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Some remarks on substitution and composition operators

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Abstract. *In this paper we study the linear substitution operator* $S_\varphi(f) := f \circ \varphi$ *generated by some function* $\varphi : [0,1] \to [0,1]$*, as well as the nonlinear composition operator* $C_g(f) := g \circ f$ *generated by some function* $g : \mathbb{R} \to \mathbb{R}$*. We will show that these operators have a very different (and sometimes quite surprising) behavior in the space of continuous functions, Lipschitz functions, functions of bounded variation, and Baire class one functions. A main emphasis is put on examples and counterexamples which illustrate this behavior.*

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1. Introduction

By C we denote the linear space of all continuous functions $f : [0,1] \to \mathbb{R}$, equipped with the usual norm

$$\|f\|_C := \max\{|f(t)| : 0 \leq t \leq 1\} \qquad (f \in C),$$

by Lip the linear subspace of C of all Lipschitz continuous functions $f : [0,1] \to \mathbb{R}$ with norm

$$\|f\|_{Lip} := |f(0)| + lip(f) \qquad (f \in Lip),$$

where

$$lip(f) := \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|},$$

and by BV the linear space of all functions $f : [0,1] \to \mathbb{R}$ of bounded (Jordan) variation $var(f)$ with norm

$$\|f\|_{BV} := |f(0)| + var(f) \qquad (f \in BV).$$

In the last section we will consider the linear space $\mathcal{B}_1$ of all Baire class one functions, i.e., pointwise limits of sequences of continuous functions. Clearly, the inclusions

$$Lip \subset C \subset \mathcal{B}_1, \qquad Lip \subset BV \subset \mathcal{B}_1$$

hold which are all strict.

It is well-known that all spaces mentioned above are linear spaces and algebras, i.e., closed under addition and multiplication of functions. More complicated, hence interesting, is the problem of compositions, and we will put the main emphasis on this problem in what follows.

More precisely, in this paper we study two operators in the above mentioned spaces. The first is the *substitution operator*

$$S_\varphi(f)(t) := f(\varphi(t)) \tag{1}$$

generated by some function $\varphi : [0,1] \to [0,1]$, the second the *composition operator*

$$C_g(f)(x) := g(f(x)) \tag{2}$$

generated by some function $g : \mathbb{R} \to \mathbb{R}$. These operators may be considered as some kind of "twin brothers": in (1) the inner function is fixed and the outer function f varies over some function space, while in (2) the outer function is fixed and the inner function f varies over some function space. There is one essential difference, however: the operator S_φ in (1) is linear, while the operator C_g in (2) is nonlinear (which makes its study pretty complicated). Thus, in contrast to S_φ we have to distinguish between boundedness and continuity for C_g, because a nonlinear operator may be bounded and discontinuous, or continuous and unbounded.

If X is some function space over $[0,1]$, the first problem consists in characterizing all $\varphi : [0,1] \to [0,1]$ such that $S_\varphi(X) \subseteq X$. In other words, we want to find the largest possible class of "changes of variable" for which the composition $f \circ \varphi$ remains in the space X if we take f from X. Moreover, we will be interested in elementary mapping properties, like injectivity, surjectivity, or bijectivity of S_φ. Here we encounter some surprises, and it turns out that these properties heavily depend on the function space we are working in.

Interestingly, there is a group of results on mapping properties of the operators (1) and (2) which are basically *independent* of the space X. We state them as Propositions 1.1 and 1.2 in this section.

Proposition 1.1. *In every function space X, the following is true.*

(a) The surjectivity of the function $\varphi : [0,1] \to [0,1]$ implies the injectivity of the operator $S_\varphi : X \to X$.

(b) Conversely, if X contains the set C^∞, then the injectivity of the operator $S_\varphi : X \to X$ implies the surjectivity of the function $\varphi : [0,1] \to [0,1]$.

(c) If X contains at least one injective function, then the surjectivity of the operator $S_\varphi : X \to X$ implies the injectivity of the function $\varphi : [0,1] \to [0,1]$.

Proof. (a) Suppose that $f \in X$ satisfies $S_\varphi(f)(t) \equiv 0$. For fixed $t \in [0,1]$ we find by assumption an $s \in [0,1]$ such that $\varphi(s) = t$. It follows that $f(t) = f(\varphi(s)) = S_\varphi(f)(s) = 0$, hence $f(t) \equiv 0$, since t was arbitrary.

(b) Assume that $S_\varphi : X \to X$ is injective, but $\varphi : [0,1] \to [0,1]$ is not surjective. Then $K := \varphi([0,1]) \subset [0,1]$, so the function $\delta_K : [0,1] \to \mathbb{R}$ defined by

$$\delta_K(t) := \operatorname{dist}(t, K) \tag{3}$$

is not identically zero on $[0,1]$, and the same is true for the function $f : [0,1] \to \mathbb{R}$ given by $f(0) := 0$ and

$$f(t) := \exp\left(-1/\delta_K(t)^2\right) \qquad (0 < t \le 1).$$

Moreover, f belongs to C^∞ and hence also to X. On the other hand, we have

$$S_\varphi(f)(t) = \exp\left(-1/\delta_K(\varphi(t))^2\right) \equiv 0,$$

because $\varphi(t) \in K$ for each t. So the injectivity of S_φ implies that $f(t) \equiv 0$, a contradiction.

(c) Let $s, t \in [0,1]$ be such that $\varphi(s) = \varphi(t)$; we have to show that $s = t$. By assumption, we find an injective function $g \in X$, as well as a function $f \in X$ satisfying $S_\varphi(f) = g$. It follows that

$$g(s) = f(\varphi(s)) = f(\varphi(t)) = g(t),$$

hence $s = t$, by the injectivity of g. □

The trivial space $X = \mathbb{R}$, containing only constant functions, shows that we cannot drop the hypothesis on the existence of an injective function in X in Proposition 1.1 (c). Later we will see that the implication in (c) cannot be inverted either in sufficiently “rich” spaces like *Lip* or *BV*.

We remark that, apart from the condition $C^\infty \subseteq X$ in Proposition 1.1 (b), one may give other conditions on X which ensure that the injectivity of S_φ implies the surjectivity of φ. A typical such condition is that the space X contains all characteristic functions of singletons. In fact, if $S_\varphi : X \to X$ is injective and $t_0 \in [0,1]$ is fixed, then $f := \chi_{\{t_0\}} \in X$ is not identically zero, and so $S_\varphi(f) = \chi_{\{\varphi(t_0)\}}$ is not identically zero either. So we may find $s_0 \in [0,1]$

such that $S_\varphi(f)(s_0) = f(\varphi(s_0))$, hence $\varphi(s_0) = t_0$. Since t_0 was arbitrary, we conclude that φ is surjective.

If X is a function space over $[0,1]$, the analogous problem for the composition operator (2) consists in characterizing all $g : \mathbb{R} \to \mathbb{R}$ such that $C_g(X) \subseteq X$. In other words, we want to find the largest possible class of "perturbations" for which the composition $g \circ f$ remains in the space X if we take f from X. Also here the problem of expressing mapping properties of C_g in terms of g exibits several surprises. The following is parallel to Proposition 1.1.

Proposition 1.2. *In every function space X, the following is true.*

(a) If X contains for each $u \in \mathbb{R}$ a function h having u in its range, then the surjectivity of the operator $C_g : X \to X$ implies the surjectivity of the function $g : \mathbb{R} \to \mathbb{R}$.

(b) The injectivity of the function $g : \mathbb{R} \to \mathbb{R}$ implies the injectivity of the operator $C_g : X \to X$.

(c) Conversely, if X contains all constant functions, then the injectivity of the operator $C_g : X \to X$ implies the injectivity of the function $g : \mathbb{R} \to \mathbb{R}$.

Proof. (a) Given $u \in \mathbb{R}$, choose a function $h \in X$ whose range contains u. Therefore, there is some $t \in [0,1]$ with $h(t) = u$. If C_g is surjective, there is a function $f \in X$ with $C_g(f) = h$. In particular, $g(f(t)) = h(t) = u$, which shows that g is surjective.

(b) Assume that C_g is not injective. Then there are functions $f, \tilde{f} \in X$ with $C_g(f) = C_g(\tilde{f})$ and some $t \in [0,1]$ with $f(t) \neq \tilde{f}(t)$. It follows that $g(f(t)) = g(\tilde{f}(t))$ and shows that g is not injective either.

(c) Fix $u, v \in \mathbb{R}$ with $g(u) = g(v)$. The constant functions $f_u(t) \equiv u$ and $f_v(t) \equiv v$ belong to X, by assumption. Moreover, $C_g(f_u) = C_g(f_v)$ as

$$C_g(f_u)(t) = g(f_u(t)) = g(u) = g(v) = g(f_v(t)) = C_g(f_v)(t) \qquad (0 \leq t \leq 1).$$

Since C_g is injective, we must have $f_u = f_v$, hence $u = v$, proving the injectivity of g. □

We point out that the condition on X used in (a), namely, that X contains for each $u \in \mathbb{R}$ a function h having u in its range, is clearly satisfied in any nontrivial *linear* function space X, and the hypothesis on X in (c) is satisfied in all spaces considered in this paper. Moreover, note that the results in Proposition 1.2 are symmetric with respect to surjectivity and injectivity, whereas in Proposition 1.1 they are antisymmetric.

We point out that the converse of (a) is in general not true, even in the quite well-behaved spaces C and *Lip*. We will present a counterexample in the corresponding sections.

Loosely speaking, our general question may be posed as follows: *does φ "feel" the properties of S_φ, and does g "feel" the properties of C_g?* The answer to this question is sometimes quite easy, sometimes surprisingly difficult, and sometimes simply unknown.

2. Continuous functions

Given $\varphi : [0,1] \to [0,1]$, suppose that $f \circ \varphi : [0,1] \to \mathbb{R}$ is continuous for each continuous function $f : [0,1] \to \mathbb{R}$. Choosing $f(t) = t$, it follows trivially that φ then must be continuous. This leads to the following elementary

THEOREM 2.1. *The following two assertions are equivalent.*

(a) $S_\varphi(C) \subseteq C$, i.e., the operator S_φ maps C into itself.

(b) The function $\varphi : [0,1] \to [0,1]$ is continuous.

Note that the operator S_φ is always bounded in C whenever it maps C into itself. In fact, we have $\|S_\varphi\|_{C\to C} \le 1$ and $\|S_\varphi e\|_C = 1$ for $e(t) \equiv 1$. The following simple example shows that the injectivity or surjectivity of φ does not imply the injectivity resp. surjectivity of S_φ in C.

EXAMPLE 2.2. Let $\varphi(t) := t/2$. Then $\varphi : [0,1] \to [0,1]$ is injective, but $S_\varphi : C \to C$ is not, because the function defined by $f(t) := t$ for $0 \le t \le 1/2$ is mapped into the function $g(t) = t/2$ for $0 \le t \le 1$, no matter how we define f on $(1/2, 1]$.

On the other hand, let $\varphi(t) := 4t(1-t)$. Then $\varphi : [0,1] \to [0,1]$ is surjective, but $S_\varphi : C \to C$ is not, because the function $g(t) = t$ is not in the range of S_φ.

Surprisingly enough, we get the correct result when we interchange the role of injectivity and surjectivity.

PROPOSITION 2.3. *Let $\varphi : [0,1] \to [0,1]$ be continuous. With S_φ given by (1), the following is true.*

(a) The operator $S_\varphi : C \to C$ is surjective if and only if the corresponding function $\varphi : [0,1] \to [0,1]$ is injective.

(b) The following three assertions are equivalent.

(i) The function $\varphi : [0,1] \to [0,1]$ is surjective.

(ii) The operator $S_\varphi : C \to C$ is an isometry, i.e.,

$$\|S_\varphi(f)\|_C = \|f\|_C \qquad (f \in C) \tag{4}$$

(iii) The operator $S_\varphi : C \to C$ is injective.

Proof. (a) If φ is injective, then the set $K := \varphi([0,1]) \subseteq [0,1]$ is a compact interval, and the map $\varphi : [0,1] \to K$ is a homeomorphism. Given $g \in C$, the function $g \circ \varphi^{-1} : K \to \mathbb{R}$ is therefore continuous, and by the Tietze Extension Theorem [17] we may find a continuous function $f : [0,1] \to \mathbb{R}$ with $f(t) = (g \circ \varphi^{-1})(t)$ for $t \in K$, hence $g = S_\varphi(f)$. So we have proved that the injectivity of φ implies the surjectivity of S_φ in C. The reverse implication follows from Proposition 1.1 (c).

(b) If φ is surjective, we have $K := \varphi([0,1]) = [0,1]$, hence

$$\|S_\varphi(f)\|_C = \max\{|f(\varphi(s))| : 0 \leq s \leq 1\} = \max\{|f(t)| : 0 \leq t \leq 1\} = \|f\|_C$$

for every $f \in C$, which means that S_φ is an isometry. The fact that (ii) implies (iii) is of course trivial. The fact that (iii) implies (i) has already been proved in Proposition 1.1 (b). □

Observe that the "crossover" between surjectivity and injectivity in our proposition is not only perfectly symmetric, but in (b) we even get the isometry property of S_φ for free. Also, this proposition shows that it was not accidental that the first function φ in Example 2.2 is not surjective, while the second one is not injective.

Proposition 2.3 shows that, whenever S_φ is injective in C, we get as a fringe benefit that it is even an isometry, which is of course much more. We will see that in all the other function spaces we are going to consider below, this is not true.

Concerning the composition operator (2), we have the following result which is parallel to Theorem 2.1.

Theorem 2.4. *The following two assertions are equivalent.*

(a) $C_g(C) \subseteq C$, i.e., the operator C_g maps C into itself.

(b) The function $g : \mathbb{R} \to \mathbb{R}$ is continuous.

It is easy to see that the operator C_g is, under the hypotheses of Theorem 2.4, always bounded and continuous. Concerning the mapping properties of $C_g : C \to C$, we state for the sake of completeness the following result which is an immediate consequence of Proposition 1.2 and therefore does not require a proof.

Proposition 2.5. *Let $g \in C(\mathbb{R})$. With C_g given by (2), the following is true.*

(a) If the operator $C_g : C \to C$ is surjective then the corresponding function $g : \mathbb{R} \to \mathbb{R}$ is surjective.

(b) The operator $C_g : C \to C$ is injective if and only if the corresponding function $g : \mathbb{R} \to \mathbb{R}$ is injective.

We point out that the converse of Proposition 2.5 (a) is not true; we will give a corresponding counterexample which simultaneously works for C and Lip in the next section.

3. Lipschitz continuous functions

Here we can make the same remark as at the beginning of the previous section, with continuity replaced by Lipschitz continuity. This leads to the following

THEOREM 3.1. *The following two assertions are equivalent.*

(a) $S_\varphi(Lip) \subseteq Lip$*, i.e., the operator* S_φ *maps* Lip *into itself.*

(b) The function $\varphi : [0,1] \to [0,1]$ *is Lipschitz continuous.*

Proof. The implication (a) $\Rightarrow$ (b) follows from the fact that the function $f(t) = t$ belongs to Lip, while the implication (b) $\Rightarrow$ (a) follows from the elementary estimate $lip(f \circ \varphi) \leq lip(f)lip(\varphi)$. □

As in the space C, the operator S_φ is always bounded in Lip whenever it maps Lip into itself. This also follows from the estimate $lip(f \circ \varphi) \leq lip(f)lip(\varphi)$. However, there is a difference: the injectivity of S_φ this time does *not* imply that S_φ is an isometry.

EXAMPLE 3.2. The substitution operator S_φ generated by $\varphi(t) := t^2$ is injective in Lip. To see this, suppose that $S_\varphi(f)(t) \equiv 0$ for some $f \in Lip$. Given $t \in [0,1]$, the point $s := \sqrt{t}$ satisfies $f(t) = f(s^2) = S_\varphi(f)(s) = 0$, hence $f(t) \equiv 0$.

On the other hand, S_φ is not an isometry, because $\|S_\varphi\|_{Lip \to Lip} = lip(\varphi) = 2$.

Observe that the function $\varphi(t) = t^2$ in the last example also shows that the injectivity of φ does not imply the surjectivity of S_φ in Lip, since $g(t) := t$ belongs to Lip, but not to the range of S_φ. The explanation for this phenomenon is given in the following

PROPOSITION 3.3. *Let* $\varphi : [0,1] \to [0,1]$ *be Lipschitz continuous. With* S_φ *given by (1) the following is true.*

(a) The function $\varphi : [0,1] \to [0,1]$ *is injective if the operator* $S_\varphi : Lip \to Lip$ *is surjective.*

(b) The operator $S_\varphi : Lip \to Lip$ *is injective if and only if the corresponding function* $\varphi : [0,1] \to [0,1]$ *is surjective.*

(c) The operator $S_\varphi : Lip \to Lip$ *is an isometry if and only if* $\varphi(t) \equiv t$ *or* $\varphi(t) \equiv 1 - t$.

Proof. Part (a) follows from Proposition 1.1 (c), while part (b) follows from Proposition 1.1 (a) and (b). So it only remains to prove part (c).

The fact that the operator S_φ generated by $\varphi(t) = t$ or $\varphi(t) = 1 - t$ is an isometry is trivial. To prove the "only if" part of (c), suppose that S_φ is an isometry, hence injective. From (b) it follows then that φ is surjective.

From $lip(\varphi) = \|S_\varphi\|_{Lip\to Lip} = 1$ it follows that φ is nonexpansive, i.e.,

$$|\varphi(s) - \varphi(t)| \leq |s - t| \qquad (0 \leq s, t \leq 1). \tag{5}$$

This implies that either $\varphi(0) = 0$ or $\varphi(0) = 1$. To see this, suppose that $\varphi(s) = 0$ and $\varphi(t) = 1$ for some $s, t \in (0, 1]$. But then $1 = |\varphi(s) - \varphi(t)| \leq |s - t| < 1$, a contradiction.

Assume first that $\varphi(0) = 0$. Then it follows from (5) that $0 \leq \varphi(t) \leq t$ for $0 \leq t \leq 1$, and the surjectivity of φ implies that $\varphi(1) = 1$. Now, if $\varphi(\tau) < \tau$ for some $\tau \in (0, 1)$, we would obtain

$$1 \geq \frac{\varphi(1) - \varphi(\tau)}{1 - \tau} > \frac{1 - \tau}{1 - \tau} = 1,$$

a contradiction. Consequently, $\varphi(t) = t$ for all $t \in [0, 1]$. Now assume that $\varphi(0) = 1$. Then it follows from (5) that $0 \leq 1 - \varphi(t) \leq t$, hence $1 \geq \varphi(t) \geq 1 - t$ for $0 \leq t \leq 1$, and the surjectivity of φ implies that $\varphi(1) = 0$. Now, if $\varphi(\tau) > 1 - \tau$ for some $\tau \in (0, 1)$, we would obtain

$$1 \geq \frac{\varphi(\tau) - \varphi(1)}{1 - \tau} > \frac{1 - \tau}{1 - \tau} = 1,$$

again a contradiction. Consequently, $\varphi(t) = 1 - t$ for all $t \in [0, 1]$ in this case. □

Proposition 3.3 perfectly explains our choice of φ in Example 3.2: in fact, any surjective function $\varphi \in Lip$ different from $\varphi(t) = t$ or $\varphi(t) = 1 - t$ could serve as an example.

Concerning the composition operator (2), we have the following result which is similar to, but not identical with, Theorem 3.1.

THEOREM 3.4. *The following two assertions are equivalent.*

(a) $C_g(Lip) \subseteq Lip$, i.e., the operator C_g maps Lip into itself.

(b) The function $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, i.e.

$$|g(u) - g(v)| \leq L(r)|u - v| \qquad (u, v \in \mathbb{R},\ |u|, |v| \leq r) \tag{6}$$

for some $L(r) > 0$ depending on $r > 0$.

The following result on the surjectivity and injectivity of C_g in *Lip* is parallel to Proposition 2.5 and follows again readily from Proposition 1.2.

PROPOSITION 3.5. *Let $g \in Lip_{loc}(\mathbb{R})$. With C_g given by (2) the following is true.*

(a) If the operator $C_g : Lip \to Lip$ is surjective then the corresponding function $g : \mathbb{R} \to \mathbb{R}$ is surjective.

(b) The operator $C_g : Lip \to Lip$ is injective if and only if the corresponding function $g : \mathbb{R} \to \mathbb{R}$ is injective.

Now we give the promised example which shows that part (a) in Proposition 2.5 and Proposition 3.5 cannot be inverted.

EXAMPLE 3.6. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(u) := \min\{u+2, |u|\} = \begin{cases} 2+u & \text{for} \quad u < -1, \\ -u & \text{for} \quad -1 \leq u \leq 0, \\ u & \text{for} \quad u > 0. \end{cases}$$

Geometrically, the graph of g consists of three linear pieces with corner points at $(-1, 1)$ and $(0, 0)$. This shows that g is (even globally) Lipschitz continuous on $\mathbb{R}$ with Lipschitz constant 1, so the operator C_g maps C into C and *Lip* into *Lip*. Moreover, g is surjective, but C_g is not, neither in C nor in *Lip*. To see this, note that for fixed $v \in \mathbb{R}$ we have

$$g^{-1}(\{v\}) = \begin{cases} \{v-2\} & \text{for} \quad v < 0, \\ \{v-2, -v, v\} & \text{for} \quad 0 \leq v \leq 1, \\ \{v\} & \text{for} \quad v > 1. \end{cases}$$

The function $h(t) = 3t - 1$ is a Lipschitz continuous homeomorphism between the intervals $[0, 1]$ and $[-1, 2]$. If $f \in C$ or $f \in Lip$ is a function satisfying $C_g(f) = h$, then f must be injective. On the other hand, since $h(0) = -1 < 0$ and $h(1) = 2 > 1$, we have $f(t) = h(t) - 2$ for $0 \leq t \leq 2/3$, but simultaneously $f(t) = h(t)$ for $1/3 \leq t \leq 1$, a contradiction.

It is again not hard to see that the operator C_g is, under the hypotheses of Theorem 3.4, always bounded in *Lip*. Remarkably, C_g need *not* be continuous; this is in contrast to the situation in C.

EXAMPLE 3.7 ([4]). On the space *Lip*, consider the composition operator C_g generated by the function

$$g(u) := \min\{|u|, 1\}.$$

Then C_g is bounded in *Lip*, but discontinuous at $f(t) = t$, because for the sequence $(f_n)_n$ with $f_n(t) := t + 1/n$ we have

$$|C_g(f_n)(1) - C_g(f)(1) - C_g(f_n)(1 - 1/n) + C_g(f)(1 - 1/n)| = \frac{1}{n},$$

hence

$$\|C_g(f_n) - C_g(f)\|_{Lip} \geq \frac{1/n}{1/n} = 1,$$

although $\|f_n - f\|_{Lip} \to 0$ as $n \to \infty$.

It is interesting to note that *the composition operator C_g is continuous in Lip if and only if $g \in C^1$;* the proof can be found in [9]. Example 3.7 shows that a function g which fails to be differentiable at only few points may generate a discontinuous composition operator in *Lip*.

4. Functions of bounded variation

It turns out that in the space BV both operators (1) and (2) behave in a strange way. The first difference with the spaces C and *Lip* we encounter here is that the composition of functions in BV need not stay in BV.

Example 4.1. Define $\varphi : [0, 1] \to [0, 1]$ and $f : [0, 1] \to \mathbb{R}$ by

$$\varphi(t) := \begin{cases} t^2 \sin^2 \dfrac{1}{t} & \text{for} \quad 0 < t \leq 1, \\ 0 & \text{for} \quad t = 0, \end{cases}$$

and $f(s) = \sqrt{s}$, respectively. Being monotone, we have $f \in BV$. Moreover, it is not hard to see that $\varphi \in BV$, since φ' exists and is bounded on $[0, 1]$. However, in every first year calculus course it is taught that $S_\varphi(f) = f \circ \varphi \notin BV$.

The problem of characterizing all functions φ for which $S_\varphi(BV) \subseteq BV$ was completely solved by Josephy in [10] and requires a new definition. Let $\mathcal{J}_n$ ($n \in \mathbb{N}$) denote the class of all functions $\varphi : [0, 1] \to [0, 1]$ with the property that the preimage $\varphi^{-1}(I)$ of any interval $I \subset [0, 1]$ can be written as union of exactly n intervals. We call a function $\varphi : [0, 1] \to [0, 1]$ *pseudomonotone* if $\varphi \in \mathcal{J}_n$ for some n. Clearly, every monotone function is pseudomonotone, since it belongs to $\mathcal{J}_1$, and every pseudomonotone function has bounded variation.

Theorem 4.2 ([10]). *The following two assertions are equivalent.*

(a) $S_\varphi(BV) \subseteq BV$, i.e., the operator S_φ maps BV into itself.

(b) The function $\varphi : [0, 1] \to [0, 1]$ is pseudomonotone.

Note that the function φ in Example 4.1 belongs to BV, but is not pseudomonotone, since

$$\varphi^{-1}(\{0\}) = \left\{0, \tfrac{1}{\pi}, \tfrac{1}{2\pi}, \tfrac{1}{3\pi}, \dots\right\}.$$

This explains why the corresponding operator S_φ does not map the space BV into itself.

In the following proposition we characterize surjectivity and injectivity of $S_\varphi : BV \to BV$ in terms of φ.

Proposition 4.3. *Let $\varphi : [0,1] \to [0,1]$ be pseudomonotone. With S_φ given by (1), the following is true.*

(a) If the operator $S_\varphi : BV \to BV$ is surjective then the corresponding function $\varphi : [0,1] \to [0,1]$ is injective.

(b) The operator $S_\varphi : BV \to BV$ is injective if and only if the corresponding function $\varphi : [0,1] \to [0,1]$ is surjective.

(c) The operator $S_\varphi : BV \to BV$ is an isometry if and only if $\varphi : [0,1] \to [0,1]$ is a homeomorphism with $\varphi(0) = 0$ and $\varphi(1) = 1$.

Proof. Again, part (a) follows from Proposition 1.1 (c), while part (b) follows from Proposition 1.1 (a) and (b). So it only remains to prove part (c). It is clear that a homeomorphism φ with $\varphi(0) = 0$ and $\varphi(1) = 1$ induces an isometric substitution operator, since such a substitution is strictly increasing and can therefore neither destroy nor generate changes in the variation. So suppose that $S_\varphi : BV \to BV$ is an isometry, which means that $\varphi(0) = 0$ and $var(f \circ \varphi) = var(f)$ for all $f \in BV_0 = \{g \in BV \mid g(0) = 0\}$.

For fixed $\tau \in (0,1]$, the function $f_\tau := \chi_{\{\tau\}} \in BV_0$ satisfies

$$var(f_\tau) = \begin{cases} 2 & \text{if} \quad 0 < \tau < 1, \\ 1 & \text{if} \quad \tau = 1. \end{cases}$$

We have $f_\tau \circ \varphi = \chi_{A_\tau}$, where $A_\tau := \{t : 0 \le t \le 1, \varphi(t) = \tau\}$. Since S_φ is an isometry, it follows that

$$var(f_\tau) = var(f_\tau \circ \varphi) = var(\chi_{A_\tau}) = 2\#(A_\tau \cap (0,1)) + \#(A_\tau \cap \{1\}).$$

We distinguish the following two cases:

<u>1st case</u>: $0 < \tau < 1$. In this case we have $\#(A_\tau \cap (0,1)) = 1$ and $\#(A_\tau \cap \{1\}) = 0$, which implies that there is precisely one $t \in (0,1)$ with $\varphi(t) = \tau$, and $\varphi(1) \neq \tau$.

<u>2nd case</u>: $\tau = 1$. In this case we have $\#(A_\tau \cap (0,1)) = 0$ and $\#(A_\tau \cap \{1\}) = 1$, which implies that $\varphi(t) = \tau$ precisely for $t = 1$.

Our reasoning shows that $\varphi : [0,1] \to [0,1]$ is a bijection with $\varphi(0) = 0$ and $\varphi(1) = 1$. Since we cannot expect φ to be continuous, we still have to show that φ is increasing. If this is false we find points t and t' with $0 \le t < t' \le 1$ and $\varphi(t) > \varphi(t')$. For the function $f(t) = t$ we obtain

$$var(\varphi) = var(f \circ \varphi) = \|S_\varphi(f)\|_{BV_0} = \|f\|_{BV_0} = var(f) = 1.$$

On the other hand, for the partition $\{0, t, t', 1\}$ we get

$$\begin{aligned} var(\varphi) &\ge |\varphi(0) - \varphi(t)| + |\varphi(t) - \varphi(t')| + |\varphi(t') - \varphi(1)| \\ &\ge \varphi(t) + \varphi(t) - \varphi(t') + 1 - \varphi(t') > 1. \end{aligned}$$

This contradiction shows that φ is increasing and, being injective, even strictly increasing. □

Now we show by means of a counterexample that the implication in Proposition 4.3 (a) cannot be inverted. To this end, we first need a lemma which shows that there exists a bijective pseudomonotone function whose inverse is not pseudomonotone.

LEMMA 4.4. *There exists a bijective pseudomonotone function* $\varphi : [0,1] \to [0,1]$ *with the property that* $\varphi^{-1} : [0,1] \to [0,1]$ *has unbounded variation.*

Proof. For $n \in \mathbb{N}$ we put

$$t_n := \frac{1}{n}, \quad I_n := (t_{n+1}, t_n], \quad J_n := [-t_n, -t_{n+1}).$$

We define a piecewise linear zigzag function ψ on each interval I_n in such a way that ψ is strictly increasing on I_{2n-1} with $\psi(I_{2n-1}) = I_n$, and strictly decreasing on I_{2n} with $\psi(I_{2n}) = J_n$. Moreover, setting $\psi(0) := 0$, it is not hard to see that $\psi : [0,1] \to [-1,1]$ is a bijection. Since the harmonic series is divergent, the function ψ is not of bounded variation, let alone pseudomonotone.

On the other hand, we show now that $\psi^{-1} : [-1,1] \to [0,1]$ *is* pseudomonotone. Let $I = (a,b) \subset [0,1]$ be an arbitrary interval with $a < b$. Then we can find $m, n \in \mathbb{N}$ with $a \in I_m$ and $b \in I_n$; in particular, $n \le m$. Writing

$$I = (a, t_m] \cup I_{m-1} \cup \ldots \cup I_{n+1} \cup (t_{n+1}, b),$$

we obtain

$$\psi(I) = \psi((a, t_m]) \cup \psi(I_{m-1}) \cup \ldots \cup \psi(I_{n+1}) \cup \psi((t_{n+1}, b)). \tag{7}$$

Suppose without loss of generality that m is odd and n is even, i.e., $m = 2k-1$ and $n = 2l$; the other cases are treated similarly. In this case we have for the first and the last term in (7)

$$\psi((a, t_m]) = \psi((a, t_{2k-1}]) = (\psi(a), \psi(t_{2k-1})] = (\psi(a), t_k]$$

and

$$\psi((t_{n+1}, b)) = \psi((t_{2l+1}, b)) = (\psi(b), -t_{l+1}),$$

while the middle terms in (7) become

$$\begin{aligned}\psi(I_{m-1}) \cup \ldots \cup \psi(I_{n+1}) &= \psi(I_{2k-2}) \cup \ldots \cup \psi(I_{2l+1}) \\ &= \bigcup_{j=l+1}^{k-1} \Big(\psi(I_{2j-1}) \cup \psi(I_{2j})\Big) = \bigcup_{j=l+1}^{k-1} I_j \cup \bigcup_{j=l+1}^{k-1} J_j \\ &= \bigcup_{j=l+1}^{k-1} (t_{j+1}, t_j] \cup \bigcup_{j=l+1}^{k-1} [-t_j, -t_{j+1}) = (t_k, t_{l+1}] \cup [-t_{l+1}, -t_k).\end{aligned}$$

We conclude that

$$\begin{aligned}(\psi^{-1})^{-1}(I) = \psi(I) &= (\psi(a), t_k] \cup (t_k, t_{l+1}] \cup [-t_{l+1}, -t_k) \cup (\psi(b), -t_{l+1}) \\ &= (\psi(a), t_{l+1}) \cup (\psi(b), -t_k)\end{aligned}$$

which shows that the preimage of I under the function ψ^{-1} is the union of two intervals. The same is true in case $I = (a, b]$, $I = [a, b)$, or $I = [a, b]$, as a analogous calculation shows.

Now we define $\varphi : [0, 1] \to [0, 1]$ by $\varphi(s) := \psi^{-1}(2s - 1)$. As before, φ is then a pseudomonotone bijection whose inverse is not pseudomonotone, which proves our claim. □

It is interesting to note that the pseudomonotone function φ in Lemma 4.4 belongs, by construction, to the class $\mathcal{J}_2$, i.e., to the minimal class of pseudomonotone functions which are not monotone. For monotone functions such a construction is not possible. By means of Lemma 4.4 it is now easy to show that the implication in Proposition 4.3 (a) is not an equivalence.

Example 4.5. The operator S_φ defined by the injective function φ from Lemma 4.4 maps BV into itself, since φ is pseudomonotone. However, S_φ is not surjective, because the function $g(t) := t$ is not in the range. In fact, any f with $S_\varphi(f) = g$ would satisfy $f = g \circ \varphi^{-1} = \varphi^{-1}$, and this function does not belong to BV.

Concerning the composition operator (2), we have the following result which is also due to Josephy.

Theorem 4.6 ([10]). *The following two assertions are equivalent.*

(a) $C_g(BV) \subseteq BV$, *i.e., the operator* C_g *maps* BV *into itself.*

(b) The function $g : \mathbb{R} \to \mathbb{R}$ *is locally Lipschitz.*

Before studying analytical properties of the composition operator (2), let us study some mapping properties like injectivity, surjectivity, and bijectivity. It follows from Theorem 4.6 that *the operator C_g is bijective in BV if and only if the function f is bijective and both g and g^{-1} satisfy the local Lipschitz condition* (6) on $\mathbb{R}$. Indeed, this is a direct consequence of the fact that C_g^{-1} (if it exists!) is the composition operator $C_{g^{-1}}$. The following simple example shows that we really need the condition $g^{-1} \in Lip_{loc}(\mathbb{R})$ to ensure the bijectivity of $C_g : BV \to BV$.

Example 4.7. The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(u) := u^3$ is a homeomorphism with $g \in Lip_{loc}(\mathbb{R})$, but $g^{-1} \notin Lip_{loc}(\mathbb{R})$. Clearly, the corresponding composition operator C_g is injective in BV. However, C_g is not surjective. To see this, observe that the function

$$h(t) := \begin{cases} \dfrac{1}{n^3} & \text{for } t = \dfrac{1}{n}, \\ 0 & \text{otherwise} \end{cases}$$

belongs to BV. The only possible preimage f of h is

$$f(t) = \begin{cases} \dfrac{1}{n} & \text{for } t = \dfrac{1}{n}, \\ 0 & \text{otherwise,} \end{cases}$$

which does not belong to BV.

Clearly, Proposition 1.2 also applies to the operator C_g in BV. So we get without any further effort the following

Proposition 4.8. *Let $g \in Lip_{loc}(\mathbb{R})$. With C_g given by (2), the following is true.*

(a) If the operator $C_g : BV \to BV$ is surjective then the corresponding function $g : \mathbb{R} \to \mathbb{R}$ is surjective.

(b) The operator $C_g : BV \to BV$ is injective if and only if the corresponding function $g : \mathbb{R} \to \mathbb{R}$ is injective.

The operator C_g in Example 4.7 is injective, but not surjective in BV. Conversely, there are composition operators in BV which are surjective, but not injective.

Example 4.9 ([14]). Define $g : \mathbb{R} \to \mathbb{R}$ as in Example 3.6. Since g is Lipschitz continuous on $\mathbb{R}$, the operator C_g maps BV into itself, by Theorem 4.6. Moreover, C_g is not injective, which follows from Proposition 4.8 (b) or may be checked directly. However, a somewhat cumbersome calculation shows that $C_g : BV \to BV$ is surjective; for details we refer to [14].

5. A comparison of spaces

In the following three tables we compare what we know about mapping properties of substitution operators in the spaces C, Lip, and BV.

$S_\varphi(C) \subseteq C$	S_φ bounded	S_φ surjective	S_φ injective	$\Leftrightarrow$	S_φ isometry
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$		$\Updownarrow$
φ continuous	φ continuous	φ injective	φ surjective		φ surjective

Table 1: Properties of $S_\varphi : C \to C$

$S_\varphi(Lip) \subseteq Lip$	S_φ bounded	S_φ surj.	S_φ inj.	$\Leftarrow$	S_φ isometry
$\Updownarrow$	$\Updownarrow$	$\Downarrow$	$\Updownarrow$		$\Updownarrow$
φ Lipschitz	φ Lipschitz	φ inj.	φ surj.		$\varphi(t) = t$ or $1 - t$

Table 2: Properties of $S_\varphi : Lip \to Lip$

$S_\varphi(BV) \subseteq BV$	S_φ bounded	S_φ surj.	S_φ inj.	$\Leftarrow$	S_φ isometry
$\Updownarrow$	$\Updownarrow$	$\Downarrow$	$\Updownarrow$		$\Updownarrow$
φ pseudomon.	φ pseudomon.	φ inj.	φ surj.		φ homeom.

Table 3: Properties of $S_\varphi : BV \to BV$

Our tables show that the situation is most satisfactory in the space C, since all conditions are both necessary and sufficient. The down implication in the third column and the updown equivalence in the fourth column of every table is a consequence of Proposition 1.1. The other conditions are in part only sufficient, or only necessary, and our counterexamples show that the implications are not equivalences.

For the isometry property of S_φ we also have necessary and sufficient criteria in terms of φ in all spaces. Interestingly, these criteria are all different in the three tables. To be specific, the function $\varphi(t) = t$ generates an isometric substitution operator in C, Lip, and BV, the function $\varphi(t) = t^2$ only in C and BV, and the function $\varphi(t) = 4t(1-t)$ only in C, see Example 2.2. In the next three tables we compare what we know about mapping properties of composition operators in the spaces C, Lip, and BV. We point out again that for C_g we have to distinguish between boundedness and continuity.

$C_g(C) \subseteq C$	C_g bounded	C_g continuous	C_g surjective	C_g injective
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Downarrow$	$\Updownarrow$
$g \in C$	$g \in C$	$g \in C$	g surjective	g injective

Table 4: Properties of C_g in C

$C_g(C) \subseteq C$	C_g bounded	C_g continuous	C_g surjective	C_g injective
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Downarrow$	$\Updownarrow$
$g \in Lip_{loc}$	$g \in Lip_{loc}$	$g \in C^1$	g surjective	g injective

Table 5: Properties of C_g in Lip

$C_g(C) \subseteq C$	C_g bounded	C_g continuous	C_g surjective	C_g injective
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Downarrow$	$\Updownarrow$
$g \in Lip_{loc}$	$g \in Lip_{loc}$	$g \in Lip_{loc}$	g surjective	g injective

Table 6: Properties of C_g in BV

Our tables above show that the nonlinear composition operator behaves in rather the same way in the three spaces C, Lip, and BV, while the behavior of the linear substitution operator is quite different in these spaces. The operator C_g has the most interesting properties in the space BV which may be summarized for $g \in Lip_{loc}(\mathbb{R})$ and $C_g : BV \to BV$ as follows:

- Injectivity of C_g implies injectivity of g, and vice versa.
- Surjectivity of C_g implies surjectivity of g, but not vice versa.
- There are injective composition operators which are not surjective.
- There are surjective composition operators which are not injective.
- Bijectivity of C_g implies bijectivity of g with $g^{-1} \in Lip_{loc}$, and vice versa.
- Bijectivity of g without $g^{-1} \in Lip_{loc}$ does not imply bijectivity of C_g.

Note that, as far as surjectivity and injectivity are concerned, for the composition operator (2) we do not have the "crossover" between g and C_g, as we have for the substitution operator (1) between φ and S_φ. We also point out that, in sharp contrast with the trivial boundedness problem for C_g in BV, the problem of proving the “automatic” continuity of $C_g : BV \to BV$ was an open problem for many years. This problem has been solved affirmatively only quite recently in [12]; a much simpler elegant proof building on convergence properties of operator sequences may be found in [15]. Apart from the mapping properties of C_g discussed in Table 4 – Table 6, in view of applications it is important to know also some topological properties of C_g. Thus, to apply the Banach-Caccioppoli fixed point principle one has to ensure that C_g satisfies a Lipschitz condition with small Lipschitz constant in the norm of X, while to apply the Schauder fixed point principle one has to impose a compactness condition on C_g. Unfortunately, both conditions for C_g lead to a strong degeneracy of g, as the following table shows. The proofs may be found in the monograph [3].

$C_g : X \to X$	C_g *bounded*	C_g *continuous*	C_g *Lipschitz*	C_g *compact*
$X = C$	always	always	$g \in Lip(\mathbb{R})$	g constant
$X = Lip$	always	not always	g affine	g constant
$X = BV$	always	always	g affine	g constant

Table 7: Topological properties of C_g

Of course, the degeneracy of g reported in the last two columns is quite disappointing: it means, roughly speaking, that one may apply the Banach-Caccioppoli fixed point principle to C_g only if the underlying problem is linear, and the Schauder fixed point principle only if every solution is constant.

6. Baire class one functions

It is well-known that the composition of two functions is Baire class one if one of them is Baire class one and the other is continuous. This implies in our setting that $S_\varphi(\mathcal{B}_1) \subseteq \mathcal{B}_1$ for continuous φ, as well as $C_g(\mathcal{B}_1) \subseteq \mathcal{B}_1$ for continuous g. However, it is not clear at all how far these conditions are from being necessary.

To analyze the situation we show first that the composition of two functions in $\mathcal{B}_1$ need not stay in $\mathcal{B}_1$. To this end, we may use the same functions which is used in every first year calculus course to show that the composition of two Riemann integrable functions need not be Riemann integrable:

EXAMPLE 6.1. Define $\varphi : [0,1] \to [0,1]$ and $f : [0,1] \to \mathbb{R}$ by

$$\varphi(t) := \begin{cases} \dfrac{1}{q} & \text{for } t = \dfrac{p}{q} \in [0,1] \cap \mathbb{Q}, \\ 0 & \text{otherwise,} \end{cases}$$

and $f(s) = \chi_{(0,1]}(s)$, respectively. Then both φ and f are Baire class one, but $f \circ \varphi = \chi_{[0,1]\cap\mathbb{Q}}$ is not.

Our discussion shows that the condition $\varphi \in C$ is too strong, and the condition $\varphi \in \mathcal{B}_1$ is too weak to ensure that $S_\varphi(\mathcal{B}_1) \subseteq \mathcal{B}_1$. The correct class of functions φ has been found in [8] and may be described as follows. A function $\varphi : [0,1] \to [0,1]$ is called *k-continuous* if for every function $\varepsilon : [0,1] \to (0,\infty)$ there exists a function $\delta : [0,1] \to (0,\infty)$ such that $|\sigma - \tau| \le \min\{\delta(\sigma), \delta(\tau)\}$ for $0 \le \sigma, \tau \le 1$ implies $|\varphi(\sigma) - \varphi(\tau)| \le \min\{\varepsilon(\varphi(\sigma)), \varepsilon(\varphi(\tau))\}$.

THEOREM 6.2 ([8]). *The following two assertions are equivalent.*

(a) $S_\varphi(\mathcal{B}_1) \subseteq \mathcal{B}_1$, i.e., the operator S_φ maps $\mathcal{B}_1$ into itself.

(b) The function φ is k-continuous.

Clearly, every continuous function is k-continuous. The converse is not true, as the following simple example shows.

EXAMPLE 6.3. Define $\varphi : [0,1] \to [0,1]$ by $\varphi(t) := \chi_{\{0\}}(t)$. Obviously, φ is not continuous. On the other hand, φ is k-continuous. In fact, it is known that a characteristic function χ_A is k-continuous if and only if A is both F_σ and G_δ, and the set $A = \{0\}$ has this property.

By Theorem 6.2, S_φ maps the space $\mathcal{B}_1$ into itself, although it does not map the space C into itself.

We remark that there exist other necessary and sufficient conditions on φ for the inclusion $S_\varphi(\mathcal{B}_1) \subseteq \mathcal{B}_1$. We recall one such condition as

PROPOSITION 6.4. *The following three assertions are equivalent.*

(a) $S_\varphi(\mathcal{B}_1) \subseteq \mathcal{B}_1$, i.e., the operator S_φ maps $\mathcal{B}_1$ into itself.

(b) For any F_σ-set $F \subseteq [0,1]$, the set $\varphi^{-1}(F)$ is F_σ.

(c) For any closed set $F \subseteq [0,1]$, the set $\varphi^{-1}(F)$ is F_σ.

The implication (a) $\Rightarrow$ (b) is clear: for any F_σ-set F, the function χ_F is Baire class one, so also $\chi_F \circ \varphi$ is Baire class one, and the set $\varphi^{-1}(F) = (\chi_F \circ \varphi)^{-1}(1/2, 3/2)$ is F_σ. The implication (b) $\Rightarrow$ (c) is trivial. The implication (c) $\Rightarrow$ (a), however, is more difficult to prove.

Let us check the criterion (c) in Proposition 6.4 for the function φ from Example 6.3. Given a closed set $F \subseteq [0,1]$, an easy calculation shows that

$$\varphi^{-1}(F) = \begin{cases} [0,1] & \text{if} \quad 0 \in F \text{ and } 1 \in F, \\ (0,1] & \text{if} \quad 0 \in F \text{ but } 1 \notin F, \\ \{0\} & \text{if} \quad 0 \notin F \text{ but } 1 \in F, \\ \emptyset & \text{if} \quad 0 \notin F \text{ and } 1 \notin F, \end{cases}$$

and each of the sets after the curly bracket is an F_σ-set.

Of course, one may also discuss conditions on $\varphi : [0,1] \to [0,1]$ under which the operator S_φ maps the class $\mathcal{B}_2$ of Baire class two functions (i.e., pointwise limits of Baire class one functions) into itself. Here we have the following result which is in a certain sense parallel to Proposition 6.4.

Proposition 6.5. *The following three assertions are equivalent.*

(a) $S_\varphi(\mathcal{B}_2) \subseteq \mathcal{B}_2$, i.e., the operator S_φ maps $\mathcal{B}_2$ into itself.

(b) For any $G_{\delta\sigma}$-set $G \subseteq [0,1]$, the set $\varphi^{-1}(G)$ is $G_{\delta\sigma}$.

(c) For any G_δ-set $G \subseteq [0,1]$, the set $\varphi^{-1}(G)$ is $G_{\delta\sigma}$.

The implication (a) $\Rightarrow$ (b) is again simple: for any $G_{\delta\sigma}$-set G, the function χ_G is Baire class two, so also $\chi_G \circ \varphi$ is Baire class two, and the set $\varphi^{-1}(G) = (\chi_G \circ \varphi)^{-1}(1/2, 3/2)$ is $G_{\delta\sigma}$. As before, the implication (b) $\Rightarrow$ (c) is trivial. Also here the implication (c) $\Rightarrow$ (a) is more difficult to prove.

Since the argument in Proposition 6.4 and Proposition 6.5 are quite similar, one could think that the same functions φ generate substitution operators in $\mathcal{B}_1$ and $\mathcal{B}_2$. However, the next example shows that this is not true.

Example 6.6. Define $\varphi : [0,1] \to [0,1]$ by $\varphi(t) := \chi_M(t)$, where $M := [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$. It is well known that M is G_δ, but not F_σ. Since $\varphi^{-1}(\{1\}) = M$, Proposition 6.4 shows that $S_\varphi(\mathcal{B}_1) \not\subseteq \mathcal{B}_1$.

On the other hand, we apply Proposition 6.5 to show that $S_\varphi(\mathcal{B}_2) \subseteq \mathcal{B}_2$. Given a G_δ-set $G \subseteq [0,1]$, an easy calculation shows that

$$\varphi^{-1}(G) = \begin{cases} [0,1] & \text{if} \quad 0 \in G \text{ and } 1 \in G, \\ [0,1] \cap \mathbb{Q} & \text{if} \quad 0 \in G \text{ but } 1 \notin G, \\ [0,1] \cap (\mathbb{R} \setminus \mathbb{Q}) & \text{if} \quad 0 \notin G \text{ but } 1 \in G, \\ \emptyset & \text{if} \quad 0 \notin G \text{ and } 1 \notin G, \end{cases}$$

and each of the sets after the curly bracket is a $G_{\delta\sigma}$-set.

By Proposition 6.5, S_φ maps the space $\mathcal{B}_2$ into itself, although it does not map the space $\mathcal{B}_1$ into itself.

The following is simply a reformulation of Proposition 1.1 for the space $X = \mathcal{B}_1$.

PROPOSITION 6.7. *Let $\varphi : [0,1] \to [0,1]$ be k-continuous. With S_φ given by (1), the following is true.*

(a) The function $\varphi : [0,1] \to [0,1]$ is injective if the operator $S_\varphi : \mathcal{B}_1 \to \mathcal{B}_1$ is surjective.

(b) The operator $S_\varphi : \mathcal{B}_1 \to \mathcal{B}_1$ is injective if and only if the corresponding function $\varphi : [0,1] \to [0,1]$ is surjective.

Remarkably, in the space $\mathcal{B}_1$ the statement of Proposition 6.7 (a) does not admit a converse, as we will show in Example 6.9 below. This means that in this respect the operator S_φ behaves in $\mathcal{B}_1$ similarly as in BV, but not as in C, although Baire one functions are defined by means of continuous functions. We start with a technical result which is parallel to Lemma 4.4.

LEMMA 6.8. *There exists an injective k-continuous function $\varphi : [0,1] \to [0,1]$ with the property that $\varphi^{-1} : \varphi([0,1]) \to [0,1]$ is not Baire class one.*

Proof. Let

$$D_0 := \{2^{-(n+1)} : n \in \mathbb{N}\}, \qquad D_1 := \{1 - 2^{-(n+1)} : n \in \mathbb{N}\}.$$

Then both $D_0 \subset [0, 1/4]$ and $D_1 \subset [3/4, 1]$ are discrete in $[0,1]$, so also $D := D_0 \cup D_1$ is discrete in $[0,1]$. Furthermore, the sets

$$A_0 := \left\{ \tfrac{k}{2^n} : n, k \in \mathbb{N}, \tfrac{2^n}{2} < k < 2^n \right\}$$

$$= \left\{ \tfrac{3}{4}, \tfrac{5}{8}, \tfrac{6}{8}, \tfrac{7}{8}, \tfrac{9}{16}, \tfrac{10}{16}, \tfrac{11}{16}, \tfrac{12}{16}, \tfrac{13}{16}, \tfrac{14}{16}, \tfrac{15}{16}, \ldots \right\}$$

and

$$A_1 := \left\{ \tfrac{k}{3^n} : n, k \in \mathbb{N}, \tfrac{3^n}{2} < k < 3^n \right\}$$

$$= \left\{ \tfrac{2}{3}, \tfrac{5}{9}, \tfrac{6}{9}, \tfrac{7}{9}, \tfrac{8}{9}, \tfrac{14}{27}, \tfrac{15}{27}, \tfrac{16}{27}, \tfrac{17}{27}, \tfrac{18}{27}, \tfrac{19}{27}, \ldots \right\}$$

are dense in the interval $(1/2, 1)$. Since all four sets are countable, we may find bijective maps $\psi_0 : D_0 \to A_0$ and $\psi_1 : D_1 \to A_1$. Taking in addition $\psi_2(x) := x/3$, we define $\varphi : [0,1] \to [0,1]$ by

$$\varphi(x) := \begin{cases} \psi_0(x) & \text{for} \quad x \in D_0, \\ \psi_1(x) & \text{for} \quad x \in D_1, \\ \psi_2(x) & \text{for} \quad x \in [0,1] \setminus D. \end{cases}$$

Since φ coincides outside the discrete set D with the continuous function ψ_2, we conclude that φ is k-continuous [8]. Observe that $\varphi : [0,1] \to [0,1]$

is not surjective, because $\varphi([0,1])$ is contained in $A_0 \cup A_1 \cup [0,1/3]$ and so omits all irrational numbers in $(1/3,1]$. We claim that φ is injective, but $\varphi^{-1} : \varphi([0,1]) \to [0,1]$ is not Baire class one. Given $x, y \in [0,1]$ with $\varphi(x) = \varphi(y)$, we must have either $x, y \in D_0$ or $x, y \in D_1$ or $x, y \in [0,1] \setminus D$, because the three sets $\varphi(D_0)$, $\varphi(D_1)$ and $\varphi^{-1}([0,1] \setminus D)$ are mutually disjoint. But all functions ψ_0, ψ_1 and ψ_2 are injective, so φ is injective as well.

A well-known characterization of the class $\mathcal{B}_1$ states that a function is Baire class one if and only if the restriction of this function to any closed subset of its domain of definition has at least one continuity point. We use this criterion to show that $\varphi^{-1} \notin \mathcal{B}_1$, being discontinuous on the whole interval $[1/2,1]$.

Fix $s \in [1/2,1]$, and suppose first that $s \in A_0 = \psi_0(D_0)$, hence $\varphi^{-1}(s) \in D_0$. Since D_0 is discrete, we find an $\varepsilon > 0$ such that $[\varphi^{-1}(s) - \varepsilon, \varphi^{-1}(s) + \varepsilon] \cap D_0 = \{\varphi^{-1}(s)\}$. Given $\delta > 0$, there exists $t \in A_0$, hence $\varphi^{-1}(t) \in D_0$, such that $t \neq s$ and $|t - s| \leq \delta$, because A_0 is dense in $[1/2,1]$. But $\varphi^{-1} : A_0 \to D_0$ is injective, so $\varphi^{-1}(t) \neq \varphi^{-1}(s)$ and $|\varphi^{-1}(t) - \varphi^{-1}(s)| > \varepsilon$. We have shown that φ^{-1} is not continuous at s. The argument in case $s \in A_1 = \psi_1(D_1)$ is similar, so it remains to consider the case $s \in [1/2,1] \setminus (A_0 \cup A_1)$.

From $s \in [1/2,1] \setminus (A_0 \cup A_1)$ it follows that $s \notin \varphi([0,1])$. Given $\delta > 0$, there exists $s_0 \in A_0$ and $s_1 \in A_1$, hence $\varphi^{-1}(s_0) \in D_0$ and $\varphi^{-1}(s_1) \in D_1$, such that both $|s - s_0| \leq \delta$ and $|s - s_1| \leq \delta$, again because A_0 and A_1 are dense in $[1/2,1]$. This implies

$$|\varphi^{-1}(s_0) - \varphi^{-1}(s_1)| \geq \operatorname{dist}(D_0, D_1) = \inf\{t_1 - t_0 : t_0 \in D_0, t_1 \in D_1\} = 1/2,$$

hence $|\varphi^{-1}(s_0) - \varphi^{-1}(s)| \geq 1/4$ or $|\varphi^{-1}(s_1) - \varphi^{-1}(s)| \geq 1/4$. Consequently, φ^{-1} is again discontinuous at s, and the claim follows. □

We may use Lemma 6.8 to construct an injective k-continuous function $\varphi : [0,1] \to [0,1]$ with the property that the corresponding operator S_φ is not surjective in $\mathcal{B}_1$.

Example 6.9. The argument is the same as in Example 4.5. The operator S_φ generated by the injective function φ constructed in Lemma 6.8 maps $\mathcal{B}_1$ into itself, since φ is k-continuous. However, S_φ is not surjective, because the function $g(t) := t$ is not in the range. In fact, any f with $S_\varphi(f) = g$ would satisfy $f = g \circ \varphi^{-1} = \varphi^{-1}$, and this function does not belong to $\mathcal{B}_1$, as we have just shown.

Now we are interested in finding conditions on g, possibly both necessary and sufficient, which ensure that C_g maps $\mathcal{B}_1$ into itself. By what we have observed above, continuity of g is sufficient, and Example 6.1 shows that even a harmless function like $g(u) = \chi_{(0,\infty)}(u)$ with only one discontinuity of first kind (jump) may destroy the inclusion $C_g(\mathcal{B}_1) \subseteq \mathcal{B}_1$. However, it could be possible that some function g with a discontinuity of second kind still generates an operator $C_g : \mathcal{B}_1 \to \mathcal{B}_1$. The following theorem shows that this cannot happen.

THEOREM 6.10. *The following two assertions are equivalent.*

(a) $C_g(\mathcal{B}_1) \subseteq \mathcal{B}_1$, *i.e., the operator* C_g *maps* $\mathcal{B}_1$ *into itself.*

(b) The function $g : \mathbb{R} \to \mathbb{R}$ *is continuous.*

Proof. We only have to prove that the inclusion $C_g(\mathcal{B}_1) \subseteq \mathcal{B}_1$ implies the continuity of g. Suppose that g is discontinuous at $u_0 \in \mathbb{R}$. Then we find a sequence $(u_n)_n$ converging to u_0 and an $\varepsilon > 0$ such that $|g(u_n) - g(u_0)| > \varepsilon$ for all n.

We use a characterization of the class $\mathcal{B}_1$ which reads as follows [16, Exercise 11.V]: *For* $n \in \mathbb{N}$, *consider the sets*

$$T_n := \{(2k-1)2^{-n} : k \in \mathbb{N}\},$$

let $(a_n)_n$ *be a sequence in* $\mathbb{R}$, *and let* $a_0 \in \mathbb{R}$. *Define a function* $f : [0,1] \to \mathbb{R}$ *by*

$$f(t) := \begin{cases} a_n & \text{if} \quad t \in T_n, \\ a_0 & \text{if} \quad t \in [0,1] \setminus \bigcup_{n=1}^{\infty} T_n. \end{cases}$$

Then f *is Baire class one if and only if* $a_n \to a_0$ *as* $n \to \infty$.

Now the proof is very easy: applying this result to $a_n := u_n$ and $a_0 := u_0$ shows that $f \in \mathcal{B}_1$, while applying it to $a_n := g(u_n)$ and $a_0 := g(u_0)$ shows that $C_g(f) \notin \mathcal{B}_1$. $\square$

7. Concluding remarks

Another space which frequently occurs in applications is the space C^1 of all continuously differentiable functions $f : [0,1] \to \mathbb{R}$ with norm

$$\|f\|_{C^1} := |f(0)| + \|f'\|_C \qquad (f \in C^1),$$

resp. the space C_0^1 of all $f \in C^1$ with $f(0) = 0$ and norm $\|f\|_{C_0^1} = \|f'\|_C$. Here we have $S_\varphi(C^1) \subseteq C^1$ if and only if $\varphi \in C^1$, and the table for S_φ looks exactly like Table 2.

Likewise, the table for C_g in C^1 is very similar to that for C_g in *Lip*, with the remarkable difference that the operator C_g is not only bounded, but also continuous whenever it maps C^1 into itself.

There is another nonlinear operator which is of utmost importance in nonlinear functional analysis, namely the so-called *Nemytskij operator*

$$N_g(f)(t) := g(t, f(t)) \qquad (0 \leq t \leq 1) \tag{8}$$

generated by some function $g : [0,1]\times\mathbb{R} \to \mathbb{R}$. Thus, the Nemytskij operator (8) is the non-autonomous version of the autonomous composition operator (2), and it is just the "interplay" between the variables t and u of the map $(t,u) \mapsto g(t,u)$ which makes the study of N_g extremely difficult. In particular, there exist many examples which show that the operator N_g behaves in a quite different way than the operator C_g; we restrict ourselves to enumerating in the following list some differences with corresponding references. A self-contained overview of these and many other facts can be found in the monograph [3].

- The condition $C_g(Lip) \subseteq Lip$ holds precisely for locally Lipschitz functions g; the condition $N_g(Lip) \subseteq Lip$ may hold even for discontinuous functions g [5].

- Whenever the operator C_g maps Lip into itself, it is automatically bounded; this is not true for the operator N_g [5].

- The condition $C_g(BV) \subseteq BV$ holds precisely for locally Lipschitz functions g; the condition $N_g(BV) \subseteq BV$ may hold even for discontinuous functions g [12].

- Whenever the operator C_g maps BV into itself, it is automatically bounded; this is not true for the operator N_g [6].

- Whenever the operator C_g maps BV into itself, it is automatically continuous; a continuous superposition operator N_g in BV may even be generated by a discontinuous function [7].

- Whenever the operator C_g maps BV into itself, it is automatically bounded; this is not true for the operator N_g [6].

- Whenever the operator C_g maps BV into itself, it is automatically continuous; this is not true for the operator N_g, even if g is very regular [12].

- Only affine functions g generate globally Lipschitz continuous operators C_g in the BV-norm; this is not true for the operator N_g [13].

- Only constant functions g generate compact operators C_g in the BV-norm; this is not true for the operator N_g [1].

- The condition $C_g(C^1) \subseteq C^1$ holds precisely for continuously differentiable functions g; the condition $N_g(C^1) \subseteq C^1$ may hold even for discontinuous functions g [11].

- Whenever the operator C_g maps C^1 into itself, it is automatically continuous; this is not true for the operator N_g [11].

To the best of our knowledge, the Nemytskij operator N_g has not been studied in the class $\mathcal{B}_1$. To conclude, we give a simple sufficient condition for the inclusion $N_g(\mathcal{B}_1) \subseteq \mathcal{B}_1$ and show then that this condition is not necessary.

PROPOSITION 7.1. *The continuity of $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ implies that the operator N_g maps $\mathcal{B}_1$ into itself.*

Proof. Given $f \in \mathcal{B}_1$, choose a sequence of continuous function $f_n : [0,1] \to \mathbb{R}$ such that $f_n(t) \to f(t)$ for each $t \in [0,1]$. Then the functions $h_n := N_g(f_n)$ are continuous, and $h_n(t) \to h(t) := g(t, f(t))$ for each $t \in [0,1]$ as $n \to \infty$. □

EXAMPLE 7.2. Define $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ by $g(t,u) := \chi_{\{0\}\times\mathbb{Q}}(t,u)$. Then for each $f \in \mathcal{B}_1$ we have $N_g(f)(t) = \chi_{\{0\}}(t)$ if $f(0) \in \mathbb{Q}$, and $N_g(f)(t) \equiv 0$ if $f(0) \notin \mathbb{Q}$. In any case, $N_g(f) \in \mathcal{B}_1$, hence $N_g(\mathcal{B}_1) \subseteq \mathcal{B}_1$. Since the section $g(0,\cdot) : \mathbb{R} \to \mathbb{R}$ is a Dirichlet-type function, g cannot be of Baire class one, let alone continuous.

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