EQUICONTINUITY AND UNIFORM BOUNDEDNESS (*)

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SOMMARIO. - *Una formulazione del Principio di uniforme limitatezza asserisce che se una famiglia di operatori lineari e continui, definiti su un B-spazio, e a valori in uno spazio normato, è limitata puntualmente, allora la famiglia è equicontinua. Semplici esempi mostrano che questa conclusione è falsa senza qualche ipotesi sul dominio di definizione. Proveremo una generalizzazione di questa formulazione del Principio di uniforme limitatezza, la quale sussiste senza alcuna ipotesi sul dominio di definizione. Mostreremo che esiste una topologia localmente connessa sul dominio di definizione con la proprietà che ogni famiglia di operatori lineari e continui limitata puntualmente è equicontinua rispetto a tale topologia.*

SUMMARY. - *One form of the Uniform Boundedness Principle asserts that if a family of continuous linear operators from a B-space into a normed space is pointwise bounded, then the family is equicontinuous. Simple examples show that this conclusion is false without some type of assumption on the domain space. We establish a generalization of this form of the Uniform Boundedness Principle which holds without any assumptions on the domain space. We show that there is a locally convex topology on the domain space with the property that any pointwise bounded family of continuous linear operators is equicontinuous with respect to this topology.*

The classical uniform boundedness principle (UBP) for normed spaces asserts that a family of continuous linear operators from a Banach space into a normed space which is pointwise bounded on the domain space is uniformly bounded on bounded subsets of the domain space. Simple examples show that this statement is false if there are no completeness or barrelledness assumptions on the domain space, i.e., in the absence of some type of completeness on the domain space the family of bounded subsets is

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too large a class of sets to insure that a pointwise bounded family of continuous linear operators is uniformly bounded on each member of this family of sets ([1] §4). In order to overcome this difficulty, P. Antosik introduced a subfamily of the bounded sets, the $\mathcal{K}$ bounded sets, and showed that the family of $\mathcal{K}$ bounded sets has the property that a pointwise bounded family of continuous linear operators is always uniformly bounded on $\mathcal{K}$ bounded subsets of the domain space ([1]). Further refinements of this general UBP are given in [2] and [3] where $\mathcal{K}$ bounded sets in weaker topologies than the original topology of the domain space are utilized.

The conclusion of uniform boundedness on bounded sets in the classical UBP for normed spaces is equivalent to the condition that the family of operators is equicontinuous with respect to the original topology of the domain space. In [7], S. Mazur and W. Orlicz generalized the UBP from normed spaces to metric linear spaces by obtaining this equicontinuity condition as the conclusion of their UBP; this is also the conclusion in the barrelled space version of the UBP of Bourbaki ([4]). In general, equicontinuity of a family of continuous linear operators implies that the family is uniformly bounded on bounded sets, but not conversely (Example 8 below). However, we show below in Theorem 2 that if a family of continuous linear operators is uniformly bounded on bounded sets, while it may not be equicontinuous with respect to the original topology of the domain space, it is equicontinuous with respect to a stronger topology on the domain, the bornological topology (see Remark 4). Indeed, we show that an analogous statement is valid for any subfamily of the bounded sets. In particular, it follows from our results that if a family of continuous linear operators between locally convex spaces is pointwise bounded, then it is always equicontinuous with respect to a locally convex topology on the domain space which is stronger than the original topology. This gives a general UBP much in the spirit of the early UBP of Mazur and Orlicz for metric linear spaces.

We first describe the topologies which will be employed. Throughout the sequel let (X, τ) be a Hausdorff locally convex (l.c.) space. Let $\mathcal{B}$ be the family of all τ bounded subsets of X. For any subfamily $\mathcal{A}$ of $\mathcal{B}$ with the property that $\bigcup_{A\in\mathcal{A}} A = X$ consider the family of subsets V of X satisfying the condition:

(#) $V \subseteq X$ is such that V is absolutely convex and absorbs every member of $\mathcal{A}$.

The family of subsets satisfying condition (#) forms a local basis for a l.c. topology, $\tau^{\mathcal{A}}$, on X ([9] 4.3.5). In particular, if $\mathcal{A} = \mathcal{B}$, then $\tau^{\mathcal{B}} = \tau^b$ is just the bornological topology of X; τ^b is the largest l.c. topology which has the same bounded sets as (X, τ) ([9] p. 117, [5] 28.2).

PROPOSITION 1. *(i) If $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{B}$, then $\tau^{\mathcal{A}} \supseteq \tau^{\mathcal{C}} \supseteq \tau^b$. (ii) $\tau^{\mathcal{A}}$ is the strongest l.c. topology on X such that all members of $\mathcal{A}$ are bounded.*

We next describe our equicontinuity result using the topology $\tau^{\mathcal{A}}$. Let Y be a l.c. space, and let $\mathcal{L}(X,Y)\,[\,L(X,Y)\,]$ be the space of linear [continuous] maps from X into Y.

THEOREM 2. *Let $\mathcal{F} \subseteq \mathcal{L}(X,Y)$. The following are equivalent:*
(1) $\mathcal{F}$ is uniformly bounded on the members of $\mathcal{A}$,
(2) $\mathcal{F}$ is $\tau^{\mathcal{A}}$ equicontinuous.

Proof: Suppose (1) holds. Let V be an absolutely convex neighbourhood of 0 in Y. Set $U = \bigcap_{T \in \mathcal{F}} T^{-1}V$. Then U is absolutely convex and absorbs every member of $\mathcal{A}$. Hence, U is a $\tau^{\mathcal{A}}$ neighborhood of 0, and $\mathcal{F}$ is $\tau^{\mathcal{A}}$ equicontinuous.

Assume (2) holds. Let $A \in \mathcal{A}$ and V be a neighborhood of 0 in Y. There exists a $\tau^{\mathcal{A}}$ basic neighborhood of 0, U, such that $TU \subseteq V$ for all $T \in \mathcal{F}$. There exists $t > 0$ such that $A \subseteq tU$. Hence $TA \subseteq tTU \subseteq tV$ for all $T \in \mathcal{F}$ and (1) holds.

REMARK 3. $\tau^{\mathcal{A}}$ is the weakest l.c. topology on X such that (1) implies (2) for every l.c. space Y and for all families $\mathcal{F} \subseteq \mathcal{L}(X,Y)$. For suppose that τ' satisfies this condition. Consider the identity map $I : X \to (X, \tau^{\mathcal{A}})$. Then $\{I\}$ is uniformly bounded on $\mathcal{A}$ since each $A \in \mathcal{A}$ is $\tau^{\mathcal{A}}$ bounded. Therefore, (1) holds and (2) for the family $\{I\}$ implies that I is $\tau' - \tau^{\mathcal{A}}$ continuous or $\tau' \supseteq \tau^{\mathcal{A}}$. This shows that in some sense, the topology $\tau^{\mathcal{A}}$ is the natural topology to use in Theorem 2.

REMARK 4. If $\mathcal{A} = \mathcal{B}$ in Theorem 2, this means that equicontinuity in (2) is with respect to the bornological topology τ^b. In particular, this means that if (X, τ) is bornological (so $\tau = \tau^b$) and $\mathcal{F} \subseteq L(X, Y)$, then $\mathcal{F}$ is τ equicontinuous if and only if $\mathcal{F}$ is uniformly bounded on bounded subsets of X. Actually, the proof of Theorem 2 shows that this statement holds for a wider class of spaces than the bornological spaces. Namely, we have

PROPOSITION 5. *Let (X, τ) be quasi-barrelled and $\mathcal{F} \subseteq L(X, Y)$. The following are equivalent:*
(3) *$\mathcal{F}$ is uniformly bounded on $\mathcal{B}$,*
(4) *$\mathcal{F}$ is τ equicontinuous.*

Proof. (4) implies (3) by the proof of (2) implies (1) above.

Also, (3) implies (4) by the proof of (1) implies (2) since we can take V to be a closed neighborhood of 0 in Y and then U is τ closed by the continuity of the elements in $\mathcal{F}$. U is therefore a τ neighborhood of 0 by the quasi-barrel assumption ([9] 10.1.7).

In particular, Proposition 5 applies to both barrelled and bornological spaces.

This result was also established in [6] (see also [8] Proposition 7). The quasi-barrelled spaces form the largest class of l.c. spaces for which (3) and (4) are equivalent since if Y is the scalar field, then (3) and (4) are equivalent if and only if X is quasi-barrelled ([9] 10.1.11).

Employing Theorem 2, we can now obtain equicontinuity versions of the UBP which require no completeness or barrelledness assumptions on the domain space. We recall the general UBP's which use $\mathcal{K}$ bounded sets. A sequence $\{x_k\}$ in a topological vector space (E, α) is $\alpha - \mathcal{K}$ convergent if every subsequence of $\{x_k\}$ has a further subsequence $\{x_{n_k}\}$ such that the series $\sum x_{n_k}$ is α convergent in E, and a subset $A \subseteq E$ is $\alpha - \mathcal{K}$ bounded if $\{t_k x_k\}$ is $\alpha - \mathcal{K}$ convergent for each sequence $\{x_k\} \subseteq A$ and scalar sequence $t_k \to 0$ ([1]; see [3] for the basic properties of $\mathcal{K}$ convergent sequences and $\mathcal{K}$ bounded sets). Let $\sigma(L(X, Y))$ be the weakest topology on X such that all of the elements of $L(X, Y)$ are continuous ([3] §4) and let $\sigma(X, X')$ be the weak topology of X.

In [3], Corollary 3, and [8], Corollary 5, it is shown that any pointwise bounded family of continuous linear operators from X into Y is uniformly bounded on the families of $\sigma(L(X, Y)) - \mathcal{K}$ bounded sets and

$\sigma(X, X') - \mathcal{K}$ bounded sets. [The UBP's in §4 of [3] are stated for metric linear spaces but the proofs are valid for l.c. spaces by considering the semi-norms which generate the l.c. topology.] From these results and Theorem 2, we obtain the following equicontinuity version of the UBP. Let $\mathcal{K}_1, [\mathcal{K}_2]$ be the family of $\sigma(L(X,Y)) - \mathcal{K}[\sigma(X,X') - \mathcal{K}]$ bounded subsets of X.

THEOREM 6. *If $\mathcal{F} \subseteq L(X,Y)$ is pointwise bounded on X, then $\mathcal{F}$ is $\tau^{\mathcal{K}_1}$ and $\tau^{\mathcal{K}_2}$ equicontinuous.*

In particular, if $\mathcal{K}$ is the class of $\mathcal{K}$ bounded subsets of X with respect to the original topology of X, then $\mathcal{F}$ in Theorem 6 is equicontinuous with respect to $\tau^{\mathcal{K}}$ since both topologies $\sigma(L(X,Y))$ and $\sigma(X,X')$ are weaker than the original topology of X.

A topological vector space (E, α) is an $\mathcal{A}$-space if every α bounded set is $\alpha - \mathcal{K}$ bounded ([6]). Recall that (E, α) is a $\mathcal{K}$-space if every sequence which is α convergent to 0 is $\alpha - \mathcal{K}$ convergent ([3] §3). A $\mathcal{K}$-space is obviously an $\mathcal{A}$-space, but there are examples of $\mathcal{A}$-spaces which are not $\mathcal{K}$-spaces (see Example 8 below). Reference [6] contains many examples of such $\mathcal{A}$-spaces. From the observation above and Remark 4, we have the following equicontinuity version of the UBP for $\mathcal{A}$-spaces.

COROLLARY 7. *Let (X, τ) be an $\mathcal{A}$-space. If $\mathcal{F} \subseteq L(X,Y)$ is pointwise bounded, then $\mathcal{F}$ is τ^b equicontinuous.*

The following example shows that the family $\mathcal{F}$ in Corollary 7 may fail to be equicontinuous with respect to the original topology of X even when X is an $\mathcal{A}$-space.

EXAMPLE 8. ℓ^2 with the weak topology $\sigma(\ell^2, \ell^2)$ is an $\mathcal{A}$-space [if $\{x_k\}$ is weakly bounded and $t_k \to 0$, then given any subsequence of $\{t_k\}$ pick a subsequence $\{t_{n_k}\}$ such that $\sum_{k=1}^{\infty} |t_{n_k}| < \infty$ and then the subseries $\sum_{k=1}^{\infty} t_{n_k} x_{n_k}$ is weakly convergent by the weak sequential completeness of ℓ^2]. If e_k is the sequence with a 1 in the k^{th} coordinate and 0 elsewhere, then $\{e_k : k = 1, 2, \ldots\} = \mathcal{F}$ pointwise bounded on ℓ^2 but is not equicontinuous with respect to the weak topology [$e_k \to 0$ in $\sigma(\ell^2, \ell^2)$ but $\langle e_k, e_k \rangle = 1$ for each k]. Thus, $\mathcal{F}$ is not equicontinuous with respect to the original topology but is equicontinuous with respect to $\sigma(\ell^2, \ell^2)^b = || \, ||_2$.

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