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A Pseudo-Measure of Fuzziness

Vicente D. Estruch and José Pastor (*)

Summary. - *In this note we give an example of a gradationof openness (a fuzzy topology in Shostak's sense) and deduce from it a pseudo-measure of fuzziness.*

1. Introduction

The question of how to measure vagueness or fuzziness has been one of issues associated with the development of the theory of fuzzy sets. In general, a measure of fuzziness is a function $f : I^X \longrightarrow \mathbb{R}$ where I^X denotes the family of all fuzzy sets of X, which must satisfy certain requirements that depend on the meaning given to the concept of the degree of fuzziness. Several measures of fuzziness proposed in the literature can be found in the interesting paper due to J. Klir [5].

General Topology was the first field of pure mathematics where concepts and ideas of fuzzy sets took strong roots. By 1968, just three years after L. A. Zadeh's pioneering paper [8], C. L. Chang [1] defined the concept of fuzzy topological spaces. However some authors criticized that his notion did not really describe fuziness with respect to the concept of openness of a fuzzy set. In the light of this

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(*) Authors' addresses: Vicente D. Estruch, Escuela Politécnica Superior de Gandia Dept. Matemática Aplicada Carretera Nazaret-Oliva, S/N, 46730 Grau de Gandia, Spain, e-mail: vdestruc@mat.upv.es
José. Pastor, Escuela Politécnica Superior de Gandia Dept. Matemática Aplicada Carretera Nazaret-Oliva, S/N, 46730 Grau de Gandia, Spain, e-mail: jpastogi@mat.upv.es
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difficulty A. Shostak [7] began his study on a fuzzy structures of topological type, and defined a fuzzy topology on X as a function $\tau : I^X \longrightarrow I$ satisfying certain axioms. On the other hand Hazra et al. [4] gave another concept of fuzzy topology that was later modified in [2], and so rediscovered the Shostak fuzzy topology concept, that called gradation of openness.

The aim of this note is to give an example of a gradation of openness and deduce from it a (pseudo)-measure of fuzziness.

2. Preliminaries

Throughtout this paper, I will denote the unit real interval $[0, 1]$. For a non-empty set X, I^X denotes the collection of all mappings from X into I. A member B of I^X is called a fuzzy set of X. The set $\{x \in X : B(x) > 0\}$ is called the support of B and is denoted by supp$B$. If B takes only the values $0, 1$, B is called a crisp set in X. An ordinary subset $A \subset X$ is identified with its characteristic function (crisp set) on X, and in consequence $\emptyset$ and X are indentical with the constant functions on X, which value is 0 and 1 respectively, and that are also denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively. The union and intersection of a family of fuzzy sets $\{A_i\}_i$ of X is $\bigvee_i A_i$ and $\bigwedge_i A_i$, respectively. The complement of $A \in I^X$, denoted by A', is defined by the formula $A'(x) = 1 - A(x)$, for $x \in X$. For $A, B \in I^X$ we write $A \subset B$, if $A(x) \leq B(x)$ for each $x \in X$.

A. P. Shostak [7] defined a fuzzy topology on X as a function $\tau : I^X \longrightarrow I$ satisfying the following axioms:

(i) $\tau(\mathbf{0}) = \tau(\mathbf{1}) = 1$.

(ii) $A, B \in I^X$, implies $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$.

(iii) $A_i \in I^X$ for all $i \in J$ implies $\tau(\bigcup_i A_i) \geq \bigwedge_i \tau(A_i)$.

The real number $\tau(A)$ is the degree of openness of the fuzzy set A.

K. C. Chattopadhyay et al. [2] rediscovered the Shostak's fuzzy topology concept and called gradation of openness the function τ, and this will be its name from now on. They also called gradation of closedness on X a function $\mathcal{F} : I^X \longrightarrow I$ satisfying the above (i)-(iii) but interchanching the intersection with the union and vice-versa.

In [3] it is shown that the mapping $\sigma : I^X \longrightarrow I$ given by: $\sigma(\mathbf{0}) = 1$ and $\sigma(A) = \inf\{A(x) : x \in \text{supp}A\}$, $\forall\ A \in I^X$, $A \neq \mathbf{0}$, is a gradation of openness (closedness), that we will call gradation support.

3. Pseudo-measure of fuzziness

Proposition 3.1. *The mapping* $\delta : I^X \longrightarrow [0,1]$ *given by* $\delta(A) = \sigma(A \cup A^{'})$*, where* σ *is the support gradation, is a gradation of openness (closedness) on* X*.*

Proof. First we will show that $\delta(A) = \frac{1}{2} + \inf\{\left|A(x) - \frac{1}{2}\right| : x \in X\}$, for each $A \in I^X$.

We have $\delta(A) = \sigma(A \cup A^{'}) = \inf\{(A \cup A^{'})(x) : (A \cup A^{'})(x) \neq \mathbf{0}\} = \inf\{(A \cup A^{'})(x) : x \in X\}$ since $(A \cup A^{'})(x) \geq \frac{1}{2}$, for each $x \in X$.

Now, it easy to verify that $(A \cup A^{'})(x) = \frac{1}{2} + \left|A(x) - \frac{1}{2}\right|$, for each $x \in X$, and then

$$\delta(A) = \inf\{\tfrac{1}{2} + \left|A(x) - \tfrac{1}{2}\right| : x \in X\} = \tfrac{1}{2} + \inf\{\left|A(x) - \tfrac{1}{2}\right| : x \in X\}.$$

It is clear that $\delta(\mathbf{0}) = \delta(\mathbf{1}) = 1$.

Now, let $\{G_i,\ i \in J\}$ be a family of fuzzy sets of X. In order to show the third axiom of fuzzy topology (gradation of openess) it is obvoiusly sufficient to verify the inequality

$$\left|\bigvee_i G_i(z) - \frac{1}{2}\right| \geqslant \bigwedge_i \left|G_i(z) - \frac{1}{2}\right| = \alpha \qquad \forall i \in J, \qquad \forall z \in X$$

Having z fixed we consider the following two cases:

(1) $\exists i$ such that $G_i(z) \geqslant \frac{1}{2}$. Then $G_i(z) - \frac{1}{2} \geqslant \alpha$ and hence $\bigvee_i G_i(z) - \frac{1}{2} \geqslant \alpha$.

(2) $\forall i \in J$ $G_i(z) < \frac{1}{2}$, then $\frac{1}{2} - G_i(z) \geq \alpha$ and hence again $\frac{1}{2} - \bigvee_i G_i(z) = \bigwedge_i (\frac{1}{2} - G_i(z)) \geq \alpha$.

This completes the proof of the third axiom.

Now, noticing that $\delta(A) = \delta(A')$, from the above we immediately get:

$$\delta(\bigcap_i G_i) = \delta((\bigcup_i G_i')') = \delta(\bigcup_i G_i') \geq \bigwedge_i \delta(G_i') = \bigwedge_i \delta(G_i).$$

This completes the proof that δ is gradation of openess and closedness (actually, it is a gradation of clopeness). □

Based on the concept of measure of fuzziness given by De Luca and Termini [6] we give the following definition.

Definition 3.2. *The function $f : I^X \longrightarrow \mathbb{R}$ is called a pseudo-measure of fuzziness on X if it satisfies the following conditions:*

(f1) $f(A) = 0$ *iff* A *is a crisp set*

(f2) If $A \prec B$ *then* $f(A) \leq f(B)$

(f3) If $M(x) = 1/2$ *for each* $x \in X$*, then* $f(M)$ *takes the maximum value of* f*, where the sharpness relation* $A \prec B$ *in (f2) is given by:*

$$A \prec B \text{ iif } \begin{cases} A(x) \leq B(x) & \text{if } B(x) \leq 1/2 \\ A(x) \geq B(x) & \text{if } B(x) \geq 1/2, \end{cases}$$

for all $x \in X$.

When the converse of (f3) is valid then f is called a measure of fuzziness.

Proposition 3.3. *Let* σ *be the support gradation on* X *and* $k \in]0, 2]$*. The mapping* $f : I^X \longrightarrow I$ *given by* $f(A) = k(1 - \sigma(A \cup A'))$ *for each* $A \in I^X$*, is a pseudo-measure of fuzziness on* X*.*

Proof. By proposition 3.1 it is clear that $f(A) \in [0, k/2] \subset I$, for each $A \in I^X$.

If A is a crisp set of X we have $A \cup A' = \mathbf{1}$ and $\sigma(A \cup A') = 1$, so $f(A) = 0$. Conversly, if $f(A) = 0$ we have $\sigma(A \cup A') = 1$ and then it is obvious that $A \cup A' = \mathbf{1}$, hence A is a crisp set of X.

Now we will show that (f2) is satisfied:

Suppose $A \prec B$, and define the following two ordinary subsets of X:

$$H = \{x \in X \ / \ A(x) \leq B(x) \leq 1/2\} \text{ and } K = \{x \in X \ / \ 1/2 \leq B(x) \leq A(x)\}.$$

For $x \in H$ we have $A^{'}(x) = 1 - A(x) \geq 1 - B(x) = B^{'}(x) \geq 1/2$, and thus for $x \in H$ we have $(A \cup A^{'})(x) = A'(x)$ and $(B \cup B')(x) = B'(x)$, and then

$$(A \cup A^{'})(x) \geq (B \cup B')(x) \geq 1/2, \text{ for each } x \in H.$$

For $x \in K$ we have $A^{'}(x) = 1 - A(x) \leq 1 - B(x) = B^{'}(x) \leq 1/2$, and thus for $x \in K$ we have $(A \cup A^{'})(x) = A'(x)$ and $(B \cup B')(x) = B'(x)$, and then

$$1/2 \leq (B \cup B')(x) \leq (A \cup A^{'})(x), \text{ for each } x \in K.$$

So, for $x \in X$, if $A \prec B$, we have $(A \cup A^{'})(x) \geq (B \cup B')(x) \geq 1/2$, and thus $\sigma(A \cup A^{'}) \geq \sigma(B \cup B^{'}) \geq 1/2$.

Hence, $f(A) = k(1 - \sigma(A \cup A^{'})) \leq k(1 - \sigma(B \cup B^{'})) = f(B)$.

Finally, if M is the constant mapping on X which takes value $\frac{1}{2}$, then

$$f(M) = k(1 - \sigma(M \cup M^{'})) = k(1 - \sigma(M))) = k/2,$$

and (f3) is satisfied. □

Notice that the converse of (f3) is not satisfied. In fact, if $f(A) = k/2$ then $k(1 - \sigma(A \cup A^{'})) = k/2$ implies $1 - \sigma(A \cup A^{'}) = 1/2$, so $\inf\{(A \cup A^{'})(x) : A(x) \neq 0\} = 1/2$, and this last condition is satisfied by all functions which take value $1/2$ in some point of X.

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