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On the existence of nontrivial solutions of differential equations subject to linear constraints

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This paper is a birthday present for Jean Mawhin, my dear friend and valued collaborator of many years. Greetings and all the best wishes from afar

Abstract. *The purpose of this paper is to consider boundary value problems for second order ordinary differential equations where the solutions sought are subject to a host of linear constraints (such as multi-point constraints) and to present a unifying framework for studying such. We show how Leray-Schauder continuation techniques may be used to obtain existence results for nontrivial solutions of a variety of nonlinear second order differential equations. A typical example may be found in studies of the four-point boundary value problem for the differential equation* $y''(t)+a(t)f(y(t)) = 0$ *on* $[0,1]$, *where the values of* y *at* 0 *and* 1 *are each some multiple of* $y(t)$ *at two interior points of* $(0,1)$. *The techniques most often used in such studies have their origins in fixed point theory. By embedding such problems into parameter dependent ones, we show that detailed information may be obtained via global bifurcation theory. Of course, such techniques, as they are consequences of properties of the topological degree, are similar in nature.*

Keywords: second order ode's; nonlinear multi-point boundary value problem; linear constraints; global bifurcation.
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1. Introduction

This paper is motivated by the paper [15] and several related ones (e.g. [7, 8, 16, 21, 42, 43, 45]), where the authors were interested in the existence of positive solutions of second-order nonlinear differential equations

$$y''(t) + a(t)f(y(t)) = 0, \quad 0 < t < 1 \tag{1}$$

subject to the four-point boundary conditions

$$y(0) = \alpha y(\xi), \quad y(1) = \beta y(\eta) \tag{2}$$

where $0 < \xi \leq \eta < 1$, $a(t)$ is a nonzero continuous, and nonnegative function on $(0, 1)$ and

$$f : \mathbb{R} \to \mathbb{R}, \; f : [0, \infty) \to [0, \infty)$$

is continuous, or other similar multi-point boundary value problems. In case $\xi = \eta$ and $\alpha + \beta \neq 2$, boundary conditions (2) were already considered Loud in [22], where Green's functions and their properties of such multi-point boundary value problems and their adjoints were discussed in great detail.

Under the assumption that the limits

$$f_0 = \lim_{u \to 0} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}. \tag{3}$$

exist and satisfy certain inequalities, it was proved [15] that (1), (2) has a positive solution. The proof was based on a use of the Krasnosel'skii compression and expansion theorems for positive completely continuous operators on a Banach space [14]. Results for the existence of solutions of nonlinear boundary value problems where the nonlinear terms behave as in (3) have a long history and such results (usually for boundary value problems subject to homogeneous end point boundary conditions, but also valid for nonlinear elliptic partial differential equations) may be found in [1, 2, 6, 9, 26, 27, 28, 44]. While the boundary conditions (2) are very much much different from those usually employed, such as Dirichlet, Neumann, Robin, or periodic ones, it is still straight forward to transform the problems into equivalent integral equations (cf. [7, 8, 13, 15, 16, 21, 23, 24, 39, 40, 45]) and thus employ fixed point theory for completely continuous operators on a Banach space of continuous functions. Further studies are also available for problems defined on time scales, see e.g. [3, 12, 46], among others.

Since the approach used here is variational and uses global bifurcation theory, the results and approach discussed here for the semilinear case should be extendable to problems of a nonlinear nature for both ordinary and elliptic partial differential equations, such as problems involving the p-Laplacian, and obtain results as in [17, 18, 25, 33].

In this paper we shall discuss a class of nonlinear boundary value problems and show, using global bifurcation techniques ([4, 30, 31, 32]), how solutions may be obtained as part of a continuum of solutions of a problem which depends upon a parameter into which the given problem has been imbedded. We shall adhere here to a prototypical example motivated by (1), (2) but want to point out that similar arguments may be used to obtain results of this type for semilinear and nonlinear elliptic problems in higher dimensions using, see e.g. [17]. We shall not attempt to consider these more general situations here, but remark that some of the work cited here will provide the tools for studying such problems.

2. Notation, assumptions, and preliminaries

We let V be a closed subspace of $H^1(0,1)$ which has the property that 0 is the only constant function that belongs to V and in addition that there exists an open set

$$\Omega \subset (0,1), \text{ such that } \bar{\Omega} = [0,1], \ m(\Omega) = 1, \ C_0^\infty(\Omega) \subset V,$$

(here $m(\cdot)$ denotes Lebesgue measure).

For example, if $L : H^1(0,1) \to \mathbb{R}^2$ is defined by the boundary conditions (2) as

$$Ly := (y(0) - \alpha y(\xi), y(1) - \beta y(\eta)), \ 0 < \xi \leq \eta < 1, \ \alpha \neq 1$$

then

$$V := \{u \in H^1(0,1) : Lu = 0\}$$

is such a subspace with

$$\Omega := (0,\xi) \cup (\xi,\eta) \cup (\eta,1).$$

For other examples of operators L defined by multipoint boundary conditions, we refer the interested reader to [7, 8, 15, 16], and the references in these papers and those in the other references given above. Of course, homogeneous Dirichlet and anti periodic boundary conditions ($y(0) = -y(1)$) yield such examples, as do the boundary conditions

$$u(0) = 0,$$

or

$$u(0) = 0, \ u(\eta) = \alpha u(1), \ \eta \in (0,1),$$

or

$$\alpha u(\eta) + \beta u(\mu) = u(1), \ 0 < \eta < \mu < 1, \ \alpha, \beta \geq 0, \ \alpha + \beta < 1,$$

whereas classical Neumann and periodic boundary conditions do not (note that these boundary conditions are natural ones imposed by minimization problems in $H^1(0,1)$, respectively in $\{u \in H^1(0,1) : u(0) = u(1)\}$).

The norm of $H^1(0,1)$ is given by

$$\|u\|_{H^1}^2 = \int_0^1 (u')^2 dt + \int_0^1 u^2 dt$$

and it is the case that

$$\|u\|^2 := \int_0^1 (u')^2 dt$$

defines an equivalent norm on such subspaces V, i.e., there exists a positive constant c such that

$$\|u\|_{L^2(0,1)} \leq c\|u'\|_{L^2(0,1)}, \forall u \in V.$$

To see this, one may use an often employed argument of Nečas [20], and assume there exists a sequence $\{u_n\} \subset V$ such that

$$\|u_n\|_{L^2(0,1)} \geq n\|u_n'\|_{L^2(0,1)}, n = 1, 2, \cdots . \tag{4}$$

Then we may assume that $\|u_n\|_{L^2(0,1)} = 1, n = 1, 2, \cdots$. So $\{u_n\}$ is bounded in $H^1(0,1)$, hence may assumed to converge weakly to say u. Hence it will converge strongly to u in $L^2(0,1)$. So, by (4) $u_n' \to 0$ in $L^2(0,1)$, which implies that $u' = 0$, i.e. u must be piecewise constant, but since u is continuous, it must be a constant throughout. On the other hand, V is closed and hence, since $u \in V$, u must equal 0, a contradiction.

Definition 2.1. *For given V, as above, we let V' denote its topological dual and for $h \in V'$, we call $u \in V$ a weak solution of the boundary value problem*

$$-u'' = h, \ u \in V, \tag{5}$$

provided that

$$\int_0^1 u'v'dt = (h, v), \ \forall v \in V, \tag{6}$$

where

$$(\cdot,\cdot) : V' \times V \to \mathbb{R}$$

is the pairing between V' and V.

The above considerations have the following immediate consequence, whose proof follows from the Lax-Milgram theorem (see [38]) and the fact that V is a Hilbert space with respect to the inner product

$$(u, v)_V := \int_0^1 u'v'dt.$$

Lemma 2.2. *Let V be as above, and let V' be its topological dual, then for every $h \in V'$ there exists a unique $u \in V$ which is a unique weak solution of (5). Further*

$$\|u\| \leq \|h\|_{V'}.$$

Remark 2.3. If $h \in L^2(0,1)$, we may deduce that the weak solution u, given above, since $C_0^\infty(\Omega) \subset V$, must satisfy

$$u'' = h, \text{ on } \Omega$$

in the sense of distributions and thus u is a solution of the differential equation on Ω, further, since $m(\Omega) = 1$, it follows that u is a solution on $(0,1)$, as well.

Let $a : [0,1] \to [0,\infty)$ be a continuous nontrivial function, then, via this lemma, we may define the mapping

$$T : L^2(0,1) \to V \subset H^1(0,1) \hookrightarrow L^2(0,1),$$

by

$$Th := u,$$

where u is the unique weak solution of

$$-u'' = ah, \ u \in V, \tag{7}$$

and hence u solves the differential equation (7) on Ω in a classical sense (viz. $C_0^\infty(\Omega) \subset V$). We note that the last inclusion is compact. Thus,

$$T : L^2(0,1) \to L^2(0,1),$$

is compact linear mapping. Thus, we have that

$$T : C[0,1] \to H^2(0,1) \hookrightarrow C^1[0,1] \hookrightarrow C[0,1],$$

i.e., we may even view T as a compact linear mapping

$$T : C[0,1] \to C[0,1],$$

and we may apply the Riesz theory for compact linear operators to obtain the spectral properties of this operator. For general multi-point boundary value problems, the study of the spectrum of the associated integral operator, has a long history, with notable contributions in [22], and recently in [5]. In fact, since the problems, in general are not self-adjoint, complex eigenvalues may exist. In the case at hand, we shall not be concerned with such complications but rather concentrate on boundary conditions (subspaces V) which have one distinguished positive eigenvalue (see below), namely a smallest positive one, called λ_1.

Remark 2.4. Since there exists $u \in V \setminus \{0\}$, such that

$$Tu = \frac{1}{\lambda_1} u,$$

we have that

$$\int_0^1 u'v' dt = \lambda_1 \int_0^1 auv\, dt, \ \forall v \in V$$

we obtain that (by normalizing)

$$0 < \lambda_1 = \inf_V \left\{ \int_0^1 (v')^2 dt \ : \ \int_0^1 av^2 dt = 1 \right\}.$$

In the given generality not much else may be asserted concerning the spectrum of T. In fact, the first example below shows that the principal eigenvalue may be of multiplicity 2.

Example 2.5. **a.** Let the space V be defined by

$$V := \left\{ u \in H^1(0,1) : u(0) = u(1), \;\; \int_0^1 u\,dt = 0 \right\}.$$

Then V is a closed subspace with 0 the only constant function. In the case that $a \equiv 1$, the eigenfunctions of the operator T satisfy

$$\int_0^1 u'v'dt = \lambda_1 \int_0^1 uv\,dt, \;\; \forall v \in V,$$

and, since $H_0^1(0,1) \subset V$ we have that

$$-u'' = \lambda_1 u,$$

in the sense of distributions. Integrating the last equality we obtain that (since $u \in H^2(0,1)$)

$$u'(0) = u'(1),$$

and so u is an eigenfunction of

$$-u'' = \lambda_1 u, \;\; u(0) = u(1), \;\; u'(0) = u'(1),$$

i.e. $\lambda_1 = 4\pi^2$, with an associated 2-dimensional eigenspace.

b. Let the space V be defined by

$$V := \left\{ u \in H^1(0,1) : \;\; \int_0^1 u\,dt = 0 \right\}.$$

Then, again, V is a closed subspace with 0 the only constant function. With $a \equiv 1$, the eigenfunctions of the operator T satisfy

$$\int_0^1 u'v'dt = \lambda_1 \int_0^1 uv\,dt, \;\; \forall v \in V,$$

and, since $H_0^1(0,1) \subset V$ we have that

$$-u'' = \lambda_1 u, \tag{8}$$

in the sense of distributions. Multiplying the equality (8) by $v \in V$ and integrating, we obtain that

$$-u'(1)v(1) + u'(0)v(0) + \int_0^1 u'v'dt = \lambda_1 \int_0^1 uv\,dt,$$

and hence, choosing v such that $v(0) = v(1) \neq 0$ we obtain

$$u'(0) = u'(1).$$

Further, choosing $v(0) = 0,\ v(1) \neq 0$, we must have $u'(0) = 0$. Hence u is an eigenfunction of the Neumann problem

$$-u'' = \lambda_1 u,\ u'(0) = u'(1) = 0$$

i.e. $\lambda_1 = \pi^2$, the second eigenvalue of the Neumann problem with an associated 1-dimensional eigenspace, spanned by $u(t) = \cos \pi t$.

Both of the above examples, of course, are examples of classical Sturm-Liouville boundary value problems, where, because of the constraints built into the space V, the eigenvalue λ_1 is actually the second eigenvalue of the problem (8) with respect to either periodic or Neumann boundary conditions in the space $H^1(0,1)$.

Next let us consider the example, related to (1)

$$-u''(t) = \lambda a(t) u, \quad 0 < t < 1 \tag{9}$$

subject to the four-point boundary conditions

$$u(0) = \alpha u(\xi), \quad u(1) = \beta u(\eta),\ 0 < \xi < \eta < 1, \tag{10}$$

where, as above, $a : [0,1] \to [0, \infty)$ is a continuous function assuming positive values.

Proposition 2.6. *Assuming that*

$$0 < \alpha, \beta < 1,$$

then the principal (weak) eigenvalue of (9), (10) is positive, simple, and has an associated eigenfunction which is positive in [0,1]. All other eigenvalues are simple, as well, and eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the L^2 inner product with weight function a.

Proof. In this case we define

$$V = \{u \in H^1(0,1) : u(0) = \alpha u(\xi), \quad u(1) = \beta u(\eta)\}.$$

Then V is a closed subspace of $H^1(0,1)$ with $C_0^\infty((0,\xi) \cup (\xi,\eta) \cup (\eta,1))$ dense in V. The principal (weak) eigenvalue is characterized by

$$0 < \lambda_1 = \inf_{v \in V} \left\{ \int_0^1 (v')^2 dt : \ \int_0^1 a v^2 dt = 1 \right\},$$

furthermore this infimum is assumed, by, say $u \in V$, and u satisfies

$$\int_0^1 u'v' dt = \lambda_1 \int_0^1 auv\, dt,\ \forall v \in V.$$

Since, for $v \in V$, we have that $|v| \in V$ and since

$$0 < \lambda_1 = \inf_{v \in V} \left\{ \int_0^1 |v|'^2 dt : \int_0^1 a|v|^2 dt = 1 \right\},$$

we may assume that the eigenfunction u is one signed, say $u \geq 0$, which implies, because of the boundary conditions that $u > 0$ in $[0,1]$. Hence, again because of the boundary conditions, and, since

$$-u'' = \lambda_1 a u,$$

u will assume its maximum in the interval $[\xi, \eta]$. If v is any other eigenfunction corresponding to λ_1, we may assume $v(0) \geq 0$. If $v(0) > 0$, we may let $w(t) = \mu v(t)$, where $\mu = \frac{u(0)}{v(0)}$. Then w is an eigenfunction with

$$w(0) = u(0)$$

and hence

$$z(t) := u(t) - w(t)$$

is an eigenfunction having zeros at 0 and ξ, which by the Sturm Separation Theorem [11] implies that u must vanish in $(0, \xi)$. Thus it must be the case that $w(t) \equiv u(t)$. If, on the other hand, $v(0) = 0$, then $v(\xi) = 0$, then we again obtain a contradiction by use of the Sturm Separation Theorem.

Next, let u_i and u_j be eigenfunctions corresponding to the eigenvalues λ_i and λ_j, $i \neq j$. Then

$$\int_0^1 u_l' v' dt = \lambda_l \int_0^1 a u_l \, v \, dt, \ \forall v \in V, \ l = i, j$$

and hence

$$\int_0^1 u_i' u_j' dt = \lambda_i \int_0^1 a u_i u_j dt = \lambda_j \int_0^1 a u_i u_j dt,$$

thus

$$(\lambda_j - \lambda_i) \int_0^1 a u_i u_j dt = 0.$$

□

3. Bifurcating continua

We shall assume that

$$a : [0,1] \to [0, \infty), \ f : \mathbb{R} \to \mathbb{R}, \ f : (0, \infty) \to (0, \infty)$$

are continuous functions such that a is nontrivial. $V \subset H^1(0,1)$ is a subspace with the property that the only constant function in V is the zero function and that the smallest positive eigenvalue λ_1 of

$$-u''(t) = \lambda a(t)u, \quad 0 < t < 1, \ u \in V \tag{11}$$

is simple and has an associated eigenfunction which is positive in $(0,1)$. This assumption holds, for example (among others), in the cases of the boundary conditions imposed in the various papers cited and related work (cf. for example Proposition 2.6).

We now consider the nonlinear problem (1). This problem we shall embed into the problem

$$-y''(t) = \mu a(t)f(y(t)), \quad 0 < t < 1, \ y \in V. \tag{12}$$

We shall prove that, under assumptions on f, spelled out below, a continuum of positive solutions (in the space $\mathbb{R} \times C[0,1]$) exists which crosses the hyperplane $\{1\} \times C[0,1]$ and thus conclude that the problem

$$-y''(t) = a(t)f(y(t)), \quad 0 < t < 1, \ y \in V, \tag{13}$$

has a nontrivial solution. To this end, let

$$f_0 = \lim_{u\to 0} \frac{f(u)}{u}, \quad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}. \tag{14}$$

We have the following theorem.

Theorem 3.1. *Let V be as above and assume that the limits in (14) exist and satisfy*

$$0 < f_0 < \lambda_1 < f_\infty \tag{15}$$

or

$$0 < f_\infty < \lambda_1 < f_0. \tag{16}$$

Then the boundary value problem (13) has a solution y which is positive in $(0,1)$.

Proof. We consider the problem (12) and apply the global bifurcation theorem of Krasnosels'kii-Rabinowitz, see [30, 32], which guarantees the existence of an unbounded continuum $\mathbb{C} := \{(\mu, y)\} \subset \mathbb{R} \times C[0,1]$ with the solution component y such that $y(t) > 0, \ t \in (0,1)$, which bifurcates from the trivial solution at the bifurcation point $(\mu f_0, 0) = (\lambda_1, 0)$ (while the application of the global bifurcation theorem also allows for the alternative that the continuum might bifurcate from another eigenvalue, this alternative may be quickly ruled out by refering to Proposition 2.6). One may further show (using arguments as in [26, 27]) that the continuum $\mathbb{C}$ is bounded in the $\mu-$ direction and hence must become unbounded in some bounded $\mu-$interval, i.e., it will bifurcate from infinity in that interval. Using results about bifurcation from infinity as in [31, 34, 35, 37], we deduce that bifurcation from infinity will take place at $\mu f_\infty = \lambda_1$. Therefore the continuum $\mathbb{C}$, projected onto the $\mu-$axis $= \mathbb{R}$ will include the open interval determined by the values $\frac{\lambda_1}{f_0}$ and $\frac{\lambda_1}{f_\infty}$. This open interval will contain the value $\mu = 1$, if either (15) or (16) hold. □

The above result and its proof may be extended to the following:

Theorem 3.2. *Under the same assumptions on the subspace V, assume that*

$$0 = f_0 < \lambda_1 < f_\infty \tag{17}$$

or

$$0 = f_\infty < \lambda_1 < f_0. \tag{18}$$

Then the boundary value problem (13) has a solution y which is positive in $(0, 1)$.

Proof. In the case of (17) there will be no bifurcation from the trivial solution, however, bifurcation from infinity will take place at $\mu = \frac{\lambda_1}{f_\infty}$ with the corresponding continuum existing for all values of $\mu > \frac{\lambda_1}{f_\infty}$, and hence (13) will have a positive solution, whereas in the case (18), bifurcation from the trivial solution occurs at $\mu = \frac{\lambda_1}{f_0}$, with the continuum existing for all values $\mu > \frac{\lambda_1}{f_0}$. □

Global bifurcation theory may also be applied at simple eigenvalues $\lambda_j > \lambda_1$, and various results may be formulated using the ideas used above; here it will be important again that bifurcating continua are global, which will follow from nodal properties of solutions inherited by the nodal properties of the eigenfunctions of the associated linearized problems.

4. Concluding Remarks

Remark 4.1. The methods developed in [26, 27, 28] may be employed to study various multi-point and nonlocal boundary value problems involving nonlinear terms f different from those considered above, as long as solution branches of positive solutions may be found which exist globally and can be shown to cross the appropriate parameter hyperplane. To this end we refer to [40, 41], where fixed point techniques have been used.

Remark 4.2. If we replace, in (2), one of boundary conditions by, say, the following

$$y(0) = \alpha y(\xi) + b \tag{19}$$

one obtains a problem from a class of problems studied in [43]. Here one may view b as a parameter and then employ homotopy continuation techniques, as done in [10], to obtain parameter intervals for the parameter b, for which solutions may be shown to exist.

Remark 4.3. The interested reader might wish to revisit the example (1), i.e.

$$y''(t) + a(t)f(y(t)) = 0, \quad 0 < t < 1$$

subject to the three-point boundary conditions

$$y(0) = \alpha y\left(\frac{1}{2}\right), \quad y(1) = \beta y\left(\frac{1}{2}\right)$$

in case $a \equiv 1$ and do the necessary computations to find that if $|\alpha+\beta| < 2$, then positive real eigenvalues exist having the properties required above, whereas if $\alpha + \beta = 2$, the problem is in fact in resonance (c.f. also [22], where it has been shown that only if $\alpha + \beta \neq 2$, a Green's function may be computed) and if $|\alpha + \beta| > 2$, no real eigenvalues exist. In the case that real eigenvalues exist, the principal eigenvalue λ_1 is given by

$$\lambda_1 = 4\mu_1^2,$$

where μ_1 is the smallest positive solution of

$$\cos \mu = \frac{\alpha + \beta}{2}.$$

Another interesting example is obtained for the same nonlinear differential equation which is subject to boundary conditions such as

$$u(0) = \int_0^{\frac{1}{2}} u(s)ds$$

(see also [41], where similar boundary conditions are considered).

Remark 4.4. For problems at resonance, such as the example in the previous remark, when $\alpha + \beta = 2$, continuation arguments based on Mawhin's continuation theorem, as was done in [29], may be used to establish existence criteria for such multi-point boundary value problems.

Remark 4.5. A useful tool to study boundary value problems for nonlinear elliptic equations has been the method of sub-supersolutions. In this regard we refer to [19], where such a theory has been developed for general variational inequalities, and hence may be applied to multi-point and nonlocal boundary value problems of the types discussed here. These methods not only apply for semilinear but nonlinear problems, as well. Here also the variational eigenvalue theory as presented in [17] may be useful.

Remark 4.6. In the case of multi-point or nonlocal boundary value problems for elliptic partial differential equations, these problems may be formulated as variational inequalities (actually equalities, since V is a subspace). Problems involving nonlinear terms f, as above, may then be analyzed using bifurcation techniques as presented in [18].

Remark 4.7. If it is the case that either of the the limits (3) does not exist, but the quotients lie asymptotically in certain non overlapping intervals, ideas, as developed in [36], may be used to develop analogous existence results for such *nondifferentiable* nonlinear problems.

References

[1] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review **18** (1976), 620–709.

[2] A. Ambrosetti and P. Hess, *Nonzero solutions of asymptotically linear elliptic eigenvalue problems*, J. Math Anal. Appl. **73** (1980), 411–422.

[3] D.Anderson, *Solutions to second order three-point problems on time scales*, J. Difference Equ. Appl. **8** (2002), 673–688.

[4] K. Deimling, *Nonlinear functional analysis*, Springer, Berlin, 1980.

[5] J. Gao, D. Sun and M. Zhang, *Structure of eigenvalues of multi-point boundary value problems*, Adv. Difference Equ. **2010** (2010), No. 381932, 1–27.

[6] L. Gong, X. Li, B. Qin and X. Xu. *Solution branches for nonlinear problems with an asymptotic oscillation property*, Electron. J. Differential Equations **2015** (2015), No. 269, 1–15.

[7] C. Gupta, *Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation*, J. Math. Anal. Appl. **168**(1992), 540–551.

[8] C. Gupta, *A note on a second order a three-point nonlinear boundary value problem*, J. Math. Anal. Appl. **186** (1994), 277-281.

[9] G. Gustafson and K. Schmitt, *Nonzero solutions of boundary value problems for second order ordinary and delay differential equations*, J. Differential Equations **12** (1972), 129-149.

[10] G. A. Harris, *The influence of boundary data on the number of solutions of boundary value problems with jumping nonlinearities*, Trans. Amer. Math. Soc. **321** (1990), 417–464.

[11] P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

[12] E. Kaufmann and Y. Raffoul, *Eigenvalue problems for a three-point boundary value problem on a time scale*, Electron. J. Qual. Theory Differ. Equ. **2004** (2004), No. 15, 1-10.

[13] N. Kosmatov, *Semipositone m-point boundary value problems*, *Electron. J. Differential Equations* **2004** (2004), No. 119, 1–7.

[14] M. A. Krasnosel'skii, *Positive solutions of operator equation*, Noordhoff, Groningen, 1964.

[15] M. K. Kwong and J. S. W. Wong, *An optimal existence theorem for positive solutions of a four point boundary value problem*, Electron. J. Differential Equations **2009** (2009), No. 165, pp. 1–8.

[16] M. K. Kwong and J. S. W. Wong, *Some remarks on three-point and four-point bvp's for second order nonlinear differential equations*, Electron. J. Qual. Theory Differ. Equ. **2009** (2009), No. 20, pp. 1–18.

[17] A. Lê and K. Schmitt, *Variational eigenvalues of degenerate eigenvalue problems for the weighted p-Laplacian*, Adv. Nonlinear Stud. **5** (2005), 573–585.

[18] V. K. Le and K. Schmitt, *Global bifurcation in variational inequalities: applications to obstacle and unilateral problems*, Springer, New York, 1997.

[19] V. K. Le and K. Schmitt, *Some general concepts of sub-supersolutions for nonlinear elliptic problems*, Topol. Methods Nonlinear Anal. **28** (2006), 87–103.

[20] J, Nečas, *Introduction to the theory of nonlinear elliptic equations*, Wiley, New York, 1986.

[21] B. Liu, *Nontrivial solutions of second order three-point boundary value problems*, Appl. Math. Comput. **132** (2002), 201–211.

[22] W. S. Loud, *Self-adjoint multi-point boundary value problems*, Pacific J. Math. **24** (1968), No. 2, 303–317.

[23] R. Ma, *Existence theorems for a second order three-point boundary value problem*, J. Math. Anal. Appl. **211** (1997), 545–555.

[24] R. Ma, *Positive solutions of a nonlinear three-point boundary value problem*, Electron. J. Differential Equations **1999** (1999), No. 34, 1–8.

[25] C. V. Pao and Y. M. Wang, *Nonlinear fourth order elliptic equations with nonlocal boundary conditions*, J. Math. Anal. Appl. **372** (2010), 351–365.

[26] H.-O. Peitgen and K. Schmitt, *Perturbations topologiques globales des problèmes non linéaires aux valeurs propres*, C. R. Acad. Sci. Paris, Serie A **291** (1980), 271–274.

[27] H.-O. Peitgen and K. Schmitt, *Global topological perturbations of nonlinear eigenvalue problems*, Math. Methods Appl. Sci. **5** (1983), 376–388.

[28] H.-O. Peitgen and K. Schmitt, *Global analysis of two-parameter elliptic eigenvalue problems*, Trans. Amer. Math. Soc. **283** (1984), 57–95.

[29] P. D. Phung and L. T. Truong, *Existence of solutions to three point boundary value problems at resonance*, Electron. J. Qual. Theory Differ. Equ. **2016** (2016), No. 115, 1–13.

[30] P. Rabinowitz, *Some global results for non linear eigenvalue problems*, J. Funct. Anal. **7** (1971), 487–513.

[31] P. Rabinowitz, *On bifurcation from infinity*, J. Differential Equations **14** (1973), 462–475.

[32] P. Rabinowitz, *Some aspects of nonlinear eigenvalue problems*, Rocky Mountain J. Math. **3** (1973), 162–202.

[33] B. P. Rynne, *Spectral properties of second-order, multi-point $p-$ Laplacian boundary value problems*, Nonlinear Anal. **72** (2010), 4244–4253.

[34] R. Schaaf and K. Schmitt, *A class of nonlinear Sturm-Liouville problems with infinitely many solutions*, Trans. Amer. Math. Soc. **306** (1988), 853–859.

[35] R. Schaaf and K. Schmitt, *Asymptotic behavior of positive solution branches of elliptic problems with linear part at resonance*, Z. Angew. Math. Phys. **43**

(1992), 645–676.

[36] K. Schmitt and H. L. Smith, *On eigenvalue problems for nondifferentiable mappings, some aspects of nonlinear eigenvalue problems*, J. Differential Equations **33** (1979), 294–319.

[37] K. Schmitt and Z. Q. Wang, *On bifurcation from infinity for potential operators*, Differential Integral Equations **4** (1991), 933–943.

[38] R. Showalter, *Hilbert space methods in partial differential equations*, Electron. J. Diff. Equations, Monograph 01, 2004.

[39] *Y. Sun and L. Liu*, *Solvability for a nonlinear second order three-point boundary value problem*, J. Math. Anal. Appl. **296** (2004), 265–275.

[40] J. R. L. Webb. *Positive solutions of some three-point boundary value problems via fixed point index theory*, Nonliner Anal. **47** (2001), 4319–4332.

[41] J. R. L. Webb and G. Infante, *Positive solutions of nonlocal boundary value problems involving integral conditions*, NoDEA, Nonlinear Differential Equations Appl. **15** (2008), 45–67.

[42] J. S. W. Wong, *Existence theorems for second order multi-point boundary value problems*, Electron. J. Qual. Theory Differ. Equ. **2010** (2010), No. 41, 1–12.

[43] J. S. W. Wong, *Positive solutions of second order multi-point boundary value problems with non homogeneous boundary conditions*, Differ. Equ. Appl. **2** (2010), No. 3, 345–375.

[44] E. Zeidler, *Nonlinear functional analysis and its applications I: fixed point theorems*, Springer, Berlin, 1986.

[45] G. Zhang and J, Sun, *Positive solutions of m-point boundary value problems*, J. Math. Anal. Appl. **291** (2004), 406-418.

[46] S. Zhu and J, Zhu, *The multiple solutions for the p-Laplacian multipoint BVP with sign changing nonlinearity on time scales*, J. Math. Anal. Appl. **344** (2008), 616–626.

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