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A Monomiality Principle Approach to the Gould-Hopper Polynomials

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Summary. - *We show how to derive properties of the Gould-Hopper polynomials using operational rules associated with the monomiality principle.*

1. Introduction

In this paper we consider the Gould-Hopper polynomials (GHP) which are a generalization of the Hermite polynomials $\{H_n(x)\}_{n\in\mathbf{N}}$ established in [6] by H.W. Gould and A.T. Hopper essentially by replacing the exponent 2 in their Rodrigues formula with an arbitrary parameter. M any authors investigated properties of these polynomials (see [10, 8, 7, 1]), using classical methods well known in the special functions theory. The GHP's fall, with suitable choices of additional parameters, in the families of polynomials given by further generalizations of the Hermite polynomials generating function introduced by M. Lahiri [8] and by R.P. Gupta and G.C. Jain in [7]. Also, in [10] H.M. Srivastava considered the general class of polynomials generated by $G[(p+1)xt - t^{p+1}]$ (where p is a positive integer and $G[z]$ has an analytical expansion at $z = 0$ with nonzero coefficients) and proved that the Gould-Hopper polynomials are contained in this class. Recently, Y. Ben Cheikh and K. Douak investigated the p-orthogonal polynomials defined by $G[(p+1)xt - t^{p+1}]$ and obtained,

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among other properties, that they are necessarily p-symmetric (see [1, Theorem 1.2]).

In this note we restrict ourselves to the Gould-Hopper generating function $e^{(p+1)xt-t^{p+1}}$, showing that the study of the properties of the GHP's is greatly simplified by the use of a new approach based on the "monomiality" principle. The outline of the paper is as follows. In Section 2 we introduce the concept of quasi-monomiality and we show that the GHP's are quasi-monomials under the action of two suitable operators. In Section 3, following the prescription of the monomiality point of view, we derive most of the properties of the GHP's: generating function, explicit expression, hypergeometric representation, differential equation, recurrence relation and p-symmetry property.

2. Quasi-monomiality of the GHP's

G. Dattoli et al. in recent papers (see e.g. [2, 3, 4]) introduced the monomiality principle. A polynomial sequence $\{p_n(x)\}_{n\in\mathbf{N}}$ can be considered a quasi-monomial sequence, if it is possible to define two operators $\hat{P}$ and $\hat{M}$ in such a way that

$$\hat{P}(p_n(x)) = np_{n-1}(x)$$

$$\hat{M}(p_n(x)) = p_{n+1}(x).$$

$\hat{M}$ and $\hat{P}$ play the role analogous to that of multiplicative and derivative operators respectively on monomials. Most of the properties of families of polynomials, recognized as quasi-monomials, can be deduced using operational rules with the $\hat{P}$ and $\hat{M}$ operators.

Namely:

i) If $\hat{M}$ and $\hat{P}$ have a differential realization, the polynomials $p_n(x)$ satisfy the differential equation

$$\hat{M}\hat{P}(p_n(x)) = np_n(x).$$

ii) If $p_0(x) = 1$, then the $p_n(x)$ can be explicitly constructed using the the Burchnall-type equation

$$p_n(x) = \hat{M}^n(1).$$

iii) The multiplicative and derivative operators satisfy the commutation bracket:

$$[\hat{P}, \hat{M}] = 1$$

and the above relation of commutation displays a Weyl group structure.

Note that the statement ii) implies that a generating function of $p_n(x)$ can always be cast in the form $e^{t\hat{M}}(1)$.

There is no theorem ensuring that it is possible to define $\hat{P}$ and $\hat{M}$ operators for any family of polynomials. In spite of that one can prove that most of the known families of polynomials can be treated as quasi-monomials.

Now, for all finite x and t, we have

$$e^{(p+1)xt-t^{p+1}} = \sum_{n=0}^{\infty} H_{p,n}(x)\frac{t^n}{n!}. \tag{1}$$

If p is equal to 1 the function e^{2xt-t^2} is the one that generates the classical Hermite polynomials $H_n(x)$.

It follows from Eq. (1):

$$\frac{\partial}{\partial x}e^{(p+1)xt-t^{p+1}} = \sum_{n=1}^{\infty} H'_{p,n}(x)\frac{t^n}{n!}. \tag{2}$$

$$\begin{aligned} \frac{\partial}{\partial x}e^{(p+1)xt-t^{p+1}} &= (p+1)te^{(p+1)xt-t^{p+1}} \\ &= (p+1)\sum_{n=0}^{\infty} H_{p,n}(x)\frac{t^{n+1}}{n!} \\ &= (p+1)\sum_{n=1}^{\infty} H_{p,n-1}(x)\frac{t^n}{(n-1)!} \\ &= (p+1)\sum_{n=1}^{\infty} nH_{p,n-1}(x)\frac{t^n}{n!}. \end{aligned} \tag{3}$$

Comparing Eqs. (2) and (3), and using the identity principle for power expansions, for $n \geq 1$, we get

$$(p+1)nH_{p,n-1}(x) = H'_{p,n}(x). \tag{4}$$

Then we obtain the operator $\hat{P}$ as follows:

$$\hat{P} = \frac{1}{p+1}\frac{d}{dx}.$$

In order to find the operator $\hat{M}$, iterative differentiation of Eq. (4) yields

$$(p+1)^p n(n-1)\cdots(n-p+1)H_{p,n-p} = H_{p,n}^{(p)}(x). \tag{5}$$

We exploit Eq. (1) again, obtaining

$$\frac{\partial}{\partial t}e^{(p+1)xt-t^{p+1}} = \sum_{n=1}^{\infty} H_{p,n}(x)\frac{t^{n-1}}{(n-1)!}. \tag{6}$$

On the other hand

$$\begin{aligned}\frac{\partial}{\partial t}e^{(p+1)xt-t^{p+1}} &= ((p+1)x-(p+1)t^p)e^{(p+1)xt-t^{p+1}}\\ &= (p+1)\sum_{n=0}^{\infty} xH_{p,n}(x)\frac{t^n}{n!} + \\ &\quad -(p+1)\sum_{n=0}^{\infty} H_{p,n}(x)\frac{t^{n+p}}{n!}.\end{aligned} \tag{7}$$

Equating Eqs. (6) and (7), with suitable shift of indices we get

$$(p+1)\sum_{n=0}^{\infty} xH_{p,n}(x)\frac{t^n}{n!} - (p+1)\sum_{n=p}^{\infty} n(n-1)\cdots(n-p+1)H_{p,n-p}(x)\frac{t^n}{n!} =$$

$$= \sum_{n=0}^{\infty} H_{p,n+1}(x)\frac{t^n}{n!},$$

by using again the identity principle, for $n \geq p$, we have

$$H_{p,n+1}(x) = (p+1)(xH_{p,n}(x) - n(n-1)\cdots(n-p+1)H_{p,n-p}(x))$$

and, recalling Eq. (5),

$$H_{p,n+1}(x) = (p+1)xH_{p,n}(x) - (p+1)^{1-p}H_{p,n}^{(p)}(x).$$

This gives us the operator $\hat{M}$:

$$\hat{M} = (p+1)x - (p+1)^{1-p}\frac{d^p}{dx^p}.$$

Note that straightforwardly from [6, Eqs. (6.4) and (6.7)], we could have deduced the quasi-monomiality of the GHP's. Nevertheless it was worth outlining the above technique in order to emphasize the effectiveness of such a procedure with the generating function as a starting point.

Observe that $\hat{M} = (p+1)x - (p+1)\hat{P}^p$. This implies that the operators satisfy, as point iii) asks for, the commutation bracket:

$$[\hat{P}, \hat{M}] = [\hat{P}, (p+1)x] = [\frac{d}{dx}, x] = 1.$$

3. Properties of the GHP's

In this section we show how to derive properties of the Gould-Hopper polynomials following the prescription of the monomiality point of view, i.e. using exponential operators, disentanglement identities and other techniques described in [5] involving the $\hat{P}$ and $\hat{M}$ operators.

3.1. Generating function

One of the rules relevant to the action of exponential operators on a given function is the Crofton identity:

$$e^{\lambda\frac{d^m}{dx^m}} f(x) = f(x + m\lambda\frac{d^{m-1}}{dx^{m-1}}),$$

where λ is a parameter and $f(x)$ is infinitely differentiable. Applying the above identity on $f(x) = e^{t(p+1)x}$ (with $\lambda = -\frac{1}{(p+1)^{p+1}}$ and $m =$

$p+1$) and the exponential expansion, we find the generating function:

$$\begin{aligned} e^{t\hat{M}}(1) &= e^{t(p+1)(x-\frac{1}{(p+1)^p}\frac{d^p}{dx^p})}(1) = e^{-\frac{1}{(p+1)^{p+1}}\frac{d^{p+1}}{dx^{p+1}}}e^{t(p+1)x}(1) \\ &= \sum_{n=0}^{\infty}\frac{(-\frac{1}{(p+1)^{p+1}})^n}{n!}\frac{d^{n(p+1)}}{dx^{n(p+1)}}e^{t(p+1)x} \\ &= \sum_{n=0}^{\infty}\frac{(-t^{p+1})^n}{n!}e^{t(p+1)x} \\ &= e^{(p+1)xt-t^{p+1}}. \end{aligned}$$

3.2. Explicit expression

Directly from point ii), it is easy to obtain the explicit representation:

$$H_{p,n}(x) = ((p+1)x - (p+1)^{1-p}\frac{d^p}{dx^p})^n(1).$$

In particular, if $n = 1$, we have, $\forall p$, $H_{p,1}(x) = (p+1)x$ (and if $p = 1$, $H_1(x) = 2x$). If $n = 2$, the operator is

$$(p+1)^2x^2 + (p+1)^{2-2p}\frac{d^{2p}}{dx^{2p}} - (p+1)^{2-p}\frac{xd^p}{dx^p} - (p+1)^{2-p}\frac{d^p}{dx^p}x,$$

where, thanks to the commutation bracket $[x, \frac{d^p}{dx^p}] = -p\frac{d^{p-1}}{dx^{p-1}}$, we get:

$$H_{p,2}(x) = ((p+1)^2x^2+(p+1)^{2-2p}\frac{d^{2p}}{dx^{2p}}-(p+1)^{2-p}(2\frac{xd^p}{dx^p}+\frac{d^{p-1}}{dx^{p-1}}))(1).$$

If $p = 1$, $H_2(x) = 4x^2 - 2$ and, $\forall p > 1$, $H_{p,2}(x) = (p+1)^2x^2$.
In a similar way we could construct theoretically all the Gould-Hopper polynomials.

Otherwise, another explicit representation of the GHP's can be derived in the classical way:

$$
\begin{aligned}
e^{(p+1)xt-t^{p+1}} &= e^{(p+1)xt}e^{-t^{p+1}} \\
&= \sum_{n=0}^{\infty} \frac{((p+1)x)^n t^n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m t^{(p+1)m}}{m!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\left[\frac{n}{(p+1)}\right]} n! \frac{(-1)^m ((p+1)x)^{n-(p+1)m}}{m!(n-(p+1)m)!} \frac{t^n}{n!},
\end{aligned}
$$

where $\left[\frac{n}{(p+1)}\right]$ denotes the integer part of $\frac{n}{(p+1)}$. Therefore

$$
H_{p,n}(x) = \sum_{m=0}^{\left[\frac{n}{(p+1)}\right]} n! \frac{(-1)^m ((p+1)x)^{n-(p+1)m}}{m!(n-(p+1)m)!}. \tag{8}
$$

If p=1, we have

$$
H_n(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]} n! \frac{(-1)^m (2x)^{n-2m}}{m!(n-2m)!}.
$$

We observe that $H_{p,n}(x)$ is a polynomial of degree n in x and that $H_{p,n}(x) = (p+1)^n x^n + \Pi_{n-(p+1)}(x)$, where $\Pi_{n-(p+1)}(x)$ is a polynomial of degree $n-(p+1)$ in x.

With the monomiality principle approach, the proof of the representation in (8) may be easily obtained, using the multiplicative operator, by induction. The first step is obvious. Supposing true the identity in (8), we get

$$
\begin{aligned}
H_{p,n+1}(x) &= ((p+1)x-(p+1)^{1-p}\frac{d^p}{dx^p})H_{p,n}(x) \\
&= \sum_{m=0}^{\left[\frac{n}{(p+1)}\right]} n!\frac{(-1)^m((p+1)x)^{n+1-(p+1)m}}{m!(n-(p+1)m)!} + \\
&\quad -(p+1)\sum_{m=0}^{\left[\frac{n-p}{(p+1)}\right]} n!\frac{(-1)^m((p+1)x)^{n-(p+1)m-p}}{m!(n-(p+1)m-p)!} \\
&= \sum_{m=0}^{\left[\frac{n+1}{(p+1)}\right]} (n+1)!\frac{(-1)^m((p+1)x)^{n+1-(p+1)m}}{m!(n+1-(p+1)m)!}.
\end{aligned}
$$

To get the last equality observe first of all that, for $m \geq 1$, the m-th term of the first sum has the same power as the $(m-1)$-th term of the second one and that

$$
\frac{n!(-1)^m}{m!(n-(p+1)m)!} - \frac{(p+1)n!(-1)^{(m-1)}}{(m-1)!(n-(p+1)(m-1)-p)!} = \\
= \frac{(n+1)!(-1)^m}{m!(n+1-(p+1)m)!}.
$$

This is enough in case $n \equiv r \bmod (p+1)$ with $r < p$, i.e. $\left[\frac{n}{(p+1)}\right] = \left[\frac{n+1}{(p+1)}\right]$, while in case $n \equiv p \bmod (p+1)$, i.e. $\left[\frac{n}{(p+1)}\right] = \left[\frac{n+1}{(p+1)}\right] - 1$, there is an extra term in the second sum:

$$
-\frac{(p+1)(-1)^{\left[\frac{n-p}{(p+1)}\right]}n!}{\left[\frac{n-p}{(p+1)}\right]!(n-(p+1)\left[\frac{n-p}{(p+1)}\right]-p)!} = \frac{(-1)^{\left[\frac{n+1}{(p+1)}\right]}(n+1)!}{\left[\frac{n+1}{(p+1)}\right]!},
$$

and this concludes the proof.

3.3. Hypergeometric representation

From the representation in (8), it is easy to derive, using the notation $(\alpha)_0 = 1,\ \alpha \neq 0,\ (\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1),\ n \geq 1$,

$$
H_{p,n}(x) = ((p+1)x)^n \sum_{m=0}^{\left[\frac{n}{(p+1)}\right]} \frac{(-n)_{(p+1)m}(-1)^m x^{-(p+1)m}}{m!(p+1)^{(p+1)m}}
$$

which gives the hypergeometric representation of the $H_{p,n}(x)$:

$$((p+1)x)^n \;{}_{p+1}F_0$$

$$\left[-\frac{n}{p+1}, -\frac{n}{p+1}+\frac{1}{p+1}, \cdots, -\frac{n}{p+1}+\frac{p}{p+1}; -; -\frac{1}{x^{p+1}}\right].$$

If $p=1$, we have $H_n(x) = (2x)^n \;{}_2F_0\left[-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}; -; -\frac{1}{x^2}\right]$.

3.4. $(p+1)$-order differential equation

We can find the differential equation by using the generating function method described in the book of Rainville [9]. In the particular case under consideration, starting by the system

$$\begin{cases} \dfrac{\partial}{\partial x} e^{(p+1)xt-t^{p+1}} & = (p+1)te^{(p+1)xt-t^{p+1}} \\ \dfrac{\partial}{\partial t} e^{(p+1)xt-t^{p+1}} & = ((p+1)x-(p+1)t^p)e^{(p+1)xt-t^{p+1}}, \end{cases}$$

we obtain

$$\frac{\partial}{\partial t} e^{(p+1)xt-t^{p+1}} = \frac{x-t^p}{t}\frac{\partial}{\partial x} e^{(p+1)xt-t^{p+1}}$$

and, by Eqs. (2) and (6),

$$\sum_{n=0}^{\infty} nH_{p,n}(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} xH'_{p,n}(x)\frac{t^n}{n!} + \sum_{n=p}^{\infty} H'_{p,n-p}(x)\frac{t^n}{(n-p)!} = 0$$

$$xH'_{p,n}(x) - nH_{p,n}(x) = \frac{n!}{(n-p)!}H'_{p,n-p}(x) = 0 \qquad (n \geq p), \tag{9}$$

finally, using p times Eq. (4), we get

$$\frac{1}{(p+1)^p}H^{(p+1)}_{p,n}(x) - xH^{(1)}_{p,n} + nH_{p,n}(x) = 0. \tag{10}$$

If $p=1$, Eq. (10) reduces to the well known second-order differential equation satisfied by the Hermite polynomials: $H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0$.

On the other hand, the monomiality principle reduces a great deal of lengthy computations. Indeed, according to property i), we get the equation:

$$((p+1)x-(p+1)^{1-p}\frac{d^p}{dx^p})\frac{1}{(p+1)}\frac{d}{dx}(H_{p,n}(x))=nH_{p,n}(x),$$

that is Eq. (10).

3.5. Recurrence relation

Examination of the defining relation (1) gives us the $(p+1)$-order recurrence relation satisfied by the GHP's.

$$\frac{\partial}{\partial t}e^{(p+1)xt-t^{p+1}}=((p+1)x-(p+1)t^p)e^{(p+1)xt-t^{p+1}},$$

$$\sum_{n=-1}^{\infty}(n+1)H_{p,n+1}(x)\frac{t^n}{(n+1)!}=(p+1)x\sum_{n=0}^{\infty}H_{p,n}(x)\frac{t^n}{n!}+$$

$$-(p+1)\sum_{n=p}^{\infty}H_{p,n-p}(x)\frac{t^n}{(n-p)!},$$

$$H_{p,n+1}(x)=(p+1)xH_{p,n}(x)-(p+1)!\binom{n}{p}H_{p,n-p}(x),\ (n\geq p). \quad (11)$$

If $p=1$, we rediscover $H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x)$. Observe that Eq. (11) alternatively follows from Eqs. (4) and (9).

Also in this case the recurrence relation can be derived, recalling the multiplicative and derivative actions of $\hat{M}$ and $\hat{P}$, in only one step:

$$H_{p,n+1}(x)=\hat{M}(H_{p,n})=(p+1)xH_{p,n}-(p+1)\hat{P}^pH_{p,n}=$$

$$(p+1)xH_{p,n}-(p+1)\frac{n!}{(n-p)!}H_{p,n-p},$$

that is Eq. (11).

3.6. p-symmetry property

Recall the following:

Definition 3.1. *A sequence* $\{P_n(x)\}_{n\geq 0}$ *is called* p*-symmetric if* $P_n(\omega x) = \omega^n P_n(x)$*, where* $\omega = \exp(2i\pi/(p+1))$.

The fact that the GHP's are p-symmetric follows essentially from the generating function itself (see [1, Lemma 3.1]) :

$$\begin{aligned} e^{(p+1)\omega x t - t^{p+1}} &= e^{(p+1)\omega x t - (\omega t)^{p+1}}, \\ \sum_{n=0}^{\infty} H_{p,n}(\omega x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} H_{p,n}(x)\frac{(\omega t)^n}{n!}, \\ H_{p,n}(\omega x) &= \omega^n H_{p,n}(x), \end{aligned} \tag{12}$$

and this gives us the p-symmetry property. If $p=1$, we get $H_n(-x) = (-1)^n H_n(x)$.

We can otherwise get the p-symmetry exploiting the monomiality principle. To do so, we need the following:

Lemma 3.2. *Let f be an analytic function and k be an integer such that $f(\omega z) = \omega^k f(z)$ where $\omega = exp(2i\pi/(p+1))$. Then $\hat{M}(f)(\omega z) = \omega^{k+1}\hat{M}(f)(z)$.*

This is easy to verify: differentiating p times $f(\omega z) = \omega^k f(z)$ gives us $\omega^p \frac{d^p f}{dz^p}(\omega z) = \omega^k \frac{d^p f}{dz^p}(z)$, that is, being ω a generator of the $(p+1)$ roots of unity, $\frac{d^p f}{dz^p}(\omega z) = \omega^{k+1}\frac{d^p f}{dz^p}(z)$ and we get

$$\hat{M}(f)(\omega z) = (p+1)\omega z f(\omega z) - (p+1)^{1-p}\frac{d^p f}{dz^p}(\omega z) =$$

$$= (p+1)\omega^{k+1} z f(z) - (p+1)^{1-p}\omega^{k+1}\frac{d^p f}{dz^p}(\omega z) = \omega^{k+1}\hat{M}(f)(z).$$

Applying recursively the above lemma to the constant analytic function $f=1$ (that satisfies the hypothesis with $k=0$), to $f = \hat{M}(1)$ (that, thanks to the lemma, satisfies the hypothesis with $k=1$) up to $f = \hat{M}^{n-1}(1)$ (with $k = n-1$), we derive $\hat{M}^n(1)(\omega z) = \omega^n \hat{M}^n(1)(z)$. Recalling point ii), we get directly Eq. (12).

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