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Explicit Parallelizations on Products of Spheres and Calabi-Eckmann Structures

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Summary. - *A classical theorem of Kervaire states that products of spheres are parallelizable if and only if at least one of the factors has odd dimension. We give explicit parallelizations. We show that the Calabi-Eckmann Hermitian structures on products of two odd-dimensional spheres are invariant with respect to these parallelizations.*

1. Introduction

It is a classical result in Algebraic Topology that spheres S^n are parallelizable only in dimension $n = 1$, 3 or 7. As for the products of two or more spheres Kervaire proved in the fifties the following (see [3]):

Theorem 1.1 (Kervaire). *The manifold $S^{n_1} \times \cdots \times S^{n_r}$, $r \geq 2$, is parallelizable if and only if at least one of the n_i is odd.*

In his article [1] Bruni provides explicit parallelizations on some products of spheres, namely, whenever one of the factors is S^1, S^3, S^5 or S^7. The general case is left as an open problem.

We solve this problem by writing down a set of orthonormal vector fields on products of two spheres, in terms of their standard

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coordinates as submanifolds of Euclidean spaces. This construction can be recursively repeated to obtain an explicit orthonormal parallelization on products of any number of spheres.

A parallelized manifold $(M, \mathcal{P})$ shares with Lie groups the property of possessing privileged finite-dimensional spaces of tensors, that is, those which are invariant with respect to the parallelization $\mathcal{P}$. In the case of two odd-dimensional spheres we consider the standard almost-Hermitian structure, and prove that it is integrable: in fact, it coincides with a Calabi-Eckmann structure.

2. The explicit parallelization

The construction of the frame is based on formula (1), where ε^k is the trivial rank k vector bundle, $\times$ denotes the cartesian product and $\oplus$ the Whitney sum of vector bundles. The equivalence sign means "isomorphic in the C^∞ category", and the proof is straightforward, see for instance [4].

$$\alpha \times (\beta \oplus \varepsilon^k) \simeq (\alpha \oplus \varepsilon^k) \times \beta. \tag{1}$$

Here and henceforth, let m and n be positive integers, and let n be odd. Denote by $x = (x_1, \ldots, x_{m+1})$, $y = (y_1, \ldots, y_{n+1})$ the coordinates of S^m, S^n respectively. It is convenient in the following to think of $T(S^m \times S^n)$ as a Riemannian subbundle of $T\mathbb{R}^{m+1}_{|S^m} \times T\mathbb{R}^{n+1}_{|S^n}$. Denote by $\{\partial_{x_1}, \ldots, \partial_{x_{m+1}}, \partial_{y_1}, \ldots, \partial_{y_{n+1}}\}$ the orthonormal frame of $T\mathbb{R}^{m+1}_{|S^m} \times T\mathbb{R}^{n+1}_{|S^n}$.

Since n is odd, the multiplication by i defines a length 1 vector field T on $S^n \subset \mathbb{R}^{n+1} = \mathbb{C}^{(n+1)/2}$, and an orthogonal splitting $T(S^n) = \eta \oplus \langle T \rangle$.

The following argument gives an elementary isomorphism ϕ between $T(S^m \times S^n)$ and $\varepsilon^{m-1} \times \varepsilon^{n+1}$, as was pointed out by Staples in [5].

Split $T(S^n)$ in $\eta \oplus \langle T \rangle$, then consider it as a subbundle of $T(S^m) \times T(S^n)$ and use formula (1) to shift on the left the trivial summand. Since $T(S^m) \oplus \varepsilon^1$ is a trivial vector bundle, a rank 2 trivial summand can be shifted on the right, again using formula (1). Now remark that $\eta \oplus \varepsilon^2$ is trivial to obtain the trivial factor ε^{n+1}.

In the above construction, choose $\langle \partial_{x_m}, \partial_{x_{m+1}} \rangle$ as rank 2 trivial summand to be shifted on the right. This way, the frame

$$\{\partial_{x_1}, \ldots, \partial_{x_{m-1}}, \partial_{y_1}, \ldots, \partial_{y_{n+1}}\}$$

is a trivialization of $\varepsilon^{m-1} \times \varepsilon^{n+1}$. Now, define the parallelization $\mathcal{P}$ as the pull-back of this trivialization by means of the isomorphism ϕ:

$$\mathcal{P} \stackrel{\text{def}}{=} \phi_*^{-1}\{\partial_{x_1}, \ldots, \partial_{x_{m-1}}, \partial_{y_1}, \dot{s}, \partial_{y_{n+1}}\}$$

In order to provide formulas for $\mathcal{P}$ we introduce the following notation:

$$M_i \stackrel{\text{def}}{=} \text{orthogonal projection of } \partial_{x_i} \text{ on } S^m \qquad i = 1, \ldots, m+1,$$
$$N_j \stackrel{\text{def}}{=} \text{orthogonal projection of } \partial_{y_j} \text{ on } S^n \qquad j = 1, \ldots, n+1.$$

and remark that

$$M_i = \partial_{x_i} - x_i M \qquad i = 1, \ldots, m+1,$$
$$N_j = \partial_{y_j} - y_j N \qquad j = 1, \ldots, n+1,$$

where M and N denote the normal versor field of $S^m \subset \mathbb{R}^{m+1}$ and $S^n \subset \mathbb{R}^{n+1}$ respectively. Finally, denote by $\{t_j\}$ the coordinates of T:

$$T = \sum_{j=1}^{n+1} t_j \partial_{y_j} \stackrel{\text{def}}{=} -y_2\partial_{y_1} + y_1\partial_{y_2} + \cdots - y_{n+1}\partial_{y_n} + y_n\partial_{y_{n+1}}.$$

A direct computation then proves the following:

Theorem 2.1. *The frame $\mathcal{P}$ on $S^m \times S^n$, for any odd n, is composed by the vector fields*
$\{p_1, \ldots, p_{m+n}\} \in \mathfrak{X}(S^m \times S^n)$ *given by*

$$\begin{aligned} p_i &\stackrel{\text{def}}{=} M_i + x_i T \qquad i = 1, \ldots, m-1, \\ p_{m-1+j} &\stackrel{\text{def}}{=} y_j M_m + t_j M_{m+1} + (t_j x_{m+1} + y_j x_m - t_j)T + N_j \\ &\qquad\qquad j = 1, \ldots, n+1. \end{aligned} \tag{2}$$

Remark 2.2. The frame $\mathcal{P}$ is orthonormal with respect to the product metric on $S^m \times S^n$, as one can check using previous theorem.

REMARK 2.3. Formula (2) can be used as a direct definition of $\mathcal{P}$. In this case, Remark 2.2 becomes a Proposition stating that $\mathcal{P}$ is orthonormal with respect to the product metric on $S^m \times S^n$.

REMARK 2.4. In the particular case $n = 1$ one can naturally define a simpler parallelization. Denote by Γ the infinite cyclic group generated by multiplication by $e^{2\pi}$ in $\mathbb{R}^{m+1} \setminus 0$. The vector fields $\{|x|\partial_{x_i}\}_{i=1,\dots,m+1}$ are Γ-equivariant, whence they define a parallelization $\mathcal{B}$ on $(\mathbb{R}^{m+1} \setminus 0)/\Gamma$, which is well-known to be diffeomorphic to $S^m \times S^1$ by

$$\begin{array}{ccc} (\mathbb{R}^{m+1} \setminus 0)/\Gamma & \longrightarrow & S^m \times S^1 \\ [x] & \longmapsto & (x/|x|, \log|x| \mod 2\pi). \end{array} \tag{3}$$

Using the above map, one obtains that $\mathcal{B} = \{b_1, \dots, b_{m+1}\}$ where

$$b_i = M_i + x_i T, \qquad i = 1, \dots, m+1. \tag{4}$$

REMARK 2.5. To obtain a parallelization in the general case, use induction in the following way: suppose that $S^{n_2} \times \dots \times S^{n_r}$, $r \geq 2$, has at least one odd-dimensional factor, whence it is parallelizable. Then using formula (1) one obtains

$$\begin{aligned} T(S^{n_1} \times \dots \times S^{n_r}) &= T(S^{n_1}) \times \varepsilon^{n_2+\dots+n_r} \\ &= (T(S^{n_1}) \oplus \varepsilon^1) \times \varepsilon^{n_2+\dots+n_r-1} \\ &= \varepsilon^{n_1+1} \times \varepsilon^{n_2+\dots+n_r-1}. \end{aligned}$$

3. Complex structures associated to $\mathcal{P}$

For any even-dimensional parallelized manifold $(M, \mathcal{P})$ denote by $I_{\mathcal{P}}$ the invariant almost-complex structure represented by the unitary matrix

$$I \stackrel{\text{def}}{=} \operatorname{diag} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Let m and n be both odd. On $(S^m \times S^n, \mathcal{P})$ the almost-Hermitian structure $I_{\mathcal{P}}$ is then defined. Moreover, on $(S^m \times S^1, \mathcal{B})$ the almost-Hermitian structure $I_{\mathcal{B}}$ is also defined.

The following is the main theorem of this paper.

THEOREM 3.1. *Let m and n be odd. The almost-Hermitian structure $I_{\mathcal{P}}$ on $(S^m \times S^n, \mathcal{P})$ is integrable.*

Proof. First, look at the simplest case $n = 1$. Recall that the Hopf complex structure on $S^m \times S^1$ is by definition the complex structure induced by the map (3) after the identification $\mathbb{R}^{m+1} \setminus 0 = \mathbb{C}^{(m+1)/2} \setminus 0$. Being $\mathcal{B}$ locally conformal to the standard frame on $\mathbb{C}^{(m+1)/2} \setminus 0$, the almost-complex structure $I_{\mathcal{B}}$ on $(S^m \times S^1, \mathcal{B})$ lifts to the standard complex structure of $\mathbb{C}^{(m+1)/2} \setminus 0$, hence $I_{\mathcal{B}}$ coincides with the Hopf complex structure on $S^m \times S^1$. Moreover, since formulas (2) and (4) imply that $\mathcal{P}$ and $\mathcal{B}$ differ by an element of $\mathrm{GL}((m+1)/2, \mathbb{C})$, we obtain that $I_{\mathcal{P}}$ coincides with the Hopf complex structure on $S^m \times S^1$, and therefore it is integrable.

We need now to recall the complex Hopf fibration. It is by definition the restriction to S^m of the canonical projection $\mathbb{C}^{(m+1)/2} \to \mathbb{CP}^{(m-1)/2}$, and the tangent bundle TS^m is decomposed under this fibration into a horizontal and a vertical subbundle. We denote by $\mathcal{H}$ the horizontal subbundle, whereas the vertical subbundle is spanned by the vector field S induced by the multiplication by i in $\mathbb{C}^{(m+1)/2}$.

The Hopf complex structure on $S^m \times S^1$ turns out to be induced on $\mathcal{H}$ by the complex structure of $\mathbb{CP}^{(m-1)/2}$, whereas S is mapped onto the unitary vector field tangent to S^1.

Using the Hopf fibrations for S^m and S^n, and collecting all these arguments, we obtain the following decomposition for the tangent bundle of $S^m \times S^n$, wherethe horizontal subbundles $\mathcal{H}_1$ and $\mathcal{H}_2$ are closed under $I_{\mathcal{P}}$:

$$T(S^m \times S^n) = \mathcal{H}_1 \oplus \langle S \rangle \oplus \mathcal{H}_2 \oplus \langle T \rangle.$$

To prove the integrability of $I_{\mathcal{P}}$ for all odd n, we consider its torsion tensor N, and we show that $N(X, Y) = 0$ for all X, $Y \in T(S^m \times S^n)$.

First case: X, Y both in $\mathcal{H}_1$ or both in $\mathcal{H}_2$. Then $N(X, Y) = 0$ since $I_{\mathcal{P}}$ is a Hopf complex structure.

Second case: X in $\mathcal{H}_1$ or in $\mathcal{H}_2$, and Y in $\langle S \rangle \oplus \langle T \rangle$. Then $N(X, Y) = 0$ since $I_{\mathcal{P}}$ is a Hopf complex structure.

Third case: X, Y both in $\langle S \rangle \oplus \langle T \rangle$. Then $N(X, Y) = 0$ since $I_{\mathcal{P}}$ is a Hopf complex structure.

Fourth case: X in $\mathcal{H}_1$ and Y in $\mathcal{H}_2$. Then $N(X,Y)=0$ since $\mathcal{H}_1$ and $\mathcal{H}_2$ are closed under $I_{\mathcal{P}}$. □

4. Calabi-Eckmann revisited

We now briefly recall the definition of Calabi-Eckmann complex structure $I^{m,n}$ on $S^m \times S^n$, for odd m and n, as given in the classical paper [2].

Denote by (z_i, z'_j) the complex coordinates of $\mathbb{C}^{(m+1)/2} \times \mathbb{C}^{(n+1)/2}$, and by $V_{\alpha,\beta}$ the open subset of $S^m \times S^n$ given by $z_\alpha z'_\beta \neq 0$. Then the maps

$$\begin{array}{rccl} \phi_{\alpha,\beta}: & V_{\alpha,\beta} & \longrightarrow & \mathbb{C}^{(m-1)/2} \times \mathbb{C}^{(n-1)/2} \times (\mathbb{C}/\mathbb{Z}^2) \\ & (z_i, z_j) & \longmapsto & (z_i/z_\alpha, z_j/z_\beta, [(\ln z_\alpha + i \ln z'_\beta)/2\pi i]) \end{array} \tag{5}$$

turn out to define complex coordinates for a complex structure $I^{m,n}$ on $S^m \times S^n$. $I^{m,n}$ is called a *Calabi-Eckmann complex structure* on $S^m \times S^n$.

THEOREM 4.1. *The complex structure $I_{\mathcal{P}}$ on $S^m \times S^n$ coincides with the Calabi-Eckmann complex structure $I^{m,n}$, namely:*

$$\begin{aligned} I^{m,n}(p_i) &= p_{i+1} \qquad \textit{if } i \textit{ is odd}, \\ I^{m,n}(p_i) &= -p_{i-1} \qquad \textit{if } i \textit{ is even}. \end{aligned}$$

Proof. It is clear from formula (5) that the Hopf fibration $S^m \times S^n \to \mathbb{CP}^{(m-1)/2} \times \mathbb{CP}^{(n-1)/2}$ is locally given by the canonical projection

$$\mathbb{C}^{(m-1)/2} \times \mathbb{C}^{(n-1)/2} \times (\mathbb{C}/\mathbb{Z}^2) \longrightarrow \mathbb{C}^{(m-1)/2} \times \mathbb{C}^{(n-1)/2}.$$

Using notation as in Theorem 3.1, this means that $I^{m,n}$ is induced on $\mathcal{H}_1 \oplus \mathcal{H}_2$ by the complex structure of $\mathbb{CP}^{(m-1)/2} \times \mathbb{CP}^{(n-1)/2}$, and a computation shows that $I^{m,n}(S) = T$. The argument of Theorem 3.1 ends the proof. □

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