

# On the Number and the Geometry of Hyperbolic Knots and Links with a Common Cyclic Branched Covering: Known Results and Open Problems

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*SUMMARY.* - We give a survey on recent progress and remaining open problems on the number and the geometry of knots and links which have a hyperbolic 3-manifold  $M$  as a common cyclic branched covering. This is strongly related to the algebra and the geometry of the finite isometry group  $G$  of  $M$ , and it naturally divides into the two cases  $G$  solvable and  $G$  non-solvable. The solvable case is relatively well understood whereas the non-solvable case remains somewhat mysterious.

## 1. Fixing the branching order

We consider the following

**PROBLEM 1.1:** Given a closed orientable hyperbolic 3-manifold  $M$  and fixing a prime  $p$ , of how many different knots and links in the 3-sphere can  $M$  be a  $p$ -fold cyclic branched covering, and how are such knots or links related?

This is, of course, a classical and much considered problem, for various classes of 3-manifolds  $M$ . In the present paper, we concentrate on the basic and most important case where  $M$  is a hyperbolic 3-manifold and the branching order  $p$  is a prime (for a dis-

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cussion for other classes of 3-manifolds, see [15]). Two (unoriented) knots or links will be considered equivalent (or equal) if there is an orientation-preserving diffeomorphism of  $S^3$  mapping one to the other.

If the hyperbolic 3-manifold is a  $p$ -fold cyclic branched covering of a knot or a link in  $S^3$  then a generator of the cyclic covering group acting on  $M$  has non-empty fixed point set, and by the orbifold geometrization one can assume that the covering group acts by isometries on  $M$  [1, 3]. Hence the problem becomes a problem about the algebra and the geometry of the finite isometry group of a hyperbolic 3-manifold, and in particular on the Sylow  $p$ -subgroups of the isometry group. We distinguish the cases  $p = 2$  and  $p > 2$ , starting with  $p > 2$ .

**THEOREM 1.2.** *Let  $p$  be an odd prime.*

- i) A hyperbolic 3-manifold  $M$  is the  $p$ -fold cyclic branched covering of at most two knots in  $S^3$ , and two is the optimal upper bound. If there are two such knots then their covering groups commute, up to conjugation [22].*
- ii) For links with at least two components, the optimal upper bound is three [16]; the covering groups commute, up to conjugation [7].*

For 2-fold branched coverings (necessarily cyclic), the situation is as follows

**THEOREM 1.3.**

- i) A hyperbolic 3-manifold  $M$  is the 2-fold branched covering of at most nine different knots or two-component links in  $S^3$  [14, 10]), and nine is the optimal upper bound [5, 11].*
- ii) For links with at least three components, the optimal upper bound is three [11].*

We note that, for 2-fold branched coverings, the homology of  $M$  determines the number of components of the link (see e.g. [16, Proposition 4]), so the different cases are mutually disjoint. The existence of the maximal number nine resp. three of knots and

links in Theorem 1.3 is obtained by applying the imitation theory of Kawauchi [5]; explicit examples of three and four different knots with the same 2-fold branched covering are constructed in [23, Theorem 4] and [16, Theorem 5]. In the case of knots, the upper bound nine in Theorem 2 can be explained as follows. Suppose that  $M$  is a hyperbolic 3-manifold which is a 2-fold branched covering of a knot in  $S^3$ . Then  $M$  is a mod 2 homology 3-sphere. Denote by  $G$  the orientation-preserving isometry group of  $G$  which is finite; each involution in  $G$ , and in particular each covering involution of a knot in  $S^3$ , is conjugate into a Sylow 2-subgroup of  $G$ . We want to determine the maximal number of conjugacy classes of involutions in a Sylow 2-subgroup of  $G$ . By [4], a finite  $p$ -group acting faithfully on a mod  $p$  homology 3-sphere acts faithfully and linearly also on  $S^3$ , with the same dimension function for the fixed point sets of its elements, and hence, in the orientation-preserving case, is a subgroup of the orthogonal group  $SO(4) \cong S^3 \times_{\mathbb{Z}_2} S^3$  (the central product of two copies of the unit quaternions  $S^3$ , with identified centers  $\mathbb{Z}_2$ ). The finite subgroups of the unit quaternions  $S^3$  are cyclic or binary polyhedral (dihedral, tetrahedral, octahedral, dodecahedral). Considering the finite 2-subgroups of  $SO(4)$  (the finite 2-subgroups of the unit quaternions  $S^3$  are cyclic or generalized quaternion (binary dihedral)), the maximal number of non-conjugate involutions with non-empty fixed point set in such a group is nine (plus one central free involutions), and the smallest 2-subgroup realizing this upper bound nine is the central product  $Q \times_{\mathbb{Z}_2} Q \subset S^3 \times_{\mathbb{Z}_2} S^3$  of two copies of the quaternion group of order eight (see [8] for a classification and the geometry of the 2-groups acting on a mod 2 homology 3-sphere and realizing the maximal number of involutions).

The group  $Q \times_{\mathbb{Z}_2} Q$ , of order 32, acts orthogonally and faithfully on  $S^3$  and has nine non-conjugate involutions with non-empty connected fixed point set and quotient  $S^3$ . The quotient  $S^3/Q \times_{\mathbb{Z}_2} Q$  is the 3-sphere, with the Kuratowski graph as singular set (the complete bi-partite graph on six vertices); the Kuratowski graph has nine edges which correspond to the nine involutions in  $Q \times_{\mathbb{Z}_2} Q$  with non-empty fixed point set. Now the imitation theory of Kawauchi permits to transfer the action of  $Q \times_{\mathbb{Z}_2} Q$  on the integer homology 3-sphere  $S^3$  to an isometric orientation-preserving action on a hyper-

bolic homology 3-sphere  $M$  such that  $Q \times_{\mathbb{Z}_2} Q$  is the full isometry group of  $M$  and the nine conjugacy classes of involutions still have non-empty connected fixed point set and quotient  $S^3$ , so  $M$  is the 2-fold branched covering of nine inequivalent knots in  $S^3$ . Dividing out the central free involution one gets a hyperbolic 3-manifold  $\bar{M}$ , with an action of  $Q \times_{\mathbb{Z}_2} Q / \mathbb{Z}_2 \cong (\mathbb{Z}_2)^4$ , which is the 2-fold branched covering of nine inequivalent 2-component links in  $S^3$  (see [11, Figure 1] for the geometry of the situation).

## 2. Different branching orders

Concentrating on the easier case of knots now, we discuss the following

**PROBLEM 2.1:** Given a closed hyperbolic 3-manifold  $M$ , for how many different primes  $p$  can  $M$  be a  $p$ -fold cyclic branched covering of a knot  $K$  in  $S^3$ , and how are these knots related? In particular, is there an upper bound on the number of such primes  $p$ ? What are the possible isometry groups of a hyperbolic 3-manifold which is a  $p$ -fold cyclic branched covering?

Problem 2.1 is more complex than Problem 1.1 since it concerns the relation between the various Sylow  $p$ -subgroups of the finite isometry group  $G$  of  $M$ , for different primes  $p$ . The problem has first been considered in [18]; it naturally divides into the two cases  $G$  solvable resp.  $G$  non-solvable. For solvable isometry groups, the following result implicit in [18] gives a complete solution.

**THEOREM 2.2.** *Suppose that the isometry group of the closed hyperbolic 3-manifold  $M$  is solvable. Then  $M$  is a  $p$ -fold cyclic branched covering of knots in  $S^3$  for at most three different odd primes  $p$ , and this upper bound is best possible. The covering groups commute, up to conjugation.*

We will give the proof of Theorem 2.2 in the next section. By similar methods, under some geometric assumption on the knot  $K$  but without the hypothesis of solvability of the isometry group of  $M$ , the following is proved in [18, Theorems 2 and 3].

**THEOREM 2.3.** *For an odd prime  $p$ , let  $M$  be a hyperbolic 3-manifold which is a  $p$ -fold cyclic branched covering of a knot  $K$  in  $S^3$ . Suppose that*

- i)  $K$  is not strongly invertible;*
- ii)  $K$  does not have symmetric period  $p$ .*

*Then  $M$  is a  $p$ -fold cyclic branched covering of a knot in  $S^3$  for at most three different odd primes  $p$ . The covering groups commute, up to conjugation.*

We will indicate the algebraic significance of the conditions i) and ii) for the isometry group  $G$  of  $M$  in the proof of the theorem sketched in the next section. A knot  $K$  has *symmetric period  $p$*  if  $K$  has a cyclic symmetry of order  $p$  (a periodic diffeomorphism  $h$  of  $(S^3, K)$  with nonempty connected fixed point set  $F$  disjoint from  $K$ ) such that the projection of the 2-component link  $K \cup F$  to the quotient  $S^3/h \cong S^3$  gives a symmetric 2-component link  $\bar{K} \cup \bar{F}$  in  $S^3$ , i.e. there is an orientation-preserving diffeomorphism of  $S^3/h$  exchanging its two components. In a sense, the two conditions in Theorem 2.3 are intrinsic to the problem, indicating the possible obstructions for an abelian situation of commuting covering transformations to occur (or, more generally, a "solvable situation" as in [19]). See [24, Section 5] for an explicit example of a non-solvable situation which seem to be quite rare; alternatively, one may apply Kawauchi's imitation theory [5, 6] to the action of the dodecahedral group  $\mathbb{A}_5$  on  $S^3$  or on the Poincaré homology 3-sphere to obtain a hyperbolic 3-manifold  $M$  with isometry group  $\mathbb{A}_5$  such that the quotients of  $M$  by the cyclic subgroups of orders 2, 3 and 5 of  $\mathbb{A}_5$  give  $S^3$ . We do not know such an example for three different odd primes, or for a simple or non-solvable group different from  $\mathbb{A}_5$ .

By Theorems 2.2 and 2.3 we are left with the situation that the isometry group  $G$  of the hyperbolic 3-manifold  $M$  is non-solvable and  $M$  is a  $p$ -fold cyclic branched covering of a knot  $K$  which is strongly invertible or has symmetric period  $p$ . One of the first questions to ask is then:

**PROBLEM 2.4:** What are the possible non-solvable isometry groups of an hyperbolic 3-manifold  $M$  which is a  $p$ -fold cyclic branched covering of a knot in  $S^3$ ?

The  $p$ -fold cyclic branched covering of a knot in  $S^3$  is a mod  $p$  homology sphere. So, if  $M$  is also a 2-fold branched covering then  $M$  is a mod 2 homology 3-sphere, and the class of finite groups acting on a mod 2 homology 3-sphere is quite restricted: a list of the non-solvable candidates is given in [12]. For the basic and less technical case of finite simple groups, the following holds.

**THEOREM 2.5.** *Let  $M$  be a mod 2 homology 3-sphere and  $G$  a finite non-abelian simple group of diffeomorphisms of  $M$ . Then  $G$  is isomorphic to a linear fractional group  $\mathrm{PSL}(2, q)$ , for an odd prime power  $q$ .*

The linear fractional groups  $\mathrm{PSL}(2, q)$ ,  $q$  odd, have dihedral Sylow 2-subgroups, and in [13] the proof of Theorem 2.5 is reduced to the Gorenstein-Walter classification of the finite simple groups with dihedral Sylow 2-subgroups (which are exactly the groups  $\mathrm{PSL}(2, q)$ ,  $q$  odd, plus the alternating group  $\mathbb{A}_7$ ). Various explicit examples of actions of  $\mathrm{PSL}(2, q)$  on mod 2-homology 3-spheres, for some small values of  $q$ , are given in [24] but the exact classification remains open (whereas the only finite simple group which acts on an integer homology 3-sphere is the alternating or dodecahedral group  $\mathbb{A}_5 \cong \mathrm{PSL}(2, 5)$ ). However, if  $M$  is also a cyclic branched covering of a knot in  $S^3$ , then the only example of such an action which we know is by the dodecahedral or alternating group  $\mathbb{A}_5 \cong \mathrm{PSL}(2, 5)$  (and there is some computational evidence that this might be, in fact, the only one).

Using the Gorenstein-Harada classification of the finite simple groups of sectional 2-rank at most four, the following is proved in [12].

**THEOREM 2.6.** *Let  $G$  be a finite non-solvable group of orientation-preserving diffeomorphisms of a mod 2 homology 3-sphere  $M$ , and let  $\mathcal{O}(G)$  denote the maximal normal subgroup of odd order of  $G$ . Then  $G/\mathcal{O}(G)$  has a normal subgroup (whose factor group is abelian or a 2-fold extension of an abelian group), isomorphic to one of the following groups:*

$$\begin{array}{l} \mathrm{PSL}(2, q) \quad \text{or} \quad \mathrm{PSL}(2, q) \times \mathbb{Z}_2, \\ \mathrm{SL}(2, q) \times_{\mathbb{Z}_2} C \quad \text{or} \quad \hat{\mathbb{A}}_7, \\ \mathrm{SL}(2, q) \times_{\mathbb{Z}_2} \mathrm{SL}(2, q') \end{array}$$

where  $C$  is solvable with a unique involution and  $q$  and  $q'$  are odd prime powers greater than four.

The main remaining problem here is the exact classification (see also the discussion in [25]):

PROBLEM 2.7: Which of the groups in Theorem 2.6 admit an action on a mod 2 homology 3-sphere? Which of them occur as isometry groups of a hyperbolic 3-manifold which is a 2-fold cyclic branched covering of a knot in  $S^3$ ?

Returning to the discussion of the conditions in Theorems 2.2 and 2.3, if an hyperbolic 3-manifold  $M$  is a  $p$ -fold cyclic branched covering of a strongly invertible knot  $K$  then the isometry group  $G$  of  $M$  contains (as in the case of a 2-fold branched covering) an involution with non-empty connected fixed point set (the lift of a strong inversion). The existence of an involution with non-empty connected fixed point set gives again strong restrictions on the possible isometry groups of the 3-manifold. For finite simple groups, the following is proved in [20] (part i) and [24] (part ii).

THEOREM 2.8. *Let  $M$  be a closed orientable 3-manifold and  $G$  a finite non-abelian simple group of diffeomorphisms of  $M$ .*

- i) If  $G$  contains an involution with non-empty connected fixed point set then  $G$  is isomorphic to a linear fractional group  $\mathrm{PSL}(2, q)$ , for an odd prime power  $q$ .*
- ii) If  $G$  contains the covering involution of a knot in  $S^3$  having  $M$  as 2-fold branched covering then  $G$  is isomorphic to the dodecahedral group  $\mathbb{A}_5 \cong \mathrm{PSL}(2, 5)$ .*

We note that, since any finite non-abelian simple group acting on a mod 2 homology sphere has an involution with non-empty connected fixed point set, Theorem 2.5 is a special case of Theorem 2.8(i). If the isometry group  $G$  of an hyperbolic 3-manifold  $M$  is isomorphic to a group  $\mathrm{PSL}(2, q)$  this gives easily some upper bound on the number of different primes  $p$  in Problem 2.1 (considering the subgroup structure of  $\mathrm{PSL}(2, q)$ ). However simple isometry groups are quite special, and by Theorem 2.3 we have to consider arbitrary non-solvable isometry groups. By a far-reaching extension of Theorem 2.8 (in analogy with Theorem 2.6 generalizing Theorem 2.5),

in [9] a list of the candidates of the finite non-solvable groups  $G$  is given which admit an orientation-preserving action on a closed orientable 3-manifold and contain an involution with non-empty connected fixed point set; the main remaining problem is again the exact classification of such groups.

By Theorems 2.2 and 2.3 and the generalization of Theorem 2.8 for non-solvable groups [9], a basic remaining case of Problem 2.1 is the following

**PROBLEM 2.9:** Let  $M$  be a closed orientable hyperbolic 3-manifold with non-solvable isometry group. Suppose that every knot  $K$  such that  $M$  is a  $p$ -fold cyclic branched covering of  $K$  is not strongly invertible but has symmetric period  $p$ . Find an upper bound on the number of such knots  $K$  and primes  $p$ . What are the possible isometry groups of such a 3-manifold  $M$ ?

This seems to be a very special situation indeed but for the moment we have no idea how to attack the problem in this case, neither geometrically in terms of the geometry of the knots nor algebraically in terms of the isometry group of  $M$ . Of course, there remains also the case of cyclic branched coverings of arbitrary links, and we close this section with the following general problem including also the case of links.

**PROBLEM 2.10:** How many different primes can occur as the orders of hyperelliptic isometries (i.e., with quotient  $S^3$ ) of a hyperbolic 3-manifold  $M$ ? Is there an upper bound on the number of such primes? What are the possible (non-solvable) isometry groups of such a hyperelliptic 3-manifold  $M$ ? (cf. [16])

We note that for a hyperbolic 3-manifold which is a  $p$ -fold cyclic branched covering of a knot or a link in  $S^3$ , the only non-solvable orientation-preserving isometry groups which we know at present are the finite non-solvable subgroups of the orthogonal group  $SO(4)$ , and in particular the only non-abelian simple group is the dodecahedral group  $\mathbb{A}_5 \cong \text{PSL}(2, 5)$ .



### 3. Proofs of Theorems 2.2 and 2.3

We remark that the  $p$ -fold cyclic branched covering of a knot in  $S^3$  is a mod  $p$  homology 3-sphere, and we start with two Lemmas concerning finite group actions on such manifolds. Low-dimensional proofs can be found in [12, Propositions 4 and 5b]; alternatively, one may apply the Borel formula [2, Theorem XIII.2.3] for actions of elementary abelian  $p$ -groups for the first Lemma and, similar as in section 1, [4] for the second.

LEMMA 3.1. *For an odd prime  $p$ , let  $G = \mathbb{Z}_p \times \mathbb{Z}_p$  be a finite group of diffeomorphisms of a mod  $p$  homology 3-sphere  $M$ . There are exactly two subgroups  $\mathbb{Z}_p$  of  $G$  with nonempty fixed point set, and each fixed point set is connected (a simple closed curve).*

LEMMA 3.2. *For an odd prime  $p$ , a finite  $p$ -group acting on a mod  $p$  homology 3-sphere is cyclic or a direct product of two cyclic groups.*

*Proof of Theorem 2.2.* The main algebraic tools for the proof are the Burnside transfer theorem for finite groups and the generalization of the Sylow theorems for finite solvable groups.

Suppose that the orientation-preserving isometry group  $G$  of the closed orientable hyperbolic 3-manifold is solvable, and that  $M$  is the  $p_i$ -fold cyclic branched covering of (inequivalent) knots  $K_i$  in  $S^3$ , for pairwise different odd primes  $p_1, \dots, p_n$ . Denote by  $S_{p_i}$  a Sylow  $p_i$ -subgroup of the isometry group  $G$  of  $M$ ; up to conjugation, we can assume that the covering group  $\mathbb{Z}_{p_i}$  of  $K_i$  is a subgroup of  $S_{p_i}$ . Each  $\mathbb{Z}_{p_i}$  has nonempty connected fixed point set  $\tilde{K}_i$ ; by the positive solution of the Smith conjecture,  $\tilde{K}_i \neq \tilde{K}_j$  for  $i \neq j$ .

By the generalization of the Sylow theorems for solvable groups [21, Chapter 4, Theorem 5.6], the solvable group  $G$  has a subgroup  $U$  of order  $|S_{p_1}| \dots |S_{p_n}|$ ; up to conjugation, we can assume that  $U$  contains all  $S_{p_i}$ . Note that  $U$  has odd order (in fact this is the main point in the following). Let  $p$  and  $q$  be two different primes among  $p_1, \dots, p_n$ .

We apply the Burnside transfer theorem; this states that if a Sylow  $p$ -subgroup  $S_p$  of a finite group  $U$  is contained in the center of its normalizer, then  $U$  has a characteristic subgroup  $V$  such that  $U = VS_p$  and  $V \cap S_p = 1$  [21, Chapter 5, Theorem 2.10]. Note that the

hypothesis is fulfilled for  $S_p$  since  $U$  has odd order; in fact, it follows from Lemmas 3.1 and 3.2 that all elements of the normalizer of  $S_p$  in  $U$  leave invariant some simple closed curve in  $M$  (the preimage of one of the knots  $K_i$ ), acting as rotations around and along this curve, and hence the normalizer is abelian.

Now  $S_p$  acts by conjugation on the set of Sylow  $q$ -subgroups of  $V$ ; by a Sylow theorem, the number of elements of this set divides the order of  $V$ . Since the number of elements of each orbit of the action of  $S_p$  is a power of  $p$  and  $p$  does not divide the order of  $V$ , some orbit must have exactly one element, so  $S_p$  normalizes a Sylow  $q$ -subgroup  $S_q$  of  $V$ . Again some non-trivial element of  $S_q$  has nonempty connected fixed point set invariant under both  $S_q$  and  $S_p$ , so these two groups commute elementwise and generate a subgroup  $S_q \times S_p$ . Then also  $U$  is the direct product  $U = V \times S_p$  of  $V$  and  $S_p$ .

By induction,  $U$  is the direct product  $U = S_{p_1} \times \dots \times S_{p_n}$  of its Sylow subgroups. Then  $U$  has a subgroup  $\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_n}$  which is the direct product of the covering groups of the knots  $K_i$ ; note that the preimages  $\tilde{K}_i$  of the knots  $K_i$  are all invariant under the action of this subgroup. Now  $\mathbb{Z}_{p_2} \times \dots \times \mathbb{Z}_{p_n}$  projects to a cyclic group  $C$  acting on  $M/\mathbb{Z}_{p_1} \cong S^3$ , and there are exactly  $n-1$  different curves in  $S^3$  fixed by a non-trivial subgroup of  $C$ . By the orbifold geometrization [1, 3], we can assume that  $C$  acts by isometries of  $S^3$ . But an orthogonal action of a finite cyclic group on  $S^3$  has at most two different great circles fixed by non-trivial elements (if there are two, they form a Hopf link), and consequently  $n \leq 3$ . This finishes the proof of the first part of Theorem 2.2.

Explicit examples of hyperbolic 3-manifolds which are the  $p$ -fold cyclic branched covering of knots in  $S^3$  for three different odd primes  $p$  can be constructed with the method in [16, Section 6]. Alternatively, one may apply the imitation theory of Kawauchi [5, 6] to the Brieskorn homology sphere  $M(p, q, r)$  which is the  $p$ -fold cyclic branched covering of the  $(q, r)$ -torus knot, the  $q$ -fold cyclic branched covering of the  $(p, r)$ -torus knot and the  $r$ -fold cyclic branched covering of the  $(p, q)$ -torus knot; the three covering groups generate a cyclic group of diffeomorphisms, and the imitation theory applied to this cyclic group produces a hyperbolic 3-manifold which is the  $p$ ,  $q$  and  $r$ -fold cyclic branched covering of knots in  $S^3$ . This finishes the

proof of Theorem 2.2.  $\square$

The *proof of Theorem 2.3* is along similar lines. Suppose that the hyperbolic 3-manifold  $M$  is a  $p$ -fold cyclic branched covering of a knot  $K$  in  $S^3$ , for an odd prime  $p$ . Now conditions i) and ii) in Theorem 2.3 are exactly the two conditions which guarantee (using Lemmas 3.1 and 3.2) that the Sylow  $p$ -subgroup  $S_p$  of the orientation-preserving isometry group  $G$  of  $M$  lies in the center of its normalizer in  $G$ , and hence the Burnside transfer theorem applies again. In fact, by Lemma 3.2 a Sylow  $p$ -subgroup  $S_p$  of  $G$  (containing the covering group  $\mathbb{Z}_p$  of  $K$ ) is abelian, hence leaves invariant the preimage  $\tilde{K}$  of the knot  $K$  in  $M$ , and by Lemma 3.1 there are at most two simple closed curves in  $M$  fixed by non-trivial elements of  $S_p$  (one of them being  $\tilde{K}$ ). By conditions ii) and i) of the theorem, any element in the normalizer of  $S_p$  in  $G$  maps  $\tilde{K}$  to itself and does not act on  $\tilde{K}$  as a strong inversion, and hence centralizes  $S_p$ .

The Burnside transfer theorem gives a splitting  $G = VS_p$  of  $G$  as a semidirect product, for a characteristic subgroup  $V$  of  $G$  with  $V \cap S_p = 1$ . Now if  $M$  is also the  $q$ -fold cyclic branched covering of a knot in  $S^3$ , for an odd prime  $q$  different from  $p$ , then one shows exactly as in the proof of Theorem 2.2 that  $S_p$  and  $S_q \subset V$  commute elementwise. Hence each such Sylow subgroup  $S_q$  maps the preimage  $\tilde{K}$  in  $M$  of the knot  $K$  to itself and consequently all these Sylow subgroups commute. Now the proof of Theorem 2.3 finishes as the proof of Theorem 2.2.  $\square$

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