ON COUNTABLY COMPACT PRODUCT SPACES (*)

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SOMMARIO. - *Si studiano in ZF forme equivalenti della seguente versione debole del teorema di Tychonov: il prodotto topologico di spazi compatti di Hausdorff è numerabilmente compatto.*

SUMMARY. - *We study in ZF equivalent forms of the following weaker version of Tychonov theorem: the topological product of compact Hausdorff spaces is countably compact.*

All topological spaces are assumed to be *Hausdorff spaces.* ZF denotes the Zermelo-Fraenkel set theory without the axiom of choice; AC^ω denotes the axiom of choice for countable families of non-empty sets and ω denotes the set of all natural numbers.

We shall consider the assertions below:

1) there exists a non-principal ultrafilter $\mathcal{U}$ on ω;

2) for any non-empty set X, the topological product space $\{0,1\}^{\{0,1\}^X}$ is countably compact ($\{0,1\}$ with the discrete topology);

3) every filter with countable base on a non-empty set X is contained in an ultrafilter on X.

DEFINITION. A topological space is countably compact if every countable open cover has a finite subcover.

THEOREM 1 ($ZF + AC^\omega$). *A topological space X is countably compact if and only if every infinite countable subset of X has an accumulation point.*

THEOREM 2 ($ZF + AC^\omega$). *Let X be a non-empty set. If (1) holds then the topological product space $\{0,1\}^{\{0,1\}^X}$ is countably compact.*

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Proof. Let $(y_n)_{n\geq 0}$ be an infinite sequence of pairwise distinct points of $\{0,1\}^{\{0,1\}^X}$. For each $n \geq 0$, consider $y_n = (y_{n,i})_{i\in\{0,1\}^X}$. Fix $\mathcal{U}$ a non-principal ultrafilter on ω.

For each $i \in \{0,1\}^X$ the intersection $\bigcap\limits_{U\in\mathcal{U}} \overline{\{y_n, i \mid n \in U\}}$ is non-empty, because each $\{y_{n,i} \mid n \in U\}$ is closed in the compact space $\{0,1\}$. Choose z_i being the smallest member of the intersection above. Then $z = (z_i)_{i\in\{0,1\}^X}$ is an accumulation point of $\{y_n \mid n = 0,1,2,\ldots\}$ in $\{0,1\}^{\{0,1\}^X}$. On the contrary, let $V = \prod\limits_{i\in\{0,1\}^X} V_i$ be an elementary open neighborhood of z such that $(V\backslash\{z\}) \cap \{y_n \mid n = 0,1,2,\ldots\} = \phi$. The set $J = \{i \in \{0,1\}^X \mid V_i \neq \{0,1\}\}$ is finite and for each $j \in J$, the set $\{n \in \omega \mid y_{n,j} \in V_j\}$ belongs to $\mathcal{U}$; hence $\{n \in \omega \mid y_{n,j} \in V_j, j \in J\} \in \mathcal{U}$ and $(V\backslash\{z\}) \cap \{y_n \mid n = 0,1,\ldots\} \neq \emptyset$ (contradiction).

Theorem 3. *Let X be a non-empty set and $\mathcal{F}$ be a filter with countable base on X. If (2) holds, then $\mathcal{F}$ is contained in an ultrafilter on X.*

Proof. This proof is a slight modification of E. Farah's proof (1953) that the axiom of ultrafilters is equivalent in ZF to the Tychonov theorem for compact Hausdorff spaces.

Define

$$\begin{array}{rcl} \varphi : X & \longrightarrow & \{0,1\}^{\{0,1\}^X} \\ t & \longmapsto & (f(t))_{f\in\{0,1\}^X} \end{array}$$

Let $\mathcal{F}$ be a filter on X with countable base $\mathcal{B}$. Then $\cap\{\overline{\varphi[B]} \mid B \in \mathcal{B}\}$ is a countable family of closed subsets of $\{0,1\}^{\{0,1\}^X}$ with the finite intersection property. By virtue of (2), $\bigcap\limits_{B\in\mathcal{B}} \overline{\varphi[B]} \neq \emptyset$ and fix y belonging to this intersection. Consider the ultrafilter on X generated by the collection

$$\{F \cap \varphi^{-1}[V] \mid F \in \mathcal{F}, V \in \mathcal{V}_y\}\ ,$$

where $\mathcal{V}_y$ is the set of all neighborhoods of y in $\{0,1\}^{\{0,1\}^X}$; it contains $\mathcal{F}$.

Theorem 4. *In ZF the following assertions are equivalent:*

i) $AC^\omega + (1)$

ii) $AC^\omega + (2)$

iii) $AC^\omega +$ (3)

Proof. $AC^\omega +$ (1) implies $AC^\omega +$ (2) by Theorem 2. $AC^\omega +$ (2) implies $AC^\omega +$ (3) by theorem 3. Finally, the filter on ω generated by the collection

$$\{\{n \in \omega \mid n \geq k\} \mid k = 0, 1, 2, \ldots\}$$

has a countable base, hence by (3) it is contained in an ultrafilter on ω which is not principal.

Assertion (4): the topological product of a non-empty family of compact Hausdorff spaces is countably compact.

It is immediate that $AC^\omega +$ (4) implies $AC^\omega +$ (2) since $\{0, 1\}$ with the discrete topology is a compact-Hausdorff space. On the other hand, let $((X_i, \tau_i))_{i \in I}$ be a non-empty family of compact Hausdorff spaces and let us show that the topological product $X = \prod_{i \in I} X_i$ is countably compact (if $\prod_{i \in I} X_i = 0$ there is nothing to prove). Let $\mathcal{C}$ be a countable open cover of $\prod_{i \in I} X_i$, without a finite subcover - say $\mathcal{C} = \{V_1, \ldots, V_n, \ldots\}$. Then $\left\{X \setminus \bigcup_{i=1}^{p} V_i \;\middle|\; p = 1, 2, \ldots\right\}$ is a *countable* base $\mathcal{B}$ of a filter $\mathcal{F}$ on the non-empty set X, that is contained in an ultrafilter $\mathcal{U}$ on X (since $AC^\omega +$ (2) implies (3)). For each $i \in I$, let $\Pi_i : X \to X_i$ be the projection; then there is a unique $b_i \in X_i$ such that the ultrafilter $\Pi_i \mathcal{U}$ converges to b_i. Finally, $(b_i)_{i \in I}$ would be in $\cap \mathcal{B}$ (contradiction).

It is easy to see that, in ZF, (2) implies (3), (3) implies (1) and (1) implies that every infinite countable subset of $\{0, 1\}^{\{0,1\}^X}$ has an accumulation point. On the other hand, (4) implies that "if $(A_n)_{n \in \omega}$ is a sequence of non-empty finite sets, then $\prod_{n \in \omega} A_n \neq \emptyset$". (Hint. For each $n \in \omega$, define $X_n = A_n \cup \{b\}$, where b does not belong to $\bigsqcup_{n \in \omega} A_n$. Consider the discrete topology on each X_n; by (4) $\prod_{n \in \omega} X_n$ is countably compact, hence $\prod_{n \in \omega} A_n \neq \emptyset$.)

In [3] Sierpinski proved that if (1) holds there is a non-measurable Lebesgue set. Solovay's model ([4]) satisfies the Dependent Choice Axiom (hence AC^ω) and every real subset is Lebesgue measurable.

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I) AC^ω does not imply (1) since in Feferman's variant of the Cohen-model AC^ω holds, but (1) fails;

II) (1) does not imply AC^ω_{fin} (= axiom of choice for countable families of non-empty finite sets).

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References

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