following

Definition 3.9. *A value $\omega_* \in \mathbb{R}^+$ is said to be resonant if, for some $\ell \geq 2$, it verifies*

$$a_\nu(\ell, \omega_*) = 0,$$

where $a_\nu(\ell, \omega)$ is defined in (37). Let us define by Λ the set of resonant values.

In the next lemma we prove some properties of the set Λ. More precisely we have the following

Lemma 3.10. *Λ is a enumerable set of $\mathbb{R}^+$, whose limit points are the values $\omega = \lambda_{1m}$, for some $m \geq 1$.*

Proof. From the definition of the set Λ, we can write $\Lambda = \bigcup_{\ell \geq 2} \Lambda_\ell$, where

$$\Lambda_\ell = \{\omega_* > 0; a_\nu(\ell, \omega_*) = 0\}.$$

By using that $a_\nu(\ell, \omega)$ are meromorphic functions of ω, we have that the set

$$[\eta^{-1}, \eta] \cap \Lambda_\ell$$

is finite for any $\ell \geq 2$, and $\eta \geq 1$. Thus, the set Λ is enumerable. Since the $\lim_{\ell \to +\infty} \lambda_{\ell 1} = +\infty$, the function $a_\nu(\ell, \omega)$ has no poles in the interval $(\lambda_{1m}, \lambda_{1(m+1)})$, for ℓ large enough. Using (40), we obtain

$$\lim_{\ell \to +\infty} \frac{\ell J_{\nu+\ell+1}(\omega)}{J_{\nu+\ell}(\omega)} = \frac{\omega}{2},$$

uniformly in the interval $(\lambda_{1m}, \lambda_{1(m+1)})$. Therefore, for any $\epsilon > 0$, there exists $\ell^* \geq 1$ such that $[\lambda_{1m} + \epsilon, \lambda_{1(m+1)} - \epsilon] \cap \Lambda_\ell = \emptyset$, for any $\ell > \ell^*$. It is easy to see that

$$\Lambda \cap [\lambda_{1m} + \epsilon, \lambda_{1(m+1)} - \epsilon\ ] = \bigcup_{2 \leq \ell \leq \ell^*} [\lambda_{1m} + \epsilon, \lambda_{\ 1(m+1)} - \epsilon] \cap \Lambda_\ell$$

is a finite set. It implies that if ω is a limit point, then $\omega \in \{\lambda_{1m} : m \geq 1\}$. For $\ell > \ell^*$, the function $a_\nu\,(\ell, \omega)$ is continuous in $(\lambda_{1m}, \lambda_{1(m+1)})$, and

$$\lim_{\omega \to \lambda_{1m}^+} a_\nu\,(\ell, \omega) = -\infty, \qquad \lim_{\omega \to \lambda_{1m}^-} a_\nu\,(\ell, \omega) = +\infty.$$

Then there exists $\xi_\ell \in (\lambda_{1m}, \lambda_{1(m+1)})$ such that $a_\nu(\ell, \xi_\ell) = 0$, and $\lim_{\ell \to +\infty} \xi_\ell = \lambda_{1m}$. Hence λ_{1m} is a limit point of the set Λ. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\omega^2 \neq (\lambda_n)_{n \geq 1}$, and let $\omega \notin \Lambda$. Since the kernel of the operator $\mathrm{d}\Phi_\omega\,(0)$ coincides with the one of $\mathrm{d}\widetilde{\Phi}_\omega\,(0)$, and

$$a_\nu(1, \omega) = 0,$$

for all $\omega > 0$, we have that the kernel $\ker \mathrm{d}\Phi_\omega\,(0)$ is given by the functions k which have frequency one, i.e.

$$\ker \mathrm{d}\Phi_\omega\,(0) = \left\{ k \in E; k = \sum_{m=1}^{n} k_{1m} Y_{1m} \right\} \cup \{0\}\,.$$

We have that the space E_0 is orthogonal to the space $\ker \mathrm{d}\Phi(0)$, and the operator $\mathrm{d}\Phi_\omega\,(0)$ is injective in E_0. In order to prove the assertion of the theorem (we observe that the image of the operator $\mathrm{d}\Phi_\omega(0) \subseteq F_0$), for $f \in F_0$, we ask if there exists a $k \in E$ such that

$$-\frac{I_1'(\omega)}{I_0(\omega)} k + \partial_{\mathbf{n}} \widetilde{k} = f \quad \text{on } \partial B_1,$$

where $\widetilde{k}$ solves (28). Now let $\overline{k} \in C^{2,\alpha}(\overline{B_1})$ solve

$$\begin{cases} \Delta \overline{k} + \omega^2 \overline{k} & = \ 0 \quad \text{in } B_1, \\ -\frac{\omega I_1'(\omega)}{I_1(\omega)} \overline{k} + \partial_{\mathbf{n}} \overline{k} & = \ f \quad \text{on } \partial B_1. \end{cases}$$

Denoting by $k = \frac{\omega I_0(\omega)}{I_1(\omega)}\overline{k}$ on ∂B_1, we have that $\widetilde{k}$ solves

$$\begin{cases} \Delta\widetilde{k} + \omega^2\widetilde{k} &= 0 \quad \text{in } B_1, \\ \widetilde{k} &= \overline{k} \quad \text{on } \partial B_1. \end{cases}$$

Then we have that $\widetilde{k} = \overline{k}$ in $\overline{B_1}$. So we obtain

$$-\frac{I_1'(\omega)}{I_0(\omega)}k + \partial_{\mathbf{n}}\widetilde{k} = -\frac{\omega I_1'(\omega)}{I_1(\omega)}\overline{k} + \partial_{\mathbf{n}}\overline{k} = f.$$

The proof of Theorem 3.1 is complete. □

We observe that for $\omega_* \in \Lambda$, the kernel of the operator $\mathrm{d}\Phi_{\omega_*}(0)$ is given by

$$\ker \mathrm{d}\Phi_{\omega_*}(0) = \left\{ k \in E; k = \sum_{\ell=1, \ell\in I} \sum_{m=1}^{d_\ell} k_{\ell m} Y_{\ell m} \right\} \cup \{0\}, \tag{41}$$

where I is a finite set of positive integers $i \geq 3$.

4. The Lipschitz Case

In this section we study the case where the domain Ω in (1) is of Lipschitz class $C^{0,1}$. More precisely let us define by

$$E = \left\{ k \in C^{0,1}(\partial B_1) \right\},$$

where $C^{0,1}(\partial B_1)$ denotes the restriction on ∂B_1 of functions of Lipschitz class $C^{0,1}$ in $\overline{B_1}$. For $k \in \mathcal{U}$, by well-known results of elliptic boundary value problems, we have that there exists a unique weak solution $u \in H_0^1(\Omega_k)$ to (1), when $\Omega = \Omega_k$. By the trace embedding, we have that $\partial_{\mathbf{n}} u \in H^{-1/2}(\partial\Omega_k)$. The operator Φ_ω is then defined as

$$\Phi_\omega : \mathcal{U} \mapsto F,$$

where F is the space

$$F = \left\{ f \in H^{-1/2}(\partial B_1) \right\}.$$

Let E_0 and F_0 be the vector spaces defined in (7) and (8) respectively. The main result of the present section is the following theorem, which is the analogous for Lipschitz domains to Theorem 3.1.

THEOREM 4.1. *Under the hypothesis of Theorem 1.1, the operator* $d\Phi_\omega(0)$ *is an isomorphism from* E_0 *into* F_0.

In analogy to Lemma 3.2, we have the following

LEMMA 4.2. *There exists a neighborhood* $\mathcal{U}$ *of the origin in* E *such that the operator* $\Phi_\omega \in C^1(\mathcal{U}, F)$.

Proof. We have that in this case the operator $L(k)$, defined in (13), becomes

$$L(k) : H_0^1(B_1) \mapsto H^{-1}(B_1).$$

Similarly we have that the matrix $G \in C^1(E, L^\infty(B_1, \mathbb{R}^{n\times n}))$. By repeating the same arguments of the regular case, it follows that the operator L is a continuously differentiable map from $\mathcal{U}$ to $\mathcal{L}(H_0^1(B_1), H^{-1}(B_1))$. Assuming that ω^2 is not a eigenvalue, $\Delta+\omega^2$ is a isomorphism, and then, reducing $\mathcal{U}$ if it is necessary, $(L(\cdot)+\omega^2)^{-1}$ is a continuously differentiable map from $\mathcal{U}$ to $\mathcal{L}(H^{-1}(B_1), H_0^1(B_1))$. We note that

$$u(k) = -(L(k) + \omega^2)^{-1}1.$$

We consider the map T of class C^1 from $\mathcal{U}$ to $\mathcal{L}(H_0^1(B_1), F)$, defined by

$$T(k)\cdot = \frac{G^{-1}(k)\nabla\cdot\cdot x}{\sqrt{G^{-1}(k)x\cdot x}}.$$

Writing $\Phi_\omega(k) = -T(k)\left(L(k)+\omega^2\right)^{-1}1$, we obtain the result. □

LEMMA 4.3. *We have that*

$$\delta\Phi_\omega(0) = \partial_{\mathbf{n}}u_1 - \partial_{\mathbf{n}}u_0(k + \partial_{\mathbf{n}}k) \quad \text{in } H^{-1/2}(\partial B_1), \tag{42}$$

where $u_1 \in H_0^1(B_1)$ *solves (15) in weak sense.*

Proof. Let $u \in H_0^1(\Omega_k)$ solve (1) in weak sense. Then we have that

$$\int_{\Omega_k} \nabla u\cdot\nabla\phi - \omega^2\int_{\Omega_k} u\phi = \int_{\Omega_k}\phi, \tag{43}$$

for all $\phi \in C_c^\infty(\Omega_k)$. By changing the coordinates, where $x = (1 + k(y))y$, denoting $\widetilde{u}(k)(y) = u((1+k)y)$, and $\widetilde{\phi}(k)(y) = \phi((1+k)y)$, we obtain, from (43), in the new coordinates y, that

$$\int_{B_1} G^{-1}\nabla u \cdot \nabla\phi\sqrt{g} - \omega^2 \int_{B_1} u\phi\sqrt{g} = \int_{B_1} \phi\sqrt{g},$$

for all $\phi \in C_c^\infty(B_1)$ (since $\nabla u = (A^T)^{-1}\nabla\widetilde{u}$, and similarly for $\nabla\phi$. We have denoted $\widetilde{u}$ and $\widetilde{\phi}$ by u and ϕ respectively). By repeating the same arguments of the regular case, we have that

$$\Phi_\omega(k) - \Phi_\omega(0) = \partial_n w - \partial_n u_0(k + \partial_n k) + o\big(\, \|k\|_{H^1(B_1)} \,\big),$$

where $w = u - u_0$ solves (24) in weak sense, i.e.

$$\int_{B_1} \nabla w \cdot \nabla\phi - \omega^2 \int_{B_1} w\phi$$
$$= \int_{B_1} \sqrt{g}(\omega^2 u + 1)\phi - \int_{B_1} K\nabla u \cdot \nabla\phi - \omega^2 \int_{B_1} u\phi - \int_{B_1} \phi$$

(we recall that the entries K_{ij} of the matrix K (which are functions of k and ∇k) are in $L^\infty(B_1)$). We have that the right hand side can be written as

$$\int_{B_1} (nk + x\cdot\nabla k)(1 + \omega^2 u_0)\phi - \int_{B_1} K\nabla(w + u_0)\cdot\nabla\phi + o\big(\, \|k\|_{H^1(B_1)} \,\big).$$

So we have that $u_1 \in H_0^1(B_1)$ solves

$$\int_{B_1} \nabla u_1 \cdot \nabla\phi - \omega^2 \int_{B_1} u_1\phi \tag{44}$$
$$= \int_{B_1} (nk + x\cdot\nabla k)(1 + \omega^2 u_0)\phi - \int_{B_1} K_1\nabla u_0 \cdot \nabla\phi$$

(i.e. u_1 solves (15) in weak sense), where the matrix K_1, the linear part of K, is given in (26). By repeating the same arguments to proving Lemma 3.3, we obtain (42). □

We observe that, in analogy to Lemma 3.4, the solution u_1 to (44) can be written as

$$u_1 = \frac{I_0'(\omega r)}{\omega I_0(\omega)} rk + \widetilde{k},$$

where $\widetilde{k} \in H^1(B_1)$ solves (28) in weak sense. The proof of Theorem 4.1 follows by using the same arguments of the proof of Theorem 3.1, and then it is omitted.

5. Proof of Theorem 1.1

For $\omega^2 \neq (\lambda_n)_{n\geq 1}$, and $\omega \notin \Lambda$, we have that the operator $\mathrm{d}\Phi_\omega(0)$ is an isomorphism from E_0 into F_0 (we recall that E_0 and F_0 are subspaces of E and F respectively, whose functions don't have the frequency 1). Now consider the following operator defined by

$$\Psi_\omega(k) = \Phi_\omega(k) + \sum_{m=1}^{n} k_{1m} Y_{1m}, \tag{45}$$

where $k_{1m} = \int_{\partial B_1} k Y_{1m}$ are the first-order Fourier-coefficients of k. We prove that the operator Ψ_ω is bijective from a neighborhood of 0 in E into a neighborhood of c in F. More precisely we have the following

THEOREM 5.1. *Under the hypothesis of Theorem 1.1, there exists a neighborhood $\mathcal{U}$ of 0 in E and a neighborhood $\mathcal{V}$ of c in F, such that the operator Ψ_ω is bijective from $\mathcal{U}$ into $\mathcal{V}$. In particular Φ_ω is injective in $E_0 \cap \mathcal{U}$.*

Proof. By Lemma 3.1 we have that the operator Ψ_ω is continuously differentiable in $\mathcal{U}$. We have that

$$< \mathrm{d}\Psi_\omega(0), k > = < \mathrm{d}\Phi_\omega(0), k > + \sum_{m=1}^{n} k_{1m} Y_{1m}.$$

By Theorem 3.1 we have that $\mathrm{d}\Psi_\omega(0)$ is an isomorphism from E into F. So by the inverse function's theorem we have that there exists a neighborhood $\mathcal{U}$ of 0 in E and a neighborhood $\mathcal{V}$ of c in F such that the operator Ψ_ω is bijective from $\mathcal{U}$ into $\mathcal{V}$. Now by (45) we have that $\Psi_{\omega|E_0} = \Phi_{\omega|E_0}$. Since Ψ_ω is bijective, it follows that Φ_ω is injective in $E_0 \cap \mathcal{U}$. □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We recall that the sphere of radius one, centered at the point $x_0 \in \mathbb{R}^n$, is parametrized by

$$\partial B_1(x_0) = \{x = (1 + k_0(y))y, y \in \partial B_1\},$$

where k_0 is given by

$$k_0(y) = x_0 \cdot y - 1 + \sqrt{1 + (x_0 \cdot y)^2 - |x_0|^2}.$$

Let $k \in \mathcal{U}$ be such that the $\Phi_\omega(k) = c$. Two cases can happen, either

(i) $k \in E_0$,

or

(ii) $k \in E_0^C$,

where E_0^C denotes the complementary of E_0, i.e. the set of functions k which have the frequency one. If case (i) occurs we have that $k \equiv 0$, since Φ_ω is injective in $E_0 \cap \mathcal{U}$, and $\Phi_\omega(0) = c$. If case (ii) occurs, we have that $\Psi_\omega(k) = c + \sum_{m=1}^{n} k_{1m} Y_{1m}$. Let us choose the center of the ball at the point $x_0 = (k_{11}, \cdots, k_{1n})$. We recall that the first-order Fourier-coefficients of k_0 are equal to x_0. So we have that $\Psi_\omega(k_0) = c + \sum_{m=1}^{n} k_{1m} Y_{1m}$. Since Ψ_ω is bijective, we have that $k = k_0$, i.e. $\partial\Omega_k$ is the circle centered at the point k_1 of radius one. The proof of Theorem 1.1 is complete. □

We conclude the paper by observing that if $\omega_* \in \Lambda$, then the orthogonal of the $\ker \mathrm{d}\Phi_{\omega_*}(0)$, defined in (41), is given by

$$E_{0*} = \left\{ k \in E; k = \sum_{\ell \neq 1, \notin I} \sum_{m=1}^{d_\ell} k_{\ell m} Y_{\ell m} \right\}.$$

Now, similarly to the case $\omega \notin \Lambda$, we can define the following operator $\Psi_{\omega_*}(k) = \Phi_{\omega_*}(k) + \sum_{\ell=1, \in I} \sum_{m=1}^{d_\ell} k_{\ell m} Y_{\ell m}$, and, by using the same argument of the proof of Theorem 5.1, we obtain that the operator Φ_{ω_*} is injective in $E_{0*} \cap \mathcal{U}$.

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Authors' addresses:

Bruno Canuto
Dpto. de Matemática, FCEyN, Univ. de Buenos Aires, Buenos Aires, Argentina
E-mail: bcanuto@hotmail.it

Diego Rial
Dpto. de Matemática, FCEyN, Univ. de Buenos Aires, Buenos Aires, Argentina
E-mail: drial@dm.uba.ar

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