

# Multiply warped products as quasi-Einstein manifolds with a quarter-symmetric connection

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**ABSTRACT.** *In this paper we study warped products and multiply warped products on quasi-Einstein manifolds with a quarter-symmetric connection. Then we apply our results to generalize Robertson-Walker spacetime with a quarter-symmetric connection.*

**Keywords:** Quasi-Einstein manifold, warped product, multiply warped product, quarter-symmetric connection.  
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## 1. Introduction

A Riemannian manifold  $(M^n, g)$ ,  $n \geq 2$ , is said to be an Einstein manifold if its Ricci tensor  $S$  satisfies the condition  $S = \frac{r}{n}g$ , where  $r$  denotes the scalar curvature of  $M$ . M. C. Chaki and R. K. Maity introduced the notion of quasi-Einstein manifold in [2]. A non-flat Riemannian manifold  $(M, g)$ ,  $n \geq 2$ , is said to be a quasi-Einstein manifold if the condition

$$S(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y),$$

is fulfilled on  $M$ , where  $\alpha$  and  $\beta$  are scalars of which  $\beta \neq 0$  and  $\eta$  is a non-zero 1-form such that  $g(X, U) = \eta(X)\eta(U)$ , for all vector field  $X$  and  $U$ , a unit vector field.

Let  $(B, g_B)$  and  $(F, g_F)$  be two Riemannian manifolds and  $f > 0$  be a differential function on  $B$ . Consider the product manifold  $B \times F$  with its projections  $\pi : B \times F \rightarrow B$  and  $\sigma : B \times F \rightarrow F$ . The warped product  $B \times_f F$  is the manifold  $B \times F$  with the Riemannian structure such that  $\|X\|^2 = \|\pi^*(X)\|^2 + f^2(\pi(p))\|\sigma^*(X)\|^2$ , for any vector field  $X$  on  $M$ . Thus we have that  $g_M = g_B + f^2 g_F$  holds on  $M$ . Here  $B$  is called the base of  $M$  and  $F$  is called the fiber. The function  $f$  is called the warping function of the warped product [7]. The concept of warped product was first introduced by Bishop

and O'Neill [1] to construct examples of Riemannian manifolds with negative curvature.

Now, we can generalize warped products to multiply warped products. A multiply warped product is the product manifold  $M = B \times_{b_1} F_1 \times_{b_2} F_2 \dots \times_{b_m} F_m$  with the metric  $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus b_3^2 g_{F_3} \dots \oplus b_m^2 g_{F_m}$ , where for each  $i \in \{1, 2, \dots, m\}$ ,  $b_i : B \rightarrow (0, \infty)$  is smooth and  $(F_i, g_{F_i})$  is a pseudo-Riemannian manifold. In particular, when  $B = (c, d)$ , the metric  $g_B = -dt^2$  is negative and  $(F_i, g_{F_i})$  is a Riemannian manifold. We call  $M$  the multiply generalized Robertson-Walker spacetime.

A multiply twisted product  $(M, g)$  is a product manifold of the form  $M = B \times_{b_1} F_1 \times_{b_2} F_2 \dots \times_{b_m} F_m$  with the metric  $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus b_3^2 g_{F_3} \dots \oplus b_m^2 g_{F_m}$ , where for each  $i \in \{1, 2, \dots, m\}$ ,  $b_i : B \times F_i \rightarrow (0, \infty)$  is smooth.

In 1924, Friedmann and Schouten introduced the notion of a semi-symmetric linear connection on a differentiable manifold [3]. The definition of metric connection with torsion on a Riemannian manifold, was given by Hayden (1932) in [5]. In 1970, K. Yano [10] considered a semi-symmetric metric connection and studied some of its properties. Then in 1975, Golab [4] introduced the definition of a quarter-symmetric linear connection on a differentiable manifold, which is a generalization of semi-symmetric connection. Later in [8], Q. Qu and Y. Wang generalized the results to warped product and multiply warped product with a quarter-symmetric connection.

In this paper we consider multiply warped products as quasi-Einstein manifolds endowed with a quarter-symmetric connection. In section 2 and 3, we discuss some preliminary concepts and results which are useful for proving our main results in the next sections 4 and 5. In Theorem 4.1, we obtain a necessary and sufficient condition for the warped product manifold to be a quasi-Einstein manifold with respect to a quarter-symmetric connection. Then in Theorem 4.2, under some assumptions on base and fiber we study quasi-Einstein manifold with respect to a quarter-symmetric connection. Next in Theorem 4.3, we establish that if  $(M, g)$  admits a metric for Robertson-Walker spacetime then it is a quasi-Einstein manifold with respect to the above mentioned connection under certain conditions. Then in Theorem 4.5, we characterize the warping function for a warped product space  $(M, g)$  with a quarter-symmetric connection. Later in Theorem 4.5, we show that for quasi-Einstein warped product with respect to a quarter-symmetric connection the complete connected  $(\bar{n} - 1)$ -dimensional base is isometric to a  $(\bar{n} - 1)$ -dimensional sphere. In the last section, we study special multiply warped product manifold with respect to a quarter-symmetric connection.

**2. Preliminaries**

Let  $(M^n, g)$  be a Riemannian manifold with the Levi-Civita connection  $\nabla$ . A linear connection  $\check{\nabla}$  on  $(M^n, g)$  is said to be a quarter-symmetric connection if its torsion tensor  $T$  with respect to the connection  $\check{\nabla}$  defined by

$$T(X, Y) = \check{\nabla}_X Y - \check{\nabla}_Y X - [X, Y],$$

satisfies

$$T(X, Y) = \omega(Y)\phi X - \omega(X)\phi Y,$$

where  $\omega$  is a 1-form on  $M^n$  with the associated vector field  $P$  defined by  $\omega(X) = g(X, P)$ , for all vector field  $X$ , and  $\phi$  is a  $(1, 1)$  tensor field.

A quarter-symmetric connection  $\check{\nabla}$  is called a quarter-symmetric metric connection if  $\check{\nabla}g = 0$ .  $\check{\nabla}$  is called a quarter-symmetric non-metric connection if  $\check{\nabla}g \neq 0$ .

The relation between a quarter-symmetric connection  $\check{\nabla}$  and the Levi-Civita connection  $\nabla$  of  $M^n$  is given by [9]

$$\check{\nabla}_X Y = \nabla_X Y + \lambda_1 \omega(Y)X - \lambda_2 g(X, Y)P, \tag{1}$$

where  $g(X, P) = \omega(X)$  and  $\lambda_1 \neq 0, \lambda_2 \neq 0$  are scalar functions.

We can easily see that:

- when  $\lambda_1 = \lambda_2 = 1$ ,  $\check{\nabla}$  is a semi-symmetric metric connection,
- when  $\lambda_1 = \lambda_2 \neq 1$ ,  $\check{\nabla}$  is a quarter-symmetric metric connection,
- when  $\lambda_1 \neq \lambda_2$ ,  $\check{\nabla}$  is a quarter-symmetric non-metric connection.

Further, a relation between the curvature tensors  $R$  and  $\check{R}$  of type (1,3) of the connections  $\nabla$  and  $\check{\nabla}$  respectively is given by [9],

$$\begin{aligned} \check{R}(X, Y)Z &= R(X, Y)Z + \lambda_1 g(Z, \nabla_X P)Y - \lambda_2 g(Z, \nabla_Y P)X, \\ &+ \lambda_2 [g(X, Z)\nabla_Y P - g(Y, Z)\nabla_X P] + \lambda_1 \lambda_2 \omega(P) [g(X, Z)Y - g(Y, Z)X] \\ &+ \lambda_2^2 [g(Y, Z)\omega(X) - g(X, Z)\omega(Y)]P + \lambda_1^2 \omega(Z) [\omega(Y)X - \omega(X)Y], \end{aligned} \tag{2}$$

for vector fields  $X, Y, Z$  on  $M$ .

**3. Warped Product Manifolds with Quarter-Symmetric Connection**

In this section we consider the following propositions from Propositions 3.5, 3.6, 3.7 and 3.8 of [8], which will be helpful to prove our main results of next section.

PROPOSITION 3.1. *Let  $M = B \times_f F$  be a warped product. Let  $S$  and  $\check{S}$  denote the Ricci tensors of  $M$  with respect to the Levi-Civita connection and a quarter-symmetric connection respectively. Let  $\dim B = n_1$ ,  $\dim F = n_2$ ,  $\dim M = \bar{n} = n_1 + n_2$ . If  $X, Y \in \chi(B)$ ,  $V, W \in \chi(F)$  and  $P \in \chi(B)$ , then*

$$(i) \quad \check{S}(X, Y) = \check{S}^B(X, Y) + n_2 \left[ \frac{H_B^f(X, Y)}{f} + \lambda_2 \frac{Pf}{f} g(X, Y) + \lambda_1 \lambda_2 \omega(P) g(X, Y) + \lambda_1 g(Y, \nabla_X P) - \lambda_1^2 \omega(X) \omega(Y) \right],$$

$$(ii) \quad \check{S}(X, V) = \check{S}(V, X) = 0,$$

$$(iii) \quad \check{S}(V, W) = S^F(V, W) + \left\{ \lambda_2 \operatorname{div}_B P + (n_2 - 1) \frac{|\operatorname{grad}_B f|_B^2}{f^2} + [(\bar{n} - 1) \lambda_1 \lambda_2 - \lambda_2^2] \omega(P) + [(\bar{n} - 1) \lambda_1 + (n_2 - 1) \lambda_2] \frac{Pf}{f} + \frac{\Delta_B f}{f} \right\} g(V, W),$$

where  $\operatorname{div}_B P = \sum_{k=1}^{n_1} \varepsilon_k \langle \nabla_{E_k} P, E_k \rangle$  and  $E_k, 1 \leq k \leq n_1$ , is an orthonormal basis of  $B$  with  $\varepsilon_k = g(E_k, E_k)$ .

PROPOSITION 3.2. *Let  $M = B \times_f F$  be a warped product,  $\dim B = n_1$ ,  $\dim F = n_2$ ,  $\dim M = \bar{n} = n_1 + n_2$ . If  $X, Y \in \chi(B)$ ,  $V, W \in \chi(F)$  and  $P \in \chi(F)$ , then*

$$(i) \quad \check{S}(X, Y) = S^B(X, Y) + [(\bar{n} - 1) \lambda_1 \lambda_2 - \lambda_2^2] \omega(P) g(X, Y) + n_2 \frac{H_B^f(X, Y)}{f} + \lambda_2 g(X, Y) \operatorname{div}_F P,$$

$$(ii) \quad \check{S}(X, V) = [(\bar{n} - 1) \lambda_1 - \lambda_2] \omega(V) \frac{Xf}{f},$$

$$(iii) \quad \check{S}(V, X) = [\lambda_2 - (\bar{n} - 1) \lambda_1] \omega(V) \frac{Xf}{f},$$

$$(iv) \quad \check{S}(V, W) = S^F(V, W) + g(V, W) \left\{ (n_2 - 1) \frac{|\operatorname{grad}_B f|_B^2}{f^2} + \frac{\Delta_B f}{f} + [(\bar{n} - 1) \lambda_1 \lambda_2 - \lambda_2^2] \omega(P) + \lambda_2 \operatorname{div}_F P \right\} + [(\bar{n} - 1) \lambda_1 - \lambda_2] g(W, \nabla_V P) + [\lambda_2^2 + (1 - \bar{n}) \lambda_1^2] \omega(V) \omega(W).$$

By Proposition 3.1 and Proposition 3.2 and by the definition of the scalar curvature, we have the following propositions.

PROPOSITION 3.3. *Let  $M = B \times_f F$  be a warped product,  $\dim B = n_1$ ,  $\dim F = n_2$ ,  $\dim M = \bar{n} = n_1 + n_2$ . If  $P \in \chi(B)$ , then*

$$\check{r}^M = \check{r}^B + \frac{r^F}{f^2} + n_2(n_2 - 1) \frac{|\operatorname{grad}_B f|_B^2}{f^2} + n_2(\bar{n} - 1)(\lambda_1 + \lambda_2) \frac{Pf}{f} + 2n_2 \frac{\Delta_B f}{f} + [n_2(\bar{n} + n_1 - 1) \lambda_1 \lambda_2 - n_2(\lambda_1^2 + \lambda_2^2)] \omega(P) + n_2(\lambda_1 + \lambda_2) \operatorname{div}_B P.$$

PROPOSITION 3.4. *Let  $M = B \times_f F$  be a warped product,  $\dim B = n_1$ ,  $\dim F = n_2$ ,  $\dim M = \bar{n} = n_1 + n_2$ . If  $P \in \chi(F)$ , then*

$$\begin{aligned} \check{r}^M = r^B + \frac{r^F}{f^2} + (\bar{n} - 1)(\lambda_1 + \lambda_2) \operatorname{div}_F P + [\bar{n}(\bar{n} - 1)\lambda_1\lambda_2 + (1 - \bar{n})(\lambda_1^2 + \lambda_2^2)]\omega(P) \\ + n_2(n_2 - 1) \frac{|\operatorname{grad}_B f|_B^2}{f^2} + 2n_2 \frac{\Delta_B f}{f}. \end{aligned}$$

#### 4. Generalized Robertson-Walker Spacetime with a Quarter-Symmetric Connection

In this section we consider a quasi-Einstein warped product manifold with respect to a quarter-symmetric connection. We prove the following theorem.

THEOREM 4.1. *Let  $(M, g)$  be a warped product  $I \times_f F$  where  $I$  is an open interval in  $\mathbb{R}$ ,  $\dim I = 1$  and  $\dim F = \bar{n} - 1$ ,  $\bar{n} \geq 3$ . Then  $(M, g)$  is a quasi-Einstein manifold with respect to a quarter-symmetric connection if and only if  $F$  is a quasi-Einstein manifold for  $P = \frac{\partial}{\partial t}$  with respect to the Levi-Civita connection or the warping function  $f$  is a constant on  $I$  for  $P \in \chi(F)$ ,  $\lambda_2 \neq (\bar{n} - 1)\lambda_1$ .*

*Proof.* Assume that  $P \in \chi(B)$  and let  $g_I$  be the metric on  $I$ . Taking  $f = e^{\frac{q}{2}}$  and using the Proposition 3.1, we get

$$\check{S} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = (1 - \bar{n}) \left[ \frac{1}{2}q'' + \frac{1}{4}q'^2 - \frac{1}{2}\lambda_2q' + \lambda_1\lambda_2 - \lambda_1^2 \right] g_I \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right), \quad (3)$$

$$\check{S} \left( \frac{\partial}{\partial t}, V \right) = 0, \quad (4)$$

$$\begin{aligned} \check{S}(V, W) = S^F(V, W) + e^q \left[ \frac{\bar{n} - 1}{4}(q')^2 + \frac{1}{2}[(\bar{n} - 1)\lambda_1 + (\bar{n} - 2)\lambda_2]q' \right. \\ \left. + \lambda_2^2 + \frac{1}{2}q'' + (1 - \bar{n})\lambda_1\lambda_2 \right] g_F(V, W), \quad (5) \end{aligned}$$

for vector fields  $V, W$  on  $F$ .

Since  $M$  is a quasi-Einstein manifold with respect to a quarter-symmetric connection, we have

$$\check{S} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = \alpha g \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) + \beta \eta \left( \frac{\partial}{\partial t} \right) \eta \left( \frac{\partial}{\partial t} \right),$$

and

$$\check{S}(V, W) = \alpha g(V, W) + \beta \eta(V)\eta(W).$$

Then the last two equations reduce to

$$\check{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \alpha g_I\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + \beta \eta\left(\frac{\partial}{\partial t}\right) \eta\left(\frac{\partial}{\partial t}\right), \quad (6)$$

and

$$\check{S}(V, W) = \alpha e^q g_F(V, W) + \beta \eta(V) \eta(W). \quad (7)$$

Decomposing the vector field  $U$  uniquely into its components  $U_I$  and  $U_F$  on  $I$  and  $F$ , respectively, we have  $U = U_I + U_F$ . Since  $\dim I = 1$ , we can take  $U_I = v \frac{\partial}{\partial t}$  which gives  $U = v \frac{\partial}{\partial t} + U_F$ , where  $v$  is a function on  $M$ . Thus, we can write

$$\eta\left(\frac{\partial}{\partial t}\right) = g\left(U, \frac{\partial}{\partial t}\right) = v. \quad (8)$$

Using equations (3) and (5), equations (6), (7) reduce to

$$\check{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \alpha + \beta v^2, \quad (9)$$

and

$$\check{S}(V, W) = \alpha e^q g_F(V, W) + \beta \eta(V) \eta(W). \quad (10)$$

Comparing the right hand sides of (3) and (9), we get

$$\alpha + \beta v^2 = (1 - \bar{n}) \left[ \frac{1}{2} q'' + \frac{1}{4} q'^2 - \frac{\lambda_2 q'}{2} + \lambda_1 \lambda_2 - \lambda_1^2 \right]. \quad (11)$$

Similarly, comparing the right hand sides of (5) and (10), we obtain

$$S^F(V, W) = e^q \left[ \alpha + \frac{1 - \bar{n}}{4} (q')^2 - \frac{1}{2} [(\bar{n} - 1)\lambda_1 + (\bar{n} - 2)\lambda_2] q' - \lambda_2^2 - \frac{1}{2} q'' + (\bar{n} - 1)\lambda_1 \lambda_2 \right] g_F(V, W) + \beta \eta(V) \eta(W), \quad (12)$$

which gives that  $F$  is a quasi-Einstein manifold with respect to the Levi-Civita connection for  $P \in \chi(B)$ .

Taking  $P \in \chi(F)$  and by the use of Proposition 3.2, we get

$$\check{S}\left(\frac{\partial}{\partial t}, V\right) = \frac{q'}{2} [(\bar{n} - 1)\lambda_1 - \lambda_2] \omega(V) \quad (13)$$

and

$$\check{S}\left(V, \frac{\partial}{\partial t}\right) = \frac{q'}{2} [\lambda_2 - (\bar{n} - 1)\lambda_1] \omega(V), \quad (14)$$

for any vector field  $V \in \chi(F)$ .

Since  $M$  is a quasi-Einstein manifold, we have

$$\check{S}\left(\frac{\partial}{\partial t}, V\right) = \check{S}\left(V, \frac{\partial}{\partial t}\right) = \alpha g\left(V, \frac{\partial}{\partial t}\right) + \beta \eta(V) \eta\left(\frac{\partial}{\partial t}\right). \tag{15}$$

Now  $g(V, \frac{\partial}{\partial t}) = 0$  as  $\frac{\partial}{\partial t} \in \chi(B)$  and  $V \in \chi(F)$ .

Hence, from the last equation, we get

$$\check{S}\left(\frac{\partial}{\partial t}, V\right) = \check{S}\left(V, \frac{\partial}{\partial t}\right) = \beta \eta(V) \eta\left(\frac{\partial}{\partial t}\right). \tag{16}$$

Therefore, we have

$$\beta \eta(V) \eta\left(\frac{\partial}{\partial t}\right) = \frac{q'}{2} [(\bar{n} - 1)\lambda_1 - \lambda_2] \omega(V), \tag{17}$$

$$\beta \eta(V) \eta\left(\frac{\partial}{\partial t}\right) = \frac{q'}{2} [\lambda_2 - (\bar{n} - 1)\lambda_1] \omega(V). \tag{18}$$

From equations (17) and (18), we get

$$q' = 0,$$

when  $\lambda_2 - (\bar{n} - 1)\lambda_1 \neq 0$ . It follows that  $q$  is a constant on  $I$ . Then  $f$  is constant on  $I$ . This completes the proof.  $\square$

Now, we consider the warped product  $M = B \times_f I$  with  $\dim B = \bar{n} - 1$ ,  $\dim I = 1$ ,  $\bar{n} \geq 3$ . Under this assumption, we obtain the following theorem.

**THEOREM 4.2.** *Let  $(M, g)$  be a warped product  $B \times_f I$ , where  $\dim I = 1$  and  $\dim B = \bar{n} - 1$ ,  $\bar{n} \geq 3$ , then*

- i) if  $(M, g)$  is a quasi-Einstein manifold with respect to a quarter-symmetric connection,  $P \in \chi(B)$  is parallel on  $B$  with respect to the Levi-Civita connection on  $B$  and  $f$  is a constant on  $B$ , then,*

$$\alpha = [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_2^2] \omega(P).$$

- ii) If  $(M, g)$  is a quasi-Einstein manifold with respect to a quarter-symmetric connection for  $P \in \chi(I)$ , and  $\lambda_2 \neq (\bar{n} - 1)\lambda_1$  then  $f$  is a constant on  $B$ .*
- iii) If  $f$  is a constant on  $B$  and  $B$  is a quasi-Einstein manifold with respect to the Levi-Civita connection for  $P \in \chi(I)$ , then  $M$  is a quasi-Einstein manifold with respect to a quarter-symmetric connection.*

*Proof.* Assume that  $(M, g)$  is a quasi-Einstein manifold with respect to a quarter-symmetric connection. Then we write

$$\check{S}(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y). \quad (19)$$

Decomposing the vector field  $U$  uniquely into its components  $U_B$  and  $U_I$  on  $B$  and  $I$ , respectively, we have

$$U = U_B + U_I. \quad (20)$$

Since  $\dim I = 1$ , we can take  $U_I = v \frac{\partial}{\partial t}$  which gives  $U = U_B + v \frac{\partial}{\partial t}$ , where  $v$  is a function on  $M$ . From (19), (20) and Proposition 3.1, we have

$$\begin{aligned} \check{S}^B(X, Y) &= \alpha g_B(X, Y) + \beta g_B(X, U_B)g_B(Y, U_B) - \left[ \frac{H^f(X, Y)}{f} \right. \\ &\left. + \lambda_2 \frac{Pf}{f} g(X, Y) + \lambda_1 \lambda_2 \omega(P)g(X, Y) + \lambda_1 g(Y, \nabla_X P) - \lambda_1^2 \omega(X)\omega(Y) \right]. \end{aligned} \quad (21)$$

By contraction over  $X$  and  $Y$ , we get

$$\begin{aligned} \check{r}^B &= \alpha(\bar{n} - 1) + \beta g_B(U_B, U_B) - \left[ \frac{\Delta_B f}{f} + \lambda_2(\bar{n} - 1) \frac{Pf}{f} \right. \\ &\left. + [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_1^2] \omega(P) + \lambda_1 \sum_{i=1}^{\bar{n}-1} g(e_i, \nabla_{e_i} P) \right]. \end{aligned} \quad (22)$$

Also from (19), we have

$$\check{r}^M = \alpha \bar{n} + \beta g_B(U_B, U_B). \quad (23)$$

Now, putting the value of (23) in (22), we get

$$\begin{aligned} \check{r}^B &= \check{r}^M - \alpha - \frac{\Delta_B f}{f} - \lambda_2(\bar{n} - 1) \frac{Pf}{f} \\ &- [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_1^2] \omega(P) - \lambda_1 \sum_{i=1}^{\bar{n}-1} g(e_i, \nabla_{e_i} P). \end{aligned} \quad (24)$$

On the other hand, from Proposition 3.3, we get

$$\begin{aligned} \check{r}^M &= \check{r}^B + (\bar{n} - 1)(\lambda_1 + \lambda_2) \frac{Pf}{f} + 2 \frac{\Delta_B f}{f} \\ &+ [2(\bar{n} - 1)\lambda_1 \lambda_2 - (\lambda_1^2 + \lambda_2^2)] \omega(P) + (\lambda_1 + \lambda_2) \sum_{i=1}^{\bar{n}-1} g(\nabla_{e_i} P, e_i). \end{aligned}$$



Then from the above two relations, we get

$$\begin{aligned} \alpha + \frac{\Delta_B f}{f} + \lambda_2(\bar{n} - 1)\frac{Pf}{f} + [(\bar{n} - 1)\lambda_1\lambda_2 - \lambda_1^2]\omega(P) + \lambda_1 \sum_{i=1}^{\bar{n}-1} g(e_i, \nabla_{e_i} P) \\ = (\bar{n} - 1)(\lambda_1 + \lambda_2)\frac{Pf}{f} + 2\frac{\Delta f}{f} + [2(\bar{n} - 1)\lambda_1\lambda_2 - (\lambda_1^2 + \lambda_2^2)]\omega(P) \\ + (\lambda_1 + \lambda_2) \sum_{i=1}^{\bar{n}-1} g(\nabla_{e_i} P, e_i). \end{aligned}$$

Since  $P \in \chi(B)$  is parallel and  $f$  is a constant on  $B$ , then we get

$$\alpha = [(\bar{n} - 1)\lambda_1\lambda_2 - \lambda_2^2]\omega(P).$$

ii) Let  $P \in \chi(I)$ . By the use of Proposition 3.2, we get

$$\check{S}(X, P) = [(\bar{n} - 1)\lambda_1 - \lambda_2]\omega(P)\frac{Xf}{f}, \tag{25}$$

and

$$\check{S}(P, X) = [\lambda_2 - (\bar{n} - 1)\lambda_1]\omega(P)\frac{Xf}{f}. \tag{26}$$

Since  $M$  is a quasi-Einstein manifold, we have

$$\check{S}(X, P) = \check{S}(P, X) = \alpha g(P, X) + \beta \eta(P)\eta(X).$$

Again, we have  $g(P, X) = 0$  for  $X \in \chi(B)$  and  $P \in \chi(I)$ .

Hence, we have

$$Xf = 0,$$

where  $\lambda_2 \neq (\bar{n} - 1)\lambda_1$ . This implies that  $f$  is a constant on  $B$ .

iii) Assume that  $B$  is a quasi-Einstein manifold with respect to the Levi-Civita connection. Then we have

$$S^B(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y), \tag{27}$$

for vector fields  $X, Y$  tangent to  $B$ .

From Proposition 3.2, we get

$$\check{S}^M(X, Y) = S^B(X, Y) + [(\bar{n} - 1)\lambda_1\lambda_2 - \lambda_2^2]\omega(P)g(X, Y) + \frac{H^f(X, Y)}{f},$$

for any vector field  $P \in \chi(I)$ . Since  $f$  is a constant,  $H^f(X, Y) = 0$  for all  $X, Y \in \chi(B)$ .

The above equation reduces to

$$\check{S}^M(X, Y) = S^B(X, Y) + [(\bar{n} - 1)\lambda_1\lambda_2 - \lambda_2^2]\omega(P)g(X, Y). \quad (28)$$

Using the value of (27) in (28), we get

$$\check{S}^M(X, Y) = \{\alpha + [(\bar{n} - 1)\lambda_1\lambda_2 - \lambda_2^2]\omega(P)\}g(X, Y) + \beta\eta(X)\eta(Y), \quad (29)$$

which shows that  $M$  is a quasi-Einstein manifold with respect to a quarter-symmetric connection.  $\square$

Next, we study  $M = I \times_f F$  with metric  $-dt^2 + f(t)^2g_F$ , where  $I$  is an open interval in  $\mathbb{R}$ , and we prove the following theorem.

**THEOREM 4.3.** *Let  $(M, g)$  be a warped product  $I \times_f F$  with the metric tensor  $-dt^2 + f(t)^2g_F$ ,  $P = \frac{\partial}{\partial t}$ ,  $\dim F = l$ . Then  $(M, g)$  is a quasi-Einstein manifold with respect to a quarter-symmetric connection  $\check{\nabla}$  with constant associated scalars  $\alpha$  and  $\beta$  if and only if the following conditions are satisfied:*

- i)  $(F, g_F)$  is a quasi-Einstein manifold with scalar  $\alpha_F, \beta_F$ ;*
- ii)  $-l \left( \lambda_2 \frac{f'}{f} - \frac{f''}{f} + \lambda_1^2 - \lambda_1\lambda_2 \right) = -\alpha + v^2\beta$ ;*
- iii)  $\alpha_F - ff'' - (l - 1)f'^2 + (\lambda_2^2 - l\lambda_1\lambda_2 - \alpha)f^2 + [l\lambda_1 + (l - 1)\lambda_2]ff' = 0$   
and  $\beta = \beta_F$ .*

*Proof.* By Proposition 3.1, we have

$$\check{S} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = -l \left( \lambda_2 \frac{f'}{f} - \frac{f''}{f} + \lambda_1^2 - \lambda_1\lambda_2 \right),$$

$$\check{S} \left( \frac{\partial}{\partial t}, V \right) = \check{S} \left( V, \frac{\partial}{\partial t} \right) = 0,$$

$$\check{S}(V, W) = S^F(V, W) + g_F(V, W) \left\{ -ff'' - (l - 1)f'^2 + (\lambda_2^2 - l\lambda_1\lambda_2)f^2 + [l\lambda_1 + (l - 1)\lambda_2]ff' \right\}.$$

Since  $M$  is a quasi-Einstein manifold, we have

$$\check{S}(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y).$$

Now,

$$\check{S} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = \alpha g \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) + \beta\eta \left( \frac{\partial}{\partial t} \right) \eta \left( \frac{\partial}{\partial t} \right).$$

We can decompose the vector field  $U$  uniquely into its components  $U_I$  and  $U_F$  on  $I$  and  $F$ , respectively. Then we have  $U = U_I + U_F$ . Since  $\dim I = 1$ , we can take  $U_I = v \frac{\partial}{\partial t}$  which gives  $U = v \frac{\partial}{\partial t} + U_F$ , where  $v$  is a function on  $M$ . Thus, we can write

$$\eta \left( \frac{\partial}{\partial t} \right) = g \left( U, \frac{\partial}{\partial t} \right) = v. \tag{30}$$

Therefore, we get

$$-l \left( \lambda_2 \frac{f'}{f} - \frac{f''}{f} + \lambda_1^2 - \lambda_1 \lambda_2 \right) = -\alpha + v^2 \beta.$$

Again,  $\check{S}(V, W) = \alpha g(V, W) + \beta \eta(V) \eta(W)$ .

Also, we have

$$\begin{aligned} \check{S}(V, W) = S^F(V, W) + g_F(V, W) \{ & -ff'' - (l-1)f'^2 \\ & + (\lambda_2^2 - l\lambda_1\lambda_2)f^2 + [l\lambda_1 + (l-1)\lambda_2]ff' \}. \end{aligned}$$

From the above two equations, we get

$$\begin{aligned} S^F(V, W) = \{ & ff'' + (l-1)f'^2 - (\lambda_2^2 - l\lambda_1\lambda_2 - \alpha)f^2 \\ & - [l\lambda_1 + (l-1)\lambda_2]ff' \} g_F(V, W) + \beta \eta(V) \eta(W). \end{aligned}$$

Hence,  $(F, g_F)$  is a quasi-Einstein manifold.

Also, we have

$$\begin{aligned} \check{S}(V, W) = S^F(V, W) + g_F(V, W) \{ & -ff'' - (l-1)f'^2 \\ & + (\lambda_2^2 - l\lambda_1\lambda_2)f^2 + [l\lambda_1 + (l-1)\lambda_2]ff' \}. \end{aligned}$$

After some calculations, we show that

$$\alpha_F - ff'' - (l-1)f'^2 + (\lambda_2^2 - l\lambda_1\lambda_2 - \alpha)f^2 + [l\lambda_1 + (l-1)\lambda_2]ff' = 0$$

and  $\beta = \beta_F$ . Thus, the proof is completed. □

Putting  $\dim F = 1$  in Theorem 4.3, we get the following corollary.

**COROLLARY 4.4.** *Let  $(M, g)$  be a warped product  $I \times_f F$  with the metric tensor  $-dt^2 + f(t)^2 g_F$ ,  $P = \frac{\partial}{\partial t}$ ,  $\dim F = 1$ . Then  $(M, g)$  is a quasi-Einstein manifold with respect to a quarter-symmetric connection if and only if*

$$f'' - \lambda_2 f' + [(\alpha - v^2 \beta) - (\lambda_1^2 - \lambda_1 \lambda_2)] f = 0.$$

By using Corollary 4.4 and elementary methods for ordinary differential equations, we obtain the following theorem.

**THEOREM 4.5.** *Let  $(M, g)$  be a warped product  $I \times_f F$  with the metric tensor  $-dt^2 + f(t)^2 g_F$ ,  $P = \frac{\partial}{\partial t}$ ,  $\dim F = 1$ . Then  $(M, g)$  is a quasi-Einstein manifold with respect to a quarter-symmetric connection if and only if*

- i)  $\alpha - v^2\beta < (\lambda_1 - \frac{\lambda_2}{2})^2$ ,  
 $f(t) = c_1 e^{\left(\frac{\lambda_2 + \sqrt{(2\lambda_1 - \lambda_2)^2 - 4(\alpha - v^2\beta)}}{2}\right)t} + c_2 e^{\left(\frac{\lambda_2 - \sqrt{(2\lambda_1 - \lambda_2)^2 - 4(\alpha - v^2\beta)}}{2}\right)t}$ ,
- ii)  $\alpha - v^2\beta = (\lambda_1 - \frac{\lambda_2}{2})^2$ ,  $f(t) = c_1 e^{(\frac{\lambda_2}{2})t} + c_2 t e^{(\frac{\lambda_2}{2})t}$ ,
- iii)  $\alpha - v^2\beta > (\lambda_1 - \frac{\lambda_2}{2})^2$ ,  $f(t) = c_1 e^{(\frac{\lambda_2}{2})t} c_1 \cos\left(\left(\frac{\sqrt{4(\alpha - v^2\beta) - (2\lambda_1 - \lambda_2)^2}}{2}\right)t\right) + c_2 e^{(\frac{\lambda_2}{2})t} \sin\left(\left(\frac{\sqrt{4(\alpha - v^2\beta) - (2\lambda_1 - \lambda_2)^2}}{2}\right)t\right)$ .

**COROLLARY 4.6.** *Let  $(M, g)$  be a warped product  $I \times_f F$  with the metric tensor  $-dt^2 + f(t)^2 g_F$ ,  $P = \frac{\partial}{\partial t}$ ,  $\dim F = 1$ , and  $\lambda_2 = 2\lambda_1$ . Then  $(M, g)$  is a quasi-Einstein manifold with respect to a quarter-symmetric connection if and only if*

- i)  $\alpha - v^2\beta < 0$ ,  $f(t) = c_1 e^{(\lambda_1 + \sqrt{-(\alpha - v^2\beta)})t} + c_2 e^{(\lambda_1 - \sqrt{-(\alpha - v^2\beta)})t}$ ,
- ii)  $\alpha - v^2\beta = 0$ ,  $f(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$ ,
- iii)  $\alpha - v^2\beta > 0$ ,  $f(t) = c_1 e^{\lambda_1 t} \cos\left(\left(\sqrt{\alpha - v^2\beta}\right)t\right) + c_2 e^{\lambda_1 t} \sin\left(\left(\sqrt{\alpha - v^2\beta}\right)t\right)$ .

Next, the following theorem shows when the base of a quasi-Einstein warped product manifold is isometric to a sphere of a particular radius.

**THEOREM 4.7.** *Let  $(M, g)$  be a warped product  $B \times_f I$  of a complete connected  $(\bar{n} - 1)$ -dimensional Riemannian manifold  $B$  where  $\bar{n} \geq 3$  and one-dimensional Riemannian manifold  $I$ . If  $(M, g)$  is a quasi-Einstein manifold with constant associated scalars  $\alpha$  and  $\beta$ ,  $U \in \chi(M)$  with respect to a quarter-symmetric connection,  $P \in \chi(B)$  and the Hessian of  $f$  is proportional to the metric tensor  $g_B$ , then  $(B, g_B)$  is a  $(\bar{n} - 1)$ -dimensional sphere of radius  $\rho = \frac{\bar{n} - 1}{\sqrt{\bar{r}^B + \alpha}}$ .*

*Proof.* Let  $M$  be a connected warped product manifold. Then from Proposition 3.1, we have

$$\begin{aligned} \check{S}^M(X, Y) &= \check{S}^B(X, Y) + \frac{H_B^f(X, Y)}{f} + \lambda_2 \frac{Pf}{f} g(X, Y) \\ &\quad + \lambda_1 \lambda_2 \omega(P)g(X, Y) + \lambda_1 g(Y, \nabla_X P) - \lambda_1^2 \omega(X)\omega(Y), \quad (31) \end{aligned}$$

for any vector field  $X, Y$  on  $B$ . Since  $M$  is a quasi-Einstein manifold with respect to a quarter-symmetric metric connection, we have

$$\check{S}^M(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y). \tag{32}$$

Decomposing the vector field  $U$  uniquely into its components  $U_B$  and  $U_I$  on  $B$  and  $I$ , respectively, we have

$$U = U_B + U_I. \tag{33}$$

Putting the values of (32), (33) in (31), we get

$$\begin{aligned} \check{S}^B(X, Y) = & \alpha g_B(X, Y) + \beta g_B(X, U_B)g_B(Y, U_B) - \left[ \frac{H_B^f(X, Y)}{f} \right. \\ & \left. + \lambda_2 \frac{Pf}{f} g(X, Y) + \lambda_1 \lambda_2 \omega(P)g(X, Y) + \lambda_1 g(Y, \nabla_X P) - \lambda_1^2 \omega(X)\omega(Y) \right]. \end{aligned} \tag{34}$$

By contraction over  $X$  and  $Y$ , we get

$$\begin{aligned} \check{r}^B = \check{r}^M - \alpha - \frac{\Delta_B f}{f} - (\bar{n} - 1)\lambda_2 \frac{Pf}{f} \\ - [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_1^2] \pi(P) - \lambda_1 \sum_{i=1}^{\bar{n}-1} g(e_i, \nabla_{e_i} P). \end{aligned} \tag{35}$$

Again from Proposition 3.1, we obtain

$$\frac{\check{r}^M}{\bar{n}} = \lambda_2 \sum_{i=1}^{\bar{n}-1} g(e_i, \nabla_{e_i} P) + (\bar{n} - 1)\lambda_1 \frac{Pf}{f} + [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_2^2] \omega(P) + \frac{\Delta_B f}{f}. \tag{36}$$

From the last two equations, it follows that

$$\begin{aligned} (\check{r}^B + \alpha)f = & (\bar{n}\lambda_2 - \lambda_1) \sum_{i=1}^{\bar{n}-1} f g(e_i, \nabla_{e_i} P) + (\bar{n} - 1)[\bar{n}\lambda_1 - \lambda_2]Pf \\ & + [(\bar{n} - 1)^2 \lambda_1 \lambda_2 + \lambda_1^2 - \bar{n}\lambda_2^2] f \omega(P) + (\bar{n} - 1)\Delta_B f. \end{aligned} \tag{37}$$

Since the Hessian of  $f$  is proportional to the metric tensor  $g_B$ , then we have

$$\begin{aligned} H^f(X, Y) = & \frac{1}{(\bar{n} - 1)^2} \left[ (\lambda_1 - \bar{n}\lambda_2) \sum_{i=1}^{\bar{n}-1} f g(e_i, \nabla_{e_i} P) + (\bar{n} - 1)[\lambda_2 - \bar{n}\lambda_1]Pf \right. \\ & \left. + (\bar{n}\lambda_2^2 - (\bar{n} - 1)^2 \lambda_1 \lambda_2 - \lambda_1^2) f \omega(P) + (1 - \bar{n})\Delta_B f \right] g_B(X, Y). \end{aligned}$$

Hence, from the above equation, we obtain

$$H^f(X, Y) + \frac{\check{r}^B + \alpha}{(\bar{n} - 1)^2} f g_B(X, Y) = 0. \quad (38)$$

So  $B$  is isometric to the  $(\bar{n} - 1)$ -dimensional sphere of radius  $\frac{\bar{n}-1}{\sqrt{\check{r}^B + \alpha}}$  [6]. Thus, the theorem is proved.  $\square$

## 5. Multiply Twisted Product Manifold with Quarter-Symmetric Connection

Now, we have the following propositions from Propositions 4.5 and 4.7 of [8], for later use.

**PROPOSITION 5.1.** *Let  $M = B \times_{b_1} F_1 \times_{b_2} F_2 \dots \times_{b_m} F_m$  be a multiply twisted product manifold with  $\dim B = n$ ,  $\dim F_i = l_i$ ,  $\dim M = \bar{n}$ . If  $X, Y \in \chi(B)$ ,  $V \in \chi(F_i)$ ,  $W \in \chi(F_j)$  and  $P \in \chi(B)$ , then*

$$(i) \check{S}(X, Y) = \check{S}^B(X, Y) + \sum_{i=1}^m l_i \left[ \lambda_1 \lambda_2 \omega(P) g(X, Y) + \frac{H_B^{b_i}(X, Y)}{b_i} + \lambda_2 \frac{P b_i}{b_i} g(X, Y) + \lambda_1 g(Y, \nabla_X P) - \lambda_1^2 \omega(X) \omega(Y) \right],$$

$$(ii) \check{S}(X, V) = \check{S}(V, X) = (l_i - 1) [V X(\ln b_i)],$$

$$(iii) \check{S}(V, W) = 0 \text{ if } i \neq j,$$

$$(iv) \check{S}(V, W) = S^{F_i}(V, W) + g(V, W) \left\{ (l_i - 1) \frac{|\text{grad}_B b_i|_B^2}{b_i^2} + \frac{\Delta_B b_i}{b_i} + [(\bar{n} - 1) \lambda_1 \lambda_2 - \lambda_2^2] \omega(P) + \lambda_2 \text{div}_F P + [(\bar{n} - 1) \lambda_1 + (l_i - 1) \lambda_2] \frac{P b_i}{b_i} + \sum_{s \neq i} l_s \frac{g_B(\text{grad}_B b_i, \text{grad}_B b_s)}{b_i b_s} + \lambda_2 \sum_{s \neq i} l_s \frac{P b_s}{b_s} \right\} \text{ if } i = j, \text{ where } \text{div}_B P = \sum_{k=1}^n \varepsilon_k \langle \nabla_{E_k} P, E_k \rangle \text{ and } E_k, 1 \leq k \leq n, \text{ is an orthonormal basis of } B \text{ with } \varepsilon_k = g(E_k, E_k).$$

**PROPOSITION 5.2.** *Let  $M = B \times_{b_1} F_1 \times_{b_2} F_2 \dots \times_{b_m} F_m$  be a multiply twisted product,  $\dim B = n$ ,  $\dim F_i = l_i$ ,  $\dim M = \bar{n}$ . If  $X, Y \in \chi(B)$ ,  $V \in \chi(F_i)$ ,  $W \in \chi(F_j)$  and  $P \in \chi(F_r)$  for a fixed  $r$ , then*

$$(i) \check{S}(X, Y) = S^B(X, Y) + \sum_{i=1}^m l_i \frac{H_B^{b_i}(X, Y)}{b_i} + [(\bar{n} - 1) \lambda_1 \lambda_2 - \lambda_2^2] \omega(P) g(X, Y) + \lambda_2 g(X, Y) \text{div}_{F_r} P,$$

$$(ii) \check{S}(X, V) = (l_i - 1)[VX(lnb_i)] + [(\bar{n} - 1)\lambda_1 - \lambda_2]\omega(V)\frac{Xb_r}{b_r},$$

$$(iii) \check{S}(V, X) = (l_i - 1)[VX(lnb_i)] + [\lambda_2 - (\bar{n} - 1)\lambda_1]\omega(V)\frac{Xb_r}{b_r},$$

$$(iv) \check{S}(V, W) = 0 \text{ if } i \neq j,$$

$$(v) \check{S}(V, W) = S^{F_i}(V, W) + g(V, W) \left\{ (l_i - 1) \frac{|grad_B b_i|^2}{b_i^2} + \frac{\Delta_B b_i}{b_i} + [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_2^2] \pi(P) + \sum_{s \neq i} l_s \frac{g_B(grad_B b_i, grad_B b_s)}{b_i b_s} \right\} + [(\bar{n} - 1)\lambda_1 - \lambda_2]g(W, \nabla_V P) + [\lambda_2^2 + (1 - \bar{n})\lambda_1^2]\omega(V)\omega(W) + \lambda_2 g(V, W)div_{F_r} P \text{ if } i = j.$$

Let  $M = B \times_{b_1} F_1 \times_{b_2} F_2 \dots \times_{b_m} F_m$  be a multiply warped product with the metric tensor  $-dt^2 \oplus b_1^2 g_{F_1} \oplus \dots \oplus b_m^2 g_{F_m}$ , and let  $I$  be an open interval in  $\mathbb{R}$  and  $b_i \in C^\infty(I)$ .

Now, we prove the following theorem for multiply generalized Robertson-Walker spacetime.

**THEOREM 5.3.** *Let  $M = I \times_{b_1} F_1 \times_{b_2} F_2 \dots \times_{b_m} F_m$  be a multiply warped product with the metric tensor  $-dt^2 \oplus b_1^2 g_{F_1} \oplus \dots \oplus b_m^2 g_{F_m}$  and  $P = \frac{\partial}{\partial t}$ . Then  $(M, g)$  is a quasi-Einstein manifold with respect to a quarter-symmetric connection  $\check{\nabla}$  with constant associated scalars  $\alpha$  and  $\beta$ , if and only if the following conditions are satisfied:*

$$i) (F_i, g_{F_i}) \text{ are quasi-Einstein manifolds with scalars } \alpha_{F_i}, \beta_{F_i}, i \in \{1, 2, \dots, m\};$$

$$ii) \sum_{i=1}^m l_i \left( \lambda_2 \frac{b'_i}{b_i} - \frac{b''_i}{b_i} + \lambda_1^2 - \lambda_1 \lambda_2 \right) = \alpha - v^2 \beta;$$

$$iii) \alpha_{F_i} - b_i b''_i - (l_i - 1)b_i'^2 + (\lambda_2 b_i^2 - b_i b'_i) \sum_{s \neq i} l_s \left( \frac{b'_s}{b_s} \right) + (\lambda_2^2 + (1 - \bar{n})\lambda_1 \lambda_2 - \alpha) b_i^2 + ((\bar{n} - 1)\lambda_1 + (l_i - 1)\lambda_2) b_i b'_i = 0 \text{ and } \beta = \beta_{F_i}.$$

*Proof.* By Proposition 5.1, we have

$$\check{S} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = \sum_{i=1}^m l_i \left( -\lambda_2 \frac{b'_i}{b_i} + \frac{b''_i}{b_i} - \lambda_1^2 + \lambda_1 \lambda_2 \right), \tag{39}$$

$$\check{S} \left( \frac{\partial}{\partial t}, V \right) = \check{S} \left( V, \frac{\partial}{\partial t} \right) = (l_i - 1)V \left( \frac{b'_i}{b_i} \right), \tag{40}$$

$$\check{S}(V, W) = 0, \text{ if } i \neq j, \tag{41}$$

$$\begin{aligned} \check{S}(V, W) = S^{F_i}(V, W) + g_{F_i}(V, W) \left\{ - (l_i - 1)b_i'^2 - b_i''b_i + [(\bar{n} - 1)\lambda_1 \right. \\ \left. + (l_i - 1)\lambda_2]b_i'b_i + (\lambda_2b_i^2 - b_i'b_i) \sum_{s \neq i} l_s \frac{b_s'}{b_s} + (\lambda_2^2 + (1 - \bar{n})\lambda_1\lambda_2)b_i^2 \right\}. \end{aligned} \quad (42)$$

Since  $M$  is a quasi-Einstein manifold, we have

$$\check{S}(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y).$$

Now,

$$\check{S} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = \alpha g \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) + \beta \eta \left( \frac{\partial}{\partial t} \right) \eta \left( \frac{\partial}{\partial t} \right).$$

Decomposing the vector field  $U$  uniquely into its components  $U_I$  and  $U_F$  on  $I$  and  $F$ , respectively, we have  $U = U_I + U_F$ . Since  $\dim I = 1$ , we can take  $U_I = v \frac{\partial}{\partial t}$  which gives  $U = v \frac{\partial}{\partial t} + U_F$ , where  $v$  is a function on  $M$ . Then we can write

$$\eta \left( \frac{\partial}{\partial t} \right) = g \left( U, \frac{\partial}{\partial t} \right) = v. \quad (43)$$

Hence, we get

$$\sum_{i=1}^m l_i \left( \lambda_2 \frac{b_i'}{b_i} - \frac{b_i''}{b_i} + \lambda_1^2 - \lambda_1\lambda_2 \right) = \alpha - v^2\beta.$$

Again,  $\check{S}(V, W) = \alpha g(V, W) + \beta \eta(V)\eta(W)$ .

From Proposition 5.1 and equation (42), we obtain that  $(F_i, g_{F_i})$  are quasi-Einstein manifolds.

After a brief calculation, we can easily prove that

$$\begin{aligned} \alpha_{F_i} - b_i b_i'' - (l_i - 1)b_i'^2 + (\lambda_2 b_i^2 - b_i b_i') \sum_{s \neq i} l_s \left( \frac{b_s'}{b_s} \right) \\ + [\lambda_2^2 + (1 - \bar{n})\lambda_1\lambda_2 - \alpha]b_i^2 + [(\bar{n} - 1)\lambda_1 + (l_i - 1)\lambda_2]b_i b_i' = 0 \end{aligned}$$

and  $\beta = \beta_{F_i}$ .

Thus, the proof of the theorem is completed.  $\square$

Next, the following theorem establishes the necessary and sufficient conditions on a multiply warped product to be a quasi-Einstein manifold with a quarter-symmetric connection whenever  $P \in \chi(F_r)$ .

**THEOREM 5.4.** *Let  $M = I \times_{b_1} F_1 \times_{b_2} F_2 \dots \times_{b_m} F_m$  be a multiply warped product with the metric tensor  $-dt^2 \oplus b_1^2 g_{F_1} \oplus \dots \oplus b_m^2 g_{F_m}$  with  $P \in \chi(F_r)$  and  $g_{F_r}(P, P) = 1$  and  $\bar{n} \geq 2$ . Then  $(M, g)$  is a quasi-Einstein manifold with respect to a quarter-symmetric connection  $\check{\nabla}$  with constant associated scalars  $\alpha$  and  $\beta$ , if and only if the following conditions are satisfied:*



- i)  $(F_i, g_{F_i})$  ( $i \neq r$ ) are quasi-Einstein manifolds with scalars  $\alpha_i, \beta_i$ ,  $i \in \{1, 2, \dots, m\}$ ;
- ii)  $b_r$  is constant and  $\sum_{i=1}^m l_i \frac{b_i''}{b_i} = \mu_0$ ,  $div_{F_r} P = \mu_1$ ,  $\mu_0 - \lambda_2 \mu_1 + \alpha - v^2 \beta = [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_2^2] b_r^2$ , where  $\mu_0, \mu_1$  are constants;
- iii)  $S^{F_r}(V, W) + \bar{\alpha} g_{F_r}(V, W) + \beta \eta(V) \eta(W) = [(\bar{n} - 1)\lambda_1^2 - \lambda_2^2] \omega(V) \omega(W) - [(\bar{n} - 1)\lambda_1 - \lambda_2] g(W, \nabla_V P)$ , for  $V, W \in \chi(F_r)$ , where  $\bar{\alpha} = b_r^2 \{ [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_2^2] b_r^2 + \lambda_2 \mu_1 - \alpha \}$ .
- iv)  $\alpha_{F_i} - b_i b_i'' + [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_2^2] b_i^2 b_r^2 - b_i b_i' \sum_{s \neq i} l_s \frac{b_s'}{b_s} - (l_i - 1)(b_i')^2 = (\alpha - \lambda_2 \mu_1) b_i^2$  and  $\beta = \beta_{F_i}$ .

*Proof.* By Proposition 5.2 (ii) and  $g_{F_r}(P, P) = 1$ , it follows that  $b_r$  is a constant. By Proposition 5.2 (i), we obtain

$$\check{S} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = \sum_{i=1}^m l_i \frac{b_i''}{b_i} + [\lambda_2^2 + (1 - \bar{n})\lambda_1 \lambda_2] b_r^2 - \lambda_2 div_{F_r} P = -\alpha + v^2 \beta.$$

By separation of variables, we have

$$\sum_{i=1}^m l_i \frac{b_i''}{b_i} = \mu_0, div_{F_r} P = \mu_1, \mu_0 - \lambda_2 \mu_1 + \alpha - v^2 \beta = [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_2^2] b_r^2.$$

Then we get ii). By proposition 5.2 (v), we have

$$\begin{aligned} \check{S}(V, W) &= S^{F_i}(V, W) + b_i^2 g_{F_i}(V, W) \left\{ (l_i - 1) \frac{-(b_i')^2}{b_i^2} + \frac{-b_i''}{b_i} \right. \\ &+ [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_2^2] \omega(P) + \sum_{s \neq i} l_s \frac{-b_i' b_s'}{b_i b_s} \left. \right\} + [(\bar{n} - 1)\lambda_1 - \lambda_2] g(W, \nabla_V P) \\ &+ [\lambda_2^2 + (1 - \bar{n})\lambda_1^2] \omega(V) \omega(W) + \lambda_2 g(V, W) div_{F_r} P, \quad \text{if } i = j. \end{aligned}$$

When  $i \neq r$ , then  $\nabla_V P = \omega(V) = 0$ , so,

$$\begin{aligned} \check{S}(V, W) &= S^{F_i}(V, W) + b_i^2 g_{F_i}(V, W) \left\{ (l_i - 1) \frac{-(b_i')^2}{b_i^2} + \frac{-b_i''}{b_i} \right. \\ &+ [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_2^2] \omega(P) + \sum_{s \neq i} l_s \frac{-b_i' b_s'}{b_i b_s} \left. \right\} + \lambda_2 \mu_1 b_i^2 g_{F_i}(V, W) \\ &= \alpha b_i^2 g_{F_i}(V, W) + \beta \eta(V) \eta(W). \end{aligned}$$

By separation of variables, it follows that  $(F_i, g_{F_i})$  ( $i \neq r$ ) are quasi-Einstein manifolds with scalars  $\alpha_i, \beta_i, i \in \{1, 2, \dots, m\}$ , and

$$\begin{aligned} \alpha_{F_i} - b_i b_i'' + [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_2^2] b_i^2 b_r^2 - b_i b_i' \sum_{s \neq i} l_s \frac{b_s'}{b_s} - (l_i - 1)(b_i')^2 \\ = (\alpha - \lambda_2 \mu_1) b_i^2 \end{aligned}$$

and  $\beta = \beta_{F_i}$ . Then we have *i)* and *iv)*.

When  $i = r$  and  $b_r$  is a constant, then we get

$$\begin{aligned} S^{F_r}(V, W) + \bar{\alpha} g_{F_r}(V, W) + \beta \eta(V) \eta(W) \\ = [(\bar{n} - 1)\lambda_1^2 - \lambda_2^2] \omega(V) \omega(W) - [(\bar{n} - 1)\lambda_1 - \lambda_2] g(W, \nabla_V P), \\ \text{for } V, W \in \chi(F_r), \end{aligned}$$

where  $\bar{\alpha} = b_r^2 \{ [(\bar{n} - 1)\lambda_1 \lambda_2 - \lambda_2^2] b_r^2 + \lambda_2 \mu_1 - \alpha \}$ , and thus we obtain *iii)*.  $\square$

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